Smooth Feshbach Map and Operator-Theoretic Renormalization Group Methods

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Abstract

A new variant of the isospectral Feshbach map defined on operators in Hilbert space is presented. It is constructed with the help of a *smooth* partition of unity, instead of projections, and is therefore called *smooth Feshbach map*. It is an effective tool in spectral and singular perturbation theory. As an illustration of its power, a novel operator-theoretic renormalization group method is described and applied to analyze a general class of Hamiltonians on Fock space. The main advantage of the new renormalization group method over its predecessors is its technical simplicity, which it owes to the use of the smooth Feshbach map.

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I Introduction

In this paper, we introduce a new mathematical tool in spectral analysis and singular perturbation theory and apply it to develop a simplified version of an operator-theoretic renormalization group method which can be used, for example, to analyze spectra of Hamiltonians on Fock space. Our new tool is a variant of the so-called Feshbach map, or Feshbach projection method, which we call *smooth Feshbach map*, for reasons that will become clear shortly.

The Feshbach projection method was originally introduced as a tool in analytic perturbation theory for operators on Hilbert space [10] (see also e. g. [11, 7, 8]) and has been generalized, in the form of the Feshbach map, to a powerful tool of spectral analysis in [2, 3, 4]; see also [5, 9, 13] for subsequent work.

"State-space decimation" (to a spectral subspace of "low-energy state vectors") is an intrinsic feature of the Feshbach map. It also is a key feature of operator-theoretic renormalization group methods. One might thus expect that the Feshbach map can be used as a central ingredient in the definition of operator-theoretic renormalization maps designed to tackle problems of spectral theory, resonance theory, and singular perturbation theory.

In [2, 3, 4], the Feshbach map has indeed been a key element in the construction of renormalization maps defined on Banach spaces of operators on Fock space which have been used, for example, to prove the existence of a groundstate and of resonances for Hamiltonians describing non-relativistic electrons bound to static nuclei and interacting with the quantized electromagnetic field.

While the Feshbach map used in [2, 3, 4], may be more general and more transparent, conceptually, than the "smooth Feshbach map" introduced in this paper, the latter leads to major technical simplifications in the construction of operator-theoretic renormalization group methods. In order to understand the structure of fixed points of a renormalization map and its linearization in the vicinity of a fixed point, the renormalization map must be shown to depend *differentiably* on the operators which it acts upon. Such differentiability properties are much more easily established if, in the definition of the renormalization map, the *smooth* Feshbach map introduced in this paper is used, because it employs a *smooth* partition of unity on Hilbert space, rather than projections.

To make things more precise, we sketch some spectral problems we are interested in, outline the definition of the smooth Feshbach map, and then explain how it can be used in the study of those spectral problems. Let H be a (generally unbounded) closed operator on a separable Hilbert space $\mathcal{H}$ of the form

$$H \; = \; T \; + \; W \, , \tag{I.1}$$

where T is a closed operator whose spectrum we understand, and W is a perturbation defined on the entire domain of T. We are interested in analyzing the part of the spectrum of the operator H located near 0. In particular, we may be interested in the spectral type near 0, or we may want to know whether 0 is an eigenvalue of H and what its multiplicity is. Problems like this can be studied with the help of the smooth Feshbach map, which we now define. We start by choosing a positive operator χ bounded above by the identity, $\mathbf{1}$, on $\mathcal{H}$, and we set

$$\overline{\chi} \;:=\; \sqrt{1-\chi^2}\;. \tag{I.2}$$

The domain of the operator T in the decomposition (I.1) is then assumed to be invariant under χ and $\overline{\chi}$, and T is assumed to commute with χ and $\overline{\chi}$. We define

$$H_\chi \;:=\; T \,+\, \chi\, W\, \chi \quad \text{and} \quad H_{\overline{\chi}} \;:=\; T \,+\, \overline{\chi}\, W\, \overline{\chi}\,. \tag{I.3}$$

We assume that $H_{\overline{\chi}}$ and its adjoint are bounded invertible on the range of $\overline{\chi}$. Let

$$H_{\overline{\chi}} \;=\; U\,|H_{\overline{\chi}}| \tag{I.4}$$

be the polar decomposition of the operator $H_{\overline{\chi}}$. We further assume that

$$|H_{\overline{\chi}}|^{-1/2} U^{-1}\, \overline{\chi}\, W\, \chi \quad \text{and} \quad \chi\, W \overline{\chi}\, |H_{\overline{\chi}}|^{-1/2} \tag{I.5}$$

extend to bounded operators on $\mathcal{H}$.

A pair of operators, H and T, possessing all these properties is called a *Feshbach pair* (corresponding to χ). We define the *smooth Feshbach map*, F_χ, on Feshbach pairs with values in operators on $\mathcal{H}$ by

$$F_\chi(H,T) \;:=\; T \,+\, \chi\, W\, \chi \,-\, \chi\, W\, \overline{\chi}\, H_{\overline{\chi}}^{-1}\, \overline{\chi}\, W\, \chi\,. \tag{I.6}$$

Notice that the operator $F_\chi(H,T)$ is defined on the domain of the operator T and leaves the range of χ, $\mathrm{Ran}\chi$, invariant. Technical details concerning the definition of Feshbach pairs and of the smooth Feshbach map can be found in Sect. II. The smooth Feshbach map has the following remarkable *isospectrality properties*:

(i) H is bounded invertible on $\mathcal{H}$ iff $F_\chi(H,T)$ is bounded invertible on $\mathrm{Ran}\chi$.

(ii) 0 is an eigenvalue of the operator H with multiplicity $n_0 = 1,2,3,\ldots$ iff 0 is an eigenvalue of $F_\chi(H,T)\big|_{\mathrm{Ran}\chi}$ with the *same* multiplicity n_0.

(iii) If T and H are *selfadjoint operators* on $\mathcal{H}$ then H and $F_\chi(H,T)$ have the same spectral type (e.g., ac, sc, pp) at 0.

For precise statements and proofs, we refer the reader to Sect. II, in particular to Theorem II.1 and its proof.

The usefulness of the Feshbach map F_χ and of Properties (i), (ii) and (iii), above, can be inferred from the following observations:

(1) It may happen that, for a clever choice of χ and T, the spectrum of $F_\chi(H,T)$ is easier to analyze than the spectrum of H. For example, if $\mathrm{Ran}\chi$ is finite-dimensional, the analysis of $F_\chi(H,T)\big|_{\mathrm{Ran}\chi}$ reduces to a *finite-dimensional matrix problem*. This situation arises in the perturbation theory of point spectra and is the realm of the original Feshbach projection method.

(2) Let A be an arbitrary, but fixed selfadjoint operator on $\mathcal{H}$, and let $\tau \equiv \tau_A$ be a map defined on some space, $\mathcal{D}_{\tau_A}$, of closed linear operators, H, on $\mathcal{H}$ into a set $\mathfrak{F}\mathfrak{P} \equiv \mathfrak{F}\mathfrak{P}_A$, of Feshbach pairs, (H,T), where $T \equiv T(A)$ is an operator-valued function of A. To an operator-valued function, $\chi \equiv \chi(A)$, of A satisfying $0 \le \chi(A) \le \mathbf{1}$ we can associate the Feshbach map, $F_{\chi(A)}$, defined on Feshbach pairs $(H, T(A))$. If, for a given operator A and a choice of $\chi \equiv \chi(A)$, $F_{\chi(A)}$ maps $\mathfrak{F}\mathfrak{P}_A$ into $\mathcal{D}_{\tau_A}$ then $\tau \circ F_\chi$ maps $\mathfrak{F}\mathfrak{P}_A$ into $\mathfrak{F}\mathfrak{P}_A$, and hence a second Feshbach map, $F_{\chi'}$, with $0 \le \chi' \equiv \chi'(A) \le \mathbf{1}$, can be applied to $\tau \circ F_\chi$; and so on.

Given a sequence, $(\chi^{(n)} \equiv \chi^{(n)}(A))_{n=0}^\infty$, of operators $\chi^{(n)}(A)$ with the property that $0 \le \chi^{(n)}(A) \le \mathbf{1}$, and such that $F_{\chi^{(n)}}$ maps $\mathfrak{F}\mathfrak{P}_A$ into $\mathcal{D}_{\tau_A}$, for $n = 0, 1, 2, \ldots$, we can recursively define Feshabch pairs by setting $(H^{(0)}, T^{(0)}) := \tau(H)$,

$$(H^{(n)}, T^{(n)}) \;:=\; \tau \circ F_{chi^{(n-1)}}\big(H^{(n-1)}, T^{(n-1)}\big) \,, \tag{I.7}$$

and we define

$$W^{(n)} \;:=\; H^{(n)} \,-\, T^{(n)} \,, \tag{I.8}$$

for $n = 0, 1, 2, \ldots$. It then can and occasionally *does* happen that the relative size of the perturbations $W^{(n)}$, as compared to the operators $T^{(n)}$, decreases in n, so that, in the limit $n \to \infty$, the spectral problem may become trivial. These remarks suggest to use the Feshbach map as a basic ingredient in the construction of an operator-theoretic renormalization map. This is the main topic addressed in Sect. III of this paper. Our renormalization group methods will be illustrated on some typical examples.

The physical systems studied in [2, 3, 4] and in Sect. III of this paper consist of a "small" system with finitely many quantum-mechanical degrees of freedom coupled to a "large" open system with infinitely many degrees of freedom. The small system may be an idealized atom with a static nucleus and electrons treated non-relativistically. To consider the simplest example, we may think of an idealized two-level atom whose quantum-mechanical state space is given by

$$\mathcal{H}_{el} \;=\; \mathbb{C}^2 \,. \tag{I.9}$$

The atomic Hamiltonian is chosen to be

$$H_{el} \;=\; \varepsilon \sigma^3 \,, \tag{I.10}$$

where $\sigma^1, \sigma^2, \sigma^3$ are the usual 2×2 Pauli matrices, and $\varepsilon > 0$ is an atomic energy.

This idealized atom is then coupled to an open, dispersive system with infinitely many quantum-mechanical degrees of freedom. To be specific, we suppose that these degrees of freedom correspond to massless, relativistic bosonic particles, e. g., photons (which are the field quanta of the quantized electromagnetic field). In order to keep our exposition as simple as possible, we neglect the polarization of photons, i. e., we consider "*scalar photons*", or "phonons". The state space of the large system is the *bosonic Fock space*, constructed as follows. Let

$$\mathfrak{h} \;:=\; L^2(\mathbb{R}^3, d^3k) \tag{I.11}$$

denote the Hilbert space of one-particle states of a bosonic particle, i. e., of a "scalar photon" or "phonon". The variable $k \in \mathbb{R}^3$ is the wave vector or momentum of the particle; (throughout this paper, the velocity of light, c, and Planck's constant, $\hbar$, are set equal to 1). Bosonic Fock space, $\mathcal{F}$, over $\mathfrak{h}$ is defined by

$$\mathcal{F} \;:=\; \bigoplus_{n=0}^{\infty} \mathcal{S}_n \, \mathfrak{h}^{\otimes n} \,, \tag{I.12}$$

where $\mathcal{S}_n$ is the orthogonal projection onto the subspace of totally symmetric n-particle wave functions contained in the n-fold tensor product of $\mathfrak{h}$; and $\mathcal{S}_0 \mathfrak{h}^{\otimes 0} := \mathbb{C}\{\Omega\}$, where Ω is the so-called *vacuum vector* in $\mathcal{F}$. Vectors $\Psi \in \mathcal{F}$ can be identified with sequences $(\psi_n)_{n=0}^{\infty}$ of n-particle wave functions, $\psi_n(k_1, \dots, k_n)$, which are totally symmetric in their n arguments, and $\psi_0 \in \mathbb{C}$. The scalar product of two vectors Ψ and Φ is given by

$$\langle \Psi \mid \Phi \rangle \;:=\; \sum_{n=0}^{\infty} \int \prod_{j=1}^{n} d^3k_j \; \overline{\psi_n(k_1, \dots, k_n)} \, \varphi_n(k_1, \dots, k_n) \,. \tag{I.13}$$

With each function $\varphi \in \mathfrak{h}$ one associates an "*annihilation operator*" $a(\varphi)$ defined as follows. For $\Psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F}$ with the property that $\psi_n = 0$, for all but finitely many n, the vector $a(\varphi)\Psi$ is defined to correspond to the sequence $(\eta_n)_{n=0}^{\infty}$ of n-particle wave functions given by

$$\eta_n(k_1, \dots, k_n) \;:=\; \sqrt{n+1} \, \int d^3k \; \overline{\varphi(k)} \, \psi_{n+1}(k, k_1, \dots, k_n) \tag{I.14}$$

and

$$a(\varphi)\Omega \;:=\; 0 \,. \tag{I.15}$$

These equations define a closable operator $a(\varphi)$ whose closure is also denoted by $a(\varphi)$. The creation operator $a^*(\varphi)$ is defined to be the adjoint of $a(\varphi)$ with respect to the scalar product defined in Eq. (I.13). Since $a(\varphi)$ is anti-linear, and $a^*(\varphi)$ is linear in φ, we may write

$$a(\varphi) \;=\; \int d^3k \; \overline{\varphi(k)} \, a(k) \,, \qquad a^*(\varphi) \;=\; \int d^3k \; \varphi(k) \, a^*(k) \,, \tag{I.16}$$

where $a(k)$ and $a^*(k)$ are unbounded, operator-valued distributions. They are well-known to obey the *canonical commutation relations* (CCR):

$$\big[a^\#(k)\,,\,a^\#(k')\big] \;=\; 0\,, \qquad \big[a(k)\,,\,a^*(k')\big] \;=\; \delta^3(k-k')\,, \tag{I.17}$$

where $a^\# = a$ or a^*.

In this paper, we are interested in the dynamics of bosonic particles of *mass zero*. The energy, $\omega(k)$, of such a particle with wave vector k is

$$\omega(k) \;=\; |k|\,. \tag{I.18}$$

The energy of a configuration of n *non-interacting* particles with wave vectors $k_1, \ldots, k_n$ is given by $\sum_{j=1}^n \omega(k_j)$. We define the *free-field Hamiltonian*, H_f, such that the vector $H_f \Psi$, with $\Psi = (\psi_n)_{n=0}^\infty$ ($\psi_n = 0$, except for finitely many n, and $\psi_n(k_1, \ldots, k_n)$ of rapid decrease in $k_1, \ldots, k_n$), corresponds to the sequence $(\varphi_n)_{n=0}^\infty$ of n-particle wave functions given by

$$\varphi_n(k_1, \ldots, k_n) \;=\; \Big(\sum_{j=1}^n \omega(k_j)\Big)\, \psi_n(k_1, \ldots, k_n)\,. \tag{I.19}$$

One verifies that H_f defines a positive, selfadjoint operator on $\mathcal{F}$ with purely absolutely continuous spectrum, except for a simple eigenvalue 0 corresponding to the eigenvector Ω, (the vacuum vector). In terms of creation- and annihilation operators, H_f is given by

$$H_f \;=\; \int d^3k \; a^*(k)\; \omega(k)\; a(k)\,, \tag{I.20}$$

where the right side can be understood as a weak integral.

The state space, $\mathcal{H}$, of the composition of the small and the large system is given by the tensor product of their state spaces, i. e.,

$$\mathcal{H} \;=\; \mathcal{H}_{el} \otimes \mathcal{F}\,. \tag{I.21}$$

As an example of a Hamiltonian, H, for the coupled system we consider

$$H_g \;=\; H_0 \,+\, V_g\,, \tag{I.22}$$

where

$$H_0 \;=\; H_{el} \,+\, H_f\,, \tag{I.23}$$

and the interaction operator is, for simplicity, chosen to be linear in a and a^*,

$$V_g \;=\; g\Big(\int \frac{\kappa(k)\, d^3k}{|k|^{1/2}}\,\big\{\sigma^-\, a^*(k) \,+\, \sigma^+\, a(k)\big\}\,. \tag{I.24}$$

Here, $\sigma^{\pm} = \frac{1}{2}(\sigma^1 \mp i\sigma^2)$ are the usual raising and lowering operators, $\kappa = \kappa(k)$ is a real function with the property that

$$\|\kappa\|_\mu \;:=\; \Big(\int \frac{d^3 k}{|k|^{3+2\mu}} \, |\kappa(k)|^2 \Big)^{1/2} \;<\; \infty \,, \tag{I.25}$$

for some (arbitrarily small, but) strictly positive $\mu > 0$, and $g \geq 0$ is a coupling constant. In the following, we fix $\|\kappa\|_\mu = 1$ and vary g. It is easy to see that, for arbitrary g, the operator V_g is a symmetric operator and small relative to H_0, in the sense of Kato [12], with an arbitrarily small constant. Thus H is self-adjoint on the domain of H_0.

We are interested in properties of the spectrum of H. The spectrum of H_0 is understood completely. It contains two simple eigenvalues, $-\varepsilon$ and ε; $-\varepsilon = \inf \sigma(H_0)$ is at the bottom of a branch of continuous spectrum covering $[-\varepsilon, \infty)$; ε is thus an eigenvalue embedded in absolutely continuous spectrum and is a treshold for a branch of continuous spectrum covering $[\varepsilon, \infty)$. We are interested in the fate of the eigenvalues $-\varepsilon, \varepsilon$ and the nature of the spectrum of H_g when the perturbation V_g is added to H_0. It is clear that standard methods of perturbation theory will not enable us to find solutions to these problems, no matter how small g is. In Sect. III, we illustrate the power of the smooth Feshbach map by solving the above problems with the help of operator-theoretic renormalization group methods based on the smooth Feshbach map.

Let A denote the generator on Fock space of momentum-space dilatations, which can be expressed in terms of creation- and annihilation operators by

$$A \;=\; \frac{i}{2} \int d^3 k \; a^*(k) \, \big\{ k \cdot \nabla_k + \nabla_k \cdot k \big\} \, a(k) \,. \tag{I.26}$$

The operator A is selfadjoint on $\mathcal{F}$. We define dilatation operators Γ_ρ by

$$\Gamma_\rho \;=\; \rho^{iA} \,, \qquad 0 < \rho < \infty \,. \tag{I.27}$$

One verifies that, for a vector $\Psi = (\psi_n)_{n=0}^\infty \in \mathcal{F}$, the image vector $\Gamma_\rho \Psi$ corresponds to the sequence $(\psi_n^\rho)_{n=0}^\infty$ of n-particle wave functions given by

$$\psi_n^\rho(k_1, \ldots, k_n) \;=\; \rho^{3n/2} \; \psi_n(\rho k_1, \ldots, \rho k_n) \,. \tag{I.28}$$

We set

$$H_0(\theta) \;:=\; \Gamma_{e^\theta} \; H_0 \; \Gamma_{e^{-\theta}} \,. \tag{I.29}$$

It is easy to see that $H_0(\theta)$ defines an analytic family in $\theta = r + i\vartheta$ (with $r, \vartheta \in \mathbb{R}$), and that, for $\vartheta \neq 0$, the spectrum of $H_0(r + i\vartheta)$ consists of two *separate* rays of continuous spectrum emanating from $-\varepsilon$ and ε, respectively, at an angle ϑ with respect to the positive real axis.

For concreteness, we choose the "form factor" $\kappa(k)$ in Eq. (I.24) to be given by

$$\kappa(k) \; = \; \pm \mathrm{const} \; |k|^{2\mu} \, e^{-\varepsilon |k|^2} \; , \tag{I.30}$$

for some constant $\varepsilon > 0$. Then

$$V_g(\theta) \; = \; \Gamma_{e^\theta} \; V_g \; \Gamma_{e^{-\theta}} \; , \tag{I.31}$$

is analytic in θ, too, and hence so is the operator $H_g(\theta) = \Gamma_{e^\theta} \, H_g \, \Gamma_{e^{-\theta}}$. We can thus hope to analyze the spectrum of H_g by using complex spectral deformation techniques in conjunction with the Feshbach map; a strategy first proposed in this context in [2, 3, 4]. The problem boils down to exploring the fate of the eigenvalues $-\varepsilon$ and ε of $H_0(r + i\vartheta)$, each attached to a ray of continuous spectrum, when the perturbation $V_g(r + i\vartheta)$ is added.

We are now prepared to describe some *basic results* concerning the spectrum of H_g, the proofs of which will be presented in Sect. III, (except for some rather elementary steps for which we refer to [2, 3, 4]). We assume that the form factor κ in (I.24) is given by (I.30), with $\|\kappa\|_\mu = 1$, and that the coupling constant $g > 0$ is sufficiently small. We then have the following results.

(a) $\varepsilon_\downarrow(g) \; := \; \inf \sigma(H_g)$ is a simple eigenvalue of H_g. There are convergent algorithms to determine $\varepsilon_\downarrow(g)$ and the corresponding eigenvector, $\psi_\downarrow(g)$, of H_g, and

$$\lim_{g \to 0} \varepsilon_\downarrow(g) \; = \; -\varepsilon \; , \qquad \lim_{g \to 0} \psi_\downarrow(g) \; = \; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Omega \, . \tag{I.32}$$

(b) The spectrum of H_g covers $[\varepsilon_\downarrow(g), \infty)$ and is absolutely continuous, except for the eigenvalue $\varepsilon_\downarrow(g)$.

(c) Let Ψ and Φ be vectors in $\mathcal{H}$ which are *entire vectors* for the dilatation generator A introduced in Eq. (I.26). Then the function

$$\big\langle \Psi \; \big| \; (z - H_g)^{-1} \, \Phi \big\rangle \tag{I.33}$$

has an analytic continuation in z from the upper half plane across the cut $[\varepsilon_\downarrow(g), \infty)$ to the lower half plane, where it has a branch-point type singularity at a point $\varepsilon_\uparrow(g)$, with $\mathrm{Im}\varepsilon_\uparrow(g) \; < \; 0$, which does *not* depend on the vectors Ψ and Φ. The interpretation of $\varepsilon_\uparrow(g)$ is of a "*resonance energy*"; it can be determined by a convergent algorithm, and

$$\lim_{g \to 0} \varepsilon_\uparrow(g) \; = \; \varepsilon \; . \tag{I.34}$$

Results (a), (b) and (c) are special cases of results first established in [2, 3]. The point of the present paper ist to describe some *major simplifications* in the proofs arising from the use of the *smooth Feshbach map*, described above, instead of the Feshbach projection method used in [2, 3, 4]. This will form the main contents of Sect. III.

The first step in the proof of (a) is to apply the smooth Feshbach map $F_{\chi_\downarrow}$ to the Feshbach pair

$$\big(H = H_g \, , \; T = H_0 \big) \, , \tag{I.35}$$

where

$$\chi_\downarrow \;:=\; P_\downarrow \otimes \sin\left[\frac{\pi}{2} \, \Theta(H_f) \right] \, , \tag{I.36}$$

$P_\downarrow$ is the orthogonal projection onto the eigenvector $\binom{0}{1}$ of H_{el} corresponding to the eigenvalue $-\varepsilon$, see (I.10), and Θ is a smooth, positive function on the interval $[0, \infty)$, with $0 \leq \Theta \leq 1$, $\Theta \equiv 1$ on $[0, \frac{3}{4}]$, and $\Theta \equiv 0$ on $[1, \infty)$.

Using arguments very similar to ones in [2, 3], see also [1] for some details, one verifies that (H_g, H_0) is indeed a Feshbach pair, in the sense that Properties (I.2)–(I.5) hold, and that $F_{\chi_\downarrow}(H_g, H_0)$, restricted to the subspace $\mathrm{Ran}(P_\downarrow \otimes \mathbf{1}[H_f < 1])$, fits into the general framework developed in Sect. III.1; see Formulas (III.1)–(III.9); (here $\mathbf{1}[x < \rho]$ is the characteristic function of the interval $(-\infty, \rho)$, for any $\rho \in \mathbb{R}$). We shall not present the details of this verification in this paper, but refer the reader to [2, 1].

To start with the proofs of (b) and (c), one applies the smooth Feshbach map $F_{\chi_\uparrow}$ to the Feshbach pair

$$\big(H = H_g(-i\vartheta) \, , \; T = H_0(-i\vartheta) \big) \, , \tag{I.37}$$

for some $\vartheta > 0$, where

$$\chi_\uparrow \;=\; P_\uparrow \otimes \sin\left[\frac{\pi}{2} \, \Theta(H_f) \right] \, , \tag{I.38}$$

and $P_\uparrow$ is the orthogonal projection onto the eigenstate, $\binom{1}{0}$, of H_{el} corresponding to the eigenvalue ε.

Again, one verifies that the resulting operator $F_{\chi_\uparrow}(H_g(-i\vartheta), H_0(-i\vartheta))$ fits into the general framework, Eqs. (III.1)–(III.9), of Sect. III.1.

Operators on Fock space like $F_{\chi_\downarrow}(H_g, H_0)$ and $F_{\chi_\uparrow}(H_g(-i\vartheta), H_0(-i\vartheta))$ will be shown to be good "*initial conditions*" for an operator-theoretic renormalization group analysis presented, in all details, in Sect. III.2

The convergence of the renormalization flow to a fixed-point operator $\propto H_f$ is established in Sect. III.3. Finally, in Sect. III.4, general results on point spectrum and eigenvectors are proven which imply, as special cases, results (a) and (c).

The *emphasis of this paper* is almost entirely on the development of *mathematical tools* and *methods*, all related to the smooth Feshbach map and its use in operator-theoretic renormalization group methods, and *not* on proving optimal results on concrete physical models. Our main concern is to present the smooth Feshbach map, to describe its general properties, and to illustrate its power on a general class of examples of operators on Fock space. In a companion paper [1], we shall apply our new tools to a technically more demanding, concrete field-theoretical model.

II The Smooth Feshbach Map

In this section, we present a new mathematical tool to analyze the spectral properties of a given closed operator H on Hilbert space, which we will refer to as the **smooth Feshbach map**. Its definition involves choosing an auxiliary operator,χ. Given χ, the smooth Feshbach map is a map defined on pairs, (H,T), of closed operators on Hilbert space. The operators (H,T) are referred to as a **Feshbach pair**. Feshbach pairs are of key importance in our analysis. Together with the **Feshbach theorem**, the smooth Feshbach map provides a general method for the spectral analysis of H by spectrally analyzing a related operator that acts on a Hilbert *subspace* which can be conveniently picked. The Feshbach map in its initial form has been successfully used for the construction of an operator-theoretic renormalization group for the rigorous analysis of Hamiltonian quantum field theories in [2, 3].

The smooth Feshbach map is a variant that uses smooth cutoff operators (in the sense of the spectral theorem) instead of projectors. It inherits all the useful properties of the standard Feshbach map, but is, in addition, significantly better behaved with respect to derivatives.

II.1 Feshbach Pairs and the Feshbach theorem

Let χ be a positive operator on a separable Hilbert space $\mathcal{H}$ whose norm is bounded by one, $0 \leq \chi \leq 1$. We assume that χ and $\overline{\chi} := \sqrt{1-\chi^2}$ are nonzero. Let H and T be two closed operators on $\mathcal{H}$ with coinciding domains, $\mathrm{dom}(H) = \mathrm{dom}(T)$, and assume that χ and $\overline{\chi}$ leave $\mathrm{dom}(T)$ invariant and commute with T. We set $W := H - T$ and define

$$W_\chi \;:=\; \chi\, W\, \chi\,, \qquad W_{\overline{\chi}} \;:=\; \overline{\chi}\, W\, \overline{\chi}\,, \tag{II.1}$$

$$H_\chi \;:=\; T \,+\, W_\chi\,, \qquad H_{\overline{\chi}} \;:=\; T \,+\, W_{\overline{\chi}}\,. \tag{II.2}$$

We remark that the domains of W_χ, $W_{\overline{\chi}}$, H_χ, and $H_{\overline{\chi}}$ all contain $\mathrm{dom}(T)$. Furthermore, we assume that $H_{\overline{\chi}}$ is bounded invertible on $\mathrm{Ran}\,\overline{\chi}$ and that $|H_{\overline{\chi}}|^{-1/2} U^{-1}$ $\overline{\chi} W \chi$ and $\chi W \overline{\chi} |H_{\overline{\chi}}|^{-1/2}$ extend to bounded operators on $\mathcal{H}$ and $\mathrm{Ran}\,\overline{\chi}$, respectively,

$$\big\| H_{\overline{\chi}}^{-1} \big\|_{\mathcal{B}[\mathrm{Ran}\,\overline{\chi}]}\,, \quad \big\| |H_{\overline{\chi}}|^{-1/2}\, U^{-1}\, \overline{\chi} W \chi \big\|_{\mathcal{B}[\mathcal{H}]}\,, \quad \big\| \chi W \overline{\chi}\, |H_{\overline{\chi}}|^{-1/2} \big\|_{\mathcal{B}[\mathrm{Ran}\overline{\chi}, \mathcal{H}]} \;<\; \infty\,, \tag{II.3}$$

where $H_{\overline{\chi}} = U |H_{\overline{\chi}}|$ is the polar decomposition on $\mathrm{Ran}\overline{\chi}$. Given χ, any pair (H,T) of operators on $\mathcal{H}$ possessing these properties is called a **Feshbach pair** corresponding to χ.

A typical example of a Feshbach pair is provided by a perturbed Hamiltonian of the form $H_g = T(A) + gW$, where A is a selfadjoint operator on $\mathcal{H}$, $T :=$

$T(A)$ and $\chi := \chi(A)$ are defined by the functional calculus, and $|g| \ll 1$ is a small coupling constant. (In abuse of notation, we denote by T and χ both the operators $T(A)$, $\chi(A)$ on $\mathcal{H}$ and the functions $T, \chi : \mathbb{R} \to \mathbb{C}$.) For a given "spectral parameter" $\lambda \in \mathbb{R}$ and a number $\rho > 0$, the function $\chi \in C^\infty(\mathbb{R}; [0,1])$ is chosen such that $\overline{\chi} := \sqrt{1 - \chi^2}$ is supported in the region where $|T(\cdot) - \lambda| \geq \rho$. Then, clearly, T is invertible on $\mathrm{Ran}\,\overline{\chi}$, and

$$\big\| (T - \lambda)^{-1} \big\|_{\mathrm{Ran}\overline{\chi}} \; = \; \sup \big\{ \, |T(r) - \lambda|^{-1} \; \big| \; r \in \mathrm{supp}\overline{\chi} \big\} \; \leq \; \rho^{-1} \, . \tag{II.4}$$

If the perturbation W is form bounded by $|T(A)|$, and $|g| \ll \rho$ is sufficiently small, then a Neumann series expansion shows that $T + W_{\overline{\chi}}$ is bounded invertible on $\mathrm{Ran}\,\overline{\chi}$, as well, and $|T + W_{\overline{\chi}}|^{-1/2} \overline{\chi} W \chi$ and $\chi W \overline{\chi} |T + W_{\overline{\chi}}|^{-1/2}$ extend to bounded operators on $\mathrm{Ran}\,\overline{\chi}$. Thus, choosing $\chi \equiv \chi(A)$, $(H_g - \lambda, T(A) - \lambda)$ defines a Feshbach pair on $\mathcal{H}$. We remark that the spectral parameter λ affects the construction of the Feshbach pair in this example, as H_g and $T(A)$ are shifted by λ, and the construction of $\chi(A)$ may have a nonlinear dependence on λ. In fact, the smooth Feshbach map is nonlinear in λ.

The **smooth Feshbach map (SFM)** maps Feshbach pairs, (H, T), to operators on $\mathcal{H}$ by

$$(H, T) \; \mapsto \; F_\chi(H, T) \; := \; T \, + \, W_\chi \, - \, \chi W \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi \, . \tag{II.5}$$

We furthermore define

$$Q_\chi(H, T) \;\; := \;\; \chi \, - \, \overline{\chi} \, H_{\overline{\chi}}^{-1} \overline{\chi} W \chi \, , \tag{II.6}$$

$$Q_\chi^\#(H, T) \;\; := \;\; \chi \, - \, \chi W \overline{\chi} \, H_{\overline{\chi}}^{-1} \overline{\chi} \, . \tag{II.7}$$

Note that $Q_\chi(H, T) \in \mathcal{B}(\mathrm{Ran}\,\chi, \mathcal{H})$ and $Q_\chi^\#(H, T) \in \mathcal{B}(\mathcal{H}, \mathrm{Ran}\,\chi)$ define two bounded operators.

The smooth Feshbach map of H is isospectral to H in the sense of the following theorem.

Theorem II.1 (SFM). *Let $0 \leq \chi \leq \mathbf{1}$ and (H, T) be a Feshbach pair corresponding to χ on a separable Hilbert space $\mathcal{H}$. Then we have the following results.*

(i) H is bounded invertible on $\mathcal{H}$ if and only if $F_\chi(H, T)$ is bounded invertible on $\mathrm{Ran}\,\chi$. In this case

$$H^{-1} \;\; = \;\; Q_\chi(H, T) \, F_\chi(H, T)^{-1} \, Q_\chi(H, T)^\# \; + \; \overline{\chi} \, H_{\overline{\chi}}^{-1} \overline{\chi} \, , \tag{II.8}$$

$$F_\chi(H, T)^{-1} \;\; = \;\; \chi \, H^{-1} \, \chi \; + \; \overline{\chi} T^{-1} \overline{\chi} \, . \tag{II.9}$$

(ii) If $\psi \in \mathcal{H} \setminus \{0\}$ solves $H\psi = 0$ then $\varphi := \chi\psi \in \mathrm{Ran}\,\chi \setminus \{0\}$ solves $F_\chi(H, T) \, \varphi = 0$.

(iii) *If* $\varphi \in \mathrm{Ran}\,\chi \setminus \{0\}$ *solves* $F_\chi(H,T)\,\varphi = 0$ *then* $\psi := Q_\chi(H,T)\varphi \in \mathcal{H}\setminus\{0\}$ *solves* $H\psi = 0$.

(iv) *The multiplicity of the spectral value* $\{0\}$ *is conserved in the sense that* $\dim \mathrm{Ker} H = \dim \mathrm{Ker} F_\chi(H,T)$.

(v) *Assume, in addition, that* $H = H^*$ *and* $T = T^*$ *are selfadjoint, and introduce the bounded operators*

$$M \;:=\; H_{\overline{\chi}}^{-1}\,\overline{\chi}\,W\,\chi \qquad \textit{and} \tag{II.10}$$

$$N \;:=\; \big(\mathbf{1} \,+\, M^*M\big)^{-1/2}\,. \tag{II.11}$$

Then, for any $\psi \in \mathcal{H}$,

$$\lim_{\varepsilon\searrow 0}\ \mathrm{Im}\big\langle\psi\big|\,(H - i\varepsilon)^{-1}\,\psi\big\rangle \;= \tag{II.12}$$

$$\lim_{\varepsilon\searrow 0}\ \mathrm{Im}\Big\langle N\,Q_\chi(H,T)^*\,\psi\Big|\,\big(N\,F_\chi(H,T)\,N \,-\, i\varepsilon\big)^{-1}\,N\,Q_\chi(H,T)^*\,\psi\Big\rangle$$

and

$$\lim_{\varepsilon\searrow 0}\ \mathrm{Im}\big\langle\psi\big|\,\big(N\,F_\chi(H,T)\,N \,-\, i\varepsilon\big)^{-1}\,\psi\big\rangle \;= \tag{II.13}$$

$$\lim_{\varepsilon\searrow 0}\ \mathrm{Im}\big\langle\chi\,N^{-1}\,\psi\big|\,(H \,-\, i\varepsilon)^{-1}\,\chi\,N^{-1}\,\psi\Big\rangle\,.$$

Remark II.2. *We remark that the original Feshbach map [2, 3], or Feshbach projection method, is included in the framework of the smooth Feshbach map as the special case in which* $\chi = \chi^2$ *and* $\overline{\chi} = \mathbf{1} - \chi$ *are projections. Namely, given* H *and projections* P, $\overline{P} = \mathbf{1} - P$, *and assuming that* $\overline{P}H\overline{P}$ *is invertible on* $\mathrm{Ran}\overline{P}$, *we set* $T := PHP + \overline{P}H\overline{P}$ *and observe that* $(H, \alpha T)$ *is a Feshbach pair corresponding to* P, *for any* $0 \le \alpha \le 1$. *We then obtain the (original)* Feshbach Map *as*

$$F_P^{(0)}(H) \;:=\; PHP \;-\; PH\overline{P}\big(\overline{P}H\overline{P}\big)^{-1}\overline{P}HP \;=\; P\,F_P(H,\alpha T)\,P\,, \tag{II.14}$$

independently of α.

Proof: Throughout the proof, we use the notation $F := F_\chi(H,T)$, $Q := Q_\chi(H,T)$, and $Q^\# := Q_\chi^\#(H,T)$. We observe the relations

$$H\,\chi \;=\; \chi\,H_\chi \;+\; \overline{\chi}^2\,W\chi\,, \qquad \text{and} \qquad H\,\overline{\chi} \;=\; \overline{\chi}\,H_{\overline{\chi}} \;+\; \chi^2\,W\overline{\chi}\,, \tag{II.15}$$

which follow from $\chi^2 + \overline{\chi}^2 = \mathbf{1}$. The next two identities,

$$H\,Q \;=\; \chi\,F \qquad \text{and} \qquad Q^\#\,H \;=\; F\,\chi\,, \tag{II.16}$$

are of key importance in the proof. They both derive from a simple computation, which we give only for the first equality in (II.16),

$$\begin{aligned} H\,Q \;&=\; H\chi \,-\, H\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}W\chi \\ &=\; \chi H_\chi \,+\, \overline{\chi}^2 W\chi \,-\, \big(\overline{\chi}H_{\overline{\chi}} + \chi^2 W\overline{\chi}\big)\, H_{\overline{\chi}}^{-1}\overline{\chi}W\chi \\ &=\; \chi F\,. \end{aligned} \tag{II.17}$$

Proof of (i). It suffices to check that (II.8) holds if F is bounded invertible on $\mathrm{Ran}\,\chi$ and that (II.9) holds if H is bounded invertible on $\mathcal{H}$.

Thus, suppose first that the restriction of F to $\mathrm{Ran}\,\chi$ is bounded invertible on $\mathrm{Ran}\,\chi$, and define

$$R \;:=\; Q\,F^{-1}\,Q^{\#} \;+\; \overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}\,. \tag{II.18}$$

Using (II.16) and (II.15), we obtain

$$\begin{aligned} H\,R \;&=\; H\,Q\,F^{-1}\,Q^{\#} \;+\; \big(\overline{\chi}H_{\overline{\chi}} + \chi^2 W\overline{\chi}\big)\, H_{\overline{\chi}}^{-1}\overline{\chi} \\ &=\; \chi\,Q^{\#} \;+\; \overline{\chi}^2 \;+\; \chi^2 W\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi} \\ &=\; \chi^2 \;+\; \overline{\chi}^2 \;=\; \mathbf{1}\,, \end{aligned} \tag{II.19}$$

and, similarly, $RH = \mathbf{1}$. Thus $R = H^{-1}$, and (II.8) holds true.

Conversely, suppose that H is bounded invertible, and define

$$\widetilde{R} \;:=\; \chi\,H^{-1}\,\chi \;+\; \overline{\chi}\,T^{-1}\overline{\chi}\,. \tag{II.20}$$

Then, using again (II.15) and (II.16), we obtain

$$\begin{aligned} F\,\widetilde{R} \;&=\; F\,\chi\,H^{-1}\,\chi \;+\; F\,\overline{\chi}\,T^{-1}\overline{\chi} \\ &=\; Q^{\#}\,\chi \;+\; H_\chi\,\overline{\chi}\,T^{-1}\overline{\chi} \;-\; \chi W\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}W\chi\,\overline{\chi}\,T^{-1}\overline{\chi} \\ &=\; \chi^2 \;-\; \chi W\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}\chi \;+\; \overline{\chi}^2 \;+\; W_\chi\overline{\chi}T^{-1}\overline{\chi} \;-\; \chi W\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}W\chi\overline{\chi}\,T^{-1}\overline{\chi} \\ &=\; \mathbf{1} \;-\; \chi W\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}\chi \;+\; W_\chi\overline{\chi}T^{-1}\overline{\chi} \;-\; \chi W\overline{\chi}\,H_{\overline{\chi}}^{-1}\overline{\chi}W\chi\overline{\chi}\,T^{-1}\overline{\chi} \\ &=\; \mathbf{1} \;-\; \chi W\overline{\chi}\big\{H_{\overline{\chi}}^{-1} \;-\; T^{-1} \;+\; H_{\overline{\chi}}^{-1}\overline{\chi}W\overline{\chi}\,T^{-1}\big\}\overline{\chi}\chi \\ &=\; \mathbf{1}\,, \end{aligned} \tag{II.21}$$

where we use the second resolvent identity and the fact that T, χ and $\overline{\chi}$ commute to derive the last equality. Similarly, one checks that $\widetilde{R}F = \mathbf{1}$. Thus F is (globally) invertible on $\mathcal{H}$ with inverse $F^{-1} = \widetilde{R}$. Denoting by P the projection onto $\mathrm{Ran}\,\chi$, we observe that

$$F \;=\; P\,F\,P \;+\; P^{\perp}\,T\,P^{\perp} \tag{II.22}$$

is block-diagonal w. r. t. $\mathcal{H} = \mathrm{Ran}\,\chi \oplus (\mathrm{Ran}\,\chi)^{\perp}$. Hence, the global invertibility of F on $\mathcal{H}$ implies the invertibility of its restriction to $\mathrm{Ran}\,\chi$, proving (II.9).

Proof of (ii). If $\psi \in \mathcal{H} \setminus \{0\}$ solves $H\psi = 0$ then (II.16) implies that

$$F(\chi\psi) \;=\; Q^{\#}\, H\, \psi \;=\; 0\,. \tag{II.23}$$

Furthermore, by (II.15)

$$0 \;=\; \overline{\chi}\, H\, \psi \;=\; H_{\overline{\chi}}\, \overline{\chi}\psi \;+\; \overline{\chi} W \chi^2 \psi\,, \tag{II.24}$$

and hence

$$Q\,\chi\psi \;=\; \chi^2\psi \;-\; \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi^2 \psi \;=\; \chi^2\psi \;+\; \overline{\chi}^2\psi \;=\; \psi\,. \tag{II.25}$$

Therefore, $\psi \neq 0$ implies $\chi\psi \neq 0$.
Proof of (iii). If $\varphi \in \mathrm{Ran}\,\chi \setminus \{0\}$ solves $F\varphi = 0$ then (II.16) implies that

$$H\,Q\varphi \;=\; \chi\, F\varphi \;=\; 0\,. \tag{II.26}$$

Denote by $\overline{P}$ the orthogonal projection onto $\mathrm{Ran}\,\overline{\chi}$, and note, in passing, that P and $\overline{P}$ commute, but $P\overline{P} \neq 0$, unless χ is a projection. Since T is invertible on $\mathrm{Ran}\,\overline{\chi}$, the trivial identity $F = T + \chi W Q$ implies that

$$\overline{P} \;=\; T^{-1}\,\overline{P}\,T \;=\; T^{-1}\,\overline{P}\,(F \;-\; \chi W Q)\,, \tag{II.27}$$

which, together with $\overline{P}^{\perp} = \overline{P}^{\perp} Q$, gives

$$\mathbf{1} \;=\; \overline{P} \;+\; \overline{P}^{\perp} \;=\; T^{-1}\overline{P}F \;+\; \big(\overline{P}^{\perp} \;-\; T^{-1}\overline{P}\chi W\big)\,Q\,. \tag{II.28}$$

Applying (II.28) to φ, we obtain that

$$\varphi \;=\; \big(\overline{P}^{\perp} \;-\; T^{-1}\overline{P}\chi W\big)\,Q\varphi\,, \tag{II.29}$$

which implies that $Q\varphi \neq 0$ provided $\varphi \neq 0$.
Proof of (iv). By (i), $\dim \mathrm{Ker} H = 0$ is equivalent to $\dim \mathrm{Ker} F = 0$, assuming that (H, T) is a Feshbach pair. We may therefore assume that $\mathrm{Ker} H \neq 0$ and $\mathrm{Ker} F \neq 0$ are both nontrivial. Eq. (II.25) shows that $\chi : \mathrm{Ker} H \to \mathrm{Ker} F$ is injective, hence $\dim \mathrm{Ker} H \leq \dim \mathrm{Ker} F$, and Eq. (II.29) shows that $Q : \mathrm{Ker} F \to \mathrm{Ker} H$ is injective, hence $\dim \mathrm{Ker} H \geq \dim \mathrm{Ker} F$. This establishes (iv) and moreover that $\chi : \mathrm{Ker} H \to \mathrm{Ker} F$ and $Q : \mathrm{Ker} F \to \mathrm{Ker} H$ are actually bijections.
Proof of (v). By assumption, $H_{\overline{\chi}} - z$ is bounded invertible on $\mathrm{Ran}\,\overline{\chi}$, for $z = 0$. Thus the first resolvent identity yields the existence of an open neighborhood $U \subset \mathbb{C}$ of 0 and a constant $C < \infty$ such that $\|\overline{\chi}(H_{\overline{\chi}} - z)^{-1}\overline{\chi}\| \leq C$, for all $z \in U$. Therefore, $(H - z, T - z)$ defines a Feshbach pair, for all $z \in U$.

In the proof, we abbreviate $F_z := F_\chi(H-z, T-z)$, $Q := Q_\chi(H,T)$, and we recall that M, N are defined as in (II.10)–(II.11), corresponding to $z = 0$. Our first goal is to show that, for any $\psi \in \mathcal{H}$,

$$\lim_{\varepsilon\searrow 0} \mathrm{Im}\langle\psi|\, F_{i\varepsilon}^{-1}\,\psi\rangle \;=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\langle N\,\psi|\, \big(N\,F_0\,N \,-\, i\varepsilon\big)^{-1}\,N\,\psi\rangle\,. \tag{II.30}$$

To this end, we compute the difference between $F_{i\varepsilon}$ and $F_0 - i\varepsilon$,

$$\begin{aligned} & F_{i\varepsilon} - F_0 + i\varepsilon \\ &= -\chi\,W\,\overline{\chi}\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,\overline{\chi}\,W\,\chi \;+\; \chi\,W\,\overline{\chi}\,H_{\overline{\chi}}^{-1}\,\overline{\chi}\,W\,\chi \\ &= -i\varepsilon\,\chi\,W\,\overline{\chi}\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,H_{\overline{\chi}}^{-1}\,\overline{\chi}\,W\,\chi \\ &= -i\varepsilon\,\chi\,W\,\overline{\chi}\,H_{\overline{\chi}}^{-2}\,\overline{\chi}\,W\,\chi \;+\; \varepsilon^2\,\chi\,W\,\overline{\chi}\,H_{\overline{\chi}}^{-1}(H_{\overline{\chi}} - i\varepsilon)^{-1}\,H_{\overline{\chi}}^{-1}\,\overline{\chi}\,W\,\chi \\ &= -i\varepsilon\,M^*\,M \;+\; \varepsilon^2\,M^*\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,M \\ &= -i\varepsilon\,M^*\,M \;+\; \varepsilon^2\,M^*\,H_{\overline{\chi}}^{-1}\,M \;+\; i\varepsilon^3\,M^*\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,H_{\overline{\chi}}^{-1}\,M\,. \end{aligned} \tag{II.31}$$

Taking into account that $N^{-2} = \mathbf{1} + M^*M$, this is equivalent to

$$F_{i\varepsilon} - F_0 + i\varepsilon N^{-2} \;=\; \varepsilon^2\,M^*\,H_{\overline{\chi}}^{-1}\,M \;+\; i\varepsilon^3\,M^*\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,H_{\overline{\chi}}^{-1}\,M\,. \tag{II.32}$$

Multiplying (II.32) by $F_{i\varepsilon}^{-1}$ from the left and by $(F_0 - i\varepsilon N^{-2})^{-1}$ from the right, we obtain

$$\begin{aligned} F_{i\varepsilon}^{-1} - \big(F_0 - i\varepsilon N^{-2}\big)^{-1} \;&=\; -\varepsilon^2\,F_{i\varepsilon}^{-1}\,M^*\,H_{\overline{\chi}}^{-1}\,M\,\big(F_0 - i\varepsilon N^{-2}\big)^{-1} \\ &\quad + i\varepsilon^3\,F_{i\varepsilon}^{-1}\,M^*\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,H_{\overline{\chi}}^{-1}\,M\,\big(F_0 - i\varepsilon N^{-2}\big)^{-1}\,. \end{aligned} \tag{II.33}$$

Since $\|F_{i\varepsilon}^{-1}\| = \mathcal{O}(\varepsilon^{-1})$ and $\|(F_0 - i\varepsilon N^{-2})^{-1}\| = \mathcal{O}(\varepsilon^{-1})$, as well, we have that

$$\big\|F_{i\varepsilon}^{-1} - \big(F_0 - i\varepsilon N^{-2}\big)^{-1}\big\| \;=\; \mathcal{O}(1) \qquad \text{and} \tag{II.34}$$

$$\big\|i\varepsilon^3\,F_{i\varepsilon}^{-1}\,M^*\,(H_{\overline{\chi}} - i\varepsilon)^{-1}\,H_{\overline{\chi}}^{-1}\,M\,\big(F_0 - i\varepsilon N^{-2}\big)^{-1}\big\| \;=\; \mathcal{O}(\varepsilon)\,. \tag{II.35}$$

Eq. (II.35) implies that the second term on the right side of (II.33) can be neglected, as $\varepsilon \to 0$, i.e.,

$$\begin{aligned} & \lim_{\varepsilon\searrow 0} \mathrm{Im}\Big\langle\psi\Big|\,\big\{F_{i\varepsilon}^{-1} \,-\, (F_0 - i\varepsilon N^{-2})^{-1}\big\}\,\psi\Big\rangle \\ &\quad=\; -\lim_{\varepsilon\searrow 0}\Big\{\varepsilon^2\,\mathrm{Im}\Big\langle\psi\Big|\,F_{i\varepsilon}^{-1}\,M^*\,H_{\overline{\chi}}^{-1}\,M\,\big(F_0 - i\varepsilon N^{-2}\big)^{-1}\,\psi\Big\rangle\Big\}\,. \end{aligned} \tag{II.36}$$

Thanks to (II.34) and the bound $\|(F_0 - i\varepsilon N^{-2})^{-1}\| = \mathcal{O}(\varepsilon^{-1})$, we may further replace $F_{i\varepsilon}^{-1}$ by $(F_0 - i\varepsilon N^{-2})^{-1}$ on the right side of (II.36). That is,

$$\begin{aligned} & \lim_{\varepsilon\searrow 0} \mathrm{Im}\Big\langle\psi\Big|\,\big\{F_{i\varepsilon}^{-1} \,-\, (F_0 - i\varepsilon N^{-2})^{-1}\big\}\,\psi\Big\rangle \\ &= -\lim_{\varepsilon\searrow 0}\Big\{\varepsilon^2\,\mathrm{Im}\Big\langle\psi\Big|\,\big(F_0 - i\varepsilon N^{-2}\big)^{-1}\,M^*\,H_{\overline{\chi}}^{-1}\,M\,\big(F_0 - i\varepsilon N^{-2}\big)^{-1}\,\psi\Big\rangle\Big\} \\ &= \lim_{\varepsilon\searrow 0} \mathrm{Im}\Big\langle M\,N\,\frac{\varepsilon}{NF_0N + i\varepsilon}\,N\,\psi\Big|\,H_{\overline{\chi}}^{-1}\,M\,N\,\frac{\varepsilon}{NF_0N - i\varepsilon}\,N\,\psi\Big\rangle\,. \end{aligned} \tag{II.37}$$

Finally, we use the fact that $\varepsilon(NF_0N \pm i\varepsilon)^{-1}$ converges strongly to $\mp i\, P_{\mathrm{Ker} NF_0N}$, where $P_{\mathrm{Ker} NF_0N}$ denotes the projection onto the kernel of NF_0N. Hence, the limit, as $\varepsilon \searrow 0$, of the matrix element on the right side of (II.37) is real, and we arrive at (II.30), i.e.,

$$\lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle \psi \big|\, F_{i\varepsilon}^{-1}\, \psi \big\rangle \;=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle N\,\psi \big|\, \big(N\, F_0\, N \,-\, i\varepsilon \big)^{-1}\, N\,\psi \big\rangle\,. \tag{II.38}$$

In order to establish Eq. (II.13), we combine (II.30) (or (II.38)) with (II.9) and obtain

$$\begin{aligned} &\lim_{\varepsilon\searrow 0} \mathrm{Im}\Big\langle \psi \Big|\, \big(N\, F_0\, N \,-\, i\varepsilon \big)^{-1}\, \psi \Big\rangle \;=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle N^{-1}\,\psi \big|\, F_{i\varepsilon}^{-1}\, N^{-1}\,\psi \big\rangle \\ &\quad=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle \chi\, N^{-1}\,\psi \big|\, (H - i\varepsilon)^{-1}\, \chi\, N^{-1}\,\psi \big\rangle \;+\; \mathrm{Im}\big\langle \overline{\chi}\, N^{-1}\,\psi \big|\, T^{-1}\, \overline{\chi}\, N^{-1}\,\psi \big\rangle \\ &\quad=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle \chi\, N^{-1}\,\psi \big|\, (H - i\varepsilon)^{-1}\, \chi\, N^{-1}\,\psi \big\rangle\,. \end{aligned} \tag{II.39}$$

The proof of Eq. (II.12) is similar. We use again (II.30) together with (II.8), which yields

$$\begin{aligned} &\lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle\, \psi \big|\, (H - i\varepsilon)^{-1}\, \psi \big\rangle \\ &\quad=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle Q^*\,\psi \big|\, F_{i\varepsilon}^{-1}\, Q^*\,\psi \big\rangle \;+\; \mathrm{Im}\big\langle \overline{\chi}\,\psi \big|\, H_{\overline{\chi}}^{-1}\, \overline{\chi}\,\psi \big\rangle \\ &\quad=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle Q^*\,\psi \big|\, F_{i\varepsilon}^{-1}\, Q^*\,\psi \big\rangle \qquad\qquad \text{(II.40)} \\ &\quad=\; \lim_{\varepsilon\searrow 0} \mathrm{Im}\big\langle N\,Q^*\,\psi \big|\, \big(N\, F_0\, N \,-\, i\varepsilon \big)^{-1}\, N\,Q^*\,\psi \big\rangle\,. \end{aligned}$$

□

II.2 Iterated Applications of the Smooth Feshbach Map

In this section we study iterated applications of the smooth Feshbach map. To motivate our study, we recall from (II.14) that the Feshbach map $F_P^{(0)}(H)$ corresponding to a projection P and an operator H is given by $F_P^{(0)}(H) = PHP - PH\overline{P}(\overline{P}H\overline{P})^{-1}\overline{P}HP$, where $\overline{P} := \mathbf{1} - P$, and we assume that $\overline{P}H\overline{P}$ is invertible on $\mathrm{Ran}\overline{P}$. In this case, Eq. (II.9) is equivalent to

$$F_P^{(0)}(H)^{-1} \;=\; P\, H^{-1}\, P\,, \tag{II.41}$$

provided H is invertible, where $F_P^{(0)}(H)$ is defined in (II.14). This identity immediately implies the following composition property

$$F_P^{(0)} \,\circ\, F_Q^{(0)} \;=\; F_{PQ}^{(0)}\,, \tag{II.42}$$

for commuting projections P and Q, assuming that the Feshbach maps are applicable.

The purpose of this section is to develop a generalization of (II.42) to the smooth Feshbach map. To this end, we consider a pair of commuting, positive operators bounded by unity, $0 \le \chi_1, \chi_2 \le 1$, and the corresponding complementary operators $\overline{\chi}_j := (\mathbf{1} - \chi_j^2)^{1/2}$. Furthermore, we set $\chi_{12} := \chi_1 \chi_2 = \chi_2 \chi_1$ and $\overline{\chi}_{12} := (\mathbf{1} - \chi_{12}^2)^{1/2}$. Thus, we have

$$\chi_1^2 + \overline{\chi}_1^2 \ = \ \chi_2^2 + \overline{\chi}_2^2 \ = \ \chi_{12}^2 + \overline{\chi}_{12}^2 \ = \ \mathbf{1} \, . \tag{II.43}$$

We suppose to be given operators H, T_1, and T_{12}, where T_1 and T_{12} commute with $\chi_1, \chi_2, \overline{\chi}_1, \overline{\chi}_2$ and thus also with $\chi_{12}, \overline{\chi}_{12}$. We assume that H, T_1, and T_{12} have the appropriate invertibility properties such that

$$(H, T_1) \quad \text{and} \quad (H, T_{12}) \tag{II.44}$$

define Feshbach pairs corresponding to χ_1 and χ_{12}, respectively. Denoting $F_1 := F_{\chi_1}(H, T_1)$, we further assume that there is a choice for an operator T_2 (possibly depending on F_1), commuting with χ_2, such that

$$\big(F_1, T_2 \big) \tag{II.45}$$

defines a Feshbach pair corresponding to χ_2. We then have the following theorem.

Theorem II.3. *Assume that (H, T_1), (H, T_{12}) and (F_1, T_2), defined in (II.44)–(II.45) above, are three Feshbach pairs corresponding to χ_1, χ_{12}, and χ_1, respectively, on a separable Hilbert space $\mathcal{H}$. Then*

$$F_{\chi_{12}}(H, T_{12}) \ = \ F_{\chi_2}\big(F_1, T_2 \big) \tag{II.46}$$

if, and only if,

$$T_{12}^{-1} \, \overline{\chi}_{12}^2 \ = \ T_1^{-1} \, \overline{\chi}_1^2 \, \chi_2^2 \ + \ \big(T_2(F_1) \big)^{-1} \, \overline{\chi}_2^2 \, . \tag{II.47}$$

In this case, we have

$$Q_{\chi_{12}}(H, T_{12}) \ = \ Q_{\chi_1}(H, T_1) \, Q_{\chi_2}\big(F_1, T_2 \big) \tag{II.48}$$

$$Q_{\chi_{12}}^{\#}(H, T_{12}) \ = \ Q_{\chi_2}^{\#}\big(F_1, T_2 \big) \, Q_{\chi_1}^{\#}(H, T_1) \, . \tag{II.49}$$

If, in addition, $\chi_1 \, \chi_2 = \chi_2$ holds then Eqs. (II.46) and (II.47) are equivalent to

$$T_{12} \ = \ T_2(F_1) \tag{II.50}$$

on $\mathrm{Ran}\, \overline{\chi}_2$*, and, for any bounded operator A with $A \overline{\chi}_1 = \overline{\chi}_1 A = 0$, we have*

$$A \, Q_{\chi_{12}}(H, T_{12}) \ = \ A \, Q_{\chi_2}\big(F_1, T_2 \big) \quad \textit{and} \quad Q_{\chi_{12}}(H, T_{12}) \ = \ Q_{\chi_2}\big(F_1, T_2 \big) \, . \tag{II.51}$$

Remark II.4. *We observe that $F_P^{(0)}(H)$ is determined by the projection P and the operator H, whereas the second "input parameter" of a Feshbach pair is a* derived *quantity in this case, $T = PHP + \overline{P}H\overline{P}$. In contrast, there is, a priori, no prefered choice for T in the smooth Feshbach map – T is merely required to commute with χ and $\overline{\chi}$. While this yields great flexibility for choosing T, the requirement (II.47) is rather restrictive, and it is not clear, in general, that a choice for T_1, T_{12}, and T_2 fulfilling (II.47) exists, at all. In this sense, the smooth Feshbach map does not possess any natural composition property.*

Proof. We may assume that H is invertible; otherwise, we pass to $H_\varepsilon := U(|H| + \varepsilon\mathbf{1})$, where $H = U\,|H|$ is a polar decomposition of H with U unitary. Clearly, H_ε is invertible, for any $\varepsilon > 0$. Since $H_\varepsilon \to H$ in norm resolvent sense, as $\varepsilon \to 0$,

$$(H_\varepsilon, T_1)\,, \quad (H_\varepsilon, T_{12})\,, \quad (F_1^\varepsilon, T_2) \tag{II.52}$$

define Feshbach pairs corresponding to χ_1, χ_{12}, and χ_2, for $\varepsilon > 0$ sufficiently small, where $F_1^\varepsilon := F_{\chi_1}(H_\varepsilon, T_1)$. Moreover, we have $F_{\chi_1}(H_\varepsilon, T_1) \to F_{\chi_1}(H, T_1)$, $F_{\chi_{12}}(H_\varepsilon, T_{12}) \to F_{\chi_{12}}(H, T_{12})$, and $F_{\chi_2}(F_1^\varepsilon, T_2) \to F_{\chi_2}(F_1, T_2)$, as $\varepsilon \to 0$. Thus, it suffices to prove (II.46) and (II.47) for $\varepsilon > 0$.

Assuming that H is invertible, we may apply (II.9) which yields

$$\begin{aligned} F_{\chi_1}(H, T_1)^{-1} &= F_1^{-1} \;=\; \chi_1\, H^{-1}\, \chi_1 \;+\; T_1^{-1}\overline{\chi}_1^2\,, && \text{(II.53)}\\ F_{\chi_{12}}(H, T_{12})^{-1} &= \chi_{12}\, H^{-1}\, \chi_{12} \;+\; T_{12}^{-1}\overline{\chi}_{12}^2\,, && \text{(II.54)}\\ F_{\chi_2}\big(F_1, T_2\big)^{-1} &= \chi_2\, F_1^{-1}\, \chi_2 \;+\; \big(T_2(F_1)\big)^{-1}\overline{\chi}_2^2 && \text{(II.55)}\\ &= \chi_2\chi_1\, H^{-1}\, \chi_1\chi_2 \;+\; \chi_2 T_1^{-1}\overline{\chi}_1^2\chi_2 \;+\; \big(T_2(F_1)\big)^{-1}\overline{\chi}_2^2 \\ &= \chi_{12}\, H^{-1}\, \chi_{12} \;+\; T_1^{-1}\overline{\chi}_1^2\chi_2^2 \;+\; \big(T_2(F_1)\big)^{-1}\overline{\chi}_2^2\,. \end{aligned}$$

Thus, the difference between $F_{\chi_{12}}(H, T_{12})^{-1}$ and $F_{\chi_2}(F_1, T_2)^{-1}$ vanishes if and only if (II.47) holds true.

To prove Identities (II.48), (II.49) we again assume invertibility of H. We first apply (II.16) to the Feshbach pairs (H, T_1) and (F_1, T_2) and obtain

$$F_1\, Q_{\chi_2}\big(F_1, T_2\big) \;=\; \chi_2\, F_{\chi_2}\big(F_1, T_2\big)\,, \qquad Q_{\chi_1}(H, T_1)\, F_1^{-1} \;=\; H^{-1}\, \chi_1\,. \tag{II.56}$$

We then derive the left identity in (II.48) by applying (II.16) to the Feshbach pair (H, T_{12}) and by using (II.46), which yields

$$\begin{aligned} Q_{\chi_{12}}(H, T_{12}) &= H^{-1}\, \chi_{12}\, F_{\chi_{12}}(H, T_{12}) \;=\; H^{-1}\, \chi_1\, \chi_2\, F_{\chi_2}\big(F_1, T_2\big) \\ &= Q_{\chi_1}(H, T_1)\, Q_{\chi_2}\big(F_1, T_2\big)\,. \end{aligned} \tag{II.57}$$

The proof of (II.49) is analogous, of course; and the remaining Eqs. (II.50) and (II.51) are obvious. $\square$

II.3 Derivations and the Smooth Feshbach Map

In a variety of applications it is necessary to discuss derivatives of the smooth Feshbach map with respect to scalar and operator-valued variables and commutators with respect to operators and operator-valued distributions (see, e.g., [6]). Both are instances of derivations, which are defined as follows.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{D} \subseteq \mathcal{H}$ a dense subspace of $\mathcal{H}$, and denote by $\mathcal{L}(\mathcal{D}, \mathcal{H})$ the vector space of (not necessarily bounded) linear operators from $\mathcal{D}$ to $\mathcal{H}$. A **derivation** δ is a linear map $\mathrm{dom}(\delta) \to \mathcal{L}(\mathcal{D}, \mathcal{H})$, defined on a subspace $\mathrm{dom}(\delta) \subset \mathcal{L}(\mathcal{D}, \mathcal{H})$ which obeys Leibniz' rule. More explicitly, if $A, B \in \mathrm{dom}(\delta)$, $\mathrm{Ran} B \subseteq \mathcal{D}$, $\mathrm{Ran}\delta(B) \subseteq \mathcal{D}$, and $A\,B \in \mathrm{dom}(\delta)$, then

$$\delta(A\,B) \ = \ \delta(A)\,B \ + \ A\,\delta(B)\,. \tag{II.58}$$

Note that Leibniz' rule (II.58) and $\mathbf{1} = \mathbf{1}^2$ implies that $\delta(\mathbf{1}) = 0$. Likewise, if $A \in \mathrm{dom}(\delta)$ is invertible then $\mathbf{1} = AA^{-1}$ implies that $A^{-1} \in \mathrm{dom}(\delta)$ and

$$\delta\big(A^{-1}\big) \ = \ -A^{-1}\,\delta(A)\,A^{-1}\,. \tag{II.59}$$

Let (H, T) be a Feshbach pair corresponding to χ.. We assume that $H, T \in \mathcal{L}(\mathcal{D}, \mathcal{H})$, where $\mathcal{D} := \mathrm{dom}(H) = \mathrm{dom}(T)$ and that $H, T, \chi, \overline{\chi}$ and the composition of operators in the definition of $F_\chi(H, T)$ are all contained in the domain, $\mathrm{dom}(\delta)$, of a given derivation δ. This implies, in particular, that $F_\chi(H, T) \in \mathrm{dom}(\delta)$, and we have the following identities.

Theorem II.5. *For a given Feshbach pair (H, T) corresponding to χ on a separable Hilbert space $\mathcal{H}$ and a derivation δ with the properties specified above, we have that*

$$\begin{aligned} \delta(F) \ &= \ Q^\#\,\delta(W)\,Q \ + \ Q^\#\,W\,\big(\delta(\chi) - \delta(\overline{\chi})\,M\big) \\ &\quad + \big(\delta(\chi) - M^\#\,\delta(\overline{\chi})\big)\,W\,Q \ + \ \delta(T) \ + \ M^\#\,\delta(T)\,M\,, \end{aligned} \tag{II.60}$$

where $F := F_\chi(H, T)$, $Q := Q_\chi(H, T)$, $Q^\# := Q^\#_\chi(H, T)$, $M := H_{\overline{\chi}}^{-1}\overline{\chi}W\chi$ and $M^\# := \chi W \overline{\chi} H_{\overline{\chi}}^{-1}$. In particular, if $\delta(\chi) = \delta(T) = 0$ then

$$\delta(F) \ = \ Q^\#\,\delta(H)\,Q\,. \tag{II.61}$$

Proof. To verify (II.60), we observe that $Q = \chi - \overline{\chi}M$ and $Q^\# = \chi - M^\#\overline{\chi}$. Applying Leibniz' rule, we obtain that

$$\begin{aligned} \delta(F) \ &= \ \delta(T) \ + \ \chi\,\delta(W)\,\chi \ + \ \delta(\chi)\,W\,\chi \ + \ \chi\,W\,\delta(\chi) \\ &\quad - \delta(\chi)\,W\,\overline{\chi}\,M \ - \ \chi\,\delta(W)\,\overline{\chi}\,M \ - \ \chi\,W\,\delta(\overline{\chi})\,M \\ &\quad - M^\#\,\delta(\overline{\chi})\,W\,\chi \ - \ M^\#\,\overline{\chi}\,\delta(W)\,\chi \ - \ M^\#\,\overline{\chi}\,W\,\delta(\chi) \\ &\quad - \chi\,W\,\overline{\chi}\,\delta\big(H_{\overline{\chi}}^{-1}\big)\,\overline{\chi}\,W\,\chi\,. \end{aligned} \tag{II.62}$$

Application of (II.59) to the last line of (II.62) gives

$$- \chi W \overline{\chi} \delta\big(H_{\overline{\chi}}^{-1}\big) \overline{\chi} W \chi \;=\; M^{\#} \delta(T) M \;+\; M^{\#} \overline{\chi} \delta(W) \overline{\chi} M \\ + M^{\#} \delta(\overline{\chi}) W \overline{\chi} M \;+\; M^{\#} \overline{\chi} W \delta(\overline{\chi}) M \,. \tag{II.63}$$

Inserting (II.63) into (II.62), we obtain that

$$\begin{aligned} \delta(F) &= (\chi - M^{\#}\overline{\chi}) \, \delta(W) \, (\chi - \overline{\chi} M) \;+\; (\chi - M^{\#}\overline{\chi}) \, W \, \delta(\chi) \\ &\quad + \delta(\chi) \, W \, (\chi - \overline{\chi} M) \;-\; (\chi - M^{\#}\overline{\chi}) \, W \, \delta(\overline{\chi}) \, M \\ &\quad - M^{\#} \, \delta(\overline{\chi}) \, W \, (\chi - \overline{\chi} M) \;+\; \delta(T) \;+\; M^{\#} \, \delta(T) \, M \\ &= Q^{\#} \, \delta(W) \, Q \;+\; Q^{\#} \, W \, \big(\delta(\chi) - \delta(\overline{\chi}) \, M\big) \\ &\quad + \big(\delta(\chi) - M^{\#} \, \delta(\overline{\chi})\big) \, W \, Q \;+\; \delta(T) \;+\; M^{\#} \, \delta(T) \, M \,, \end{aligned} \tag{II.64}$$

as asserted. Here we use that $\delta(\chi) = 0$ and $\overline{\chi} = (\mathbf{1} - \chi^2)^{1/2}$ implies that $\delta(\overline{\chi}) = 0$, as well. $\square$

Typical examples of derivations are commutators, $\delta_B(A) := [B, A]$, for a fixed operator B, and derivatives with respect to parameters. In particular, if $z \mapsto H(z)$ is an analytic family, and T and χ are independent of z, then (II.61) implies that $F_\chi(H(z), T)$ is analytic, too, because

$$\partial_{\bar z} F_\chi(H(z), T) \;=\; Q_\chi^{\#}(H(z), T) \, \big(\partial_{\bar z} H(z)\big) \, Q_\chi(H(z), T) \;=\; 0 \,, \tag{II.65}$$

in this case. For examples of applications of derivations, we refer the reader to [6]. In a forthcoming publication, derivations play a crucial role in the analysis of the *Nelson model* at total momentum $P_{\rm tot} = 0$.

III The Renormalization Group based on the Smooth Feshbach Map

In this section we develop an operator-theoretic renormalization group (RG) transformation based on the smooth Feshbach map and compare it to the one in [2, 3], which was based on the Feshbach (projection) map. It has been applied to spectral problems of quantum electrodynamics with nonrelativistic matter in [2, 3, 4].

The renormalization transformation is homothetic to an isospectral map defined on a subset of a suitable Banach space of Hamiltonians with a certain contraction property. Its contraction property insures that the iteration of the renormalization transformation converges to a fixed-point (limiting) Hamiltonian, whose spectral analysis is particularly simple. Thanks to the isospectrality of the renormalization map, certain properties of the spectrum of the initial Hamiltonian can be studied by analyzing the limiting Hamiltonian.

In the following sections, we make use of the definitions and notation introduced in Sect. I, Eqs. (I.11)–(I.20).

III.1 A Banach space of Hamiltonians

To define the renormalization transformation we shall first construct a Banach space of Hamiltonians on which it can be defined. Let

$$\mathcal{H}_{\mathrm{red}} \; := \; \mathrm{Ran}\, \mathbf{1}[H_f < 1] \; \equiv \; \mathbf{1}[H_f < 1]\mathcal{F} \; \subseteq \; \mathcal{F} \tag{III.1}$$

be the spectral subspace corresponding to photon energies less than 1. Our goal is to study Hamiltonians $H \in \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ of the form

$$H \; = \; T[H_f] \; + \; W \; - \; E \cdot \mathbf{1} \, , \tag{III.2}$$

where $E \in \mathbb{C}$ is a complex number, $T \in C^1(I)$ is a continuously differentiable function on $I := [0,1]$ with $T[0] = 0$, and $T[H_f] \in \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ is defined by the functional calculus. Moreover, $W = \sum_{m+n \geq 1} W_{m,n}$, with

$$\begin{aligned} W_{m,n} \;\; &\equiv \;\; W_{m,n}[w_{m,n}] \\ &:= \;\; P_{\mathrm{red}} \int_{B_1^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \, a^*(k^{(m)}) \, w_{m,n}\big[H_f; K^{(m,n)}\big] \, a(\tilde{k}^{(n)}) \, P_{\mathrm{red}} \, , \end{aligned} \tag{III.3}$$

where B_1 denotes the unit ball in $\mathbb{R}^d$, and P_{red} is a shorthand for $\mathbf{1}[H_f < 1]$. We use the notation

$$k^{(m)} := (k_1, \ldots, k_m) \in \mathbb{R}^{dm}, \quad \tilde{k}^{(n)} := (\tilde{k}_1, \ldots, \tilde{k}_n) \in \mathbb{R}^{dn}, \tag{III.4}$$

$$K^{(m,n)} := (k^{(m)}, \tilde{k}^{(n)}), \quad dK^{(m,n)} := \prod_{i=1}^m d^d k_i \prod_{i=1}^n d^d \tilde{k}_i, \tag{III.5}$$

$$a^*(k^{(m)}) := \prod_{i=1}^m a^*(k_i), \quad a(\tilde{k}^{(n)}) := \prod_{i=1}^n a(\tilde{k}_i), \tag{III.6}$$

$$|K^{(m,n)}| := |k^{(m)}| \cdot |\tilde{k}^{(n)}|, \quad |k^{(m)}| := |k_1| \cdots |k_m|, \tag{III.7}$$

$$\Sigma[k^{(m)}] := |k_1| + \ldots + |k_m|. \tag{III.8}$$

The function $w_{m,n}[\,\cdot\,, K^{(m,n)}] \in C^1(I)$ is continuously differentiable on I, pointwise, for almost every $K^{(m,n)} \in B_1^{(m+n)}$, its derivative being denoted by $\partial_r w_{m,n}$. As a function of $K^{(m,n)}$, it is totally symmetric w. r. t. the variables $k^{(m)} = (k_1, \ldots, k_m)$ and $\tilde{k}^{(n)} = (\tilde{k}_1, \ldots, \tilde{k}_n)$, and obeys the L^2-norm bound

$$\|w_{m,n}\|_\mu^\# := \|w_{m,n}\|_\mu + \|\partial_r w_{m,n}\|_\mu < \infty, \tag{III.9}$$

where

$$\|w_{m,n}\|_\mu := \left(\int_{B_1^{m+n}} \frac{dK^{(m,n)}}{\left|K^{(m,n)}\right|^{3+2\mu}} \sup_{r \in I} \left| w_{m,n}[r; K^{(m,n)}] \right|^2 \right)^{1/2}, \tag{III.10}$$

for some $\mu > 0$. The Banach space of these functions is denoted by $\mathcal{W}_{m,n}^\#$. Moreover, $W_{m,n}[w_{m,n}]$ stresses the dependence of $W_{m,n}$ on $w_{m,n}$.

The following basic bound links the L^2-norm defined in (III.10) to the operator norm on $\mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ and hence gives an unambiguous meaning to H.

Theorem III.1. *Fix $\mu > 0$ and $m, n \in \mathbb{N}_0$ such that $m + n \geq 1$. Suppose that $w_{m,n} \in \mathcal{W}_{m,n}$, and let $W_{m,n} := W_{m,n}[w_{m,n}]$ be as defined in (III.3). Then on $\mathcal{H}_{\mathrm{red}}$*

$$\left\|W_{m,n}\right\|_{\mathrm{op}} \leq \left\|(P_\Omega^\perp H_f)^{-m/2}\, W_{m,n}\, (P_\Omega^\perp H_f)^{-n/2}\right\|_{\mathrm{op}} \leq \frac{1}{\sqrt{m^m\, n^n}} \|w_{m,n}\|_\mu, \tag{III.11}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on $\mathcal{B}[\mathcal{H}_{\mathrm{red}}]$, $P_\Omega := |\Omega\rangle\langle\Omega|$ is the orthogonal projection onto the vacuum vector in $\mathcal{F}$, and $P_\Omega^\perp := \mathbf{1} - P_\Omega$.

Before turning to the proof of Theorem III.1 we recall a very useful identity, the **Pull-Through Formula** (and its adjoint relation),

$$a(k)\, F[H_f] = F[H_f + |k|]\, a(k) \quad \text{and} \quad F[H_f]\, a^*(k) = a^*(k)\, F[H_f + |k|], \tag{III.12}$$

which holds on $\mathcal{H}_{\mathrm{red}}$ in the sense of operator-valued distributions, for every bounded, measurable function F. The Pull-Through Formula (III.12) has the following consequence.

Lemma III.2. *For $\psi \in \mathcal{H}_{\rm red}$, $m \geq 1$, $\alpha > 0$, and $k^{(m)} in B_1^m$,*

$$\big\| a(k^{(m)})\, \psi \big\| \;\leq\; m^{-\alpha m} \, \big| k^{(m)} \big|^{-\alpha} \, \big\| a(k^{(m)})\, \psi \big\| \, . \tag{III.13}$$

Proof. For arbitrary $\psi = \mathbf{1}[H_f < 1]\psi = \mathbf{1}[H_f < 1]^2\psi$ in $\mathcal{H}_{\rm red}$, the Pull-Through Formula (III.12) yields

$$\begin{aligned} a(k^{(m)})\psi &= a(k^{(m)})\, \mathbf{1}[H_f < 1]\, \psi \;=\; \mathbf{1}\big[H_f + \Sigma[k^{(m)}] < 1 \big]\, a(k^{(m)})\, \psi \\ &= \mathbf{1}\big[\Sigma[k^{(m)}] < 1 \big]\, a(k^{(m)})\, \psi \, , \end{aligned} \tag{III.14}$$

where $\Sigma[k^{(m)}]$ is defined in (III.8). Thus (III.13) follows by showing that, for $a_1, \ldots, a_m \in (0,1]$ with $a_1 + \ldots + a_m \leq 1$, we have that $a_1 \cdots a_m \leq m^{-m}$, with $a_i = |k_i|$. Indeed, this inequality is equivalent to $\ln(1/a_1) + \ldots + \ln(1/a_m) \geq m \, \ln(m)$, and we may assume that $a_1 + \ldots + a_m = 1$. This inequality is a consequence of Jensen's inequality applied to the convex function $F(x) := x \, \ln(x)$; more explicitly

$$\sum_{j=1}^m \ln\big(a_j^{-1} \big) \;=\; \sum_{j=1}^m a_j F\big(a_j^{-1} \big) \;\geq\; F\Big(\sum_{j=1}^m a_j \, a_j^{-1} \Big) \;=\; F(m) \;=\; m \, \ln(m) \, . \tag{III.15}$$

□

Proof of Theorem III.1: For $\psi, \varphi \in \mathcal{H}_{\rm red}$, we estimate the matrix element $\langle \psi |\, W_{m,n} \varphi \rangle$ by means of the Cauchy-Schwarz inequality and Lemma III.2 with $\alpha := (1/2) + \mu$,

$$\begin{aligned} |\langle \psi |\, W_{m,n}\, \varphi \rangle| \;&=\; \bigg| \int_{B_1^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \, \big\langle a(k^{(m)})\, \psi \,\big|\, w_{m,n}\big[H_f; K^{(m,n)}\big] \, a(\tilde{k}^{(n)})\, \varphi \big\rangle \bigg| \\ \leq&\; \int_{B_1^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \, \sup_{r \in I} \big| w_{m,n}[r; K^{(m,n)}] \big| \, \big\| a(k^{(m)})\, \psi \big\| \, \big\| a(\tilde{k}^{(n)})\, \varphi \big\| \qquad \text{(III.16)} \\ \leq&\; \frac{1}{\sqrt{m^m \, n^n}} \int_{B_1^{m+n}} dK^{(m,n)} \, \frac{\sup_{r \in I} \big| w_{m,n}[r; K^{(m,n)}] \big|}{|K^{(m,n)}|^{1+\mu}} \, \big\| a(k^{(m)})\, \psi \big\| \, \big\| a(\tilde{k}^{(n)})\, \varphi \big\| \\ \leq&\; \frac{D_m(\psi)^{1/2} \, D_n(\varphi)^{1/2}}{\sqrt{m^m \, n^n}} \bigg(\int_{B_1^{m+n}} \frac{dK^{(m,n)}}{\big| K^{(m,n)} \big|^{3+2\mu}} \, \sup_{r \in I} \big| w_{m,n}[r; K^{(m,n)}] \big|^2 \bigg)^{1/2} \, , \end{aligned}$$

where

$$D_m(\psi) \;:=\; \int_{B_1^m} dk^{(m)} \, \big| k^{(m)} \big| \, \big\| a(k^{(m)})\, \psi \big\|^2 \, . \tag{III.17}$$

Clearly, $D_0(\psi) = \|\psi\|^2$, and we observe that

$$D_m(\psi) \;:=\; \int_{B_1^{m-1}} dk^{(m-1)} \, \big| k^{(m-1)} \big| \, \big\| H_f^{1/2} \, a(k^{(m-1)})\, P_\Omega^\perp \psi \big\|^2 \, , \tag{III.18}$$

for $m \geq 1$. The Pull-Through Formula (III.12) implies that

$$\big\| H_f^{1/2}\, a(k^{(m-1)})\, P_\Omega^\perp \psi \big\| \;=\; \big\| H_f^{1/2}\, a(k^{(m-1)})\, H_f^{-1/2}\, H_f^{1/2}\, P_\Omega^\perp \psi \big\| \tag{III.19}$$

$$= \;\; \bigg\| \Big(\frac{H_f}{H_f + \Sigma[k^{(m-1)}]} \Big)^{1/2} a(k^{(m-1)})\, H_f^{1/2}\, P_\Omega^\perp \psi \bigg\| \;\leq\; \big\| a(k^{(m-1)})\, H_f^{1/2}\, P_\Omega^\perp \psi \big\| \, .$$

Hence, for $m \geq 1$, we have

$$D_m(\psi) \;\leq\; D_{m-1}(H_f^{1/2} P_\Omega^\perp \psi) \;\leq\; \cdots \;\leq\; D_0(H_f^{m/2} P_\Omega^\perp \psi) \;=\; \big\| (H_f P_\Omega^\perp)^{m/2} \psi \big\|^2 \, , \tag{III.20}$$

where the last expression is an upper bound when $m = 0$ and $m \geq 1$. Inserting this into (III.16), we obtain

$$|\langle \psi | \, W_{m,n}\, \varphi \rangle| \;\leq\; m^{-m/2}\, n^{-n/2}\, \|w_{m,n}\|_\mu \, \big\| (H_f P_\Omega^\perp)^{m/2} \psi \big\| \, \big\| (H_f P_\Omega^\perp)^{n/2} \varphi \big\| \, , \tag{III.21}$$

from which the right inequality in (III.11) follows immediately. Using, moreover, that $0 \leq H_f \leq 1$ on $\mathcal{H}_{\rm red}$, the left inequality in (III.11) is also a direct consequence of (III.21). □

Theorem III.1 says that the finiteness of $\|w_{m,n}\|_\mu$ insures that $W_{m,n}$ defines a bounded operator on $\mathcal{B}[\mathcal{H}_{\rm red}]$. In order to be able to add up all components $W_{m,n}$ of W, we fix two numbers $\mu > 0$, $0 < \xi < 1$, and define the Banach space

$$\mathcal{W}_{\geq 1}^{\#} \;:=\; \bigoplus_{m+n \geq 1} \mathcal{W}_{m,n}^{\#} \, , \tag{III.22}$$

to consist of all sequences $\underline{w} := (w_{m,n})_{m+n \geq 1}$ obeying

$$\|\underline{w}\|_{\mu,\xi}^{\#} \;:=\; \sum_{m+n \geq 1} \xi^{-(m+n)} \, \|w_{m,n}\|_\mu^{\#} \;<\; \infty \, . \tag{III.23}$$

Note that $\mathcal{W}_{\geq 1}^{\#}$ depends on the two parameters $\mu > 0$ and $0 < \xi < 1$.

Next we define the Banach spaces

$$\mathcal{T} \;\;:=\;\; \Big\{ f \in C^1(I) \;\Big|\; f(0) = 0 \, , \; \|f\|_{\mathcal{T}} := \sup_{r \in I} |f'(r)| < \infty \Big\} \, , \tag{III.24}$$

$$\mathcal{W}_{0,0}^{\#} \;\;:=\;\; \Big\{ w_{0,0} \in C^1(I) \;\Big|\; \|w_{0,0}\|_\mu^{\#} := \|w_{0,0}\|_\mu + \|\partial_r w_{0,0}\|_\mu < \infty \Big\}, \tag{III.25}$$

where $\|w_{0,0}\|_\mu := \sup_{r \in I} |w_{0,0}[r]|$. We observe that there is a natural bijection

$$\mathcal{W}_{0,0}^{\#} \;\to\; \mathbb{C} \oplus \mathcal{T} \, , \qquad w_{0,0} \;\mapsto\; w_{0,0}[0] \oplus (w_{0,0} - w_{0,0}[0]) \, . \tag{III.26}$$

We shall henceforth not distinguish between $\mathcal{W}_{0,0}^{\#}$ and $\mathbb{C} \oplus \mathcal{T}$. We write $W_{0,0}[w_{0,0}] := w_{0,0}[H_f] \in \mathcal{B}[\mathcal{H}_{\rm red}]$.

Defining

$$\mathcal{W}^{\#}_{\geq 0} \;:=\; \bigoplus_{m+n\geq 0} \mathcal{W}^{\#}_{m,n} \;=\; \mathbb{C} \,\oplus\, \mathcal{T} \,\oplus\, \bigoplus_{m+n\geq 1} \mathcal{W}^{\#}_{m,n}\,, \tag{III.27}$$

with the norm

$$\big\| \underline{w} \big\|^{\#}_{\mu,\xi} \;:=\; \sum_{m+n\geq 0} \xi^{-(m+n)} \, \| w_{m,n} \|^{\#}_{\mu} \;<\; \infty\,, \tag{III.28}$$

we observe that any sequence $\underline{w} := (w_{m,n})_{m+n\geq 0}$ in $\mathcal{W}^{\#}_{\geq 0}$ defines an operator of the form (III.2) on $\mathcal{H}_{\mathrm{red}}$ by setting

$$\begin{aligned} H(\underline{w}) \;&:=\; \sum_{m+n\geq 0} W_{m,n}[\underline{w}] \qquad\qquad \text{(III.29)} \\ &=\; \big(w_{0,0}[H_f] - w_{0,0}[0] \big) \;+\; w_{0,0}[0]\cdot \mathbf{1} \;+\; \sum_{m+n\geq 1} W_{m,n}[\underline{w}]\,, \end{aligned}$$

where we write $W_{m,n}[\underline{w}] := W_{m,n}[w_{m,n}]$. Next, we derive a corollary of Theorem III.1 which gives meaning to (III.29) and shows that $H : \mathcal{W}^{\#}_{m+n\geq 0} \to \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ is an embedding. Furthermore, we demonstrate that this embedding is injective.

Theorem III.3. *For any $\mu > 0$ and $0 < \xi < 1$, the map $H : \mathcal{W}^{\#}_{\geq 0} \to \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ defines an injective embedding, the subspace $H(\mathcal{W}^{\#}_{\geq 0}) \subseteq \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ is closed, and we have the operator norm estimates*

$$\big\| \, H(\underline{w}) \, \big\|_{\mathrm{op}} \;\leq\; \big\| \, \underline{w} \, \big\|^{\#}_{\mu,\xi}\,, \tag{III.30}$$

for any $\underline{w} = (w_{m,n})_{m+n\geq 0} \in \mathcal{W}^{\#}_{\geq 0}$, and

$$\big\| \, H(\underline{w}) \, \big\|_{\mathrm{op}} \;\leq\; \xi \, \big\| \, \underline{w} \, \big\|^{\#}_{\mu,\xi}\,, \tag{III.31}$$

for any $\underline{w} = (w_{m,n})_{m+n\geq 1} \in \mathcal{W}^{\#}_{\geq 1}$.

Proof. The estimates (III.30) and (III.31) follow directly from Theorem III.1, using that $0 < \xi < 1$. Eq. (III.30) immediately implies the closedness of $H(\mathcal{W}^{\#}_{\geq 0}) \subseteq \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$,

The injectivity of H is more subtle. We observe that by linearity, it suffices to show that if $H(\underline{w}) = 0$ then $\underline{w} = 0$.

We first prove that if $H(\underline{w}) = 0$ then $w_{0,0} = 0$. To this end, we pick a function $f \in C_0^\infty[B_1; \mathbb{R}^+]$, with $\int |f(x)|^2 \, d^d x = 1$, and define $f_{\varepsilon,k}(x) := \varepsilon^{-d/2} f\big(\varepsilon^{-1}(x -$

$k)\big)$. A simple computation gives

$$\begin{aligned}
&\big\langle a^*(f_{\varepsilon,k})\Omega \big|\, H(\underline{w})\; a^*(f_{\varepsilon,k})\Omega \big\rangle \\
&= \big\langle a^*(f_{\varepsilon,k})\Omega \big|\, w_{0,0}[H_f]\; a^*(f_{\varepsilon,k})\Omega \big\rangle \;+\; \big\langle a^*(f_{\varepsilon,k})\Omega \big|\, W_{1,1}[w_{1,1}]\; a^*(f_{\varepsilon,k})\Omega \big\rangle \\
&= \int_{B_1} d^d x |f_{\varepsilon,k}(x)|^2 \; w_{0,0}[\omega(x)] \qquad \text{(III.32)} \\
&\quad + \int_{B_1^2} \frac{d^d x \, d^d x'}{|x|^{1/2}\,|x'|^{1/2}} \; \overline{f_{\varepsilon,k}(x)}\, f_{\varepsilon,k}(x') \; w_{1,1}[0; x, x'] \, .
\end{aligned}$$

The second term in the last line of (III.32) tends to 0, because $f_{\varepsilon,k} \to 0$ weakly in $L^2(B_1)$, and the first term tends to $w_{0,0}[\omega(k)]$ since $|f_{\varepsilon,k}(x)|^2 \to \delta(x-k)$, as $\varepsilon \to 0$. By assumption, $H(\underline{w}) = 0$, and we obtain

$$w_{0,0}[\omega(k)] \;=\; \lim_{\varepsilon \to 0} \big\langle a^*(f_{\varepsilon,k})\Omega \big|\, H(\underline{w})\; a^*(f_{\varepsilon,k})\Omega \big\rangle \;=\; 0 \, , \qquad \text{(III.33)}$$

for any $k \in B_1$.

To proceed in the proof, we use induction in $J := m + n \geq 1$. Although technically somewhat more involved, the idea for the proof is the same as for $m = n = 0$. We prove that $H(\underline{w}) = 0$ and $w_{m,n} = 0$, for all $(m,n) \in \mathbb{N}_0^2$ with $m + n \leq J - 1$, imply $w_{m,n} = 0$, for all $(m,n) \in \mathbb{N}_0^2$ with $m + n = J$. Thus, fix (m,n) with $m + n = J$, pick $g_1, \ldots, g_m, h_1, \ldots, h_n \in L^2(B_1)$ such that $\int |g_i(k)|^2 \, |k|^{-3-2\mu} \, d^d k \;=\; \int |h_j(k)|^2 \, |k|^{-3-2\mu} \, d^d k \;=\; 1$. We set $\underline{g} := g_1, \ldots g_m$, $\underline{h} := h_1, \ldots h_n$ and furthermore

$$\Psi_{\underline{g},\varepsilon,k} \;:=\; a^*(g_1) \cdots a^*(g_m)\, a^*(f_{\varepsilon,k})\, \Omega \;\in\; \mathcal{F}^{(m+1)} \, , \qquad \text{(III.34)}$$

$$\Phi_{\underline{h},\varepsilon,k} \;:=\; a^*(h_1) \cdots a^*(h_n)\, a^*(f_{\varepsilon,k})\, \Omega \;\in\; \mathcal{F}^{(n+1)} \, . \qquad \text{(III.35)}$$

Then we observe that $\langle \Psi_{\underline{g},\varepsilon,k} \mid W_{m+i,n+j}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \rangle$ vanishes, unless the following three conditions are fulfilled,

$$i \leq 1 \, , \qquad j \leq 1 \, , \quad \text{and} \quad i = j \, . \qquad \text{(III.36)}$$

Moreover, $w_{m+i,n+j} = 0$, unless $i + j \geq 0$, by the induction hypothesis. Thus

$$\begin{aligned}
\big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, H(\underline{w})\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \;&=\; \Big\langle \Psi_{\underline{g},\varepsilon,k} \,\Big|\, \sum_{i+j \geq 0} W_{m+i,n+j}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \Big\rangle \qquad \text{(III.37)} \\
&=\; \big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, W_{m+1,n+1}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \;+\; \big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, W_{m,n}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \, .
\end{aligned}$$

To analyse the first term in the sum in the last line of (III.37), we note that

$$\begin{aligned}
&\big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, W_{m+1,n+1}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \;=\; (m+1)!\,(n+1)! \qquad \text{(III.38)} \\
&\int_{B_1^{m+n+2}} \frac{dX^{(m+1,n+1)}}{|X^{(m+1,n+1)}|^{1/2}} \; w_{m+1,n+1}[0;\; X^{(m+1,n+1)}] \, \big(\Psi_{\underline{g},\varepsilon,k} \otimes \Phi_{\underline{h},\varepsilon,k}\big)[X^{(m+1,n+1)}] \, ,
\end{aligned}$$

where

$$\big(\Psi_{\underline{g},\varepsilon,k} \otimes \Phi_{\underline{h},\varepsilon,k}\big)[X^{(m+1,n+1)}] \ := \qquad \text{(III.39)}$$
$$g_1(x_1)\cdots g_m(x_m)\, f_{\varepsilon,k}(x_{m+1})\, h_1(\tilde{x}_1)\cdots h_n(\tilde{x}_n)\, f_{\varepsilon,k}(\tilde{x}_{n+1})\,.$$

Since $\Psi_{\underline{g},\varepsilon,k} \otimes \Phi_{\underline{h},\varepsilon,k} \to 0$ weakly in $L^2(B_1^{m+n+2})$, as $\varepsilon \to 0$, we have that

$$\lim_{\varepsilon\to 0} \big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, W_{m+1,n+1}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \ = \ 0\,. \qquad \text{(III.40)}$$

The second term in the sum in the last line of (III.37) equals

$$\big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, W_{m,n}[\underline{w}]\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \ = \ m!\, n! \Big\{ A(\varepsilon) + \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} B_{\mu\nu}(\varepsilon) \Big\}\,, \qquad \text{(III.41)}$$

where

$$A(\varepsilon) \ := \ \int_{B_1^{m+n+1}} \frac{dX^{(m,n)}}{|X^{(m,n)}|^{1/2}}\, dy\, |f_{\varepsilon,k}(y)|^2 \qquad \text{(III.42)}$$
$$\overline{g_1(x_1)\cdots g_m(x_m)}\, h_1(\tilde{x}_1)\cdots h_n(\tilde{x}_n)\ \ w_{m,n}[\omega(y);\, X^{(m,n)}]\,,$$

and

$$B_{\mu\nu}(\varepsilon) \ := \qquad \text{(III.43)}$$
$$\int_{B_1^{m+n}} \frac{dX^{(m,n)}}{|X^{(m,n)}|^{1/2}}\, \widetilde{w}_{m,n}[\mu,\nu;\ 0;\ X^{(m,n)}] \big(\Psi_{\underline{g}(\mu),\varepsilon,k} \otimes \Phi_{\underline{h}(\nu),\varepsilon,k}\big)[X^{(m,n)}]\,,$$

$$\widetilde{w}_{m,n}[\mu,\nu;\ 0;\ X^{(m,n)}] \ := \ \int_{B_1} dy\, \overline{g_\mu(y)}\, h_\nu(y)\, w_{m,n}[\omega(y);\, X^{(m,n)}]\,, \qquad \text{(III.44)}$$

and $\underline{g}(\mu) := (g_1,\ldots,g_{\mu-1},g_{\mu+1},\ldots,g_m)$, $\underline{h}(\nu) := (h_1,\ldots,h_{\nu-1},h_{\nu+1},\ldots,h_n)$.

Again, $\Psi_{\underline{g}(\mu),\varepsilon,k} \otimes \Phi_{\underline{h}(\nu),\varepsilon,k} \to 0$ weakly in $L^2(B_1^{m+n+2})$, as $\varepsilon \to 0$, and thus $\lim_{\varepsilon\to 0} B_{\mu\nu}(\varepsilon) = 0$. Since $|f_{\varepsilon,k}(y)|^2 \to \delta(k-y)$, as $\varepsilon \to 0$, and the function $w_{m+1,n+1}[\omega(y);\, X^{(m+1,n+1)}]$ is continuous in y, we obtain

$$\lim_{\varepsilon\to 0} A(\varepsilon) \ = \ \int_{B_1^{m+n+1}} \frac{dX^{(m,n)}}{|X^{(m,n)}|^{1/2}}\, \overline{g_1(x_1)\cdots g_m(x_m)}\, h_1(\tilde{x}_1)\cdots h_n(\tilde{x}_n)$$
$$w_{m,n}[\omega(k);\, X^{(m+1,n+1)}] \qquad . \text{(III.45)}$$

Using that $H(\underline{w}) = 0$, we conclude that

$$0 \ = \ \lim_{\varepsilon\to 0} \big\langle \Psi_{\underline{g},\varepsilon,k} \,\big|\, H(\underline{w})\, \Phi_{\underline{h},\varepsilon,k} \big\rangle \ = \ \lim_{\varepsilon\to 0} A(\varepsilon) \ = \qquad \text{(III.46)}$$
$$\int_{B_1^{m+n}} \frac{dX^{(m,n)}}{|X^{(m,n)}|^{1/2}}\, \overline{g_1(x_1)\cdots g_m(x_m)}\, h_1(\tilde{x}_1)\cdots h_n(\tilde{x}_n)\, w_{m,n}[\omega(k);\, X^{(m,n)}]\,,$$

for all $\underline{g}, \underline{h}$, and all $k \in \mathbb{R}^d \setminus 0$. Thus $w_{m,n} = 0$. □

Note that our estimate on the second term in the last line of (III.32) would have been better if we had assumed some higher regularity of $w_{1,1}[r, k, \tilde{k}]$ in k and $\tilde{k}$. For instance, if we had assumed that $w_{1,1}[r, k, \tilde{k}]$ is locally bounded, as the assumptions made in [2, 3] would imply, the second term in the last line of (III.32) would have been $\mathcal{O}(\varepsilon^d)$. It is clear that we cannot expect to have injectivity of H without minimal regularity assumptions on $K^{(m,n)} \mapsto w_{m,n}[r; K^{(m,n)}]$, because $H_f = \int |k|\, a^*(k) a(k)\, d^d k$ itself could then be regarded as $W_{1,1}[\delta(k - \tilde{k})\, |k|^2]$.

III.2 The Renormalization Transformation $\mathcal{R}_\rho$

The operators we study in this paper depend nonlinearly, but holomorphically on a spectral parameter. To accommodate this parameter, we introduce the Banach space, $\mathcal{W}_{\geq 0}$, of analytic functions on $D_{1/2} := \{z \in \mathbb{C} : |z| \leq 1/2\}$ with values in $\mathcal{W}_{\geq 0}^{\#}$, equipped with the norm

$$\| \underline{w}[\cdot] \|_{\mu,\xi} := \sup_{z \in D_{1/2}} \| \underline{w}[z] \|_{\mu,\xi}^{\#} . \tag{III.47}$$

Likewise, the Banach space of analytic families, $D_{1/2} \to H(\mathcal{W}_{\geq 0}^{\#})$, $z \mapsto H(\underline{w}[z])$, of operators is denoted by $H(\mathcal{W}_{\geq 0})$. It is this Banach space on which the renormalization transformation constructed below acts.

The renormalization map is denoted by $\mathcal{R}_\rho$. It depends on a scaling parameter ρ, $0 < \rho < 1$, which must be chosen sufficiently small. The transformation $\mathcal{R}_\rho$ is defined on a *subset* $\mathrm{dom}\, \mathcal{R}_\rho \subseteq \mathcal{W}_{\geq 0}$ with values in $\mathcal{W}_{\geq 0}$. The Banach space $\mathcal{W}_{\geq 0}$ depends on the two parameters $\mu > 0$ and $0 < \xi < 1$. $\mathcal{R}_\rho$ depends implicitly on μ and ξ and explicitly on ρ.

Given an element $\underline{w} \in \mathcal{W}_{\geq 0}$, the renormalization transformation $\mathcal{R}_\rho$ is defined as the composition of three operations:

(F) A decimation of the degrees of freedom in the range of field energies between ρ and 1 which is implemented through the smooth Feshbach map $F_{\chi_\rho[H_f]}$ applied to the Feshbach pair $(H(\underline{w}), T[H_f] - E)$, where $T[H_f] - E = W_{0,0}[\underline{w}]$ is the $\mathcal{W}_{0,0}$-component of $\underline{w}$, and $\chi_\rho[H_f] := \sin[\frac{\pi}{2}\Theta(H_f/\rho)]$, with $\Theta \in C_0^\infty([0,1);[0,1])$ and $\Theta \equiv 1$ on $[0, 3/4)$, is a smooth characteristic function of $[0, \rho]$ (so that $\overline{\chi}_\rho[H_f] := \sqrt{1 - \chi_\rho[H_f]^2}$ is smooth, as well);

(S) a unitary scaling transformation S_ρ which maps $\mathbf{1}[H_f < \rho] \mapsto \mathbf{1}[H_f < 1]$ and $\chi_\rho[H_f] \mapsto \chi_1[H_f]$, followed by multiplication by ρ^{-1};

(E) an analytic transformation E_ρ of the spectral parameter $z \in D_{1/2}$ in $\underline{w}[z]$.

(S) In Subsect. III.3 we show that a small ball in $\mathcal{W}_{m+n\geq 0}$ centered at $\underline{w}_0 := 0 \oplus r \oplus (0)_{m+n\geq 1}$ is contained in $\mathrm{dom}\, \mathcal{R}_\rho$ and is mapped into itself. To this end, we first introduce the **scaling transformation** $S_\rho : \mathcal{B}[\mathcal{F}] \to \mathcal{B}[\mathcal{F}]$, for $A \in \mathcal{B}[\mathcal{F}]$, by

$$S_\rho(A) \ := \ \frac{1}{\rho} \, \Gamma_\rho \ A \ \Gamma_\rho^* \, , \tag{III.48}$$

where Γ_ρ is the unitary dilatation on $\mathcal{F}$ defined by $\Gamma_\rho \Omega := \Omega$, and

$$\Gamma_\rho \, a^*(k) \, \Gamma_\rho^* \ := \ \rho^{-d/2} \, a^*(\rho^{-1} k) \ , \quad \Gamma_\rho \, a(k) \, \Gamma_\rho^* \ := \ \rho^{-d/2} \, a(\rho^{-1} k) \ , \tag{III.49}$$

for all $k \in \mathbb{R}^d$. It is easy to check that $\Gamma_\rho H_f \Gamma_\rho^* = \rho H_f$, and hence

$$\Gamma_\rho \, \chi_\rho[H_f] \, \Gamma_\rho^* \ = \ \chi_1[H_f] \quad \text{and} \quad S_\rho\big(T[H_f]\big) \ = \ \rho^{-1} \, T[\rho \, H_f] \ , \tag{III.50}$$

for any $T \in \mathcal{T}$. For $H(\underline{w}) = T(H_f) + E \cdot \mathbf{1} + W$, smoothly projected by $\chi_\rho[H_f]$, we find that

$$\begin{aligned} & S_\rho\big(\chi_\rho[H_f] \, H(\underline{w}) \chi_\rho[H_f]\big) \\ & \quad = \ \chi_1[H_f] \Big\{ \rho^{-1} T[\rho \, H_f] \ + \ \rho^{-1} \, E \cdot \mathbf{1} \ + \ S_\rho(W) \Big\} \, \chi_1[H_f] \, . \end{aligned} \tag{III.51}$$

Note that $S_\rho(H_f) = H_f$, which means that the operator $H_f = H[\underline{w}_0]$ corresponding to the center of the small ball in $\mathcal{W}_{\geq 0}$ mentioned above is a *fixed point* of S_ρ. Further note that $S_\rho(E \cdot \mathbf{1}) = \rho^{-1} E$ *expands* under the scaling map at a rate ρ^{-1}. (To control this expansion it is necessary to suitably project the spectral parameter by means of the analytic transformation E_ρ alluded to above). The key idea of the renormalization group is that the interaction contracts under the scaling transformation, and it is instructive to check that S_ρ contracts W.

To this end we remark that the scaling map S_ρ restricted to $H[\mathcal{W}_{\geq 0}^{\#}] \subseteq \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ induces a scaling map s_ρ on $\mathcal{W}_{\geq 0}$ by

$$S_\rho\big(H(\underline{w})\big) \ =: \ H\big(s_\rho(\underline{w})\big) \ =: \ H\big((s_\rho(w_{m,n}))_{m+n\geq 0}\big) \, , \tag{III.52}$$

and it is easy to verify that, for all $(m,n) \in \mathbb{N}_0^2$,

$$s_\rho(w_{m,n})\big[r, K^{(m,n)}\big] \ = \ \rho^{\frac{d-1}{2}(m+n)-1} \, w_{m,n}\big[\rho \, r \ , \ \rho \, K^{(m,n)}\big] \, . \tag{III.53}$$

Inserting this into (III.51), we obtain

$$\begin{aligned} & S_\rho\big(\chi_\rho[H_f] \, H(\underline{w}) \chi_\rho[H_f]\big) \\ & = \ \chi_1[H_f] \Big\{ \rho^{-1} T[\rho \, H_f] \ - \ \rho^{-1} \, E \cdot \mathbf{1} \ + \sum_{m+n\geq 1} W_{m,n}\big[s_\rho(w_{m,n})\big] \Big\} \, \chi_1[H_f] \, , \end{aligned} \tag{III.54}$$

and we note that by Theorem III.1, the operator norm of $W_{m,n}\big[s_\rho(w_{m,n})\big]$ is controlled by

$$\begin{aligned} & \|s_\rho(w_{m,n})\|_\mu && \text{(III.55)} \\ = \;& \rho^{\frac{d-1}{2}(m+n)-1} \left(\int_{B_1^{m+n}} \frac{dK^{(m,n)}}{\big|K^{(m,n)}\big|^{3+2\mu}} \sup_{r \in I} \big| w_{m,n}[\rho\, r\,;\, \rho\, K^{(m,n)}] \big|^2 \right)^{1/2} \\ = \;& \rho^{(1+\mu)(m+n)-1} \left(\int_{B_\rho^{m+n}} \frac{dK^{(m,n)}}{\big|K^{(m,n)}\big|^{3+2\mu}} \sup_{r \in [0,\rho]} \big| w_{m,n}[r; K^{(m,n)}] \big|^2 \right)^{1/2} \\ \leq \;& \rho^{(1+\mu)(m+n)-1} \, \|w_{m,n}\|_\mu \;\; \leq \;\; \rho^{\mu(m+n)} \, \|w_{m,n}\|_\mu \, , \end{aligned}$$

for $m+n \geq 1$. Since $\mu > 0$, this estimate shows that S_ρ contracts $\|w_{m,n}\|_\mu$ by at least a factor of $\rho^\mu < 1$. We show in Subsect. III.3 below that this contraction actually dominates the modifications the other two ingredients of the smooth Feshbach map impose on the argument $\underline{w}$ of the renormalization map $\mathcal{R}_\rho$.

(E) The expansion of multiples of the identity operator under S_ρ suggests to *renormalize the spectral parameter* of the problem. Given $\underline{w} \in \mathcal{W}_{\geq 0}$ with $E[z] := -w_{0,0}[z; 0]$, this is accomplished by means of an **analytic map** defined by

$$E_\rho \; : \; \mathcal{U}[\underline{w}] \; \to \; D_{1/2} \, , \quad z \; \mapsto \; \rho^{-1} \, E[z] \, , \tag{III.56}$$

where

$$\mathcal{U}[\underline{w}] \; := \; \left\{ z \in D_{1/2} \;\middle|\; |E[z]| \leq \frac{\rho}{2} \right\} . \tag{III.57}$$

We now show that E_ρ is a bijection and that $\mathcal{U}[\underline{w}]$ is close to the disc of radius ρ centered at 0, provided $\underline{w}$ is close to $\underline{w}_0$. To this end, we define a polydisc

$$\begin{aligned} \mathcal{D}(\varepsilon, \delta) \;\; := \;\; \Big\{ \underline{w} \in \mathcal{W}_{m+n \geq 0} \;\Big|\; & \sup_{z \in D_{1/2}} \big\| T[z; r] - r \big\|_{\mathcal{T}} \leq \varepsilon \, , && \text{(III.58)} \\ & \sup_{z \in D_{1/2}} |E[z] - z| \leq \delta \, , \quad \sup_{z \in D_{1/2}} \big\| (w_{m,n}[z])_{m+n \geq 1} \big\|_{\mu,\xi}^{\#} \leq \delta \Big\} \, , \end{aligned}$$

in $\mathcal{W}_{\geq 0}$, for all $\varepsilon, \delta > 0$. Note that

$$\begin{aligned} \Big\{ \underline{w} \in \mathcal{W}_{\geq 0} \;\Big|\; \|\underline{w} - \underline{w}_0\|_{\mu,\xi} \leq \varepsilon \Big\} \; & \subseteq \; \mathcal{D}(\varepsilon, \varepsilon) && \text{(III.59)} \\ & \subseteq \; \Big\{ \underline{w} \in \mathcal{W}_{\geq 0} \;\Big|\; \|\underline{w} - \underline{w}_0\|_{\mu,\xi} \leq 3\varepsilon \Big\} \, , \end{aligned}$$

is comparable to the ball of radius ε about $\underline{w}_0[z] := -z \oplus r \oplus (0)_{m+n \geq 1}$ in $\mathcal{W}_{m+n \geq 0}$, i.e., $H(\underline{w})$ is close to

$$H(\underline{w}_0[z]) \; = \; H_f \; - \; z \cdot \mathbf{1} \, . \tag{III.60}$$

Lemma III.4. *Fix $0 < \rho \le 1/2$, $\mu > 0$, and $0 < \xi < 1$. Then*

$$D_{3\rho/8} \;\subseteq\; \mathcal{U}[\underline{w}] \;\subseteq\; D_{5\rho/8} \;, \tag{III.61}$$

for all $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$, and

$$\big|\rho\, \partial_z E_\rho[z] - 1\big| \;\le\; 1/5 \,, \tag{III.62}$$

for all $z \in \mathcal{U}[\underline{w}]$. In particular, $E_\rho : \mathcal{U}[\underline{w}] \to D_{1/2}$ is a bijection, for all $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$.

Proof. If $\underline{w} \in \mathcal{D}(\varepsilon, \varepsilon)$ then $\sup_{z \in D_{1/2}} |E[z] - z| \le \varepsilon$. Thus, if $z \in \mathcal{U}[\underline{w}]$ then $|z| \le \frac{\rho}{2} + |E[z] - z| \le \frac{\rho}{2} + \varepsilon$, and $|z| \in D_{(\rho/2)+\varepsilon}$. Conversely, if $|z| \in D_{(\rho/2)-\varepsilon}$ then $|E[z]| \le |E[z] - z| + |z| \le \rho/2$, and $z \in \mathcal{U}[\underline{w}]$. This proves (III.61) with $\varepsilon := \rho/8$.

To show that E_ρ is a bijection, we use Cauchy's estimate with contour on $\partial D_{1/2} = \{|z| = 1/2\}$. Since $\mathcal{U}[\underline{w}] \subseteq D_{(\rho/2)+\varepsilon}$, we have that $\mathrm{dist}\big(\mathcal{U}[\underline{w}]\,,\, \partial D_{1/2}\big) \ge \frac{1}{2}(1-\rho) - \varepsilon$. Thus, we obtain the following estimate on the complex derivative of E_ρ,

$$\big|\rho\, \partial_z E_\rho[z] - 1\big| \;=\; \big|\partial_z E[z] - 1\big| \;\le\; \frac{2\,\varepsilon}{1 - \rho - 2\varepsilon} \;<\; 1 \,, \tag{III.63}$$

provided $\varepsilon := \rho/8$ and $\rho < 2/3$, which proves (III.62) and hence that $E_\rho : \mathcal{U}[\underline{w}] \to D_{1/2}$ is invertible. □

(F) We now describe the construction of the third element of $\mathcal{R}_\rho$, the *decimation of degrees of freedom* by the smooth Feshbach map. In order to apply the smooth Feshbach map as indicated above, we need to verify, for a given $\underline{w}$, that $(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$ defines a Feshbach pair corresponding to $\chi_\rho[H_f]$, for all $z \in \mathcal{U}[\underline{w}]$.

Lemma III.5. *Fix $0 < \rho < 1/2$, $\mu > 0$, and $0 < \xi < 1$. Then it follows that $(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$ defines a Feshbach pair corresponding to $\chi_\rho[H_f]$, for all $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$ and all $z \in \mathcal{U}[\underline{w}]$.*

Proof. We remark that $H(\underline{w}[z])$ and $W_{0,0}[\underline{w}[z]]$ define bounded operators on $\mathcal{H}_{\mathrm{red}}$, and we only need to check the invertibility of $T[z; H_f] - E[z] = W_{0,0}[\underline{w}[z]]$ on $\mathrm{Ran}\, \overline{\chi}_\rho[H_f]$, which follows if we can show that $T[z; r] - E[z]$ is away from 0 for all $r \in [3\rho/4, 1]$. Indeed, we have

$$\begin{aligned} |T[z;r] - E[z]| \;&\ge\; r \,-\, |T[z;r] - r| \,-\, |E[z]| \\ &\ge\; r\big(1 \,-\, \sup_{z,r} |T'[z;r] - 1|\big) \,-\, \sup_{z \in \mathcal{U}[\underline{w}]} |E[z]| \\ &\ge\; \frac{3\,\rho}{4}(1 - \rho/8) \,-\, \frac{\rho}{2} \;\ge\; \frac{5\,\rho}{32} \;\ge\; \frac{\rho}{8} \,. \end{aligned} \tag{III.64}$$

□

Having described the smooth Feshbach map $F_{\chi_\rho[H_f]}(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$ corresponding to the Feshbach pair $(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$, the scaling transformation S_ρ, and the analytic transformation E_ρ of the spectral parameter, we are prepared to define the **renormalization transformation** $\mathcal{R}_\rho$.

For $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$ and $\zeta \in D_{1/2}$, we set

$$\mathcal{R}_\rho^H(\underline{w})[\zeta] \ := \ S_\rho\Big(F_{\chi_\rho[H_f]}\big(H(\underline{w}[E_\rho^{-1}(\zeta)]), \ W_{0,0}[\underline{w}[E_\rho^{-1}(\zeta)]] \big) \Big) . \qquad \text{(III.65)}$$

Eq. (III.65) requires some comment. We assume that $\zeta \in D_{1/2}$ and $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$ which, by Lemma III.4, implies that $z := E_\rho^{-1}(\zeta) \in \mathcal{U}[\underline{w}]$. Furthermore, Lemma III.5 then insures that $\big(H(\underline{w}[z]) - z, W_{0,0}[\underline{w}[z]] - z\big)$ is a Feshbach pair corresponding to $\chi_\rho[H_f]$, and hence the smooth Feshbach map $F_{\chi_\rho[H_f]}(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$ defines an operator on $\mathrm{Ran}\, \chi_\rho[H_f] \subset \mathrm{Ran}\, \mathbf{1}[H_f < \rho]$. The application of the scaling map S_ρ to $F_{\chi_\rho[H_f]}(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$ yields a bounded operator $\mathcal{R}_\rho^H(\underline{w})[\zeta] = S_\rho(F_{\chi_\rho[H_f]}(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]]))$ on $\mathcal{H}_{\mathrm{red}}$.

In the next subsection we demonstrate that S_ρ actually maps $F_{\chi_\rho[H_f]}(H(\underline{w}[z]), W_{0,0}[\underline{w}[z]])$ into the subspace $H[\mathcal{W}_{\geq 0}^{\#}] \subseteq \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ by computing $\underline{\hat{w}} \in \mathcal{W}_{\geq 0}$ such that $H(\underline{\hat{w}}[\zeta]) = \mathcal{R}_\rho^H(\underline{w})[\zeta]$. By Theorem III.3, $\underline{\hat{w}}$ is the unique element in $\mathcal{W}_{\geq 0}$ with this property, and we may thus set $\mathcal{R}_\rho(\underline{w}) := \underline{\hat{w}}$.

Our next goal is to determine $\underline{\hat{w}} = \mathcal{R}_\rho(\underline{w})$. To this end, we write out $\mathcal{R}_\rho^H(\underline{w})[\zeta]$ explicitly. According to the definition (II.5) of the smooth Feshbach map, we have that

$$\begin{aligned} F_{\chi_\rho[H_f]}\big(H(\underline{w}[z]), \ W_{0,0}[\underline{w}[z]]\big) \ &= \ T[z; H_f] - E[z] \ + \ W_{\chi_\rho[H_f]}[z] \qquad \text{(III.66)} \\ - \ \chi_\rho[H_f]\, W[z]\, \overline{\chi}_\rho[H_f] &\big(T[z; H_f] - E[z] \ + \ W_{\overline{\chi}_\rho[H_f]}[z]\big)^{-1} \overline{\chi}_\rho[H_f]\, W[z]\, \chi_\rho[H_f] , \end{aligned}$$

where

$$T[z; r] - E[z] \ := \ W_{0,0}\big[w_{0,0}[z; r]\big], \ \ W[z] \ = \ \textstyle\sum_{m+n \geq 1} W_{m,n}\big[w_{m,n}[z]\big], \qquad \text{(III.67)}$$

$$W_{\chi_\rho[H_f]}[z] \ = \ \chi_\rho[H_f] W[z] \chi_\rho[H_f], \ \ W_{\overline{\chi}_\rho[H_f]}[z] \ = \ \overline{\chi}_\rho[H_f] W[z] \overline{\chi}_\rho[H_f]. \qquad \text{(III.68)}$$

Note that

$$\big\| \overline{\chi}_\rho[H_f]^2 |T[z; H_f] - E[z]|^{-1} \big\| \ \leq \ \frac{8}{\rho} \quad \text{and} \quad \|W[z]\| \ \leq \ \frac{\xi\, \rho}{8} , \qquad \text{(III.69)}$$

because of $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$, (III.31), and (III.64). Eq. (III.69) implies that the Neumann series expansion in $W_{\overline{\chi}}$ of the resolvent in (III.66) is norm convergent and yields

$$\begin{aligned} F_{\chi_\rho[H_f]}\big(H(\underline{w}[z]), \ W_{0,0}[w_{0,0}[z]]\big) \ &= \ T[z; H_f] - E[z] \qquad \text{(III.70)} \\ + \ \sum_{L=1}^{\infty} (-1)^{L-1}\, \chi_\rho[H_f] W[z] &\left(\frac{\overline{\chi}_\rho[H_f]^2}{T[z; H_f] - E[z]} \ W[z] \right)^{L-1} \chi_\rho[H_f] . \end{aligned}$$

The Neumann series on the right side of (III.70) is not yet normal-ordered, in the sense that all annihilation operators are moved to the right of functions of H_f and all creation operators are moved to the left of functions of H_f. The right side (III.70) can be normal ordered by systematically using the Pull-Through formula (III.12) and Wick's theorem; see Theorem III.6. To formulate it, we introduce

$$W_{p,q}^{m,n}\big[\underline{w}\big|r;\, K^{(m,n)}\big] \;:=\; P_{\mathrm{red}} \cdot \tag{III.71}$$
$$\int_{B_1^{p+q}} \frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}}\, a^*(x^{(p)})\, w_{m+p,n+q}\big[H_f + r;\, k^{(m)}, x^{(p)}\,,\, \tilde{k}^{(n)}, \tilde{x}^{(q)}\big]\, a(\tilde{x}^{(q)})\, P_{\mathrm{red}}\,,$$

for $m+n \geq 1$, which defines an operator for each $K^{(m,n)} \in B_1^{m+n}$, a.e. Here we use notation for $X^{(p,q)}$, $x^{(p)}$, $\tilde{x}^{(q)}$, etc. similar to the one introduced in Eqs. (III.4)–(III.8). For $m = n = 0$, we set

$$W_{p,q}\big[\underline{w}\big|r\big] \;:= \tag{III.72}$$
$$P_{\mathrm{red}} \int_{B_1^{p+q}} \frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}}\, a^*(x^{(p)})\, w_{p,q}\big[H_f + r;\, x^{(p)}\,,\, \tilde{x}^{(q)}\big]\, a(\tilde{x}^{(q)})\, P_{\mathrm{red}}\,.$$

Theorem III.6 (Wick Ordering). *Let $\underline{w} = (w_{m,n})_{m+n\geq 1} \in \mathcal{W}_{\geq 1}^{\#}$, write $W_{m,n} := W_{m,n}[w_{m,n}]$, $W = \sum_{m+n\geq 1} W_{m,n}$, and $F_0, \ldots, F_L \in \mathcal{W}_{0,0}$, and denote by S_M the group of permutations of M elements. Then*

$$F_0\, W\, F_1\, W \cdots W\, F_{L-1}\, W\, F_L \;=\; H[\underline{\tilde{w}}]\,, \tag{III.73}$$

where $\underline{\tilde{w}} \in (\widetilde{w}_{M,N}^{(\mathrm{sym})})_{M+N\geq 0} \in \mathcal{W}_{\geq 0}^{\#}$ is given by the symmetrization w. r. t. $k^{(M)}$ and $\tilde{k}^{(N)}$,

$$\widetilde{w}_{M,N}^{(\mathrm{sym})}[r;\, K^{(M,N)}] \tag{III.74}$$
$$:= \;\frac{1}{M!\,N!} \sum_{\pi\in S_M} \sum_{\tilde{\pi}\in S_N} \widetilde{w}_{M,N}[r;\, k_{\pi(1)}, \ldots, k_{\pi(M)}\,;\, \tilde{k}_{\tilde{\pi}(1)}, \ldots, \tilde{k}_{\tilde{\pi}(N)}\,]\,,$$

of

$$\widetilde{w}_{M,N}[r;\, K^{(M,N)}] \;= \tag{III.75}$$
$$\sum_{\substack{m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N}} \;\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1}} \;\prod_{\ell=1}^{L} \left\{ \binom{m_\ell + p_\ell}{p_\ell} \binom{n_\ell + q_\ell}{q_\ell} \right\}$$
$$F_0[r + \tilde{r}_0] \left\langle \Omega \middle| \widetilde{W}_1\big[r + r_1;\, K_1^{(m_1,n_1)}\big]\, F_1[H_f + r + \tilde{r}_1]\, \widetilde{W}_2\big[r + r_2;\, K_2^{(m_2,n_2)}\big] \right.$$
$$\left. \cdots F_{L-1}[H_f + r + \tilde{r}_{L-1}]\, \widetilde{W}_L\big[r + r_L;\, K_L^{(m_L,n_L)}\big]\, \Omega \right\rangle F_L[r + \tilde{r}_L]\,,$$

where

$$\widetilde{W}_\ell\big[r + r_\ell;\, K_\ell^{(m_\ell,n_\ell)}\big] \;:=\; W_{p_\ell,q_\ell}^{m_\ell,n_\ell}\big[\underline{w}\big|\, r + r_\ell;\, K_\ell^{(m_\ell,n_\ell)}\big]\,, \qquad \text{(III.76)}$$

$$K^{(M,N)} \;=\; (K_1^{(m_1,n_1)}, \ldots, K_L^{(m_L,n_L)})\,, \quad K_\ell^{(m_\ell,n_\ell)} \;=\; (k_\ell^{(m_\ell)}, \tilde{k}_\ell^{(n_\ell)})\,, \qquad \text{(III.77)}$$

$$r_\ell \;:=\; \Sigma[\tilde{k}_1^{(n_1)}] + \ldots + \Sigma[\tilde{k}_{\ell-1}^{(n_{\ell-1})}] \;+\; \Sigma[k_{\ell+1}^{(m_{\ell+1})}] + \ldots + \Sigma[k_L^{(m_L)}]\,, \qquad \text{(III.78)}$$

$$\tilde{r}_\ell \;:=\; \Sigma[\tilde{k}_1^{(n_1)}] + \ldots + \Sigma[\tilde{k}_\ell^{(n_\ell)}] \;+\; \Sigma[k_{\ell+1}^{(m_{\ell+1})}] + \ldots + \Sigma[k_L^{(m_L)}]\,. \qquad \text{(III.79)}$$

Proof. See [3, Theorem A.4]. □

As a direct consequence of Theorem III.6 and Eqs. (III.52)–(III.53) and (III.71)–(III.72), we find a sequence $\hat{\underline{w}}$ such that $H(\hat{\underline{w}}[\zeta]) = \mathcal{R}_\rho^H(\underline{w})[\zeta]$ $= S_\rho(F_{\chi_\rho[H_f]}\{\, H(\underline{w}[E_\rho^{-1}(\zeta)]),\, W_{0,0}[\underline{w}[E_\rho^{-1}(\zeta)]]\,)\}$ as follows.

Theorem III.7. *Let $\zeta \in D_{1/2}$, $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$, and set $z := E_\rho^{-1}(\zeta) \in \mathcal{U}[\underline{w}]$. Then $H(\hat{\underline{w}}[\zeta]) = \mathcal{R}_\rho^H(\underline{w})[\zeta]$ determines $\hat{\underline{w}} = (\hat{w}_{M,N}^{(sym)})_{M+N\geq 0}$ to be the symmetrization w. r. t. $k^{(M)}$ and $\tilde{k}^{(N)}$ (as in Eq. (III.74)) of*

$$\begin{aligned}
&\hat{w}_{M,N}[\zeta;\, r;\, K^{(M,N)}] \;=\; \rho^{\frac{d-1}{2}(M+N)-1} \sum_{L=1}^{\infty} (-1)^{L-1} \qquad \text{(III.80)}\\
&\sum_{\substack{m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N}} \;\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1}} \prod_{\ell=1}^{L} \left\{ \binom{m_\ell+p_\ell}{p_\ell} \binom{n_\ell+q_\ell}{q_\ell} \right\} \chi_1[r+\tilde{r}_0]\, \chi_1[r+\tilde{r}_L]\\
&\Big\langle \Omega \Big|\, \widetilde{W}_1\big[z;\, \rho(r+r_1);\, \rho K_1^{(m_1,n_1)}\big] \left(\frac{\overline{\chi}_1^2[H_f + r + \tilde{r}_1]}{T[z;\, \rho(H_f + r + \tilde{r}_1)] - E[z]} \right)\\
&\quad \widetilde{W}_2\big[z;\, \rho(r+r_2);\, \rho K_2^{(m_2,n_2)}\big] \left(\frac{\overline{\chi}_1^2[H_f + r + \tilde{r}_2]}{T[z;\, \rho(H_f + r + \tilde{r}_2)] - E[z]} \right) \cdots\\
&\quad \cdots \left(\frac{\overline{\chi}_1^2[H_f + r + \tilde{r}_{L-1}]}{T[z;\, \rho(H_f + r + \tilde{r}_{L-1})] - E[z]} \right) \widetilde{W}_L\big[z;\, \rho(r+r_L);\, \rho K_L^{(m_L,n_L)}\big]\, \Omega \Big\rangle,
\end{aligned}$$

for $M + N \geq 1$, and

$$\begin{aligned}
&\hat{w}_{0,0}[\zeta;\, r] \;-\; \rho^{-1}\, w_{0,0}[z;\, \rho\, r] \qquad \text{(III.81)}\\
&= \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{p_1+q_1\geq 1} \cdots \sum_{p_L+q_L\geq 1} \chi_1[r]\\
&\quad \Big\langle \Omega \Big|\, W_{p_1,q_1}\big[\underline{w}[z]\big|\, \rho r\big] \left(\frac{\overline{\chi}_1^2[H_f + r]}{T[z;\, \rho(H_f + r)] - E[z]} \right) W_{p_2,q_2}\big[\underline{w}[z]\big|\, \rho r\big] \cdots\\
&\quad \cdots \left(\frac{\overline{\chi}_1^2[H_f + r]}{T[z;\, \rho(H_f + r)] - E[z]} \right) W_{p_L,q_L}\big[\underline{w}[z]\big|\, \rho r\big]\;\; \Omega \Big\rangle\, \chi_1[r]\,,
\end{aligned}$$

where the notation introduced in Eqs. (III.71)–(III.72) and (III.76)–(III.79) is used. In particular, the summation on the right side of (III.81) starts with $L = 2$.

We remark that Theorem III.7 determines $\underline{\hat{w}}$ from $\underline{w} \in \mathcal{D}(\rho/8, \rho/8)$ only as a sequence of integral kernels that define an operator in $\mathcal{B}[\mathcal{H}_{\rm red}]$. In order to have $\underline{\hat{w}} \in \mathcal{W}_{\geq 0}$, we also need to show that its norm is finite, $\|\underline{\hat{w}}\|_{\mu,\xi} < \infty$. This is done in the next subsection.

III.3 The Codimension-1 Contractivity of $\mathcal{R}_\rho$

In Eq. (III.65) we have defined the renormalization map $\mathcal{R}_\rho^H$ as a map from the polydisc $\mathcal{D}(\rho/8, \rho/8)$ into $\mathcal{B}[\mathcal{H}_{\rm red}]$, and in Theorem III.7 we have identified a sequence $\mathcal{R}_\rho(\underline{w}) := \underline{\hat{w}}$ of integral kernels such that (formally) $H(\mathcal{R}_\rho(\underline{w})) = \mathcal{R}_\rho^H(\underline{w})$. By showing that $\underline{\hat{w}} \in \mathcal{W}_{\geq 0}$, it would follow that $\mathcal{R}_\rho : \mathcal{D}(\rho/8, \rho/8) \to \mathcal{W}_{\geq 0}$. The purpose of this subsection is to prove a much stronger result, namely, that $\mathcal{R}_\rho$ posesses a contraction property on a subspace of codimension one.

Theorem III.8 (Codimension-1 Contractivity). *For any $\mu > 0$, there exist numbers $\rho, \xi, \varepsilon_0 > 0$ such that, for all $0 \leq \varepsilon \leq \varepsilon_0$ and $0 \leq \delta \leq \varepsilon_0$,*

$$\mathcal{R}_\rho \ : \ \mathcal{D}\Big(\varepsilon \, , \, \delta\Big) \ \to \ \mathcal{D}\Big(\varepsilon + \frac{\delta}{2} \, , \, \frac{\delta}{2}\Big) \, . \tag{III.82}$$

We remark that a choice for $\rho, \xi, \varepsilon_0 > 0$ with the desired properties is given by $\rho := (16\, C_\Theta)^{-2/\mu}$, $\xi := (4\, C_\Theta)^{-1} \rho^{1/2}$, and $\varepsilon_0 := (8\, C_\Theta)^{-1} \rho$, where $1 \leq C_\Theta < \infty$ is the constant only depending on the smooth cutoff function Θ, which appears in the proof of Lemma III.10.

Remark III.9. *We emphasize that the main advantage of using the smooth Feshbach map, rather than the (projection) Feshbach map, becomes manifest in the proof of Theorem III.8, where we have to take derivatives of $\chi_1[r]$ and $\overline{\chi}_1[r]$; see Eq. (III.87). If these were projections, i.e., characteristic functions of intervals, we would inevitably encounter δ-distributions. In fact, the appearance of these δ-distributions are the reason for using (a rather involved mixture of) supremum and L^1-norms in [2, 3]. In contrast, our proof of Theorem III.8 below, i.e., of the codimension-1 contractivity of $\mathcal{R}_\rho$, is quite straightforward and merely requires summation of geometric series.*

In this subsection we use the notation introduced in Eqs. (III.71)–(III.72) and (III.76)–(III.79). For fixed $L \in \mathbb{N}$ and with $\underline{m, p, n, q} := (m_1, p_1, n_1, q_1, \ldots, m_L, p_L, n_L, q_L) \in \mathbb{N}_0^{4L}$, $M := m_1 + \ldots + m_L$, and $N := n_1 + \ldots + n_L$, we define

$$V_{\underline{m,p,n,q}}[r; K^{(M,N)}] \ := \tag{III.83}$$

$$\Big\langle \Omega \Big| \ F_0[H_f + r] \prod_{\ell=1}^{L} \Big\{ \widetilde{W}_\ell\big[z; \ \rho(r + r_\ell); \ \rho K_\ell^{(m_\ell, n_\ell)}\big] \ F_\ell[H_f + r] \Big\} \ \Omega \Big\rangle ,$$

where $F_0[r] := \chi_1[r + \tilde{r}_0]$, $F_L[r] := \chi_1[r + \tilde{r}_L]$ and

$$F_\ell[r] \;:=\; \frac{\overline{\chi}_1[r+\tilde{r}_\ell]^2}{T[\rho(r+\tilde{r}_\ell)] - E}\,, \tag{III.84}$$

for $\ell = 1, \ldots, L-1$. We note that

$$\hat{w}_{M,N}[\zeta;\, r;\, K^{(M,N)}] \;=\; \sum_{L=1}^{\infty} (-1)^{L-1}\, \rho^{\frac{d-1}{2}(M+N)-1} \tag{III.85}$$
$$\sum_{\substack{m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N}} \;\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1}} \prod_{\ell=1}^{L} \left\{ \binom{m_\ell+p_\ell}{p_\ell} \binom{n_\ell+q_\ell}{q_\ell} \right\} V_{\underline{m,p,n,q}}[r; K^{(M,N)}]\,.$$

To estimate $\underline{\hat{w}}$, we start with the following preparatory lemma.

Lemma III.10. *For fixed* $L \in \mathbb{N}$ *and* $\underline{m,p,n,q} \in \mathbb{N}_0^{4L}$, *the kernel* $V_{\underline{m,p,n,q}} \in \mathcal{W}_{M,N}^{\#}$ *and*

$$\rho^{\frac{d-1}{2}(M+N)-1}\, \big\| V_{\underline{m,p,n,q}} \big\|_\mu^{\#} \tag{III.86}$$
$$\leq\;\; 2(L+1)\, C^{(L+1)}\, \rho^{(1+\mu)(M+N)-L} \prod_{\ell=1}^{L} \frac{\big\| w_{m_\ell+p_\ell,n_\ell+q_\ell}[z] \big\|_\mu^{\#}}{\sqrt{p_\ell^{p_\ell}\, q_\ell^{q_\ell}}}\,,$$

with the convention that $p^p := 1$, *for* $p = 0$.

Proof. We first note that $|F_0[r]| \leq C_\Theta$, $|\partial_r F_0[r]| \leq C_\Theta$, $|F_L[r]| \leq C_\Theta$, and $|\partial_r F_L[r]| \leq C_\Theta$, and that, for $\ell = 1, \ldots, L-1$,

$$\begin{aligned} |F_\ell[r]| + |\partial_r F_\ell[r]| \;&\leq\; \left| \frac{\overline{\chi}_1^2[r+\tilde{r}_\ell]}{T[\rho(r+\tilde{r}_\ell)] - E} \right| + \left| \frac{2\overline{\chi}_1[r+\tilde{r}_\ell]\, \partial_r \overline{\chi}_1[r+\tilde{r}_\ell]}{T[\rho(r+\tilde{r}_\ell)] - E} \right| \\ &\quad + \left| \frac{\overline{\chi}_1^2[r+\tilde{r}_\ell]\; \rho\, \partial_r T[z; \rho(r+\tilde{r}_\ell)]}{(T[\rho(r+\tilde{r}_\ell)] - E)^2} \right| \\ &\leq\; \frac{C_\Theta}{\rho}\,, \end{aligned} \tag{III.87}$$

for some constant $1 \leq C_\Theta < \infty$ only depending on the smooth cutoff function Θ.

Remark III.11. *Estimate (III.87) holds uniformly in* $r \in I$, *as follows from Lemma III.5, which insures that* $|T[\rho(r+\tilde{r}_\ell)] - E| \geq \rho/8$. *Note that it is Estimate (III.87) that simplifies the analysis considerably, as compared to [3], because the functions* $F_\ell[r]$ *are continuously differentiable in* r.

Next we estimate $|V_{\underline{m,p,n,q}}|$, using that $|\langle\Omega|A\Omega\rangle| \le \|A\|_{\rm op}$, for any $A \in \mathcal{B}[\mathcal{H}_{\rm red}]$. We have that

$$\begin{aligned} & \big|V_{\underline{m,p,n,q}}[r;K^{(M,N)}]\big| \\ & \le \prod_{\ell=0}^{L} \big\|F_\ell[H_f+r]\big\|_{\rm op} \prod_{\ell=1}^{L} \Big\|\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\rm op} . \end{aligned} \tag{III.88}$$

We employ Leibniz' rule to compute $\partial_r V_{\underline{m,p,n,q}}$, with the result

$$\partial_r V_{\underline{m,p,n,q}}[r;K^{(M,N)}] \;=\; \widetilde{V}^{(i)}_{\underline{m,p,n,q}}[r;K^{(M,N)}] \;+\; \widetilde{V}^{(ii)}_{\underline{m,p,n,q}}[r;K^{(M,N)}]\,, \tag{III.89}$$

where

$$\begin{aligned} & \widetilde{V}^{(i)}_{\underline{m,p,n,q}}[r;K^{(M,N)}] \;:= \\ & \sum_{j=0}^{L}\Big\langle\Omega\Big|\prod_{\ell=1}^{j-1}\Big\{F_{\ell-1}[H_f+r]\,\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\}\,\big(\partial_r F_j[H_f+r]\big) \\ & \qquad \prod_{\ell=j+1}^{L}\Big\{\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\;F_\ell[H_f+r]\Big\}\;\Omega\Big\rangle\,, \end{aligned} \tag{III.90}$$

and

$$\begin{aligned} & \widetilde{V}^{(ii)}_{\underline{m,p,n,q}}[r;K^{(M,N)}] \;:= \\ & \sum_{j=1}^{L}\Big\langle\Omega\Big|F_0[H_f+r]\prod_{\ell=1}^{j-1}\Big\{\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\;F_\ell[H_f+r]\Big\} \\ & \qquad \rho\,\big(\partial_r\widetilde{W}_j\big[z;\,\rho(r+r_j);\,\rho K_j^{(m_j,n_j)}\big]\big) \\ & \qquad \prod_{\ell=j+1}^{L}\Big\{F_{\ell-1}[H_f+r]\,\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\}\;F_L[H_f+r]\;\Omega\Big\rangle\,. \end{aligned} \tag{III.91}$$

Proceeding in the same way as in Eq. (III.88) and using (III.87), we obtain

$$\begin{aligned} \big|\widetilde{V}^{(i)}_{\underline{m,p,n,q}}[r;K^{(M,N)}]\big| \;\le\; & \Big\{\sum_{j=0}^{L}\big\|\partial_r F_j[H_f+r]\big\|_{\rm op}\prod_{\substack{\ell=0,\\ \ell\neq j}}^{L}\big\|F_\ell[H_f+r]\big\|_{\rm op}\Big\} \\ & \prod_{\ell=1}^{L}\Big\|\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\rm op} \qquad \text{(III.92)} \\ \le\;\; & (L+1)\,C_\Theta^{L+1}\,\rho^{-L+1}\prod_{\ell=1}^{L}\Big\|\widetilde{W}_\ell\big[z;\,\rho(r+r_\ell);\,\rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\rm op} \end{aligned}$$

and

$$\begin{aligned}
&\big|\widetilde{V}^{(ii)}_{\underline{m,p,n,q}}[r;K^{(M,N)}]\big| \\
&\leq \prod_{\ell=0}^{L} \big\|F_\ell[H_f+r]\big\|_{\mathrm{op}} \Big\{ \sum_{j=1}^{L} \big\|\partial_r \widetilde{W}_j\big[z;\, \rho(r+r_j);\, \rho K_j^{(m_j,n_j)}\big]\big\|_{\mathrm{op}} \\
&\qquad \prod_{\substack{\ell=1,\\ \ell\neq j}}^{L} \Big\|\widetilde{W}_\ell\big[z;\, \rho(r+r_\ell);\, \rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}} \Big\} \\
&\leq C_\Theta^{L+1}\, \rho^{-L+2} \Big\{ \sum_{j=1}^{L} \big\|\widetilde{W}_j'\big[z;\, \rho(r+r_j);\, \rho K_j^{(m_j,n_j)}\big]\big\|_{\mathrm{op}} \\
&\qquad \prod_{\substack{\ell=1,\\ \ell\neq j}}^{L} \Big\|\widetilde{W}_\ell\big[z;\, \rho(r+r_\ell);\, \rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}} \Big\}\,, \qquad \text{(III.93)}
\end{aligned}$$

where

$$\widetilde{W}_j'\big[z;\, r+r_j;\, K_j^{(m_j,n_j)}\big] \;:=\; W^{m_j,n_j}_{p_j,q_j}\big[\partial_r \underline{w}[z]\big|\, r+r_j;\, K_j^{(m_j,n_j)}\big]\,, \qquad \text{(III.94)}$$

with $\partial_r \underline{w} := (\partial_r w_{m,n})_{m+n\geq 0}$, similar to (III.71) and (III.76). Inserting (III.92)–(III.93) into (III.89) and adding the result to (III.88), we obtain

$$\begin{aligned}
&\sup_{r\in I} \big|V_{\underline{m,p,n,q}}[r;K^{(M,N)}]\big|^2 \;+\; \sup_{r\in I} \big|\partial_r V_{\underline{m,p,n,q}}[r;K^{(M,N)}]\big|^2 \qquad \text{(III.95)} \\
&\leq (L+1)^2\, C_\Theta^{2(L+1)}\, \rho^{-2L+2} \\
&\prod_{\ell=1}^{L} \Big\{ \sup_{r\in I} \Big\|\widetilde{W}_\ell\big[z;\, r;\, \rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}}^2 \;+\; \rho \sup_{r\in I} \Big\|\widetilde{W}_\ell'\big[z;\, r;\, \rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}}^2 \Big\}\,,
\end{aligned}$$

and integrating this inequality against the measure $|K^{(M,N)}|^{-3-2\mu}\, dK^{(M,N)}$, we arrive at

$$\begin{aligned}
\big\|V_{\underline{m,p,n,q}}\big\|_{M,N}^2 \;&=\; \int_{B_1^{M+N}} \frac{dK^{(M,N)}}{|K^{(M,N)}|^{3+2\mu}} \sup_{r\in I} \big|V_{\underline{m,p,n,q}}[r;K^{(M,N)}]\big|^2 \qquad \text{(III.96)} \\
&\leq (L+1)^2\, C_\Theta^{2(L+1)}\, \rho^{-2L+2} \prod_{\ell=1}^{L} \Big\{ \int_{B_1^{m_\ell+n_\ell}} \frac{dK^{(m_\ell,n_\ell)}}{|K^{(m_\ell,n_\ell)}|^{3+2\mu}} \\
&\qquad \Big\{ \sup_{r\in I} \Big\|\widetilde{W}_\ell\big[z;\, r;\, \rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}}^2 \;+\; \rho \sup_{r\in I} \Big\|\widetilde{W}_\ell'\big[z;\, r;\, \rho K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}}^2 \Big\}\Big\} \\
&= (L+1)^2\, C_\Theta^{2(L+1)}\, \rho^{(3+2\mu-d)(M+N)-2L+2} \prod_{\ell=1}^{L} \Big\{ \int_{B_\rho^{m_\ell+n_\ell}} \frac{dK^{(m_\ell,n_\ell)}}{|K^{(m_\ell,n_\ell)}|^{3+2\mu}} \\
&\qquad \Big\{ \sup_{r\in I} \Big\|\widetilde{W}_\ell\big[z;\, r;\, K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}}^2 \;+\; \rho \sup_{r\in I} \Big\|\widetilde{W}_\ell'\big[z;\, r;\, K_\ell^{(m_\ell,n_\ell)}\big]\Big\|_{\mathrm{op}}^2 \Big\}\Big\}\,.
\end{aligned}$$

As a key step, we now convert the operator norms on the right side of (III.96) into L^2-norms by means of Inequality (III.11) in Theorem III.1. Indeed, inserting this estimate into the ℓ^{th} factor on the right side of (III.96), pointwise in $K_\ell^{(m_\ell,n_\ell)}$ a.e., and integrating over the variables $X^{(p_\ell,q_\ell)}$, we obtain

$$\begin{aligned}
&\int_{B_1^{m_\ell+n_\ell}} \frac{dK^{(m_\ell,n_\ell)}}{|K^{(m_\ell,n_\ell)}|^{3+2\mu}} \sup_{r\in I} \Big\| \widetilde{W}_\ell\big[z;\, r;\, \rho K_\ell^{(m_\ell,n_\ell)}\big] \Big\|_{\mathrm{op}}^2 \qquad \text{(III.97)} \\
&\leq \frac{1}{\sqrt{p_\ell^{p_\ell}\, q_\ell^{q_\ell}}} \int_{B_1^{m_\ell+n_\ell}} \frac{dK^{(m_\ell,n_\ell)}}{|K^{(m_\ell,n_\ell)}|^{3+2\mu}} \Big\| w_{m_\ell+p_\ell,n_\ell+q_\ell}\big[z;\, \cdot\,;\, k^{(m_\ell)},\, \cdot\,;\, \tilde{k}^{(n_\ell)},\, \cdot\,\big] \Big\|_\mu^2 \\
&= \frac{1}{\sqrt{p_\ell^{p_\ell}\, q_\ell^{q_\ell}}} \Big\| w_{m_\ell+p_\ell,n_\ell+q_\ell}\big[z\big] \Big\|_\mu^2 .
\end{aligned}$$

This estimate, inserted into (III.96), yields

$$\begin{aligned}
&\rho^{\frac{d-1}{2}(M+N)-1} \, \big\| V_{\underline{m,p,n,q}} \big\|_\mu^\# \;\leq \qquad \text{(III.98)} \\
&\qquad 2(L+1)\, C_\Theta^{(L+1)}\, \rho^{(1+\mu)(M+N)-L} \prod_{\ell=1}^{L} \frac{\big\| w_{m_\ell+p_\ell,n_\ell+q_\ell}\big[z\big] \big\|_\mu^\#}{\sqrt{p_\ell^{p_\ell}\, q_\ell^{q_\ell}}} .
\end{aligned}$$

□

We are now prepared for the proof of the *main theorem* about the renormalization transformation $\mathcal{R}_\rho$, namely, the codimension-1 contractivity.

Proof of Theorem III.8. First, we apply Lemma III.10 to (III.85) and use that $\binom{m+p}{p} \leq 2^{m+p}$. This yields

$$\begin{aligned}
&\big\| \hat{w}_{M,N}[\zeta] \big\|_\mu^\# \;\leq\; \sum_{L=1}^{\infty} 2\, C_\Theta\, (L+1) \Big(\frac{C_\Theta}{\rho} \Big)^L \big(2\, \rho^{(1+\mu)} \big)^{M+N} \qquad \text{(III.99)} \\
&\sum_{\substack{m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N}} \; \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1}} \prod_{\ell=1}^{L} \Big\{ \Big(\frac{2}{\sqrt{p_\ell}} \Big)^{p_\ell} \Big(\frac{2}{\sqrt{q_\ell}} \Big)^{q_\ell} \big\| w_{m_\ell+p_\ell,n_\ell+q_\ell}[z] \big\|_\mu^\# \Big\} .
\end{aligned}$$

Summing over the indices $\underline{m,p,n,q}$, as required in (III.85), we derive the following bound,

$$
\begin{aligned}
\big\|(\hat{w}_{M,N}[\zeta])_{M+N\geq 1}\big\|^{\#}_{\mu,\xi} \;&=\; \sum_{M+N\geq 1} \xi^{-(M+N)} \, \big\|\hat{w}_{M,N}[\zeta]\big\|^{\#}_{\mu} \qquad \text{(III.100)}\\
\leq \;& 4\, C_\Theta\, \rho^{(1+\mu)} \sum_{L=1}^{\infty} (L+1) \left(\frac{C_\Theta}{\rho}\right)^{L} \sum_{M+N\geq 1} \; \sum_{\substack{m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N}} \; \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1}} \\
& \prod_{\ell=1}^{L} \left\{ \Big(\frac{2\,\xi}{\sqrt{p_\ell}}\Big)^{p_\ell} \Big(\frac{2\,\xi}{\sqrt{q_\ell}}\Big)^{q_\ell} \xi^{-(m_\ell+p_\ell+n_\ell+q_\ell)} \, \big\|w_{m_\ell+p_\ell, n_\ell+q_\ell}[z]\big\|^{\#}_{\mu} \right\} \\
\leq \;& 4\, C_\Theta\, \rho^{(1+\mu)} \sum_{L=1}^{\infty} (L+1) \left(\frac{C_\Theta}{\rho}\right)^{L} \\
& \left\{ \sum_{m+n\geq 1} \left(\sum_{p=0}^{m} \Big(\frac{2\,\xi}{\sqrt{p}}\Big)^{p} \right) \left(\sum_{q=0}^{n} \Big(\frac{2\,\xi}{\sqrt{q}}\Big)^{q} \right) \xi^{-(m+n)} \, \|w_{m,n}[z]\|^{\#}_{\mu} \right\}^{L} \\
\leq \;& 4\, C_\Theta\, \rho^{(1+\mu)} \sum_{L=1}^{\infty} (L+1) \left(\frac{C_\Theta}{\rho}\right)^{L} A^{2L} \left(\big\|(w_{m,n}[z])_{m+n\geq 1}\big\|_{\mu,\xi} \right)^{L} ,
\end{aligned}
$$

where

$$
A \;:=\; \sum_{p=0}^{\infty} \Big(\frac{2\,\xi}{\sqrt{p}}\Big)^{p} \;\leq\; \sum_{p=0}^{\infty} \big(2\,\xi\big)^{p} \;=\; \frac{1}{1\,-\,2\,\xi}\,, \qquad \text{(III.101)}
$$

since $\xi \;=\; \sqrt{\rho}/(4C_\Theta) \;\leq\; 1/4$, by assumption. Note that in (III.101) we have dropped the factor $p^{-p/2}$ gained in Theorem III.1. Our assumption also insures that

$$
B \;:=\; \frac{C_\Theta}{\rho\,(1-2\xi)^2} \, \big\|(w_{m,n}[z])_{m+n\geq 1}\big\|_{\mu,\xi} \;\leq\; \frac{4\,C_\Theta\,\delta}{\rho} \;\leq\; \frac{1}{2}\,, \qquad \text{(III.102)}
$$

since $\delta \leq \varepsilon_0 := (8\,C_\Theta)^{-1}\rho$. Thus the geometric series in the last line of (III.100) is convergent. We obtain

$$
\sum_{L=1}^{\infty} (L+1)\, B^{L} \;=\; \frac{d}{dB}\Big[\sum_{L=0}^{\infty} B^{L}\Big] \;-\; 1 \;=\; \frac{1}{(1-B)^2} \;-\; 1 \;\leq\; 8\,B\,. \qquad \text{(III.103)}
$$

Inserting (III.101)–(III.103) into (III.100), we arrive at

$$
\begin{aligned}
\big\|(\hat{w}_{M,N}[\zeta])_{M+N\geq 1}\big\|^{\#}_{\mu,\xi} \;&\leq\; 32\, C_\Theta\, \rho^{(1+\mu)}\, B \;=\; 128\, C_\Theta^2\, \rho^{\mu} \, \big\|(w_{m,n}[z])_{m+n\geq 1}\big\|^{\#}_{\mu,\xi} \\
&\leq\; \frac{\delta}{2}\,, \qquad \text{(III.104)}
\end{aligned}
$$

as $128 C_\Theta^2 \rho^{\mu} = 128 C_\Theta^2 (16\, C_\Theta)^{-2} = 1/2$.

Next, we estimate $\hat{w}_{0,0}[\zeta]$, which, according to (III.81) and (III.83)–(III.85), is given by

$$\hat{w}_{0,0}[\zeta;\, r] \;=\; \rho^{-1}\, w_{0,0}[z;\, \rho\, r] + \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 1}} V_{\underline{0,p,0,q}}[r]\,. \quad \text{(III.105)}$$

The function $V_{\underline{0,p,0,q}}[r]$ and its partial derivative $\partial_r V_{\underline{0,p,0,q}}[r]$ are estimated as in Eqs. (III.95)–(III.98), with the result that

$$\rho^{-1} \, \big\| V_{\underline{0,p,0,q}} \big\|_\mu^{\#} \;\leq\; 2(L+1)\, C_\Theta^{(L+1)}\, \rho^{-L} \prod_{\ell=1}^{L} \frac{\big\| w_{p_\ell,q_\ell}[z] \big\|_\mu^{\#}}{\sqrt{p_\ell^{p_\ell}\, q_\ell^{q_\ell}}}\,. \quad \text{(III.106)}$$

Writing $\widehat{T}[\zeta; r] := \hat{w}_{0,0}[\zeta; r] - \hat{w}_{0,0}[\zeta; 0]$, we therefore find that

$$\begin{aligned}
\big\| \widehat{T}[\zeta; r] - r \big\|_{\mathcal{T}} \;&=\; \sup_{r\in I} \big| \partial_r \hat{w}_{0,0}[\zeta;\, r] \,-\, 1 \big| \qquad \text{(III.107)}\\
&\leq\; \sup_{r\in I} \big| \partial_r\, w_{0,0}[z;\, \rho\, r] \,-\, 1 \big| \,+\, \rho^{-1} \sum_{L=2}^{\infty} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 1}} \sup_{r\in I} \big| \partial_r V_{\underline{0,p,0,q}}[r] \big|\,.\\
&\leq\; \big\| T[z; r] - r \big\|_{\mathcal{T}} \,+\, 2\, C_\Theta \sum_{L=2}^{\infty} (L+1) \Big(\frac{C_\Theta}{\rho} \Big)^L \Big\{ \sum_{p+q\geq 1} \big\| w_{p,q}[z] \big\|_\mu^{\#} \Big\}^L\\
&\leq\; \big\| T[z; r] - r \big\|_{\mathcal{T}} \,+\, 2\, C_\Theta \sum_{L=2}^{\infty} (L+1) \Big(\frac{C_\Theta\, \xi}{\rho} \Big)^L \Big(\big\| (w_{m,n}[z])_{m+n\geq 1} \big\|_{\mu,\xi}^{\#} \Big)^L\\
&\leq\; \big\| T[z; r] - r \big\|_{\mathcal{T}} \,+\, 24\, C_\Theta \Big(\frac{C_\Theta\, \xi}{\rho} \, \big\| (w_{m,n}[z])_{m+n\geq 1} \big\|_{\mu,\xi}^{\#} \Big)^2\\
&\leq\; \varepsilon \,+\, \frac{\delta}{2}\,,
\end{aligned}$$

where we are using that $\sum_{L=2}^{\infty} (L+1) D^L \leq 12 D^2$, for D defined by $D := C_\Theta \xi \rho^{-1} \| (w_{m,n}[z])_{m+n\geq 1} \|_{m+n\geq 1}^{\#} \leq C_\Theta \xi \rho^{-1} \varepsilon_0 = \xi/8 \leq 1/2$, and $24 C_\Theta^3 \xi^2 \varepsilon_0 \rho^{-2} = 3 C_\Theta^2 \xi^2 \rho^{-1} = 3/16 \leq 1/2$, similar to (III.103).

In the third step we set $\widehat{E}[\zeta] := \hat{w}_{0,0}[\zeta; 0]$. Since $E[z] = w_{0,0}[z; 0]$ and $\zeta = E_\rho[z] = \rho^{-1} E[z]$, Eq. (III.105) yields

$$\widehat{E}[\zeta] \;=\; \zeta \,+\, \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 1}} V_{\underline{0,p,0,q}}[0]\,. \quad \text{(III.108)}$$

Inserting the estimates (III.106)–(III.107), we finally obtain

$$\big| \widehat{E}[\zeta] \,-\, \zeta \big| \;\leq\; 24\, C_\Theta \Big(\frac{C_\Theta\, \xi}{\rho} \, \big\| (w_{m,n}[z])_{m+n\geq 1} \big\|_{m+n\geq 1}^{\#} \Big)^2 \;\leq\; \frac{\delta}{2}\,, \quad \text{(III.109)}$$

which completes the proof. □

III.4 Construction of Eigenvalues and Eigenvectors

In this section we apply the renormalization transformation $\mathcal{R}_\rho$ to Hamiltonian operators $H \in \mathcal{B}[\mathcal{H}_{\mathrm{red}}]$ of the form

$$H \;=\; T[H_f] \;+\; W \;-\; E \cdot \mathbf{1}\,, \tag{III.110}$$

assumed to be small perturbations of $H(\underline{w}_0) \;=\; H_f - z$, in the sense that H is represented as $H = H(\underline{w})$, with

$$\underline{w} \;\in\; \mathcal{D}\Big(\frac{\varepsilon_0}{2}\,,\; \frac{\varepsilon_0}{2} \Big)\,, \tag{III.111}$$

where $\varepsilon_0 := (8\,C_\Theta)^{-1}(16\,C_\Theta)^{-2/\mu}$, as in Theorem III.8. In Eqs. (III.59)–(III.60) it is shown that (III.111) implies that

$$\big\| \underline{w} - \underline{w}_0 \big\|_{m+n \geq 0} \;\leq\; \frac{3\,\varepsilon_0}{2}\,. \tag{III.112}$$

The main result of this section is the following theorem.

Theorem III.12 (Construction of eigenvalues and eigenvectors). *Fix $\mu > 0$, and choose $\rho, \xi, \varepsilon_0 > 0$ sufficiently small such that (III.82) holds, for all $0 \leq \varepsilon \leq \varepsilon_0$ and $0 \leq \delta \leq \varepsilon_0$. Suppose that $\underline{w} \in \mathcal{D}(\varepsilon_0/2\,,\, \varepsilon_0/2)$. Then the complex number $e_{(0,\infty)} \in D_{1/2}$ defined in (III.126) is an eigenvalue of $H(\underline{w})$, in the sense that*

$$\dim \mathrm{Ker}\{ H(\underline{w}[e_{(0,\infty)}]) \} \;\geq\; 1\,. \tag{III.113}$$

Moreover, the vector $\Psi_{(\infty)} \in \mathcal{H}_{\mathrm{red}}$ defined in (III.146) is a corresponding eigenvector, i.e., it is a nonzero element of $\mathrm{Ker}\{ H(\underline{w}[e_{(0,\infty)}]) \}$.

Proof. First, we observe that thanks to Eq. (III.82) in Theorem III.8,

$$\mathcal{R}_\rho \;:\; \mathcal{D}\Big(2^{-1}\,\varepsilon_0\,,\; 2^{-1}\,\varepsilon_0 \Big) \;\to\; \mathcal{D}\Big(\big[2^{-1} + 2^{-2} \big]\varepsilon_0\,,\; 2^{-2}\,\varepsilon_0 \Big)\,, \tag{III.114}$$

$$\mathcal{R}_\rho^2 \;:\; \mathcal{D}\Big(2^{-1}\,\varepsilon_0\,,\; 2^{-1}\,\varepsilon_0 \Big) \;\to\; \mathcal{D}\Big(\big[2^{-1} + 2^{-2} + 2^{-3} \big]\varepsilon_0\,,\; 2^{-3}\,\varepsilon_0 \Big)\,, \tag{III.115}$$

$$\vdots$$

$$\mathcal{R}_\rho^\alpha \;:\; \mathcal{D}\Big(2^{-1}\,\varepsilon_0\,,\; 2^{-1}\,\varepsilon_0 \Big) \;\to\; \mathcal{D}\Big(\big[2^{-1} + \ldots + 2^{-\alpha} \big]\varepsilon_0\,,\; 2^{-\alpha}\,\varepsilon_0 \Big)\,. \tag{III.116}$$

It follows that the iteration of $\mathcal{R}_\rho$ on $\underline{w}$,

$$\underline{w}^{(\alpha)} \;:=\; \mathcal{R}_\rho^\alpha(\underline{w}) \;\in\; \mathcal{D}\Big(\varepsilon_0\,,\; 2^{-\alpha}\,\varepsilon_0 \Big)\,, \tag{III.117}$$

is defined, for all $\alpha \in \mathbb{N}_0$. We set $E_{(\alpha)}[z] := w_{0,0}^{(\alpha)}[z;0]$,

$$\mathcal{U}_{(\alpha)} := \mathcal{U}[\underline{w}^{(\alpha)}] = \left\{ z \in D_{1/2} \,\middle|\, |E_{(\alpha)}[z]| \le \frac{\rho}{2} \right\}, \tag{III.118}$$

and

$$J_{(\alpha)} := E_{\rho,(\alpha)} : \mathcal{U}_{(\alpha)} \to D_{1/2}, \quad z \mapsto \rho^{-1} E_{(\alpha)}[z]. \tag{III.119}$$

We recall from Lemma III.4 that $J_{(\alpha)} : \mathcal{U}_{(\alpha)} \to D_{1/2}$ is an analytic bijection and that

$$\left|\rho\, \partial_z J_{(\alpha)}[z] - 1\right| \le 1/5, \tag{III.120}$$

for all $z \in \mathcal{U}_{(\alpha)}$, which implies that

$$\left|\rho^{-1}\, \partial_\zeta J_{(\alpha)}^{-1}[\zeta] - 1\right| \le 1/4, \tag{III.121}$$

for all $\zeta \in D_{1/2}$, where $J_{(\alpha)}^{-1}$ denotes the inverse of $J_{(\alpha)}$,

$$J_{(\alpha)}^{-1} : D_{1/2} \to \mathcal{U}_{(\alpha)}, \quad \zeta \mapsto E_{(\alpha)}^{-1}[\rho\,\zeta]. \tag{III.122}$$

Thus, for any $\zeta, \zeta' \in D_{1/2}$,

$$\frac{3\,\rho}{4} |\zeta - \zeta'| \le \left|J_{(\alpha)}^{-1}[\zeta] - J_{(\alpha)}^{-1}[\zeta']\right| \le \frac{5\,\rho}{4} |\zeta - \zeta'|. \tag{III.123}$$

Here we use that (III.121) applies to all points of the line segment joining ζ and ζ', since $D_{1/2}$ is convex.

For $0 \le \alpha \le \beta$, we define

$$e_{(\alpha,\beta)} := J_{(\alpha)}^{-1} \circ \cdots \circ J_{(\beta)}^{-1}[0]. \tag{III.124}$$

Our next goal is to prove that $(e_{(\alpha,\beta)})_{\beta\in\mathbb{N}}$ converges, as $\beta \to \infty$. Indeed, an iterated application of (III.123) yields

$$\begin{aligned} |e_{(\alpha,\beta)} - e_{(\alpha,\beta+1)}| &= \left|J_{(\alpha)}^{-1}\big[e_{(\alpha+1,\beta)} - e_{(\alpha+1,\beta+1)}\big]\right| \\ &\le \frac{5\,\rho}{4} \left|e_{(\alpha+1,\beta)} - e_{(\alpha+1,\beta+1)}\right| \le \cdots \\ &\le \left(\frac{5\,\rho}{4}\right)^{\beta-\alpha} \left|e_{(\beta,\beta)} - e_{(\beta,\beta+1)}\right| \le \left(\frac{5\,\rho}{4}\right)^{\beta-\alpha}, \end{aligned} \tag{III.125}$$

using that $e_{(\beta,\beta)}, e_{(\beta,\beta+1)} \in D_{1/2}$. Since $5\rho/4 < 1$, Eq. (III.125) implies that the sequences $\big(e_{(\alpha,\beta)}\big)_{\beta=\alpha}^{\infty}$ are convergent, for all $\alpha \in \mathbb{N}_0$. Hence, the limits

$$e_{(\alpha,\infty)} := \lim_{\beta\to\infty} e_{(\alpha,\beta)} \in \mathcal{U}_{(\alpha)} \tag{III.126}$$

exist and obey

$$\rho^{-1} E_{(\alpha)}[e_{(\alpha,\infty)}] \; = \; J_{(\alpha)}[e_{(\alpha,\infty)}] \; = \; e_{(\alpha+1,\infty)} \, . \tag{III.127}$$

We observe that this identity and the bound $|E_{(\alpha)}[z] - z| \leq 2^{-\alpha}\varepsilon_0$, which follows from (III.117), imply that

$$\big| e_{(\alpha,\infty)} - \rho \, e_{(\alpha+1,\infty)} \big| \; = \; \big| e_{(\alpha,\infty)} - E_{(\alpha)}[e_{(\alpha,\infty)}] \big| \; \leq \; 2^{-\alpha} \, \varepsilon_0 \, . \tag{III.128}$$

Our next task is to show that $e_{(\alpha,\infty)} \to 0$, as $\alpha \to \infty$. To this end, we remark that, clearly, $\rho^j \, e_{(\alpha+j,\infty)} \to 0$, as $j \to \infty$, since $\rho < 1$. Therefore,

$$\begin{aligned} |e_{(\alpha,\infty)}| \; &= \; \Big| \sum_{j=0}^{\infty} \rho^j \big(e_{(\alpha+j,\infty)} - \rho \, e_{(\alpha+j+1,\infty)} \big) \Big| \\ &\leq \; \sum_{j=0}^{\infty} \rho^j \, 2^{-\alpha-j} \, \varepsilon_0 \; \leq \; 2^{-\alpha+1} \, \varepsilon_0 \, , \end{aligned} \tag{III.129}$$

using (III.128), which implies that $(e_{(\alpha,\infty)})_{\alpha \in \mathbb{N}_0}$ converges to 0.

Next, we define a sequence of operators $(H_{(\alpha)})_{\alpha=0}^{\infty}$ in $H(\mathcal{W}_{\geq 0}^{\#}) \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ by

$$H_{(\alpha)} \; := \; T_{(\alpha)}[H_f] \, + \, W_{(\alpha)} \, + \, E_{(\alpha)} \cdot \mathbf{1} \; := \; H(\underline{w}^{(\alpha)})[e_{(\alpha,\infty)}] \, , \tag{III.130}$$

with

$$T_{(\alpha)}[r] \;\; := \;\; w_{0,0}^{(\alpha)}[e_{(\alpha,\infty)}; r] - w_{0,0}^{(\alpha)}[e_{(\alpha,\infty)}; 0] \, , \tag{III.131}$$

$$W_{(\alpha)} \;\; := \;\; \sum_{m+n \geq 1} W_{m,n}\big[\underline{w}^{(\alpha)}[e_{(\alpha,\infty)}]\big] \, , \tag{III.132}$$

$$E_{(\alpha)} \;\; := \;\; w_{0,0}^{(\alpha)}[e_{(\alpha,\infty)}; 0] \;\; = \;\; E_{(\alpha)}[e_{(\alpha,\infty)}] \;\; = \;\; e_{(\alpha+1,\infty)} \, , \tag{III.133}$$

for all $\alpha \in \mathbb{N}_0$, noting that $H_{(0)} = H(\underline{w})[e_{(0,\infty)}]$. Morerover, we observe that the definition (III.65) of $\mathcal{R}_\rho^H$ then implies that, for all $\alpha \in \mathbb{N}$,

$$\begin{aligned} H_{(\alpha)} \;\; &= \;\; H(\underline{w}^{(\alpha)})[e_{(\alpha,\infty)}] \; = \; \mathcal{R}_\rho^H \big(\underline{w}^{(\alpha-1)} \big)[e_{(\alpha,\infty)}] \\ &= \;\; S_\rho \Big(F_{\chi_\rho[H_f]} \big(\, H(\underline{w}^{(\alpha-1)}[e_{(\alpha-1,\infty)}]), \; W_{0,0}[w_{0,0}^{(\alpha-1)}[e_{(\alpha-1,\infty)}]] \, \big) \Big) \\ &= \;\; \frac{1}{\rho} \, \Gamma_\rho \, F_{(\alpha-1)} \, \Gamma_\rho^* \, , \end{aligned} \tag{III.134}$$

where $F_{(\alpha-1)}$ denotes

$$F_{(\alpha-1)} \;\; := \;\; F_{\chi_\rho[H_f]} \big(\, H(\underline{w}^{(\alpha-1)}[e_{(\alpha-1,\infty)}]), \; W_{0,0}[\underline{w}^{(\alpha-1)}[e_{(\alpha-1,\infty)}]] \, \big) \, , \tag{III.135}$$

$$Q_{(\alpha-1)} \;\; := \;\; Q_{\chi_\rho[H_f]} \big(\, H(\underline{w}^{(\alpha-1)}[e_{(\alpha-1,\infty)}]), \; W_{0,0}[\underline{w}^{(\alpha-1)}[e_{(\alpha-1,\infty)}]] \, \big) \, , \tag{III.136}$$

and Γ_ρ is the unitary dilatation introduced in (III.49). Then Eq. (II.16) reads

$$H_{(\alpha-1)}\, Q_{(\alpha-1)} \;=\; \chi_\rho[H_f]\; F_{(\alpha-1)}\,, \tag{III.137}$$

which, together with (III.134), implies the intertwining property

$$H_{(\alpha-1)}\, Q_{(\alpha-1)}\, \Gamma_\rho^* \;=\; \rho\, \Gamma_\rho^*\, \chi_1[H_f]\; H_{(\alpha)}\,. \tag{III.138}$$

Eq. (III.138) is the *key identity* for the proof of the *existence of an eigenvector* at $e_{(0,\infty)}$.

For the construction of this eigenvector, we fix $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \leq \beta$, and we define vectors $\Psi_{(\alpha,\beta)}$ in $\mathcal{H}_{\rm red}$ by setting $\Psi_{(\alpha,\alpha)} := \Omega$ and

$$\Psi_{(\alpha,\beta)} \;:=\; Q_{(\alpha)}\, \Gamma_\rho^*\, Q_{(\alpha+1)}\, \Gamma_\rho^* \cdots Q_{(\beta-1)}\, \Omega\,. \tag{III.139}$$

We first show that this sequence is convergent, as $\beta \to \infty$. To this end, we observe that $\Omega = \Gamma_\rho^*\, \chi_\rho[H_f]\, \Omega$ and hence

$$\Psi_{(\alpha,\beta+1)} - \Psi_{(\alpha,\beta)} \;=\; Q_{(\alpha)}\, \Gamma_\rho^*\, Q_{(\alpha+1)}\, \Gamma_\rho^* \cdots Q_{(\beta-1)}\, \Gamma_\rho^* \big(Q_{(\beta)} - \chi_\rho[H_f]\big)\, \Omega\,. \tag{III.140}$$

Since $\|\chi_\rho[H_f]\| \leq 1$, this implies that

$$\big\|\Psi_{(\alpha,\beta+1)} - \Psi_{(\alpha,\beta)}\big\| \;\leq\; \big\|Q_{(\beta)} - \chi_\rho[H_f]\big\|_{\rm op} \prod_{j=\alpha}^{\beta-1} \Big\{1 \,+\, \big\|Q_{(j)} - \chi_\rho[H_f]\big\|_{\rm op}\Big\}\,. \tag{III.141}$$

Recalling the definition (II.6) of Q_Φ, we have

$$\chi_\rho[H_f] - Q_{(j)} \;= \tag{III.142}$$
$$\overline{\chi}_\rho[H_f]\big(T_{(j)}[H_f] - E_{(j)}[z] \,+\, \overline{\chi}_\rho[H_f]\, W_{(j)}\, \overline{\chi}_\rho[H_f]\big)^{-1} \overline{\chi}_\rho[H_f]\, W_{(j)}\, \chi_\rho[H_f]\,.$$

Furthermore, for $j \in \mathbb{N}$ sufficiently large,

$$\|W_{(j)}\|_{\rm op} \;\leq\; 2^{-j}\, \varepsilon_0 \;\leq\; \frac{\rho}{16}\,. \tag{III.143}$$

Indeed, since $\varepsilon_0 = (8C_\Theta)^{-1}\rho \leq \rho/8$, Eq. (III.143) holds true for all $j \geq 1$. Thus, for all $j \in \mathbb{N}$, we may estimate

$$\|\chi_\rho[H_f] - Q_{(j)}\|_{\rm op} \;\leq\; \Big(\frac{\rho}{8} \,-\, \|W_{(j)}\|_{\rm op}\Big)^{-1} \|W_{(j)}\|_{\rm op} \;\leq\; \frac{16\,\varepsilon_0}{\rho}\, 2^{-j}\,. \tag{III.144}$$

Inserting this estimate into (III.141), we obtain

$$\begin{aligned} \big\|\Psi_{(\alpha,\beta+1)} - \Psi_{(\alpha,\beta)}\big\| \;&\leq\; \frac{16\,\varepsilon_0}{\rho}\, 2^{-\beta} \prod_{j=\alpha}^{\beta-1} \Big\{1 \,+\, \frac{16\,\varepsilon_0}{\rho}\, 2^{-j}\Big\} \\ &\leq\; 2^{-\beta}\, \frac{16\,\varepsilon_0}{\rho}\, \exp\big[2^{-\alpha}\, 32\, \varepsilon_0\, \rho^{-1}\big]\,, \end{aligned} \tag{III.145}$$

using that $\prod_{j=0}^{\infty}(1+\lambda_j) \leq \exp\Big[\sum_{j=0}^{\infty}\lambda_j\Big]$, for $\lambda_j \geq 0$. Hence, for fixed $\alpha \in \mathbb{N}_0$, the sequence $(\Psi_{(\alpha,\beta)})_{\beta\in\mathbb{N}_0}$ of vectors in $\mathcal{H}_{\rm red}$ is convergent, and defining

$$\Psi_{(\alpha,\infty)} \;:=\; \lim_{\beta\to\infty} \Psi_{(\alpha,\beta)} \,, \tag{III.146}$$

we have the norm estimate

$$\big\|\Psi_{(\alpha,\infty)} - \Omega\big\| \;=\; \big\|\Psi_{(\alpha,\infty)} - \Psi_{(\alpha,\alpha-1)}\big\| \;\leq\; 2^{-\alpha}\,\frac{32\,\varepsilon_0}{\rho}\,\exp\big[2^{-\alpha}\,32\,\varepsilon_0\,\rho^{-1}\big]\,, \tag{III.147}$$

which guarantees that $\Psi_{(\alpha_0,\infty)} \neq 0$, provided $\alpha_0 \in \mathbb{N}$ is chosen sufficiently large.

The vector $\Psi_{(\alpha,\infty)}$ constructed above is an element of the kernel of $H_{(\alpha)}$, for every $\alpha \in \mathbb{N}_0$, as we will now demonstrate. For $\beta > \alpha$, observe that, thanks to (III.138),

$$\begin{aligned} H_{(\alpha)}\,\Psi_{(\alpha,\beta)} \;&=\; \big(H_{(\alpha)}\,Q_{(\alpha)}\,\Gamma_\rho^*\big)\,\big(Q_{(\alpha+1)}\,\Gamma_\rho^*\cdots Q_{(\beta-1)}\,\Omega\big) \\ &=\; \rho\,\Gamma_\rho^*\,\chi_1[H_f]\,\big(H_{(\alpha+1)}\,Q_{(\alpha+1)}\,\Gamma_\rho^*\big)\big(Q_{(\alpha+2)}\,\Gamma_\rho^*\cdots Q_{(\beta-1)}\,\Omega\big) \\ &\;\;\vdots \\ &=\; \rho^{\beta-\alpha}\,\big(\Gamma_\rho^*\,\chi_1[H_f]\big)^{\beta-\alpha}\,H_{(\beta)}\,\Omega\,. \end{aligned} \tag{III.148}$$

Since $T_{(\beta)}[H_f]\Omega = 0$, we furthermore observe that

$$\begin{aligned} \big\|\chi_1[H_f]\,H_{(\beta)}\,\Omega\big\| \;&=\; \big\|\chi_1[H_f]\,W_{(\beta)}\,\Omega \;+\; E_{(\beta)}\,\Omega\big\| \\ &\leq\; \big\|(w_{m,n}^{(\beta)}[e_{(\beta,\infty)}])_{m+n\geq 1}\big\|_{\beta,\xi}^{\#} \;+\; \big|E_{(\beta)}\big| \\ &\leq\; 2^{-\beta}\,\varepsilon_0 \;+\; \rho\,|e_{(\beta+1,\infty)}| \;\leq\; 2^{-\beta+1}\,\varepsilon_0\,. \end{aligned} \tag{III.149}$$

Here we also use Theorem III.3, the fact that $\underline{w}^{(\beta)} \in \mathcal{D}(\varepsilon_0,\, 2^{-\beta}\varepsilon_0)$, which, in particular, implies that $\|(w_{m,n}^{(\beta)}[e_{(\beta,\infty)}])_{m+n\geq 1}\|_{\beta,\xi}^{\#} \leq 2^{-\beta}\varepsilon_0$, and Eq. (III.133). Summarizing (III.148)–(III.149), we have that

$$\big\|H_{(\alpha)}\,\Psi_{(\alpha,\beta)}\big\| \;\leq\; 2^{-\beta+1}\,\varepsilon_0 \;\to\; 0\,, \tag{III.150}$$

as $\beta \to \infty$, and we have used that the operator norm of $\Gamma_\rho^*\,\chi_1[H_f]$ is bounded by 1. Since $H_{(\alpha)} \in \mathcal{B}(\mathcal{H}_{\rm red})$ is continuous, (III.150) implies that

$$H_{(\alpha)}\,\Psi_{(\alpha,\infty)} \;=\; \lim_{\beta\to\infty}\big\{H_{(\alpha)}\,\Psi_{(\alpha,\beta)}\big\} \;=\; 0\,, \tag{III.151}$$

for all $\alpha \in \mathbb{N}_0$. Thus $H_{(0)}\Psi_{(0,\infty)} = 0$ and $H_{(\alpha_0)}\Psi_{(\alpha_0,\infty)} = 0$, as well. Moreover, $H_{(\alpha_0)}$ derives from $H_{(0)}$ by $\alpha_0 < \infty$ applications of the Feshbach map and scaling transformations. Since $\Psi_{(\alpha_0,\infty)} \neq 0$, Theorem II.1(iii) implies that there is a vector

$\Psi' \neq 0$ such that $H_{(0)}\Psi' = 0$, and a closer look at the proof of this statement reveals that $\Psi_{(0,\infty)} = \Psi' \neq 0$. $\square$

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