Chaotic zone in the Bogdanov-Takens bifurcation for diffeomorphisms *

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Abstract

We consider a two-parametric analytic family of diffeomorphisms near the Bogdanov-Takens bifurcation. It is known that if the parameters belong to a homoclinic zone, the map has homoclinic points. The width of the homoclinic zone is exponentially small. We derive an asymptotic formula for the width of the homoclinic zone. An analytic invariant associated with a parabolic fixed point is an important ingredient of this formula. The proof of the asymptotic formula is not complete. Additionally we provide results of computations of the invariant for model families.

Keywords: Bogdanov-Takens bifurcation, homoclinic orbits, exponentially small phenomena

1 Introduction

There is a remarkable similarity between bifurcations of equilibria for planar vector fields and bifurcations of periodic orbits for 3 dimensional vector fields.

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This similarity is due to the fact that in many cases the Poincaré map near a bifurcating periodic orbit can be formally written as a time-1 map of a planar vector field. Although all Taylor coefficients of the interpolating vector field can be determined, there is no reason to expect convergence of the corresponding series since the dynamics in the 3 dimensional space can be much richer than the dynamics on the plane. On the other hand the interpolating planar vector field can be used to construct normal forms for bifurcations of the periodic orbits. Thus the difference between these two different types of bifurcations is moved beyond all algebraic orders. In real-analytic theory this difference is typically exponentially small [2, 6, 9].

The Bogdanov-Takens bifurcation [1, 15, 5] has codimension 2. In the case of a planar vector field it corresponds to an equilibrium with two vanishing eigenvalues. In the case of a periodic orbit in the 3 dimensional space both multipliers of the periodic orbit equal 1. It is more convenient to study two dimensional Poincaré maps instead of 3 dimensional vector fields.

The Bogdanov-Takens bifurcation for planar diffeomorphisms has been intensively studied in the recent time [3, 2]. It was shown that, similar to planar vector fields, there are both Hopf and homoclinic bifurcations in a neighborhood of the Bogdanov-Takens bifurcation. In the vector field case the values of parameters, which correspond to the homoclinic bifurcations, form a line on the plane of parameters. In the case of diffeomorphisms transversal homoclinic orbits are possible. The latter are stable with respect to small changes of parameters, so one can expect that there is a sector (instead of the line) on the parameter plane, where the map has homoclinic points. We call this sector a *homoclinic zone*. An exponentially small upper bound for its width was established in [2]. The aim of this paper is to present a lower bound. Of course this lower bound is exponentially small, and we conjecture that it is positive for a generic map. In order to prove this conjecture it is necessary to show that a certain constant Θ (see Section 3) does not vanish. We checked this conjecture numerically for several examples.

Let us formulate our main result more precisely. We consider a two-parametric analytic family $F_{\mu\nu} : (x, y) \mapsto (x_1, y_1)$:

$$\begin{aligned} x_1 &= x + y, \\ y_1 &= y + f(x, y, \mu, \nu) \end{aligned} \tag{1}$$

with analytic f. We assume f to be of the form

$$f(x, y, \mu, \nu) = -\mu + x^2 + \nu y + \gamma x y + \dots \tag{2}$$

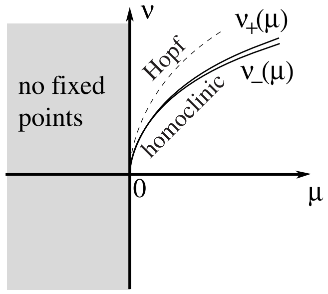

Figure 1: Bifurcation diagram

It is convenient to order terms of Taylor series according to values of the calibration function, which assigns orders to monomials:

$$\delta(\mu^j \nu^k x^l y^m) = 4j + 2k + 2l + 3m.$$

The coefficients μ, ν and γ are invariant under close-to-identity transformations, which preserve the form of the map $F_{\mu\nu}$.

This form of the map is not exceptional: a generic two-parametric family of maps near its Bogdanov-Takens bifurcation can be transformed to the form (1), (2) by a change of coordinates and parameters [3].

If $\mu = \nu = 0$ the map $F_{\mu\nu}$ has a parabolic fixed point at the origin. Following [3, 2] we study study the bifurcation of this point on the plane of (μ, ν). The corresponding bifurcation diagram is shown on Fig. 1.

If a point (μ, ν) is situated between the lines $\nu = \nu_-(\mu)$ and $\nu = \nu_+(\mu)$, $0 < \mu < \mu_0$, then the map $F_{\mu\nu}$ has a homoclinic trajectory. On the lower and upper boundary the homoclinic trajectory disappears. The first and the last homoclinic points are shown on Fig. 2. The pictures correspond to $\nu = \nu_-(\mu)$ and $\nu = \nu_+(\mu)$ respectively.

It is not very difficult to show that $\nu_\pm(\mu) = \frac{5}{7}(\gamma - 2)\mu^{1/2} + O(\mu^{3/4})$. A much more difficult problem is to prove the following estimate:

$$\nu_+(\mu) - \nu_-(\mu) = \frac{A_\gamma}{\mu^{5/4}} \cdot \mathrm{e}^{-\sqrt{2}\pi/\sqrt[4]{\mu}}, \tag{3}$$

$$A_\gamma = \frac{5|\Theta|}{6\sqrt{2}}\, \mathrm{e}^{-6\pi^2(\gamma-2)/7} + O(\mu^{1/4} \log \mu). \tag{4}$$

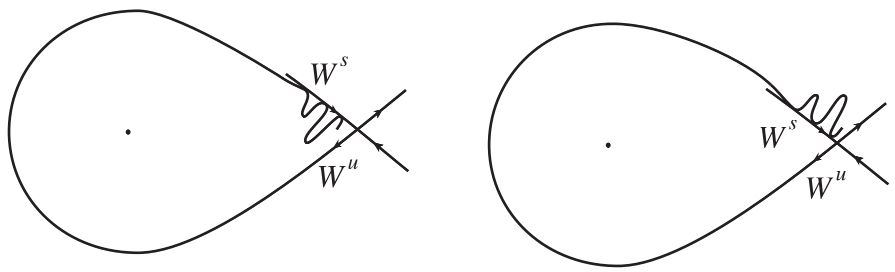

Figure 2: The first and last homoclinic points of the map $F_{\mu\nu}$

Here Θ is a constant defined later in Section 3. It depends on γ and on all other Taylor coefficients of the function $f(x,y,0,0)$. If the map F_{00} is exactly the time-1 map of a planar flow, the corresponding constant Θ vanishes. Each F_{00} can be interpolated by a flow up to an error, which is smaller than any power of x and y. Thus all Taylor coefficients of $f(x,y,0,0)$ are relevant for Θ.

The formula (3) implies a lower bound for the width of the zone provided Θ does not vanish. It was proved analytically that Θ does not vanish in the cases of the Hénon family [10]; numerically $|\Theta(\text{Hénon})| = 2.47\ldots \cdot 10^6$. One should expect that Θ does not vanish generically, since it depends analytically on F_{00}. We recall that the Hénon map

$$\begin{aligned} u_1 &= v, \\ v_1 &= av^2 - bu + 1 \end{aligned}$$

can be transformed to the form (1) with

$$f(x,y,\mu,\nu) = (x+y)^2 - \mu + \nu y.$$

The coefficients μ and ν are in one-to-one correspondence with the original parameters:

$$a = (1+\nu/2)^2 - \mu, \qquad b = 1 + \nu.$$

The Hénon family has the special property: it is area-preserving for $\nu = 0$. The separatrices of the area-preserving Hénon map split transversally [9, 10, 4].

The rest of the paper contains an informal derivation of the asymptotic formula (3). It can not be considered as a complete proof. The approach is based on Lazutkin's method originally proposed for a study of the standard map [13, 8]. Section 5 provides a summary of numerical experiments used to evaluate the splitting constant Θ for some model families and to check the correctness of the asymptotic formula (3).

2 Formal interpolation by a flow

In this section we discuss the formal interpolation of the family $F_{\mu\nu}$ by an autonomous vector-field

$$X = X^1 \partial_x + X^2 \partial_y,$$

where ∂_x and ∂_y stand for derivatives in x and y respectively; X^1 and X^2 are power series in x, y, μ and ν. The vector field is a differential operator and can be expanded into a sum of homogeneous operators:

$$X = \sum_{k \geq 1} X_k,$$

where X_k increases the order of a δ-homogeneous polynomial by k.

It is not too difficult to check that there is a unique formal vector-field, such that

$$F_{\mu\nu}(\underline{x}) = \mathrm{e}^X \underline{x}, \qquad \underline{x} = (x, y)^T.$$

In general one expects that the series for X diverges. Comparing the first two orders of the series on both sides we obtain

$$\begin{aligned} X_1 &= y\partial_x + (x^2 - \mu)\partial_y, \\ X_2 &= -\tfrac{1}{2}(x^2 - \mu)\partial_x + (\nu + (\gamma - 1)x)y\partial_y. \end{aligned}$$

The first oder vector-field X_1 is Hamiltonian with the Hamiltonian function

$$h_0 = \frac{y^2}{2} + \mu x - \frac{x^3}{3}.$$

If $\mu > 0$ it has a homoclinic loop defined by $h_0(x, y) = \frac{2}{3}\mu^{3/2}$. The perturbation by X_2 can destroy this loop. Let us study the homoclinic bifurcation of the vector field $X_1 + X_2$ using the Melnikov method [11]. The unperturbed homoclinic orbit can be find explicitly:

$$\begin{aligned} x_0(t,\mu) &= \sqrt{\mu}\left(1-\frac{3}{\cosh^2(t\sqrt[4]{\mu}/\sqrt{2})}\right),\\ y_0(t,\mu) &= \dot{x}_0 = 3\sqrt{2}\mu^{3/4}\frac{\sinh(t\sqrt[4]{\mu}/\sqrt{2})}{\cosh^3(t\sqrt[4]{\mu}/\sqrt{2})}\,. \end{aligned}$$

The Melnikov function describes the distance between the stable and unstable separatrices of the flow generated by X_1+X_2 measured in units of h_0:

$$\begin{aligned} M(\mu,\nu) &= \int_{-\infty}^{+\infty}(X_2h_0)|_{x_0(t),y_0(t)}\,\mathrm{d}t\\ &= (\gamma-2)\int_{-\infty}^{+\infty}x_0(t)y_0^2(t)\,\mathrm{d}t+\nu\int_{-\infty}^{+\infty}y_0^2(t)\,\mathrm{d}t. \end{aligned}$$

It is not too difficult to calculate this integral explicitly:

$$M(\mu,\nu)=\frac{24\sqrt{2}}{35}\,\mu^{5/4}\left(-5(\gamma-2)\sqrt{\mu}+7\nu\right).$$

The Melnikov function vanishes along the line $\nu=5(\gamma-2)\sqrt{\mu}/7$, which gives an asymptotic of the actual line of homoclinic bifurcations. In the following we will need the following derivative:

$$\frac{\partial M}{\partial\nu}(\mu,\nu)=\frac{24\sqrt{2}}{5}\mu^{5/4}, \tag{5}$$

which describes how fast the distance between separatrices changes as a function of ν for a fixed μ.

Of course, Melnikov-type computations do not give the width of the homoclinic zone for the map $F_{\mu\nu}$, because for a flow it reduces to a line.

Let $\nu=5(\gamma-2)\sqrt{\mu}/7$ and find the first correction to the unperturbed homoclinic: $x=x_0(t)+x_1(t)+\ldots$, $y=y_0(t)+y_1(t)+\ldots$ We have

$$\begin{aligned} \dot{x}_1 &= y_1-\tfrac{1}{2}(x_0^2-\mu),\\ \dot{y}_1 &= 2x_0x_1+(\gamma-1)x_0y_0+\nu y_0. \end{aligned}$$

For the purposes of the the present paper it is sufficient to compute x_1 only:

$$x_1(t)=\tfrac{3}{7}(\gamma-2)y_0(t)\log\left(\sqrt{\mu}-x_0(t)\right).$$

This calculation can be continued and approximations for the separatrix loop for all δ-orders can be constructed. All these approximations give a closed homoclinic loop. In order to detect the exponentially small splitting of separatrices we analytically continue in t and study the approximations of the separatrices near a complex singularity.

It is clear that x_0 and x_1 have a common singularity at $t = \pm \mathrm{i}\rho$ with

$$\rho = \frac{\pi}{\sqrt{2}\mu^{1/4}} \,. \tag{6}$$

Both x_0 and x_1 are analytic in the complex strip $|\Im t| < \rho$. The main parts of x_0 and x_1 at the singularity are:

$$\begin{aligned} x_0(t) &= \frac{6}{(t-\mathrm{i}\rho)^2} + O(1)\,, \\ x_1(t) &= \tfrac{36}{7}(\gamma-2)\frac{\mathrm{i}\pi + \log 6 - \log(t-\mathrm{i}\rho)^2}{(t-\mathrm{i}\rho)^3} + O\left((t-\mathrm{i}\rho)^{-1}\right)\,. \end{aligned} \tag{7}$$

We will use this expansions for the complex matching method described later in Section 4.

3 Splitting constant Θ

Let us consider the map $F_0 \equiv F_{00}$, which has a parabolic fixed point at the origin. The fixed point has stable and unstable invariant manifold, which we parameterize by solutions of the finite-difference equation:

$$\underline{x}_0^{\pm}(\tau+1) = F_0(\underline{x}_0^{\pm}(\tau)).$$

We supply this equation with asymptotic boundary conditions for the first components of $\underline{x}_0^+$ and $\underline{x}_0^-$:

$$x_0^{\pm}(\tau) = \frac{6}{\tau^2} - \tfrac{36}{7}(\gamma-2)\frac{\log\tau^2}{\tau^3} + O\left(\tau^{-4}\log^2\tau\right). \tag{8}$$

The unstable solution $\underline{x}_0^-$ is analytic in a sectorial domain

$$D_- = \{\, \tau \in \mathbb{C} : \delta < \arg(\tau + c) < 2\pi - \delta \,\}$$

with $\delta \in (0, \frac{\pi}{2})$ and sufficiently large $c > 0$. The stable solution $\underline{x}_0^+$ is analytic in the domain D_+, which is a reflection of D_- with respect to the imaginary axis. These conditions define $\underline{x}_0^+$ and $\underline{x}_0^-$ uniquely.

Consider the auxiliary scalar equation

$$w_0(\tau+1) = J_0(\tau)w_0(\tau), \qquad \text{where } J_0(\tau) = \det F_0'(\underline{x}_0^-(\tau)).$$

In D_- this linear finite-difference equation has a unique analytic solution, which satisfies the asymptotic boundary condition $w_0(\tau) = 1 + o(\tau^{-1})$. Then

$$\Theta = \lim_{\Im\tau\to-\infty} \mathrm{e}^{2\pi\mathrm{i}\tau} \frac{\tilde{\theta}(\tau)}{w_0(\tau)}$$

gives the same value of Θ; but the expression under the limit sign converges exponentially as $|\Im\tau| \to -\infty$. Let us define the splitting function

$$\tilde{\theta}(\tau) = \frac{1}{w_0(\tau)} \det\left(\frac{\mathrm{d}\underline{x}_0^-}{\mathrm{d}\tau}(\tau);\ \underline{x}_0^+(\tau) - \underline{x}_0^-(\tau)\right) \tag{9}$$

for $\tau \in \Omega_-$, the lower component of $D_- \cap D_+$. Then the splitting constant Θ is defined by

$$\Theta = \lim_{\Im\tau\to-\infty} \mathrm{e}^{2\pi\mathrm{i}\tau}\tilde{\theta}(\tau).$$

This definition generalizes the definition from [9]. The existence of the limit can be proved by methods of [8, 10]. Given a map F_0, the constant Θ can be numerically computed directly from the definition.

Finally we rewrite the definition of Θ in the form

$$\tilde{\theta}(\tau) \approx w_0(\tau)\Theta\,\mathrm{e}^{-2\pi\mathrm{i}\tau} \tag{10}$$

in oder to use it later in Section 4.

4 Flow box theorem

Now let us come back to the study of separatrices of the original map $F_{\mu\nu}$. We assume that parameters μ and ν are both small and close to the unperturbed homoclinic line $\nu = \frac{5}{7}(\gamma-2)\mu^{1/2}$. So the stable and unstable separatrices of the map are close to each other.

We parameterize the stable and unstable separatrices of its hyperbolic fixed point by solutions of finite-difference equation

$$\underline{x}^{u,s}(t+1) = F_{\mu\nu}(\underline{x}^{u,s}(t)),$$

which satisfy appropriate boundary conditions [8, 9]. These functions are close to the separatrix solution of the differential equations of Section 2.

In a neighborhood of the first intersection of the curve $\underline{x}^u(t)$ with the y-axes we introduce the time-energy coordinates (E, T) [7]. In these coordinates the map $F_{\mu\nu}$ takes the form of the translation

$$(E, T) \mapsto (E, T + 1).$$

The coordinates can be normalized in such a way that

$$E(\underline{x}^u(t)) = 0, \qquad T(\underline{x}^u(t)) = t.$$

Moreover the energy E is close to the Hamiltonian function h_0. In the (E, T) coordinates a segment of the stable manifold is a graph of a periodic function $E = A(T, \mu, \nu)$, $A(T + 1, \mu, \nu) = A(T, \mu, \nu)$. We can expand this function in Fourier series

$$A(T, \mu, \nu) = A_0(\mu, \nu) + A_1(\mu, \nu) \sin 2\pi(T - T_0(\mu, \nu)) + h.o.t.$$

In [7] it was shown the h.o.t. is an exponentially small quantity of higher order. The zeroes of $A(T, \mu, \nu)$ correspond to homoclinic points. Moreover, if this function have nor zeroes, it can be shown that the separatrices do not intersect at all.

The first term A_0 is approximated by the Melnikov function described in Section 2:

$$A_0(\mu, \nu) = M(\mu, \nu) + O(\mu^2).$$

The amplitude A_1 of the second term is exponentially small. The product $A_1 \sin 2\pi T$ can be large for non-real T. In order to approximate it we use the following complex matching method. We expand $E(\underline{x}^s(t)) - E(\underline{x}^u(t))$ in powers of $\underline{x}^s(t) - \underline{x}^u(t)$ and use

$$\partial_x E(\underline{x}^u(t)) = \frac{\dot{y}^u(t)}{w(t)}, \qquad \partial_y E(\underline{x}^u(t)) = -\frac{\dot{x}^u(t)}{w(t)},$$

where $1/w(t)$ is the Jacobian of the map $(x, y) \mapsto (E, T)$ computed at the point $\underline{x}^u(t)$. We obtain that

$$A(t, \mu, \nu) = \frac{1}{w(t)} \det(\underline{\dot{x}}^u(t); \underline{x}^s(t) - \underline{x}^u(t)) + O(|\underline{x}^s(t) - \underline{x}^u(t)|^2) \,. \tag{11}$$

In oder to approximate this function we consider its analytic continuation in the variable t. The functions $\underline{x}^u(t)$ and $\underline{x}^s(t)$ are approximated by $\underline{x}_0^-(\tau)$ and $\underline{x}_0^+(\tau)$ respectively, where

$$\tau = t - \mathrm{i}\rho - \rho_1,$$

ρ is defined in (6); and $\rho_1 = \frac{3}{7}(\gamma - 2)(\mathrm{i}\pi + \log 6)$ takes into account the matching of the expansions (8) and (7).

Comparing (10), (9) and (11) we obtain

$$A_1 \approx 2|\Theta|\,|\mathrm{e}^{-2\pi\mathrm{i}\rho_1}|\,\mathrm{e}^{-2\pi\rho} = 2|\Theta|\,\mathrm{e}^{-6\pi^2(\gamma-2)/7}\,\mathrm{e}^{-\sqrt{2}\pi^2/\sqrt[4]{\mu}}.$$

Now we calculate the width of the homoclinic zone:

$$\nu_+(\mu) - \nu_-(\mu) = \left.\frac{2A_1(\mu,\nu)}{\frac{\partial A_0}{\partial\nu}(\mu,\nu)}\right|_{\nu:A_0(\mu,\nu)=0} = \frac{5|\Theta|\mathrm{e}^{-6\pi^2(\gamma-2)/7}}{6\mu^{5/4}}\,\mathrm{e}^{-\sqrt{2}\pi^2/\sqrt[4]{\mu}},$$

where we used the approximations for A_0 and A_1. Thus we obtained the desired asymptotic (3) for the width of the homoclinic zone on the parameter plane of the Bogdanov-Takens bifurcation for diffeomorphisms. The error term comes from the consideration of next order corrections to this asymptotic.

5 Numerical experiments

In the most cases there is not known ways to find the splitting constant Θ analyticly. Given a map F_0, this constant is not too difficult to compute with any reasonable precision. For the area-preserving Hénon map this constant was computed with several thousands digits [14]. We did high-precision computations of Θ for the family (1) with $f = x^2 + \gamma xy$ for several γ. The method is essentially the same as in [9]. Numerically the splitting constant Θ does not vanish on the interval $[-0.5, 2]$. Several values are given in the following table:

γ	Θ
0	$5.202169191369226 \cdot 10^{-5}$
1	$6.9898737i51682470943...$
2	$1.203229677075980622 \cdot 10^{8}$

The estimates for the width of the homoclinic zone were derived non-rigorously. So it is interesting to compare them with results of direct numerical computations. We choose the family (1) with

$$f(x, y, \mu, \nu) = -\mu + x^2 + \nu y + \gamma xy,$$

and the Hénon map. There is a reasonably good agreement between our theory and experiments. For $\mu^{1/4} = 0.08$ the relative error of the asymptotic formula is typically around 20%.

It is difficult to perform computations for essentially smaller values of μ. So we took the data for several μ and then extrapolated towards $\mu = 0$. The values of Θ obtained in this way coincide in several digits with the values from the table. This provides numerical evidence for the correctness of the asymptotic formula (3).

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References

[1] R.I. Bogdanov, Versal deformations of a singular point on the plane in the case of zero eigenvalues, *Functional Analysis and Its Applications* 9(2), 144–145 (1975)

[2] H.W. Broer, R. Roussarie, *Exponential confinement of chaos in the bifurcation set of real analytic diffeomorphisms.* In H.W. Broer et. al. (Eds.), Global Analysis of Dynamical Systems (dedicated to Floris Takens for his 60th birthday). IOP, Bristol (2001)

[3] H. Broer, R. Roussarie, C. Simó. Invariant circles in the Bogdanov-Takens bifurcation for diffeomorphisms. *Ergodic Th. Dyn. Syst.* 16, no.6, 1147-1172 (1996).

[4] V. Chernov, Splitting of separatrices for the Hénon map, preprint (2001)

[5] S.-N.Chow, C.Li, D.Wang, *Normal forms and bifurcations of planar vector fields*, Cambridge University Press: Cambridge, 1994

[6] E. Fontich, C. Simó. The splitting of separatrices for analytic diffeomorphisms. *Ergodic Theory Dyn. Syst.* 10, No.2, 295-318 (1990)

[7] V.G. Gelfreich. Conjugation to a shift and splitting of separatrices, *Applicationes Mathematicae* 24, 2, pp. 127–140 (1996)

[8] V.G. Gelfreich. A proof of the exponentially small transversality of the separatrices for the Standard Map *Comm. Math. Phys.* 201, pp. 155–216 (1999)

[9] V.G. Gelfreich. Splitting of a small separatrix loop near the saddle-center bifurcation in area-preserving maps. *Physica D* 136, pp. 266–279 (2000)

[10] V. Gelfreich, D. Sauzin. Borel summation and the splitting of separatrices for the Hénon map. Annales l'Institut Fourier, vol. 51, fasc. 2, pp. 513–567 (2001)

[11] J. Guckenheimer, Ph. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Appl. Math. Sciences, 42. Springer, 453 p. (1983)

[12] A.Yu. Kuznetsov, *Elements of applied bifurcation theory*, Springer, 1998

[13] V.F. Lazutkin. Splitting of separatrices for the Chirikov's standard map. VINITI no. 6372/84 (1984) (Russian). Revised English version: Mathematical Physics Preprint Archive 98-421, `http://www.ma.utexas.edu/mp_arc`

[14] C. Simó. Analytical and numerical detection of exponentially small phenomena. Fiedler, B. (ed.) et al., *International conference on differential equations.* Proceedings of the conference, Equadiff '99, Berlin, Germany, August 1-7, 1999. Vol. 2. World Scientific. 967-976 (2000).

[15] F. Takens. Forced oscillations and bifurcations, in *Applications of global analysis*, I (Sympos., Utrecht State Univ., Utrecht, 1973), pp. 1–59. Comm. Math. Inst. Rijksuniv. Utrecht, No. 3-1974.