Phase-averaged transport for quasi-periodic Hamiltonians

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Abstract

For a class of discrete quasi-periodic Schrödinger operators defined by covariant representations of the rotation algebra, a lower bound on phase-averaged transport in terms of the multifractal dimensions of the density of states is proven. This result is established under a Diophantine condition on the incommensuration parameter. The relevant class of operators is distinguished by invariance with respect to symmetry automorphisms of the rotation algebra. It includes the critical Harper (almost-Mathieu) operator. As a by-product, a new solution of the frame problem associated with Weyl-Heisenberg-Gabor lattices of coherent states is given.

1 Introduction

This work is devoted to proving a lower bound on the diffusion exponents of a class of quasiperiodic Hamiltonians in terms of the multifractal dimensions of their density of states (DOS). The class of models involved describes the motion of a charged particle in a perfect two-dimensional crystal with 3-fold, 4-fold or 6-fold symmetry, submitted to a uniform irrational magnetic field. Irrationality means that the magnetic flux through each lattice cell is equal to an irrational number θ in units of the flux quantum. As shown by Harper [Har] in the specific case of a square lattice with nearest neighbor hopping, the Landau gauge allows to reduce such models to a family of Hamiltonians each describing the motion of a particle on a $1D$ chain with quasiperiodic potential. The latter representation gives a strongly continuous family $H = (H_\omega)_{\omega\in\mathbb{T}}$ of self-adjoint bounded operators on the Hilbert space $\ell^2(\mathbb{Z})$ of the chain indexed by a phase $\omega \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. This family satisfies the covariance relation $\widehat{T} H_\omega \widehat{T}^{-1} = H_{\omega+2\pi\theta}$ (here $\widehat{T}$ represents the operator of translation by one site along the chain).

The phase-averaged diffusion exponents $\beta(q)$, $q>0$, of H are defined by:

$$\int_{\mathbb{T}} d\omega \int_{-T}^{T} \frac{dt}{2T} \langle\phi|e^{\imath H_\omega t}|\widehat{X}|^q e^{-\imath H_\omega t}|\phi\rangle \underset{T\uparrow\infty}{\sim} T^{q\beta(q)} ,$$

where $\widehat{X}$ denotes the position operator on the chain. The DOS of the family H is the Borel measure $\mathcal{N}$ defined by phase-averaging the spectral measure with respect to any site. Its generalized multifractal dimensions $D_{\mathcal{N}}(q)$ for $q\neq 1$ are formally defined by

$$\int_{\mathbb{R}} d\mathcal{N}(E) \left(\int_{E-\varepsilon}^{E+\varepsilon} d\mathcal{N}(E')\right)^{q-1} \underset{\varepsilon\downarrow 0}{\sim} \varepsilon^{(q-1)D_{\mathcal{N}}(q)} .$$

A somewhat imprecise statement of the main result of this work is: whenever $\theta/2\pi$ is a Roth number [Her] (namely, for any $\epsilon>0$, there is $c>0$ such that $|\theta-p/q|\geq c/q^{2+\epsilon}$ for all $p/q\in\mathbb{Q}$), and for the class of models mentioned above, the following inequality holds for all $0<q<1$

$$\beta(q) \geq D_{\mathcal{N}}(1-q) . \tag{1}$$

This result can be reformulated in terms of two-dimensional magnetic operators on the lattice and then gives an improvement of the general Guarneri-Combes-Last lower bound [Gua, Com, Las] by a factor 2. More precise definitions and statements will be given in Section 2.

The inequality (1) has been motivated by work by Piéchon [Pie], who gave heuristic arguments and numerical support for $\beta(q)=D_{\mathcal{N}}(1-q)$ for $q>0$, valid for the Harper model and the Fibonacci chain (for the latter case, a perturbative argument was also given). It was theoretically and numerically demonstrated by Mantica [Man] that the same *exact* relation between spectral and transport exponents is also valid for the Jacobi matrices associated with a Julia set. This result was rigorously proven in [GSB1, BSB]. For the latter operators, the DOS and the local density of states (LDOS) coincide.

Numerous works [Gua, Com, Las, GSB2, GSB3, BGT] yield lower bounds on the quantum diffusion of a given wave packet in terms of the fractal properties of the corresponding LDOS. These rigorous lower bounds are typically not optimal as shown by numerical simulations [GM, KKKG]. Better lower bounds are obtained if the behaviour of generalized eigenfunctions is taken into account [KKKG]. Kiselev and Last have proven general rigorous bounds in terms of upper bounds for the algebraic decay of the eigenfunctions [KL].

However, in most models used in solid state physics, the Hamiltonian is a covariant strongly continuous family of self-adjoint operators [Bel] indexed by a variable which represents the phase or the configuration of disorder. The measure class of the singular part of the LDOS may sensitively depend on the phase [DS]. In addition, the multifractal dimensions are not even measure class invariants [SBB] (unlike the Hausdorff and packing dimensions). This raises concerns about the practical relevance of bounds based on multifractal dimensions of the LDOS in this context. The bound (1) has a threefold advantage: (i) it involves the DOS, which is phase-averaged; (ii) it does not require information about eigenfunctions; (iii) the exponent of phase-averaged transport is the one that determines the low temperature behaviour of the conductivity [SBB].

The present formulation uses the C*-algebraic framework introduced by one of the authors for the study of homogeneous models of solid state physics. While referring to [Bel, SBB] for motivations and details, in the opening Section 2 we briefly recall some of the basic notions. A precise statement of our main results is also given in Section 2, along with an outline of the logical structure of their proofs. In the subsequent sections we present more results and proofs.

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2 Notations and results

A number $\alpha \in \mathbb{R}$ is of Roth type if and only if, for any $\epsilon > 0$, there is a constant $c_\epsilon > 0$ such that for all rational number p/q the following inequality holds

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c_\epsilon}{q^{2+\epsilon}} . \tag{2}$$

Most properties of numbers of Roth type can be found in [Her]. They form a set of full Lebesgue measure containing all algebraic numbers (Roth's theorem). $\theta > 0$ will be called a Roth angle if $\theta/2\pi$ is a number of Roth type.

The *rotation algebra* $\mathcal{A}_\theta$ [Rie] is the smallest C^*-algebra generated by two unitaries U and V, such that $UV = e^{\imath\theta} VU$. It is convenient to set $W_\theta(\mathbf{m}) = e^{-\imath\theta m_1 m_2/2} U^{m_1} V^{m_2}$, whenever $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$. The $W_\theta(\mathbf{m})$'s are unitary operators satisfying $W_\theta(\mathbf{l})W_\theta(\mathbf{m}) = e^{\imath(\theta/2)\,\mathbf{l}\wedge\mathbf{m}} W_\theta(\mathbf{l}+\mathbf{m})$ where $\mathbf{l}\wedge\mathbf{m} = l_1 m_2 - l_2 m_1$. The unique trace on $\mathcal{A}_\theta$ ($\theta/2\pi$ irrational) is defined by $\mathcal{T}_\theta(W_\theta(\mathbf{m})) = \delta_{\mathbf{m},0}$. A strongly continuous action of the torus $\mathbb{T}^2$ on $\mathcal{A}_\theta$ is given by $((k_1, k_2), W_\theta(\mathbf{m})) \in \mathbb{T}^2 \times \mathcal{A}_\theta \mapsto e^{\imath(m_1 k_1 + m_2 k_2)} W_\theta(\mathbf{m})$. The associated $*$-derivations are denoted by δ_1, δ_2. For $n \in \mathbb{N}$, one says $A \in \mathcal{C}^n(\mathcal{A}_\theta)$ if $\delta_1^{m_1}\delta_2^{m_2} A \in \mathcal{A}_\theta$ for all positive integers m_1, m_2 satisfying $m_1 + m_2 \leq n$.

$\mathcal{A}_\theta$ admits three classes of representations that will be considered in this work. The $1D$-*covariant representations* is a faithful family $(\pi_\omega)_{\omega\in\mathbb{R}}$ of representations on $\ell^2(\mathbb{Z})$ defined by $\pi_\omega(U) = \widehat{T}$ and $\pi_\omega(V) = e^{\imath(\omega - \theta\widehat{X})}$ where $\widehat{T}$ and $\widehat{X}$ are the shift and the position operator respectively, namely

$$\widehat{T}\, u(n) = u(n-1) , \qquad \widehat{X}\, u(n) = n\, u(n) , \qquad \forall \;\; u \in \ell^2(\mathbb{Z}) .$$

It follows that $\pi_{\omega+2\pi} = \pi_\omega$ (*periodicity*) and that $\widehat{T}\pi_\omega(\cdot)\widehat{T}^{-1} = \pi_{\omega+\theta}(\cdot)$ (*covariance*). Moreover $\omega \mapsto \pi_\omega(\cdot)$ is strongly continuous. In the sequel, it will be useful to denote by $|n\rangle = u_n$ $(n \in \mathbb{Z})$ the canonical basis of $\ell^2(\mathbb{Z})$ defined by $u_n(n') = \delta_{n,n'}$. The $2D$-*representation* (or the GNS-representation of $\mathcal{T}_{theta}$) is given by the magnetic translations on $\ell^2(\mathbb{Z}^2)$ (in symmetric gauge):

$$\pi_{2D}(W_\theta(\mathbf{m}))\psi(\mathbf{l}) = e^{\imath\theta\mathbf{m}\wedge\mathbf{l}/2}\psi(\mathbf{l}-\mathbf{m}) , \qquad \psi \in \ell^2(\mathbb{Z}^2) .$$

The position operators on $\ell^2(\mathbb{Z}^2)$ are denoted by (X_1, X_2). The *Weyl representation* π_W acts on $L^2(\mathbb{R})$. Let Q and P denote the position and momentum operators defined by $Q\phi(x) = x\phi(x)$

and $P\phi = -\imath d\phi/dx$ whenever ϕ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. It is known that Q and P are essentially selfadjoint and satisfy the canonical commutation rule $[Q,P]=\imath\mathbf{1}$. Then π_W is defined by

$$\pi_W(U) \;=\; e^{\imath\sqrt{\theta}P}\;, \qquad\qquad \pi_W(V) \;=\; e^{\imath\sqrt{\theta}Q}\;.$$

For every $\theta > 0$, π_W and π_{2D} are unitarily equivalent and faithful. More results about $\mathcal{A}_\theta$ are reviewed in Section 3.2.

The group $SL(2,\mathbb{Z})$ acts on $\mathcal{A}_\theta$ through the automorphisms $\widehat{\eta}_S(W_\theta(\mathbf{m})) = W_\theta(S\mathbf{m})$, $S \in SL(2,\mathbb{Z})$. S is called a *symmetry* if $S \neq \pm\mathbf{1}$ and $\sup_{n\in\mathbb{N}} \|S^n\| < \infty$. Of special interest are the 3-fold, 4-fold and 6-fold symmetries

$$S_3 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad S_4 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S_6 \;=\; \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively generating the symmetry groups of the hexagonal (or honeycomb), square and triangular lattices in dimension 2.

In this work, the Hamiltonian $H = H^*$ is an element of $\mathcal{A}_\theta$. Of particular interest are Hamiltonians invariant under some symmetry $S \in SL(2,\mathbb{Z})$, that is $\widehat{\eta}_S(H) = H$. The most prominent among such operators is the (critical) Harper Hamiltonian on a square lattice $H_4 = U + U^{-1} + V + V^{-1}$. For the sake of concreteness, let us write out its covariant representations

$$\pi_\omega(H_4)u(n) \;=\; u(n+1) + u(n-1) + 2\cos(n\theta+\omega)u(n)\;, \qquad u \in \ell^2(\mathbb{Z})\;.$$

Its Weyl representation is $\pi_W(H_4) \;=\; 2\,\cos(\sqrt{\theta}\,Q) + 2\,\cos(\sqrt{\theta}\,P)$. Further examples are the magnetic operator on a triangular lattice $H_6 = U + U^{-1} + V + V^{-1} + e^{-\imath\theta/2}UV + e^{-\imath\theta/2}U^{-1}V^{-1}$ as well as on a hexagonal lattice (which reduces to two triangular ones [Ram]).

For $H = H^* \in \mathcal{A}_\theta$ let us introduce the notations $H_\omega = \pi_\omega(H)$ and $H_{2D} = \pi_{2D}(H)$. Its *density of states* (DOS) is the measure $\mathcal{N}$ defined by (see, *e.g.*, [Bel])

$$\int_{\mathbb{R}} d\mathcal{N}(E)\, f(E) \;=\; \mathcal{T}_\theta\,(f(H)) \;=\; \langle \mathbf{0}|\, f(H_{2D})\,|\mathbf{0}\rangle \;=\; \lim_{\Lambda\to\infty} \frac{1}{\Lambda}\mathrm{Tr}_\Lambda(H_\omega)\;, \qquad f \in \mathcal{C}_0(\mathbb{R})\;. \tag{3}$$

Here $|\mathbf{0}\rangle$ denotes the normalized state localized at the origin of $\mathbb{Z}^2$, $\mathrm{Tr}_\Lambda(A) = \sum_{n=1}^{\Lambda}\langle n|A|n\rangle$ and the last equality in (3) holds almost surely. For a Borel set $\Delta \subset \mathbb{R}$ and a Borel measure μ, the family of *generalized multifractal dimensions* is defined by

$$D_\mu^{\pm}(\Delta;q) \;=\; \frac{1}{1-q}\,\lim_{T\to\infty}{}^{\pm}\, \frac{\log\left(\int_\Delta d\mu(E)\, \left(\int_\Delta d\mu(E')\, \exp(-(E-E')^2T^2)\right)^{q-1}\right)}{\log(T)}\;, \tag{4}$$

where $\lim^+$ and $\lim^-$ denote $\limsup$ or $\liminf$ respectively. The gaussian $\exp(-(E-E')^2T^2)$ may be replaced by the indicator function on $[E-\frac{1}{T}, E+\frac{1}{T}]$ without changing the values of the generalized dimensions [GSB3].

Let now $H \in \mathcal{C}^2(\mathcal{A}_\theta)$. The diffusion exponents of H_{2D} are defined by

$$\beta_{2D}^{\pm}(H,\Delta;q) \;=\; \lim_{T\to\infty}{}^{\pm}\, \frac{\log(\langle M_{2D}(H,\Delta;q,\cdot)\rangle_T)}{q\log(T)}\;, \qquad q \in (0,2]\;, \tag{5}$$

where

$$M_{\rm 2D}(H,\Delta;q,t) \;=\; \langle \mathbf{0}|\chi_\Delta(H_{\rm 2D})\, e^{\imath H_{\rm 2D}t}(|X_1|^q+|X_2|^q)e^{-\imath H_{\rm 2D}t}\chi_\Delta(H_{\rm 2D})|\mathbf{0}\rangle\,, \tag{6}$$

and $\langle f(\cdot)\rangle_T$ denotes the average $\int_{-T}^{+T} dt\, f(t)/2T$ of a measurable function $t\in\mathbb{R}\mapsto f(t)\in\mathbb{R}$. The *phase-averaged diffusion exponents* of the covariant family $(H_\omega)_{\omega\in\Omega}$ are defined as in (5) as growth exponents of

$$M_{\rm 1D}(H,\Delta;q,t) \;=\; \int_{\mathbb{T}} \frac{d\omega}{2\pi}\,\langle 0|\chi_\Delta(H_\omega)\, e^{\imath H_\omega t}|\hat{X}|^q e^{-\imath H_\omega t}\chi_\Delta(H_\omega)|0\rangle\,, \tag{7}$$

Because $H\in\mathcal{C}^2(\mathcal{A}_\theta)$ and $q\in(0,2]$, $M_{\rm 2D}(H,\Delta;q,t)$ and $M_{\rm 1D}(H,\Delta;q,t)$ are finite. Moreover, $\beta^\pm_{\rm 2D}(H,\Delta;q)$ and $\beta^\pm_{\rm 1D}(H,\Delta;q)$ take values in the interval $[0,1]$ [SBB].

Main Theorem *Let θ be a Roth angle and $H=H^*\in\mathcal{C}^2(\mathcal{A}_\theta)$.*
(i) *For any Borel subset $\Delta\subset\mathbb{R}$ and $q\in(0,1)$*

$$\beta^\pm_{\rm 2D}(H,\Delta;q)\;\geq\; D^\pm_{\mathcal{N}}(\Delta;1-q)\,. \tag{8}$$

(ii) *Let H be invariant under some symmetry $S\in SL(2,\mathbb{Z})$. Then, for any Borel subset $\Delta\subset\mathbb{R}$ and $q\in(0,1)$*

$$\beta^\pm_{\rm 1D}(H,\Delta;q)\;\geq\; D^\pm_{\mathcal{N}}(\Delta;1-q)\,. \tag{9}$$

Remark 1 Existing lower bounds (inequalities proved in [GSB3, BGT]) yield $\beta^\pm_{\rm 2D}(H,\Delta;q)\geq\frac{1}{2}D^\pm_{\mathcal{N}}(\Delta;1/(1+q))$ where the factor $\frac{1}{2}$ stems from the dimension of physical space. In addition, $D^\pm_{\mathcal{N}}(\Delta;1-q)\geq D^\pm_{\mathcal{N}}(\Delta;1/(1+q))$, so inequality (8) substantially improves such bounds. The same is true of the inequality in Theorem 1 below which is actually the key to the bounds (8) and (9). This crucial improvement follows from an almost-sure estimate on the growth of the generalized eigenfunctions in the Weyl representation (*cf.* Proposition 4 below) which in turn follows from number-theoretic estimates. As in [KL], a control on the asymptotics of the generalized eigenfunctions then leads to an improved lower bound on the diffusion coefficients (here by a factor 2 at $q=0$).

Remark 2 The bound (8) is of practical interest especially if H is invariant under some symmetry. Non-symmetric Hamiltonians may lead to ballistic motion and absolutely continuous spectral measures (as it is generically the case for the non-critical Harper Hamiltonian, see [Jit] and references therein). In this situation, the bound becomes trivial because both sides in (9) are equal to 1.

Remark 3 Numerical results [TK, RP] as well as the Thouless property [RP] support that $D_{\mathcal{N}}(-1)=\frac{1}{2}$ in the case of the critical Harper Hamiltonian H_4 for Diophantine $\theta/(2\pi)$. According to (9), one thus expects $\beta_{\rm 1D}(H_4,\mathbb{R};2)\geq\frac{1}{2}$.

Remark 4 Numerical simulations by Piéchon [Pie] for the Harper model with some strongly incommensurate $\theta/(2\pi)$ indicate that (9) may actually be an exact estimate. Piechon also gave a perturbative argument supporting the equality $\beta_{1D}(H;q) = D_{\mathcal{N}}(1-q)$ in the case of the Fibonacci Hamiltonian, and verified it numerically. The techniques of the present article do not apply to the Fibonacci model which has no phase-space symmetry.

Remark 5 Our proof forces $q \in (0,1)$ (see Lemma 3). If $D^+_{\mathcal{N}}(\Delta;q) = D^-_{\mathcal{N}}(\Delta;q)$ for all $q \neq 1$, the large deviation technique of [GSB3] leads to (8) for all $q > 0$ (if $H \in \mathcal{C}^\infty(\mathcal{A}_\theta)$) and (9) for all $q \in (0,2]$. Numerical results [TK, RP] suggest that the upper and lower fractal dimensions indeed coincide for Diophantine $\theta/(2\pi)$. This is hardly to be expected for Liouville $\theta/(2\pi)$: the study in [Las] can be taken as an indicator for such *bad* scaling behavior.

Remark 6 Two-sided time averages are used for technical convenience.

Important intermediate steps of the proof are summarized below. Associated with the symmetry S there is a harmonic oscillator Hamiltonian $\mathfrak{H}_S$ invariant under $\widehat{\eta}_S$ with ground state $\phi_S \in \mathcal{S}(\mathbb{R})$, see Section 3.3. In the case of S_4 (relevant to the critical Harper model) this is the conventional harmonic oscillator hamiltonian $\mathfrak{H}_{S\,4} = (P^2+Q^2)/2$, and ϕ_S is the gaussian state. Let ρ_S be the spectral measure of $H_W = \pi_W(H)$ with respect to ϕ_S.

Proposition 1 *Let $\theta > 2\pi$. There are two positive constants $c_\pm$ such that for any Borel subset $\Delta \subset \mathbb{R}$*

$$c_-\,\mathcal{N}(\Delta) \;\leq\; \rho_S(\Delta) \;=\; \langle \phi_S\,|\chi_\Delta(H_W)|\phi_S\,\rangle \;\leq\; c_+\,\mathcal{N}(\Delta)\,.$$

In particular, $\mathcal{N}$ and ρ_S have same multifractal exponents.

The Hamiltonian $\mathfrak{H}_S$ will be used to study transport in phase space. Similarly to eqs. (5) and (6), moments of the phase space distance and growth exponents thereof can be defined in the Weyl representation as follows:

$$M_W(H,\Delta;q,t) \;=\; \langle \phi_S\,|\chi_\Delta(H_W)\,e^{\imath t H_W}\,\mathfrak{H}_S{}^{q/2}e^{-\imath t H_W}\,\chi_\Delta(H_W)|\phi_S\,\rangle\,,$$

$$\beta^\pm_W(H,\Delta;q) \;=\; \lim_{T\to\infty}{}^{\pm}\;\frac{\log(\langle M_W(H,\Delta;q,\cdot)\rangle_T)}{q\log(T)}\,.$$

Proposition 2 *Let $\theta > 2\pi$ and $H = H^* \in \mathcal{C}^2(\mathcal{A}_\theta)$. For $q \in (0,2]$,*

$$\beta^\pm_W(H,\Delta;q) \;=\; \beta^\pm_{2D}(H,\Delta;q)\,.$$

Proposition 3 *Let $\theta > 2\pi$ and $H = H^* \in \mathcal{C}^2(\mathcal{A}_\theta)$ be invariant under $\widehat{\eta}_S$ for some symmetry $S \in SL(2,\mathbb{Z})$. Then*

$$\beta^\pm_W(H,\Delta;q) \;\leq\; \beta^\pm_{1D}(H,\Delta;q)\,, \qquad q \in (0,2]\,.$$

Thanks to Propositions 1, 2 and 3 and since θ may be replaced by $\theta + 2\pi$ without changing the $1D$ and $2D$-representations, the Main Theorem is a direct consequence of the following:

Theorem 1 *Let $H = H^* \in \mathcal{C}^2(\mathcal{A}_\theta)$ and $\theta > 2\pi$ be a Roth angle. Then, for any Borel subset $\Delta \subset \mathbb{R}$*

$$\beta^\pm_W(H,\Delta;q) \;\geq\; D^\pm_{\rho_S}(\Delta;1-q)\,, \qquad\qquad \forall\;\, q \in (0,1)\,.$$

The proof of Theorem 1 will require two technical steps that are worth being mentioned here. The first one requires some notations. Given a symmetry S, let Π_S be the projection onto the H_W-cyclic subspace $\mathcal{H}_S \subset \mathcal{H}$ of ϕ_S. Using the spectral theorem, there is an isomorphism between $\mathcal{H}_S$ and $L^2(\mathbb{R}, d\rho_S)$. If $(\phi_S^{(n)})_{n\in\mathbb{N}}$ denotes the orthonormal basis of eigenstates of $\mathfrak{H}_S$ in $\mathcal{H}$, let $\Phi_{n,S}(E)$ be the representative of $\Pi_S\, \phi_S^{(n)}$ in $L^2(\mathbb{R}, d\rho_S)$. Then:

Proposition 4 *Let $H = H^* \in \mathcal{C}^2(\mathcal{A}_\theta)$ and let θ be a Roth angle. Then for any $\epsilon > 0$ there is $c_\epsilon > 0$ such that*

$$\sum_{n=0}^{\infty} |\Phi_{n,S}(E)|^2\, e^{-\delta\,(n+1/2)} \;\leq\; c_\epsilon\, \delta^{-(1/2+\epsilon)}\,, \qquad \forall\;\; 0 < \delta < 1\,, \qquad \rho_S - a.\,e.\;\; E \in \mathbb{R}\,.$$

Remark 7 This result is uniform (ρ_S-almost surely) with respect to the spectral parameter E and to δ. In particular, integrating over E with respect to ρ_S shows that $\sum_{n=0}^{N-1} \|\Pi_S\, \phi_S^{(n)}\|^2 = O(N^{1/2+\epsilon})$. This is possible because of the following complementary result proved in the Appendix:

Proposition 5 *Let $H = H^* \in \mathcal{A}_\theta$. Then H_W has infinite multiplicity and no cyclic vector.*

The second technical result concerns the so-called *Mehler kernel* of the Hamiltonian $\mathfrak{H}_S$, notably the integral kernel of the operator $e^{-t\,\mathfrak{H}_S}$ in the Q-representation:

$$\mathcal{M}_S(t; x, y) \;=\; \langle x|\, e^{-t\,\mathfrak{H}_S}\; |y\rangle\,, \tag{10}$$

Proposition 6 *Let θ be a Roth angle. Then, for all $\epsilon > 0$,*

$$\sup_{0\leq x\leq 2\pi\theta^{-1/2},\, 0\leq y\leq\theta^{1/2}} \;\sum_{m\in\mathbb{Z}^2} |\mathcal{M}_S(t; x + 2\pi m_1\theta^{-1/2}, y + \theta^{1/2} m_2)| \;=\; \mathcal{O}(t^{-1/2-\epsilon})\,, \qquad as\;\; t \downarrow 0\,.$$

3 Weyl's calculus

This chapter begins with a review of basic facts about Weyl operators, the rotation algebra and implementation of symmetries therein. The formulas are well-known (*e.g.* [Per, Bel94] and mainly given in order to fix notations, but for the convenience of the reader their proofs are nevertheless given in the Appendix. The chapter also contains a new and compact solution of the frame problem for coherent states (Section 3.4).

3.1 Weyl operators

Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbb{R})$. Given a vector $\mathfrak{a} = (a_1, a_2) \in \mathbb{R}^2$, the associated Weyl operator is defined by:

$$\mathfrak{W}(\mathfrak{a}) \;=\; e^{\imath(a_1 P + a_2 Q)} \qquad \Leftrightarrow \qquad \mathfrak{W}(\mathfrak{a})\, \psi(x) \;=\; e^{\imath a_1 a_2/2}\, e^{\imath a_2 x}\, \psi(x + a_1)\,, \qquad \forall\;\; \psi \in \mathcal{H}\,. \tag{11}$$

The Weyl operators are unitaries, strongly continuous with respect to $\mathfrak{a}$ and satisfy

$$\mathfrak{W}(\mathfrak{a})\,\mathfrak{W}(\mathfrak{b}) \;=\; e^{\imath\,\mathfrak{a}\wedge\mathfrak{b}/2}\,\mathfrak{W}(\mathfrak{a}+\mathfrak{b})\,, \qquad \mathfrak{a}\wedge\mathfrak{b} \;=\; a_1b_2-a_2b_1\,. \tag{12}$$

The following weak-integral identities are verified in the Appendix:

$$\langle\psi|\mathfrak{W}(\mathfrak{a})^{-1}\,|\psi\rangle\;\mathfrak{W}(\mathfrak{a}) \;=\; \int_{\mathbb{R}^2}\frac{d^2\mathfrak{b}}{2\pi}\,e^{\imath\,\mathfrak{a}\wedge\mathfrak{b}}\;\mathfrak{W}(\mathfrak{b})\,|\psi\rangle\langle\psi|\mathfrak{W}(\mathfrak{b})^{-1}\,, \tag{13}$$

$$\mathfrak{W}(\mathfrak{b})\,|\psi\rangle\langle\psi|\mathfrak{W}(\mathfrak{b})^{-1} \;=\; \int_{\mathbb{R}^2}\frac{d^2\mathfrak{a}}{2\pi}\,e^{\imath\,\mathfrak{b}\wedge\mathfrak{a}}\;\langle\psi|\mathfrak{W}(\mathfrak{a})^{-1}\,|\psi\rangle\;\mathfrak{W}(\mathfrak{a})\,. \tag{14}$$

Applying (13) to ϕ and setting $\mathfrak{a}=0$ leads to

$$\phi \;=\; \int_{\mathbb{R}^2}\frac{d^2\mathfrak{b}}{2\pi}\;\langle\psi|\mathfrak{W}(\mathfrak{b})^{-1}|\phi\rangle\;\mathfrak{W}(\mathfrak{b})\,\psi\;\;, \qquad \phi,\;\psi\;\in\;\mathcal{H}\;\;, \qquad \|\psi\|=1\;\;. \tag{15}$$

In particular, any non zero vector in $\mathcal{H}$ is cyclic for the Weyl algebra $\{\mathfrak{W}(\mathfrak{a})|\mathfrak{a}\in\mathbb{R}^2\}$. If $\psi\in\mathcal{H}$, the map $\mathfrak{a}\in\mathbb{R}^2\;\mapsto\;\langle\psi|\mathfrak{W}(\mathfrak{a})|\psi\rangle\;\in\;\mathbb{C}$ is continuous, tends to zero at infinity and belongs to $L^2(\mathbb{R}^2)$, whereas $\psi\in\mathcal{S}(\mathbb{R})$ if and only if this map belongs to $\mathcal{S}(\mathbb{R}^2)$.

3.2 The rotation algebra

The *rotation algebra* $\mathcal{A}_\theta$, its representations $(\pi_\omega)_{\omega\in\mathbb{R}}$, $\pi_{\rm 2D}$ and π_W as well as the tracial state $\mathcal{T}_\theta$ and $*$-derivations δ_1,δ_2 were defined in Section 2. Here we give some complements, further definitions and the short proof of Proposition 5. The trace is faithful and satisfies the Fourier formula:

$$A \;=\; \sum_{\mathbf{l}\in\mathbb{Z}^2} a_{\mathbf{l}}\,W_\theta(\mathbf{l})\,, \qquad a_{\mathbf{l}} \;=\; \mathcal{T}_\theta(W_\theta(\mathbf{l})^{-1}\,A)\;\;. \tag{16}$$

In addition,

$$\mathcal{T}_\theta(A) \;=\; \int^{2\pi}\frac{d\omega}{2\pi}\,\langle m|\pi_\omega(A)|m\rangle \;\;=\;\; \langle\mathbf{l}|\pi_{\rm 2D}(A)|\mathbf{l}\rangle\,, \qquad \forall\;A\in\mathcal{A}_\theta\,,\;\forall\;m\in\mathbb{Z},\;\forall\;\mathbf{l}\in\mathbb{Z}^2. \tag{17}$$

The $*$-derivations satisfy $\delta_j\,W_\theta(\mathbf{m}) \;=\; \imath\,m_jW_\theta(\mathbf{m})$, $j\;=\;1,2$. It follows from (16) that $A\in\mathcal{C}^\infty(\mathcal{A}_\theta)$ if and only if the sequence of its Fourier coefficients is fast decreasing. If $A\in\mathcal{C}^\infty(\mathcal{A}_\theta)$ and A is invertible in $\mathcal{A}_\theta$, then $A^{-1}\in\mathcal{C}^\infty(\mathcal{A}_\theta)$. The position operator (X_1,X_2) defined on the space $\mathfrak{s}(\mathbb{Z}^2)$ of Schwartz sequences in $\ell^2(\mathbb{Z}^2)$ forms a *connection* [Co] in the following sense

$$X_j(\pi_{\rm 2D}(A)\phi) \;=\; \pi_{\rm 2D}(\delta_jA)\phi+\pi_{\rm 2D}(A)X_j\psi \qquad \forall\;\;A\in\mathcal{C}^\infty(\mathcal{A}_\theta)\,, \qquad \phi\in\mathfrak{s}(\mathbb{Z}^2)\,. \tag{18}$$

Similarly, if (∇_1,∇_2) is defined on $\mathcal{S}(\mathbb{R})$ by $\nabla_1=-\imath Q/\sqrt{\theta}$, $\nabla_2=\imath P/\sqrt{\theta}$, then

$$\nabla_j(\pi_W(A)\psi) \;=\; \pi_W(\delta_jA)\psi+\pi_W(A)\nabla_j\psi \qquad \forall\;\;A\in\mathcal{C}^\infty(\mathcal{A}_\theta)\,, \qquad \psi\in\mathcal{S}(\mathbb{R})\,. \tag{19}$$

Then $\mathcal{S}(\mathbb{R})$ is exactly the set of $\mathcal{C}^\infty$-elements of $\mathcal{H}$ with respect to $\vec{\nabla}$. In particular, if $\psi \in \mathcal{S}(\mathbb{R})$ and $A \in \mathcal{C}^\infty(\mathcal{A}_\theta)$, then $\pi_W(A)\psi \in \mathcal{S}(\mathbb{R})$.

For the Weyl representation, let us use the notations

$$\pi_W(W_\theta(\mathbf{m})) \;=\; \mathcal{W}_\theta(\mathbf{m}) \;:=\; \mathfrak{W}(\sqrt{\theta}\mathbf{m}) \;, \qquad \forall \;\; \mathbf{m} \in \mathbb{Z}^2 \;. \tag{20}$$

It can be seen as a direct integral of $1D$-representations by introducing the family $(\mathcal{G}_\omega)_{\omega\in\mathbb{R}}$ of transformations from $\mathcal{H}$ into $\ell^2(\mathbb{Z})$

$$(\mathcal{G}_\omega\phi)(n) \;=\; \theta^{-1/4}\,\phi\left(\frac{\omega - n\theta}{\sqrt{\theta}}\right)\;, \qquad \forall \;\; \phi \in \mathcal{H} \;. \tag{21}$$

Then a direct computation (given in the Appendix) shows that:

$$\langle\phi|\pi_W(A)|\phi\rangle \;=\; \int_0^\theta d\omega\,\langle\mathcal{G}_\omega\phi|\pi_\omega(A)|\mathcal{G}_\omega\psi\rangle \;\;,\; A \in \mathcal{A}_\theta \;,\; \phi,\psi \in \mathcal{H} \;. \tag{22}$$

In particular, $\|\phi\|^2 \;=\; \int_0^\theta d\omega\,\|\mathcal{G}_\omega\phi\|^2_{\ell^2}$. The link between π_W and π_{2D} will be established in Section 4.2.

It follows from a theorem by Rieffel [Rie] that the commutant of $\pi_W(\mathcal{A}_\theta)$ is the von Neumann algebra generated by $\pi_W(\mathcal{A}_{\theta'})$ where $\theta'/2\pi \;=\; 2\pi/\theta$ and $\pi_W(W_{\theta'}(\mathbf{l})) \;=\; \mathcal{W}_{\theta'}(\mathbf{l})$. The following result is proven in the Appendix:

Proposition 7 (The generalized Poisson summation formula):

$$T^\theta_\psi \;:=\; \sum_{\mathbf{l}\in\mathbb{Z}^2} \mathcal{W}_{\theta'}(\mathbf{l})\,|\psi\rangle\,\langle\psi|\mathcal{W}_{\theta'}(\mathbf{l})^{-1} \;=\; \frac{\theta}{2\pi}\sum_{\mathbf{m}\in\mathbb{Z}^2}\langle\psi|\mathcal{W}_\theta(\mathbf{m})^{-1}|\psi\rangle\,\mathcal{W}_\theta(\mathbf{m}) \;. \tag{23}$$

By eq. (23), $\psi \in \mathcal{S}(\mathbb{R})$ implies $T^\theta_\psi \in \mathcal{C}^\infty(\mathcal{A}_\theta)$. It follows immediately from eq. (23) that, given $\psi \in \mathcal{S}(\mathbb{R})$, there is a positive element in $\mathcal{A}_\theta$, denoted F^θ_ψ, such that $T^\theta_\psi = (\theta/2\pi)\;\pi_W\left(F^\theta_\psi\right)$. Moreover

$$\langle\psi|\,\pi_W(A)\,|\psi\rangle \;=\; \mathcal{T}_\theta\left(A\,F^\theta_\psi\right)\;, \qquad \forall A \in \mathcal{A}_\theta \,. \tag{24}$$

3.3 Symmetries

It is well-known that $S \in SL(2,\mathbb{R})$ can be uniquely decomposed in a torsion, a dilation and a rotation as follows :

$$S \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}\;,$$

with $\kappa \;=\; (ac+db)/(a^2+b^2)$, $\lambda \;=\; (a^2+b^2)^{1/2}$, $e^{\imath s} \;=\; (a-\imath b)(a^2+b^2)^{-1/2}$. Moreover, if $S \in SL(2,\mathbb{R})$, then there is a unitary transformation $\mathcal{F}_S$ acting on $\mathcal{H}$ such that

$$\mathfrak{W}(S\mathfrak{a}) \;=\; \mathcal{F}_S\,\mathfrak{W}(\mathfrak{a})\,\mathcal{F}_S^{-1}\;, \qquad \mathfrak{a} \in \mathbb{R}^2 \;, \tag{25}$$

as shows the above decomposition as well as the following result, the proof of which is deferred to the Appendix:

Proposition 8 *For any $\kappa, \lambda, s \in \mathbb{R}$, $\lambda \neq 0$, up to a phase*

$$\mathcal{F}_{\left(\begin{smallmatrix} 1 & 0 \\ \kappa & 1 \end{smallmatrix}\right)} = e^{-\imath\,\kappa\, Q^2/2}\,, \quad \mathcal{F}_{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)} = e^{-\imath\,\ln(\lambda)\,(QP+PQ)/2}\,, \quad \mathcal{F}_{\left(\begin{smallmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{smallmatrix}\right)} = e^{-\imath\, s\,(Q^2+P^2-1)/2}\,. \tag{26}$$

Note in particular that $\mathcal{F}_S\,\mathcal{F}_{S'} = z\,\mathcal{F}_{S\,S'}$ for $z \in \mathbb{C}, |z| = 1$. Furthermore, if $0 < s < \pi$,

$$\mathcal{F}_{\left(\begin{smallmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{smallmatrix}\right)}\,\phi(x) \;=\; \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi\,\sin s}}\; e^{\imath\left(\cos s\,(x^2+y^2)-2xy\right)/2\sin s}\;\phi(y)\;. \tag{27}$$

In the special case $s = \pi/2$, namely for the matrix S_4 (see Section 2), this gives the usual Fourier transform

$$\mathcal{F}_{S_4}\,\phi(x) \;=\; \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}}\; e^{-\imath xy}\;\phi(y)\,. \tag{28}$$

For the case of the 3-fold and 6-fold symmetries S_3 and S_6, acting on a hexagonal or a triangular lattice (see Section 2), eqs. (26) and (27) give

$$\mathcal{F}_{S_3}\,\phi(x) \;=\; e^{\imath\pi/12}\int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}}\; e^{-\imath x(x+2y)/2}\;\phi(y)\,, \qquad \mathcal{F}_{S_6}\,\phi(x) \;=\; e^{-\imath\pi/12}\int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi}}\; e^{-\imath y(2x-y)/2}\;\phi(y)\,. \tag{29}$$

Now suppose that $S \in SL(2,\mathbb{R})$ satisfies $S^r = \mathbf{1}$ for some $r \in \mathbb{N}$, $r \geq 2$ and $S^n \neq \mathbf{1}$ for $n < r$. It will be convenient to introduce the following operator acting on $\mathcal{H}$

$$\mathfrak{H}_S \;=\; \frac{1}{2r}\sum_{n=0}^{r-1} \mathcal{F}_S^n\, Q^2 \mathcal{F}_S^{-n} \;=\; \frac{1}{2}\,\langle K|M_S|K\rangle\,, \qquad M_S \;=\; \frac{1}{r}\sum_{n=0}^{r-1} S^n\,|e_2\rangle\langle e_2|\,(S^t)^n\,,$$

where $K = (P,Q)$ and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Note that $\mathfrak{H}_{S_4} = (P^2+Q^2)/2$. There is $0 \leq n \leq r-1$ such that $S^n e_2 \wedge e_2 \neq 0$, so M_S is positive definite and can be diagonalized by a rotation:

$$M_S \;=\; \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \mu_S^+ & 0 \\ 0 & \mu_S^- \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}^{-1}\,.$$

Hence $\mathfrak{H}_S$ is unitarily equivalent to the harmonic oscillator Hamiltonian $(\mu_S^+ P^2 + \mu_S^- Q^2)/2$. Therefore,

$$\mathfrak{H}_S \;=\; \mu\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\,|\phi_S^{(n)}\rangle\langle\phi_S^{(n)}|\,, \qquad \mu \;=\; (\mu_S^+\mu_S^-)^{1/2}\;, \qquad \lambda \;=\; \left(\frac{\mu_S^+}{\mu_S^-}\right)^{1/4}\;, \tag{30}$$

where the $\phi_S^{(n)}$ are the eigenstates. The ground state is denoted $\phi_S \equiv \phi_S^{(0)}$.

Proposition 9 *Up to a phase, the ground state is given by*

$$\phi_S\,(x) \;=\; \left(\frac{\Re e(\sigma_S)}{\pi}\right)^{1/4} e^{-\sigma_S\,x^2/2}\,, \qquad \sigma_s \;=\; \frac{\sqrt{\mu_S^-}\,\cos\gamma + \imath\sqrt{\mu_S^+}\sin\gamma}{\sqrt{\mu_S^+}\,\cos\gamma + \imath\sqrt{\mu_S^-}\sin\gamma}\;, \tag{31}$$

and the Mehler kernel (10) *by*

$$\mathcal{M}_S(t;x,y) \;=\; \frac{e^{-\frac{(x-y)^2\tanh(t\mu)^{-1}+(x+y)^2\tanh(t\mu)}{4(\lambda^2\cos\gamma^2+\lambda^{-2}\sin\gamma^2)}}}{\lambda\sqrt{2\pi\sinh(t\mu)(\lambda^2\cos\gamma^2+\lambda^{-2}\sin\gamma^2)}}\; e^{\imath(x^2-y^2)\frac{\sin(2\gamma)(\lambda^2-\lambda^{-2})}{4(\lambda^2\cos\gamma^2+\lambda^{-2}\sin\gamma^2)}}\,. \tag{32}$$

By construction, $\mathcal{F}_S\mathfrak{H}_S\,\mathcal{F}_S^* = \mathfrak{H}_S$, so that $\mathcal{F}_S\phi_S \;= e^{\imath\delta_S}\phi_S$ for some phase δ_S. Thus, it is possible to choose the phase of $\mathcal{F}_S$ such that $\mathcal{F}_S\phi_S = \phi_S$. Such is the case for $\mathcal{F}_{S_i}$ in eqs. (28) and (29).

Recall from Section 2 that $\pm\mathbf{1} \neq S \in SL(2,\mathbb{Z})$ is called a symmetry of $\mathcal{A}_\theta$ if $\sup_{n\in\mathbb{Z}}\|S^n\| < \infty$. Since the set of $M \in SL(2,\mathbb{Z})$ with $\|M\| \leq c$ is finite (for any $0 < c < \infty$), and since $S \neq \pm\mathbf{1}$, there is an integer $r \in \mathbb{N}_*$ such that $S^r = \mathbf{1}$ and $S^n \neq \mathbf{1}$ for $0 < n < r$. So the two eigenvalues are $\{e^{\pm\imath\varphi_s}\}$, with $r\varphi_s = 0 \pmod{2\pi}$ and $\varphi_s \neq 0,\pi$. In particular $\mathrm{Tr}(S) = 2\cos\varphi_s \in \mathbb{Z}$, implying $r \in \{3,4,6\}$ and $\varphi_s \in \{\pm\pi/3\,,\pm\pi/2\,,\pm2\pi/3\}$. Any $S \in SL(2,\mathbb{Z})$ defines a $*$-automorphism $\widehat{\eta}_S$ of $\mathcal{A}_\theta$ through $\widehat{\eta}_S(W_\theta(\mathbf{m})) = W_\theta(S\mathbf{m})$. According to the above, $\pi_W(\widehat{\eta}_S(W_\theta(\mathbf{m}))) = \mathcal{F}_S\pi_W(W_\theta(S\mathbf{m}))\mathcal{F}_S^{-1}$.

3.4 θ-traces and θ-frames

Definition 1 *A vector $\psi \in \mathcal{H}$ will be called* θ-tracial *if $\langle\psi|\mathcal{W}_\theta(\mathbf{l})\,|\psi\rangle = \mathcal{T}_\theta(W_\theta(\mathbf{l})) = \delta_{\mathbf{l},0}$ for all $\mathbf{l} \in \mathbb{Z}^2$. Equivalently, the family $(\mathcal{W}_\theta(\mathbf{l})\psi)_{\mathbf{l}\in\mathbb{Z}^2}$ is orthonormal.*

Using the commutation rules (12), it is possible to check that ψ is θ-tracial if and only if $\mathfrak{W}(\mathfrak{a})\psi$ is θ-tracial for any $\mathfrak{a} \in \mathbb{R}^2$. It also follows from eq. (23) that ψ is θ-tracial if and only if $T_\psi^\theta = (\theta/2\pi)\mathbf{1}$. Such θ-tracial states exist under the following condition:

Theorem 2 *There is a θ-tracial vector $\psi \in \mathcal{H}$ if and only if $\theta \geq 2\pi$. If $\theta > 2\pi$ there is a θ-tracial vector in $\mathcal{S}(\mathbb{R})$. For $\theta \geq 2\pi$, denote by Π_ψ the projection on the orthocomplement of the ψ-cyclic subspace $\overline{\pi_W(\mathcal{A}_\theta)\psi} \subset \mathcal{H}$. There is a projection $P_\psi \in \mathcal{A}_{\theta'}$ satisfying $\pi_W(P_\psi) = \Pi_\psi$ and $\mathcal{T}_{\theta'}(P_\psi) = 1 - 2\pi/\theta$. In particular, ψ is also $\mathcal{A}_\theta$-cyclic for $\theta = 2\pi$.*

Proof: If ψ is θ-tracial, then $(\theta/2\pi) = \langle\psi|T_\psi^\theta|\psi\rangle = \sum_{\mathbf{l}\in\mathbb{Z}^2}|\langle\mathcal{W}_{\theta'}(\mathbf{l})\psi\,|\psi\rangle|^2 \geq \|\psi\|^2 = 1$.

If $\theta > 2\pi$, for $0 < \varepsilon < \min(2\pi,\theta-2\pi)$, let ϕ be a C^∞ function on $\mathbb{R}$ such that $0 \leq \phi \leq 1$, with support in $[0,2\pi+\varepsilon]$, such that $\phi = 1$ on $[\varepsilon,2\pi]$, and $\phi(x)^2 + \phi(x+2\pi)^2 = 1$ whenever $0 \leq x \leq \varepsilon$. Using (22), ϕ is θ-tracial (after normalization), and belongs to $\mathcal{S}(\mathbb{R})$. If $\theta = 2\pi$, the same argument holds with $\varepsilon = 0$. Then $\phi \in \mathcal{H}$, but it is not smooth anymore.

Let ψ be θ-tracial. Exchanging the rôles of θ and θ', the Poisson summation formula implies

$$T_\psi^{\theta'} \;=\; \sum_{\mathbf{m}\in\mathbb{Z}^2}\mathcal{W}_\theta(\mathbf{m})|\psi\rangle\langle\psi|\mathcal{W}_\theta(\mathbf{m})^{-1} \;=\; \frac{2\pi}{\theta}\sum_{\mathbf{l}\in\mathbb{Z}^2}\langle\psi|\mathcal{W}_{\theta'}(\mathbf{l})^{-1}|\psi\rangle\,\mathcal{W}_{\theta'}(\mathbf{l})\,.$$

Hence $\Pi_\psi = 1 - T_\psi^{\theta'}$ is the desired orthonormal projection which, due to the r.h.s., is the Weyl representative of an element $P_\psi \in \mathcal{A}_{\theta'}$. Its trace is $\mathcal{T}_{\theta'}(P_\psi) = 1 - 2\pi/\theta$. If $\theta = 2\pi$, since the trace is faithful, $T_\psi^{\theta'} = \mathbf{1}$, so that ψ is cyclic. $\square$

Definition 2 *A vector $\psi \in \mathcal{H}$ is called a θ'-*frame*, if there are constants $0 < c < C < \infty$ such that $c\mathbf{1} \leq T_\psi^\theta \leq C\,\mathbf{1}$.*

This definition is in accordance with the literature ([Sei] and references therein) where the complete set $(\mathcal{W}_{\theta'}(\mathbf{l})\psi)_{\mathbf{l}\in\mathbb{Z}^2}$ is called a frame. The principal interest of frames is due to the following: any vector $\phi \in \mathcal{H}$ can be decomposed as $\phi = T_\psi^\theta (T_\psi^\theta)^{-1}\phi = \sum_{\mathbf{l}\in\mathbb{Z}^2} c_\mathbf{l}\, \mathcal{W}_{\theta'}(\mathbf{l})\psi$ where $c_\mathbf{l} = \langle\psi|\mathcal{W}_{\theta'}(\mathbf{l})^*(T_\psi^\theta)^{-1}|\phi\rangle$. If $\psi \in \mathcal{S}(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$, then $(c_\mathbf{l})_{\mathbf{l}\in\mathbb{Z}^2} \in \mathfrak{s}(\mathbb{Z}^2)$. Further note that, if ψ is a θ'-frame, then $\hat{\psi} = (\theta/2\pi)^{1/2}(T_\psi^\theta)^{-1/2}\psi$ is θ-tracial. In addition, if $\psi \in \mathcal{S}(\mathbb{R})$ then $\hat{\psi} \in \mathcal{S}(\mathbb{R})$.

The next result shows that so-called Weyl-Heisenberg or Gabor lattices constructed with a gaussian mother state are frames if only the volume of the chosen phase-space cell is sufficiently small. This was proved in [Sei], but the present proof is new and covers more general cases.

Suppose $S \in SL(2,\mathbb{R})$ satisfies $S^r = 1$ for some r. Using the results of Section 3.3 and eq. (11), it is possible to compute

$$\langle\phi_S\,|\mathfrak{W}(\mathfrak{a})|\phi_S\,\rangle \;=\; e^{-|\mathfrak{a}\,|_S^2/4}\;, \qquad |\mathfrak{a}\,|_S^2 \;=\; \frac{\mu_S^+\,a_1^2 + \mu_S^-\,a_2^2}{\mu}\;. \tag{33}$$

Theorem 3 *For $\theta > 2\pi$, ϕ_S is a θ'-frame in $\mathcal{S}(\mathbb{R})$.*

Proof: The proof below is given for $\phi_0 \equiv \phi_{S_4}$, but the same strategy works for any ϕ_S.

Thanks to Poisson's formula (23) and eq. (33), $T_{\phi_0}^\theta \leq (\theta/2\pi)\sum_\mathbf{m} e^{-\theta|\mathbf{m}|^2/4}$. It is therefore enough to find a positive lower bound. Since π_W is faithful, it is enough to show that $T_0 = \sum_\mathbf{m} e^{-\theta|\mathbf{m}|^2/4}W_\theta(\mathbf{m})$ is itself bounded from below in $\mathcal{A}_\theta$. Writing $\theta = 2\pi + \delta$ with $\delta > 0$, there is a $*$-isomorphism between $\mathcal{A}_\theta$ and the closed subalgebra of $\mathcal{A}_{2\pi}\otimes\mathcal{A}_\delta$ generated by $(W_{2\pi}(\mathbf{m})\otimes W_\delta(\mathbf{m}))_{\mathbf{m}\in\mathbb{Z}^2}$. It is enough to show that $\hat{T}_0 = \sum_\mathbf{m} e^{-\theta|\mathbf{m}|^2/4}W_{2\pi}(\mathbf{m})\otimes W_\delta(\mathbf{m})$ is bounded from below in $\mathcal{A}_{2\pi}\otimes\mathcal{A}_\delta$. $\mathcal{A}_{2\pi}$ is abelian and $*$-isomorphic to $C(\mathbb{T}^2)$, provided $W_{2\pi}(\mathbf{m})$ is identified with the map $\kappa = (\kappa_1,\kappa_2) \in \mathbb{T}^2 \mapsto (-1)^{m_1m_2}e^{\imath\kappa\cdot\mathbf{m}} \in \mathbb{C}$. Hence it is enough to show that $\hat{T}_0(\kappa) = \sum_\mathbf{m}(-1)^{m_1m_2}\,e^{-\theta|\mathbf{m}|^2/4+\imath\kappa\cdot\mathbf{m}}\,W_\delta(\mathbf{m})$ is bounded from below in $\mathcal{A}_\delta$ uniformly in κ. Since the Weyl representation is faithful, $W_\delta(\mathbf{m})$ can be replaced by $\mathcal{W}_\delta(\mathbf{m})$. Using eq. (13) with $\psi = \phi_0$ and $\mathfrak{a} = \sqrt{\delta}\mathbf{m}$, it is thus enough to show that

$$\tilde{T}_0(\kappa) \;=\; \int_{\mathbb{R}^2}\frac{d^2\mathfrak{b}}{2\pi}\,\Theta(\kappa_1+\sqrt{\delta}b_2,\kappa_2-\sqrt{\delta}b_1)\,\mathfrak{W}(\mathfrak{b})|\phi_0\,\rangle\langle\phi_0\,|\mathfrak{W}(\mathfrak{b})^{-1}\,,$$

where

$$\Theta(\kappa) \;=\; \sum_{\mathbf{m}\in\mathbb{Z}^2}(-1)^{m_1m_2}\,e^{-\pi|\mathbf{m}|^2/2+\imath(\kappa\cdot\mathbf{m})}\;, \tag{34}$$

is bounded from below. Clearly the function Θ is 2π-periodic in both of its arguments. Hence, decomposing the integral into a sum of integrals over the shifted unit cell $C = [0, 2\pi) \times [0, 2\pi)$ and using $\mathcal{W}_{\delta'}(\mathfrak{a}) = \mathfrak{W}(2\pi\,\mathfrak{a}/\sqrt{\delta})$ gives

$$\tilde{T}_0(\kappa) \;=\; \sum_{\mathbf{l}\in\mathbb{Z}^2} \int_C \frac{d^2\mathfrak{a}}{2\pi\delta}\,\Theta(\mathfrak{a})\,\mathcal{W}_{\delta'}\left(\mathbf{l}+\frac{\mathfrak{a}+\hat{\kappa}}{2\pi}\right)|\phi_0\rangle\langle\phi_0|\mathcal{W}_{\delta'}\left(\mathbf{l}+\frac{\mathfrak{a}+\hat{\kappa}}{2\pi}\right)^{-1}\,,$$

where $\hat{\kappa} = (\kappa_2, -\kappa_1)$. The Poisson summation formula applied to the summation over m_1 in (34) gives a sum over an index n_1. Changing summation indexes $n_2 = m_2 - n_1$ shows that $\Theta(\kappa) = \sqrt{2}\,e^{-\kappa_1^2/2\pi}\,|f(\kappa_1 + \imath\kappa_2)|^2$, where f is the holomorphic entire function given by $f(z) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 - nz}$. It can be checked that $f(z+2\imath\pi) = f(z)$ and that $f(z+2\pi) = e^{z+\pi}\,f(z)$. Moreover, using the Poisson summation formula, f does not vanish on γ, the boundary of C oriented clockwise. As Θ has no poles, the number of zeros of f within C counted with their multiplicity is given by $\oint_\gamma df/2\imath\pi f$. Using the periodicity properties of f, this integral equals 1. Moreover, a direct calculation shows that the unique zero with multiplicity 1 of f lies at the center $\pi(1+\imath)$ of C. Hence there is a constant $c_1 > 0$ such that $|f(\pi + \imath\pi + re^{\imath\varphi})| \geq c_1 r^2$ for all $\varphi \in [0, 2\pi)$. Let B_r denote the ball of size r around $\pi(1+\imath)$. Replacing this shows

$$\tilde{T}_0(\kappa) \;\geq\; \frac{c_1 r^2}{\delta}\left(\mathbf{1} - \int_{B_r} \frac{d^2\mathfrak{a}}{2\pi}\,\mathcal{W}_{\delta'}\left(\frac{\mathfrak{a}+\hat{\kappa}}{2\pi}\right)\,T^{\delta'}_{\phi_0}\;\mathcal{W}_{\delta'}\left(\frac{\mathfrak{a}+\hat{\kappa}}{2\pi}\right)^{-1}\right)\,.$$

As $T^{\delta'}_{\phi_0} \leq c_2\mathbf{1}$, $\tilde{T}_0(\kappa) \geq \mathbf{1}\,c_1 r^2(1 - c_2 r^2/2)/\delta$. Choosing r small enough, $\tilde{T}_0(\kappa)$ is bounded from below by a positive constant uniformly in κ. □

4 Comparison theorems

4.1 Proof of Proposition 1

Let $H = H^* \in \mathcal{A}_\theta$ and set $H_W = \pi_W(H)$. For normalized $\phi \in \mathcal{H}$, ρ_ϕ denotes the spectral measure of H_W relative to ϕ. Proposition 1 is a corollary of the following result:

Theorem 4 *For $\theta \geq 2\pi$, for any normalized θ'-frame $\phi \in \mathcal{H}$ and any Borel subset Δ of $\mathbb{R}$,*

$$\frac{2\pi}{\theta}\|(T^\theta_\phi)^{-1}\|^{-1}\,\mathcal{N}(\Delta) \;\leq\; \rho_\phi(\Delta) \;\leq\; \frac{2\pi}{\theta}\|T^\theta_\phi\|\,\mathcal{N}(\Delta)\,. \tag{35}$$

Proof: Eq. (24) leads to

$$\rho_\phi(\Delta) \;=\; \mathcal{T}_\theta\left(\chi_\Delta(H)\,F^\theta_\phi\right) \;\leq\; \|F^\theta_\phi\|\,\mathcal{N}(\Delta)\,,$$

and to

$$\mathcal{N}(\Delta) \;=\; \mathcal{T}_\theta\left(\chi_\Delta(H)\,F^\theta_\phi(F^\theta_\phi)^{-1}\right) \;\leq\; \rho_\phi(\Delta)\,\|(F^\theta_\phi)^{-1}\|\,.$$

Since $T^\theta_\phi = \theta/2\pi\;\pi_W(F^\theta_\phi)$, the theorem follows. □

4.2 Proof of Proposition 2

Let $\theta > 2\pi$. The ground state ϕ_S of $\mathfrak{H}_S$ is a θ'-frame according to Theorem 3. Let $\psi_S = (\theta/2\pi)^{1/2}(T^{\theta}_{\phi_S})^{-1/2}\phi_S$ be the associated θ-tracial vector. Further set $\mathcal{H}_S = \overline{\pi_W(\mathcal{A}_\theta)\psi_S}$. In this section, π_W denotes the restriction of the Weyl representation to $\mathcal{H}_S$. A unitary transformation $\mathcal{U} : \mathcal{H}_S \to \ell^2(\mathbf{Z}^2)$ is defined by

$$(\mathcal{U}\phi)(\mathbf{l}) \;=\; \langle \psi_S|\mathcal{W}_\theta(\mathbf{l})^{-1}|\phi\rangle\;, \qquad \phi \in \mathcal{H}_S\;,\quad \mathbf{l} \in \mathbb{Z}^2\;.$$

Then $\mathcal{U}\pi_W(A)\mathcal{U}^* = \pi_{\rm 2D}(A)$ for all $A \in \mathcal{A}_\theta$. Moreover $\mathcal{U} : \mathcal{S}(\mathbb{R}) \cap \mathfrak{H}_S \to \mathfrak{s}(\mathbb{Z}^2)$. As $\mathcal{U}\psi_S = |\mathbf{0}\rangle$,

$$M_W(H,\Delta;q,t) \;=\; \langle \mathbf{0}|\chi_\Delta(H_{\rm 2D})\, e^{\imath H_{\rm 2D}t}(\mathcal{U}\mathfrak{H}_S\,\mathcal{U}^*)^{q/2}e^{-\imath H_{\rm 2D}t}\chi_\Delta(H_{\rm 2D})|\mathbf{0}\rangle\;.$$

Recall that $\mathfrak{H}_S$ is a polynomial of second degree in Q and P. From (19) follows

$$\mathcal{U}Q\mathcal{U}^* \;=\; -\theta^{1/2}X_1 + A_1\;, \qquad \mathcal{U}P\mathcal{U}^* \;=\; -\theta^{-1/2}X_2 + A_2\;,$$

where $\langle \mathbf{l}|A_1|\mathbf{m}\rangle = \langle\psi_S|\mathcal{W}_\theta(\mathbf{l}-\mathbf{m})|Q\psi_S\rangle$ and $\langle \mathbf{l}|A_2|\mathbf{m}\rangle = \langle\psi_S|\mathcal{W}_\theta(\mathbf{l}-\mathbf{m})|P\psi_S\rangle$. Because ψ_S, $Q\psi_S$ and $P\psi_S$ are in $\mathcal{S}(\mathbb{R})$, A_1 and A_2 are bounded operators. Using the standard operator inequalities $|AB| \leq \|A\|\,|B|$ and $|A+B| \leq 2(|A|+|B|)$ and the commutation relation $[X_1,X_2] = 0$, it is now possible to deduce $M_W(H,\Delta;q,t) \leq c_1 M_{\rm 2D}(H,\Delta;q,t) + c_2$ for two positive constants c_1 and c_2. An inequality $M_{\rm 2D}(H,\Delta;q,t) \leq c_1 M_W(H,\Delta;q,t) + c_2$ is obtained similarly. This implies Proposition 2.

4.3 Proof of Proposition 3

Lemma 1 *Let $Y_1,\ldots,Y_N$ be selfadjoint operators on $\mathcal{H}$ with common domain which satisfy $[Y_m,Y_n] = \imath\, c_{m,n}\mathbf{1}$. Then, if $c = \max_{m,n}(|c_{m,n}|) > 0$ and if $0 \leq \alpha \leq 1$,*

$$\frac{1}{N}\sum_{n=1}^{N} Y_n^{2\alpha} \;\leq\; \left(\sum_{n=1}^{N} Y_n^2\right)^{\alpha} \;\leq\; \sum_{n=1}^{N} Y_n^{2\alpha} \;+\; 2N(N-1)c^{\alpha}\,. \tag{36}$$

Proof: For $\alpha = 0,1$ both inequalities are trivial. For $0 < \alpha < 1$ the following identity holds

$$A^\alpha \;=\; \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{dv}{v^{1-\alpha}}\,\frac{A}{v+A}\;, \tag{37}$$

for a positive operator A. If $A = \sum_{n=1}^N Y_n^2$ then the left-hand inequality in (36) follows from $Y_n^2 \leq A$ and from the operator monotonicity of $A/(v+A) = \mathbf{1} - v/(v+A)$. On the other hand

$$\frac{A}{v+A} \;=\; \sum_{n=1}^{N}\left(Y_n\frac{1}{v+A}Y_n \;+\; Y_n\left[Y_n,\frac{1}{v+A}\right]\right)\,.$$

The first term of each summand is bounded by $Y_n^2/(v+Y_n^2)$. Noting $Y_n\,[Y_n,(v+A)^{-1}] = Y_n(v+A)^{-1}\,[A,Y_n]\,(v+A)^{-1}$, and using the commutation rules for the Y_n's, the second term in the r.h.s. is estimated by

$$\left\|-2\imath\sum_{m,n} c_{m,n}\,Y_n\frac{1}{v+A}Y_m\frac{1}{v+A}\right\| \;\leq\; 2\,\frac{1}{v+c_0}\sum_{m,n}|c_{m,n}|\,,$$

where c_0 is the infimum of the spectrum of A. In the latter inequality $Y_n^2 \leq A$ has been used. By definition, there are m, n such that $c_{m,n} = c > 0$ so that $Y_n^2 + Y_m^2 = (Y_m - \imath Y_n)(Y_m + \imath Y_n) + c\mathbf{1} \geq c\mathbf{1}$. Hence $c_0 \geq c$. Integrating over v, using the eq. (37), and remarking that $\sum_{m,n} |c_{m,n}| \leq N(N-1)c$ gives the result. □

If $S \in SL(2, \mathbb{Z})$ is a symmetry such that $S^r = 1$, the operators $Y_n = \mathcal{F}_S^n Q \mathcal{F}_S^{-n}$ satisfy the hypothesis of Lemma 1, because calculating the derivative of (25) at $\mathfrak{a} = 0$ shows that each Y_n is linear in P and Q. Clearly $\mathfrak{H}_S = 1/(2r) \sum_{n=1}^r Y_n^2$. If $H \in \mathcal{A}_\theta$ is S-invariant, then $\mathfrak{H}_S(t) = 1/(2r) \sum_{n=1}^r \mathcal{F}_S^n Q^2(t) \mathcal{F}_S^{-n}$, where $A(t) = e^{\imath t H_W} A e^{-\imath t H_W}$ whenever A is an operator on $\mathcal{H}$. Therefore, if $0 \leq q \leq 2$, the inequality (36) leads to (with $\chi_\Delta = \chi_\Delta(H_W)$)

$$\langle \phi_S \,|\, \chi_\Delta \mathfrak{H}_S(t)^{q/2} \chi_\Delta |\phi_S \rangle \;\leq\; r(2r)^{-q/2} \langle \phi_S \,|\, \chi_\Delta |Q(t)|^q \chi_\Delta \,|\phi_S \rangle \;+\; 2r(r-1) \left(\frac{c}{2r}\right)^{q/2} ,$$

where $\mathcal{F}_s \phi_S = \phi_S$ has been used. Proposition 3 is then a direct consequence of the definitions of the exponents $\beta^\pm_{\rm 1D}(H, \Delta; q)$, $\beta^\pm_W(H, \Delta; q)$ and of the following lemma:

Lemma 2 *Let $\phi \in \mathcal{S}(\mathbb{R})$, $\theta \geq 2\pi$ and $q \geq 0$. Then, there are two positive constants c_0, c_1 such that, for any element $B \in \mathcal{A}_\theta$*

$$\langle \phi | B_W^* \, |Q|^q \, B_W \, |\phi\rangle \;\leq\; c_0 \int_0^{2\pi} \frac{d\omega}{2\pi} \, \langle 0 | B_\omega^* \, |\widehat{X}|^q \, B_\omega \, |0\rangle \;+\; c_1 \, ,$$

where $B_W = \pi_W(B)$ and $B_\omega = \pi_\omega(B)$.

Proof: Definition (21) and identity (22) of Section 3.2 lead to

$$\langle \phi | B_W^* \, |Q|^q \, B_W \, |\phi\rangle \;=\; \theta^{(q-1)/2} \int_0^\theta d\omega \sum_{n,n' \in \mathbb{Z}} \overline{\phi\left(\frac{\omega - n\theta}{\sqrt{\theta}}\right)} \, \phi\left(\frac{\omega - n'\theta}{\sqrt{\theta}}\right) \langle n | K_\omega | n' \rangle \, ,$$

with $K_\omega = B_\omega^* |(\omega/\theta) - \widehat{X}|^q B_\omega$. Since K_ω is a positive operator, the Schwarz inequality gives $|\langle n | K_\omega | n' \rangle| \leq (\langle n | K_\omega | n \rangle + \langle n' | K_\omega | n' \rangle)/2$. Both terms can be bounded similarly. The covariance property of π_ω (see Section 3.2) gives $\langle n | K_\omega | n \rangle = \langle 0 | K_{\omega - n\theta} | 0 \rangle$. Since $\phi \in \mathcal{S}(\mathbb{R})$, summing up over n' first, then over n, there are constants C, c_1 such that

$$\langle \phi | B_W^* \, |Q|^q \, B_W \, |\phi\rangle \;\leq\; C \int_{\mathbb{R}} dx \, |\phi(x)| \, \langle 0 | \, K_{x\sqrt{\theta}} \, |0\rangle \;\leq\; C \int_{\mathbb{R}} dx \, |\phi(x)| \, \langle 0 | \, B_{x\sqrt{\theta}}^* |\widehat{X}|^q B_{x\sqrt{\theta}} \, |0\rangle \;+\; c_1 \, ,$$

where the inequality $|x - \widehat{X}|^q \leq C_q(|x|^q + |\widehat{X}|^q)$, valid for $q \geq 0$ and some suitable constant C_q, has been used. Thanks to the periodicity of π_ω, the r.h.s. of the latter estimate can be written as

$$\text{r.h.s.} \;\leq\; \int_0^{2\pi} \frac{d\omega}{\sqrt{\theta}} \langle 0 | \, B_\omega^* |\widehat{X}|^q B_\omega \, |0\rangle \sup_{0 < \omega < 2\pi} \sum_n \left| \phi\left(\frac{\omega - 2\pi n}{\sqrt{\theta}}\right) \right| \;+ c_1 \, ,$$

completing to the proof of the lemma. □

5 Bounds on phase-space transport

Section 5.1 is devoted to the proof of Theorem 1 assuming Propositions 4 and 6 which in turn are proven in the subsequent sections.

5.1 Proof of Theorem 1

The proof goes along the lines of [GSB3] and is reproduced here for the sake of completeness. As shown in [GSB3], the time average $\langle f(\cdot)\rangle_T$ of a non-negative function can be replaced by the gaussian average

$$\langle f(\cdot)\rangle_T^g \;=\; \int_{\mathbf{R}} \frac{dt}{2T\sqrt{\pi}}\, e^{-t^2/4T^2} f(t)\,,$$

without changing the values of the growth exponents, provided f has at most powerlaw increase. Let $\Delta \subset \mathbb{R}$ be a Borel set and $\psi_\Delta(t) = e^{-\imath t\, H_W}\chi_\Delta(H_W)\phi_S$. Since $x^\alpha \geq (1-e^{-x})$ whenever $0 \leq \alpha \leq 1$ and $x \geq 0$, for any $\delta > 0$ one has

$$\langle M_q(H,\Delta;T)\rangle_T^g \;\geq\; \delta^{-q/2}\left(\|\psi_\Delta\|^2 \;-\; \big\langle \langle\psi_\Delta(t)|e^{-\delta\,\mathfrak{H}_S}\;|\psi_\Delta(t)\rangle\big\rangle_T^g\right)\,.$$

For $\Delta_1 \subset \Delta$, Δ_1^c will denote the complement $\Delta\setminus\Delta_1$. The decomposition of ψ_Δ into $\psi_{\Delta_1}+\psi_{\Delta_1^c}$ gives rise to the following lower bound

$$\langle M_q(H,\Delta;T)\rangle_T^g \;\geq\; \delta^{-q/2}\left(\|\psi_{\Delta_1}\|^2 \;-\; A_{\Delta_1,\Delta_1}(T,\delta) \;-\; 2\,\Re e\, A_{\Delta_1,\Delta_1^c}(T,\delta)\right)\,,$$

where $A_{\Delta_1,\Delta_2}(T,\delta) := \big\langle \langle\psi_{\Delta_1}(t)|e^{-\delta\,\mathfrak{H}_S}\;|\psi_{\Delta_2}(t)\rangle\big\rangle_T^g$. Using the spectral decomposition of $\mathfrak{H}_S$ (see eq. (30) in Section 3.3), it is easy to get

$$A_{\Delta_1,\Delta_2}(T,\delta) \;=\; \int_{\Delta_1} d\rho_S(E) \int_{\Delta_2} d\rho_S(E')\, e^{-(E-E')^2T^2} \sum_{n=0}^{\infty} \Phi_{n,S}(E)\,\overline{\Phi_{n,S}(E')}\, e^{-\delta\mu(n+1/2)}\,.$$

The Schwarz inequality $2\,|\langle\psi_1|\psi_2\rangle| \leq \|\psi_1\|^2 + \|\psi_2\|^2$ applied to the sum on the r.h.s., together with Proposition 4, lead to

$$|A_{\Delta_1,\Delta_2}(T,\delta)| \;\leq\; c_\epsilon\,\delta^{-(1/2+\epsilon)} \int_{\Delta_1} d\rho_S(E) \int_{\Delta_2} d\rho_S(E')\, e^{-(E-E')^2T^2}\,,$$

for a suitable constant c_ϵ. For $\alpha > 0$, let $\Delta_1 = \Delta(\alpha,T)$ be chosen as

$$\Delta(\alpha,T) \;=\; \left\{ E\in\Delta \;\middle|\; T^{-\alpha-1/\log(T)} \;\leq\; \int_\Delta d\rho_S(E')\, e^{-(E-E')^2T^2} \;\leq T^{-\alpha} \right\}\,.$$

By definition of ρ_S it follows then that

$$\langle M_q(H,\Delta;T)\rangle_T^g \;\geq\; \delta^{-q/2}\,\rho_S(\Delta(\alpha,T))\,\left(1-c_\epsilon\delta^{-(1/2+\epsilon)}T^{-\alpha}\right) \;\geq\; c\,T^{q\alpha/(1+2\epsilon)}\,\rho_S(\Delta(\alpha,T))\,,$$

for suitable c_ϵ, c, and the choice $\delta = (2cT^{-\alpha})^{2/(1+2\epsilon)}$. The final step uses Lemma 3 below, which is a variation of a result in [BGT]. Choosing $p = 1-q/(1+2\epsilon)$ therein, the definition of the multifractal dimensions completes the proof of Theorem 1. $\Box$

Lemma 3 *Let ρ be a positive measure on $\mathbb{R}$ with compact support I and define for $T > 0$*

$$I_\alpha(T) \;=\; \left\{ E\in I \;\middle|\; T^{-\alpha-1/\log(T)} \;\leq\; \int_I d\rho(E')\, e^{-(E-E')^2T^2} \;=\; \rho(B_T^g(E)) \;\leq\; T^{-\alpha} \right\}\,.$$

Then, for all $p\in[0,1]$, there is $\alpha = \alpha(p,T)$ and a constant c such that

$$\rho(I_\alpha(T)) \;\geq\; \frac{c\,T^{(p-1)\alpha}}{\log(T)} \int_I d\rho(E)\;\left(\rho(B_T^g(E))\right)^{p-1}\,.$$

Proof: Let $\kappa > 0$ and set $\Omega_0 = \{E \in \text{supp}(\rho) | \rho(B_T^g(E)) \leq T^{-\kappa}\}$. In addition, for $j = 1, \ldots, \kappa \log(T)$ let $\Omega_j = \{E \in \text{supp}(\rho) \,|\, T^{-\kappa+(j-1)/\log(T)} \leq \rho(B_T^g(E)) \leq T^{-\kappa+j/\log(T)}\}$. Then

$$\int d\rho(E)\, \rho(B_T^g(E))^{p-1} \;\leq\; \int_{\Omega_0} d\rho(E)\, \rho(B_T^g(E))^{p-1} + \kappa \log(T) \max_{j=1\ldots\kappa\log(T)} \int_{\Omega_j} d\rho(E)\, \rho(B_T^g(E))^{p-1} \tag{38}$$

Let $j = j(T,p)$ be the index where the maximum is taken, and then set $\alpha = \alpha(T,p) = \kappa - j\log(T)$. It only remains to show that the Ω_0-term is subdominant if only κ is chosen sufficiently big. To do so, the support of ρ is covered with intervals $(A_k)_{k=1\ldots K}$ of length $1/T$. Then $K \leq T\,|\text{supp}(\rho)|$ (where $|A|$ denotes the diameter of A). If $a_k = \inf\{\rho(B_T^g(E))|E \in A_k \cap \Omega_0\}$, then $a_k \leq T^{-\kappa}$ by definition of Ω_0. Moreover $\rho(B_T^g(E)) \geq \int_{A_k\cap\Omega_0} d\rho(E') e^{-(E-E')^2T^2}$. In particular, if $E \in A_k \cap \Omega_0$, then $|E - E'|T \leq 1$ implying $\rho(B_T^g(E)) \geq e^{-1}\rho(A_k \cap \Omega_0)$ and thus, $\rho(A_k \cap \Omega_0) \leq e a_k$. Hence $(p - 1 \leq 0)$:

$$\int_{\Omega_0} d\rho(E)\; \rho(B_T^g(E))^{p-1} \;\leq\; \sum_{k\leq K} \rho(A_k \cap \Omega_0) a_k^{p-1} \;\leq\; e \sum_{k\leq K} a_k^p \;\leq\; e\, T^{1-\kappa p}\, |\text{supp}(\rho)|\;.$$

Hence choosing $\kappa = 2/p$, for example, provides a subdominant contribution in (38) such that (38) fulfills the desired bound. □

5.2 Proof of Proposition 4

This section is devoted to the proof of Proposition 4 assuming Proposition 6. Since $\widehat{U} = e^{2\imath\pi Q/\sqrt{\theta}} = \mathcal{W}_{\theta'}(0,1)$ commutes with $\pi_W(\mathcal{A}_\theta)$, it commutes, in particular, with H_W. Therefore the pair $(H_W, \widehat{U})$ has a joint spectrum contained in $\mathbb{R} \times \mathbb{T}$. Let m_S denote the spectral measure of the pair relative to ϕ_S defined by

$$\int_{\mathbb{R}\times\mathbb{T}} dm_S(E,\eta)\, F(E, e^{\imath\eta}) \;=\; \langle \phi_S \,|\, F(H_W, \widehat{U})\, |\phi_S\rangle\,, \qquad\qquad \forall\, F \in \mathcal{C}_0(\mathbb{R}\times\mathbb{T})\,.$$

The marginal probabilities associated with m_S are respectively $d\rho_S(E)$, the spectral measure of H_W, and $d\eta\, \|\mathcal{G}_{\theta\,\eta/2\pi}\phi_S\|_{\ell^2}^2\, \theta/(2\pi)$ for $\eta \in \mathbb{T}$, the spectral measure of $\widehat{U}$. Thanks to the Radon-Nikodym theorem, m_S can be written either as

$$\int_{\mathbb{R}\times\mathbb{T}} dm_S(E,\eta)\; F(E, e^{\imath\eta}) \;=\; \frac{\theta}{2\pi} \int_0^{2\pi} d\eta \int_{\mathbb{R}} d\mu_{(\theta\eta/2\pi)}(E)\; F(E, e^{\imath\eta})\,, \tag{39}$$

(where μ_ω is the spectral measure of H_ω relative to $\mathcal{G}_\omega\phi_S$), or as

$$\int_{\mathbb{R}\times\mathbb{T}} dm_S(E,\eta)\; F(E, e^{\imath\eta}) \;=\; \int_{\mathbb{R}} d\rho_S(E) \int_0^{2\pi} d\nu_E(\eta)\; F(E, e^{\imath\eta})\,, \tag{40}$$

for some probabilty measure ν_E depending ρ_S-measurably upon E. Due to the spectral theorem, for every $n \in \mathbb{Z}$, there is a function $g_n(\omega, \cdot) \in L^2(\mathbb{R}, \mu_\omega)$ such that

$$\langle \mathcal{G}_\omega\phi_S \,|\, f(H_\omega)\, |n\rangle \;=\; \int_{\mathbb{R}} d\mu_\omega(E)\; f(E)\, g_n(\omega, E)\,. \tag{41}$$

In the following lemma, $\tilde{g}_n(\eta, E)$ stands for $\theta^{-1/4} g_n(\theta\eta/2\pi, E)$:

Lemma 4 *Let $\psi \in \mathcal{S}(\mathbb{R})$. Then the representative in $L^2(\mathbb{R}, \rho_S)$ of the projection of ψ on the H_W-cyclic component of ϕ_S is given by*

$$\tilde{\psi}(E) \;=\; \int_0^{2\pi} d\nu_E(\eta) \sum_{n\in\mathbb{Z}} \tilde{g}_n(\eta, E)\, \psi\left((\eta - 2\pi n)\, \theta^{1/2}/2\pi\right)$$

Proof: $\tilde{\psi}$ is defined by $\langle \phi_S \,|\, f(H_W)\,|\psi\rangle = \int_{\mathbb{R}} d\rho_S(E) f(E) \tilde{\psi}(E)$ for every $f \in \mathcal{C}_0(\mathbb{R})$. On the other hand, thanks to eq. (22),

$$\langle \phi_S \,|\, f(H_\omega)\,|\psi\rangle \;=\; \int_0^\theta d\omega\, \langle \mathcal{G}_\omega \phi_S \,| f(H_\omega) | \mathcal{G}_\omega \psi\rangle \;=\; \sum_{n\in\mathbb{Z}} \int_0^\theta d\omega\, \langle \mathcal{G}_\omega \phi_S \,| f(H_\omega)\,|n\rangle\, (\mathcal{G}_\omega \psi)(n)\,.$$

Then, using the definition (41) of g_n together with eqs. (39) and (40), and changing from ω to η, gives the result. □

Proof of Prop. 4: Let $\Delta \subset \mathbb{R}$ be a Borel set and, for $\delta > 0$, let $Q(\Delta, \delta)$ be defined by

$$Q(\Delta, \delta) \;=\; \int_\Delta d\rho_S(E) \sum_{n=0}^\infty e^{-\delta(n+1/2)} \left|\Phi_{n,S}(E)\right|^2.$$

Thanks to Lemma 4 applied to the eigenstates $\phi_S^{(n)}$ of $\mathfrak{H}_S$ (see eq. (30)), it can be written as

$$\begin{aligned} Q(\Delta, \delta) \;&=\; \textstyle\int_\Delta d\rho_S \int d\nu_E(\eta) d\nu_E(\eta') \;\sum_{m,m'} \tilde{g}_m(\eta, E)\, \overline{\tilde{g}_{m'}(\eta, E)} \cdots \\ &\qquad \cdots \textstyle\sum_{n=0}^\infty e^{-\delta(n+1/2)}\; \phi_S^{(n)}((\eta - 2\pi m)\, \theta^{1/2}/2\pi)\, \overline{\phi_S^{(n)}((\eta' - 2\pi m')\, \theta^{1/2}/2\pi)}\,. \end{aligned}$$

The last sum on the r.h.s. of this identity reconstructs the Mehler kernel of eq. (32) with $t = \delta/\mu$. It will be convenient to define

$$G_\delta(E; x) \;=\; \int d\nu_E(\eta') \sum_{m'} \left|\mathcal{M}_S(\delta/\mu; x, (\eta - 2\pi m')\, \theta^{1/2}/2\pi)\right|\,. \tag{42}$$

Since the Mehler kernel decays fastly, this sum converges. Using the Schwarz inequality together with the symmetry $(m, \eta) \leftrightarrow (m', \eta')$, $Q(\Delta, \delta)$ can be bounded from above by

$$Q(\Delta, \delta) \;\le\; \sum_m \int_\Delta d\rho_S \int d\nu_E(\eta)\; \left|\tilde{g}_m(\eta, E)\right|^2\; G_\delta\left(E; (\eta - 2\pi m)\, \theta^{1/2}/2\pi\right)\,.$$

Thanks to eqs. (39) and (40), and changing again from η to ω, this bound can be written as

$$Q(\Delta, \delta) \;\le\; \sum_m \int_0^\theta \frac{d\omega}{\theta^{1/2}} \int_\Delta d\mu_\omega(E)\; \left|g_m(\omega, E)\right|^2\; G_\delta\left(E; (\omega - m\theta)/\theta^{1/2}\right)\,.$$

If now P_ω is the projection on the H_ω-cyclic component of $\mathcal{G}_\omega \phi_S$ in $\ell^2(\mathbb{Z})$, the definition (41) of g_m and the covariance lead to the following inequality

$$\int d\mu_\omega(E)\; \left|g_m(\omega, E)\right|^2\, f(E) \;=\; \langle m| P_\omega f(H_\omega) P_\omega |m\rangle \;\le\; \langle 0| f(H_{\omega - m\theta}) |0\rangle\,,$$

valid for $f \in \mathcal{C}_0(\mathbb{R})$, $f \geq 0$, because H_ω commutes with P_ω and the latter is a projection. Let then $\mu_\omega^{(0)}$ be the spectral measure of H_ω relative to the vector $|0\rangle$. The previous estimate implies

$$\begin{aligned} Q(\Delta,\delta) \;&\leq\; \textstyle\sum_m \theta^{-1/2} \int_0^\theta d\omega \int_\Delta d\mu^{(0)}_{\omega-m\theta}(E)\; G_\delta\left(E;(\omega-m\theta)/\theta^{1/2}\right) \\ &\leq\; \theta^{-1/2} \textstyle\int_\infty^\infty d\omega \int_\Delta d\mu^{(0)}_\omega(E)\; G_\delta\left(E;\omega/\theta^{1/2}\right) \,. \end{aligned}$$

Since $\mu_\omega^{(0)}$ is 2π-periodic with respect to ω, the latter integral can be decomposed into a sum over intervals of length 2π leading to the following estimate

$$Q(\Delta,\delta) \;\leq\; \theta^{-1/2} \int_0^{2\pi} d\omega \int_\Delta d\mu^{(0)}_\omega(E) \sum_{k\in\mathbb{Z}} G_\delta\left(E;(\omega+2\pi k)/\theta^{1/2}\right) \,.$$

Definitions (17) of the trace on $\mathcal{A}_\theta$, (3) of the DOS and (42) of G_δ give

$$Q(\Delta,\delta) \;\leq\; \frac{2\pi}{\theta^{1/2}} \int_{\Delta\times[0,2\pi]} d\mathcal{N}(E)\, d\nu_E(\eta) \sum_{(k,m)\in\mathbb{Z}^2} \left| \mathcal{M}_S\left(\delta/\mu;\frac{\omega+2\pi k}{\theta^{1/2}},\frac{(\eta-2\pi m)\,\theta^{1/2}}{2\pi}\right)\right| \,.$$

The result of Proposition 6 can now be used. Remarking that ν_E is a probability, and using the equivalence between ρ_S and the DOS (Theorems 3 and 4 combined), the last estimate implies

$$Q(\Delta,\delta) \;\leq\; c_\epsilon\, \rho_S(\Delta)\, \delta^{-(1/2+\epsilon)} \,,$$

for some suitable constant c_ϵ. Since this inequality holds for all Borel subset Δ of $\mathbb{R}$, the Proposition 4 is proven. □

5.3 Proof of Proposition 6

If $\alpha = \theta/2\pi \in [0,1]$ is an irrational number, a rational approximant is a rational number p/q, with p,q prime to each other, such that $|\alpha - p/q| < q^{-2}$. The continued fraction expansion $[a_1,\cdots,a_n,\cdots]$ of α [Her], provides an infinite sequence p_n/q_n of such approximants, the *principal convergents*, recursively defined by $p_{-1}=1, q_{-1}=0, p_0=0, q_0=1$ and $s_{n+1}=a_{n+1}s_n+s_{n-1}$ if $s=p,q$. It can be proved (see [Her] Prop. 7.8.3) that α is a number of Roth type (see eq. (2) in Section 2) if and only if $\sum_{n=1}^\infty a_{n+1}/q_n^\epsilon < \infty$ for all $\epsilon > 0$.

The proof of Proposition 6 relies upon the so-called *Denjoy-Koksma inequality* [Her]. Let φ be a periodic function on $\mathbb{R}$ with period 1, of bounded total variation $\mathrm{Var}(\varphi)$ over a period interval. Then (see [Her], Theo. 3.1)

Theorem [Denjoy-Koksma inequality] *Let $\alpha \in [0,1]$ be irrational and let φ be a real valued function on $\mathbb{R}$ of period one. Then, if p/q is a rational approximant of α*

$$\left| \sum_{j=1}^q \varphi(x+j\alpha) - q\int_0^1 dy\,\varphi(y)\right| \;\leq\; \mathrm{Var}(\varphi)\,.$$

Proposition 6 is a direct consequence of the definition of the Mehler kernel (see eq. (32)) and of the following result

Lemma 5 *If $\delta > 0$, let F_δ be the function on $\mathbb{R}^2$ defined by*

$$F_\delta(x,y) \;=\; \delta\,(x+y)^2 + \delta^{-1}\,(x-y)^2\,.$$

If α is a number of Roth type, then for any $a > 0$, $\epsilon > 0$, there is $c_\epsilon > 0$ such that

$$\sup_{x,y\in\mathbb{R}} \sum_{(k,m)\in\mathbb{Z}^2} e^{-a\,F_\delta(x+k,y+m\alpha)} \;\leq\; c_\epsilon\,\delta^{-\epsilon}\,, \qquad \forall\;\delta\in(0,1)\,.$$

Proof: Let $(x_0,y_0)\in\mathbb{R}^2$ be fixed and set $\mathcal{L} = \{(x_0+k, y_0+m\alpha)\in\mathbb{R}^2\,|\,(k,m)\in\mathbb{Z}^2\}$. If $S(x_0,y_0) = \sum_{k,m} e^{-a\,F_\delta(x_0+k,y_0+m\alpha)}$ then S is periodic of period 1 in x_0 and of period α in y_0. Therefore, it is enough to assume $0\leq x_0<1$ and $0\leq y_0<1$ (since $0<\alpha<1$). For $0<\sigma<1$ and for $j\in\mathbb{N}$, let $\mathcal{L}_j$ be the set of points $(x,y)\in\mathcal{L}$ for which $j^2\delta^{-\sigma}\leq F_\delta(x,y)<(j+1)^2\delta^{-\sigma}$. Thus

$$S(x_0,y_0) \;\leq\; \sum_{j=0}^{\infty} e^{-aj^2\delta^{-\sigma}}\;|\mathcal{L}_j|\,, \tag{43}$$

where $|A|$ denotes the number of points in A. $\mathcal{L}_j$ is contained in an elliptic crown with axis along the two diagonals $x=\pm y$. In particular,

$$(x,y)\in\mathcal{L}_j \;\;\Rightarrow\;\; \max\{|x|,|y|\} \;\leq\; (j+1)\,\delta^{-(1+\sigma)/2} \quad\text{and}\quad |x-y| \;\leq\; (j+1)\,\delta^{(1-\sigma)/2}\,. \tag{44}$$

If $j\geq 1$, the number of points contained in $\mathcal{L}_j$ can be estimated by counting the number of rectangular cells of sizes $(1,\alpha)$ centered at points of $\mathcal{L}$ and meeting the elliptic crown. Since this crown is included inside the square $\max\{|x|,|y|\}\leq(j+1)\delta^{-(1+\sigma)/2}$ it is enough to count such cells meeting this square. Such cells are all included inside the square $C = \{(x,y)\in\mathbb{R}^2\,|\,\max\{|x|,|y|\}\leq(j+2)\delta^{-(1+\sigma)/2}\}$ (since $\delta\leq 1$). Hence the number of such cells is certainly dominated by the ratio of the area of C to the area of each cell, namely

$$|\mathcal{L}_j| \;\leq\; \frac{(j+2)^2}{\alpha}\;\delta^{-(1+\sigma)}\,.$$

Therefore, the part of the sum in (43) coming from $j\geq 1$ converges to zero as $\delta\downarrow 0$. In particular, it is bounded by a constant c_1 that is independent of (x_0,y_0). Thus, it is sufficient to consider the term $j=0$ only.

Let φ be the function on $\mathbb{R}$ defined by $\varphi(x) \;=\; \sum_{k\in\mathbb{Z}}\chi_I(x+y_0-x_0+k)$ where I is the interval $I=[-\delta^{(1-\sigma)/2},\delta^{(1-\sigma)/2}]\subset\mathbb{R}$. It is a periodic function of period 1 with $\mathrm{Var}(\varphi)=2$. Moreover, using (44) it can be checked easily that

$$S(x_0,y_0) \;\leq\; c_1 + \sum_{|m|<M}\varphi(m\alpha) \;\leq\; c_1 + \sum_{m=0}^{M-1}\left(\varphi(m\alpha)+\varphi(-m\alpha)\right)\,,$$

provided $M\geq 3\,\delta^{-(1+\sigma)/2}/\alpha$. For indeed, $(x,y)\in\mathcal{L}_0$ only if $|y_0+m\alpha|\leq\delta^{-(1+\sigma)/2}$ for some $m\in\mathbb{Z}$. Let then $n\in\mathbb{N}$ be such that $q_n\leq M<q_{n+1}$, where the p_n/q_n's are the principal convergents of α. Replacing M by q_{n+1} in the r.h.s. gives an upper bound. By the Denjoy-Koksma inequality, the r.h.s. is therefore bounded from above by $c_1+4q_{n+1}\delta^{(1+\sigma)/2}$. Since α

is a number of Roth type, $q_{n+1} \leq (a_{n+1}+1)q_n \leq c_2 \cdot q_n^{1+\sigma}$, thanks to Prop. 7.8.3 in [Her] (see above). It is important to notice that c_2 only depends upon α and the choice of the exponent σ. Collecting all inequalities, gives

$$S(x_0,y_0) \ \leq \ c_1 + \frac{12 \cdot c_2}{\alpha}\, \delta^{-2\sigma}\,.$$

Choosing $\sigma = \epsilon/2$ and remarking that none of the constants on the r.h.s. depends on (x_0, y_0) leads to the result. □

Appendix: Proofs of various results on Weyl operators

Proof of eqs. (13) and (14): Due to the polarization principle, (13) is equivalent to

$$\langle \phi | \mathfrak{W}(\mathfrak{a})\, | \phi \rangle\, \overline{\langle \psi | \mathfrak{W}(\mathfrak{a})\, | \psi \rangle} \ = \ \int_{\mathbb{R}^2} \frac{d^2 \mathfrak{b}}{2\pi}\, e^{\imath\, \mathfrak{a} \wedge \mathfrak{b}}\ \left| \langle \psi | \mathfrak{W}(\mathfrak{b})\, | \psi \rangle \right|^2 \ . \tag{45}$$

By inverse Fourier transform, (45) is equivalent to

$$\left| \langle \phi | \mathfrak{W}(\mathfrak{b})\, | \psi \rangle \right|^2 \ = \ \int_{\mathbb{R}^2} \frac{d^2 \mathfrak{a}}{2\pi}\, e^{\imath\, \mathfrak{b} \wedge \mathfrak{a}}\, \langle \phi | \mathfrak{W}(\mathfrak{a})\, | \phi \rangle\, \overline{\langle \psi | \mathfrak{W}(\mathfrak{a})\, | \psi \rangle}\ , \tag{46}$$

which is equivalent to (14), so that it is sufficient to prove (45). Using (11),

$$\text{r.h.s. of (45)} \ = \ \int_{\mathbb{R}^2} \frac{db_1\, db_2}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, \overline{\phi(x)}\, \phi(y)\, \psi(x+b_1)\, \overline{\psi(y+b_1)}\, e^{\imath (b_2(x-y+a_1) - a_2 b_1)}\ .$$

The integral over b_2 can be immediately evaluated by $\int_{\mathbb{R}} db_2\, e^{\imath b_2 (x-y+a_1)} \ = \ 2\pi\, \delta(y-x-a_1)$. Thus the integration over y is elementary. Changing variable from b_1 to $x' = x + b_1$ therefore gives

$$\text{r.h.s. of (45)} \ = \ \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx'\, \overline{\phi(x)}\, \phi(x+a_1)\, e^{\imath a_2 x + \imath \frac{a_1 a_2}{2}}\ \psi(x')\, \overline{\psi(x'+a_1)}\, e^{-\imath a_2 x' - \imath \frac{a_1 a_2}{2}}\ ,$$

which is precisely the l.h.s. of (45). □

Proof of eq. (22): It is sufficient to verify (22) for the generators $A = W_\theta(m)$, $m \in \mathbb{Z}^2$, of $\mathcal{A}_\theta$. For such A,

$$\text{r.h.s. of (22)} \ = \ \int_0^\theta \frac{d\omega}{\sqrt{\theta}} \sum_{n,l \in \mathbb{Z}} \overline{\phi\left(\frac{\omega - n\theta}{\sqrt{\theta}}\right)}\, \langle n | \pi_\omega (W_\theta(m) | l \rangle\ \psi\left(\frac{\omega - l\theta}{\sqrt{\theta}}\right)\ .$$

As $\langle n | \pi_\omega (W_\theta(m) | l \rangle = e^{\imath \theta m_1 m_2/2}\, e^{\imath(\omega - l\theta) m_2} \delta_{n, l+m_1}$, the sum over n can be immediately computed, and the one over l can be combined with the integral over ω in order to give

$$\text{r.h.s. of (22)} \ = \ \int_{\mathbb{R}} \frac{dx}{\sqrt{\theta}}\, \overline{\phi\left(\frac{x - m_1\theta}{\sqrt{\theta}}\right)}\, e^{\imath \theta \frac{m_1 m_2}{2}}\, e^{\imath x m_2} \psi\left(\frac{x}{\sqrt{\theta}}\right)\ .$$

Changing variable $y=(x-m_1\theta)/\sqrt{\theta}$ and identifying $\mathfrak{W}(\sqrt{\theta}m)$ shows

$$\text{r.h.s. of (22)} \;=\; \int_{\mathbb{R}} dy\,\overline{\phi(y)}\,\left(\mathfrak{W}(\sqrt{\theta}m)\psi\right)(y)\;,$$

namely the l.h.s. of (22) $\square$

Proof of Proposition 7: For $f\in\mathcal{S}(\mathbb{R}^2)$, let $\tilde{f}$ be its symplectic Fourier transform defined by ($\mathfrak{l},\mathfrak{m}\in\mathbb{R}^2$):

$$\tilde{f}(\mathfrak{l}) \;=\; \int_{\mathbb{R}^2}\frac{d^2\mathfrak{m}}{2\pi}\,e^{\imath\mathfrak{l}\wedge\mathfrak{m}}\,f(\mathfrak{m})\;, \qquad \Leftrightarrow \qquad f(\mathfrak{m}) \;=\; \int_{\mathbb{R}^2}\frac{d^2\mathfrak{l}}{2\pi}\,e^{\imath\mathfrak{m}\wedge\mathfrak{l}}\,\tilde{f}(\mathfrak{m})\;.$$

Then the classical Poisson summation formula reads

$$\sum_{m\in\mathbb{Z}^2} f(m) \;=\; 2\pi\sum_{l\in\mathbb{Z}^2}\tilde{f}(2\pi\,l)\;.$$

Setting $f(\mathfrak{m})=\langle\phi|\,\mathfrak{W}(\sqrt{\theta}\mathfrak{m})\,|\phi\rangle\,\overline{\langle\psi|\mathfrak{W}(\sqrt{\theta}\mathfrak{m})\,|\psi\rangle}$, equation (46) leads to

$$\tilde{f}(\mathfrak{l}) \;=\; \frac{1}{\theta}\,\left|\langle\psi|\,\mathfrak{W}\left(\frac{2\pi}{\sqrt{\theta}}\,\mathfrak{l}\right)\,|\phi\rangle\right|^2\;.$$

Inserting this into the Poisson summation formula and recalling the notation (20) gives (23). $\square$

Proof of eq. (24): By (16) and (20), $\pi_W(A)=\sum_{\mathbf{l}\in\mathbb{Z}^2}a_{\mathbf{l}}\,\mathcal{W}_\theta(\mathbf{l})$ with $a_{\mathbf{l}}\;=\;\mathcal{T}_\theta(W_\theta(\mathbf{l})^{-1}A)$. Thus

$$\langle\psi|\,\pi_W(A)\,|\psi\rangle \;=\; \sum_{\mathbf{l}\in\mathbb{Z}^2}a_{\mathbf{l}}\,\langle\psi|\,\mathcal{W}_\theta(\mathbf{l})\,|\psi\rangle \;=\; \mathcal{T}_\theta\left(\sum_{l\in\mathbb{Z}^2}\langle\psi|\,\mathcal{W}_\theta(\mathbf{l})\,|\psi\rangle W_\theta(\mathbf{l})^{-1}A\right)\;.$$

Comparing with the Poisson summation formula (23) shows (24). $\square$

Proof of Proposition 8: Because of the freedom of phase and relation (12), it is sufficient to verify all implementation formulas (25) for the Weyl operators $e^{\imath Q}$ and $e^{\imath P}$ or equivalently (on the domain of) their generators Q and P. Concerning the first formula in (26), it thus follows from the identities

$$e^{-\imath\kappa Q^2/2}\,Q\,e^{\imath\kappa Q^2/2} \;=\; Q\;, \qquad e^{-\imath\kappa Q^2/2}\,P\,e^{\imath\kappa Q^2/2} \;=\; \kappa\,Q+P\;.$$

Next let us consider the dilations on $L^2(\mathbb{R})$ defined by $(D(a)\phi)(x)=\sqrt{e^a}\,\phi(e^a\,x)$. It generators are computed by

$$\left.\frac{d}{da}(D(a)\phi)(x)\right|_{a=0} \;=\; \frac{\imath}{2}(QP+PQ)\,\phi(x)\;,$$

so that for $a=-\ln(\lambda)$

$$e^{-\imath\,\ln\lambda\,(QP+PQ)/2}\,\phi(x) \;=\; \sqrt{\frac{1}{\lambda}}\,\phi\left(\frac{x}{\lambda}\right)\;.$$

This immediately allows to verify

$$e^{-\imath \ln\lambda\,(QP+PQ)/2}\, Q\, e^{\imath \ln\lambda\,(QP+PQ)/2} \;=\; \frac{1}{\lambda} Q \;, \qquad e^{-\imath \ln\lambda\,(QP+PQ)/2}\, P\, e^{\imath \ln\lambda\,(QP+PQ)/2} \;=\; \lambda\, P \;,$$

which proves the second formula in (26). To prove the last one, we use the annihiliation-creation operators $a = (Q - \imath P)/\sqrt{2}$ and $a^* = (Q + \imath P)/\sqrt{2}$. As $(P^2 + Q^2 - 1)/2 = a^*a$ and $e^{-\imath s a^* a} a e^{\imath s a^* a} = e^{\imath s} a$, the formula follows after decomposing $\mathfrak{W}(\mathfrak{a})$ into a and a^*. Finally we search the integral kernel for $K = e^{-\imath s a^* a}$, notably $(K\phi) = \int dy\, k(x,y)\phi(y)$. If $\phi^{(n)}_{S_4}$ are the Hermite functions, then $K\phi^{(n)}_{S_4} = e^{\imath s n}\phi^{(n)}_{S_4}$. Equivalently, k has to satisfy $a_y k = e^{\imath s} a^*_x k$ and $K\phi^{(0)} = \phi^{(0)}$ (here the index on the a's indicate with respect to which variable the operator acts). An Ansatz $k(x,y) = e^{-b(x^2+y^2)+cxy+d}$ leads to the integral kernel in (27). $\Box$

Proof of Proposition 32: Let us set

$$R \;=\; \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \;, \qquad D \;=\; \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \;.$$

Then, using the notations and formulas in Subsection 3.3,

$$\mathfrak{H}_S \;=\; \frac{\mu}{2}\, \langle (RD)^t K\, |(RD)^t K\rangle \;=\; \mu\, \mathcal{F}_R\, \mathcal{F}_D\, \mathfrak{H}_{S_4}\, \mathcal{F}_D^{-1}\, \mathcal{F}_R^{-1} \;, \qquad \phi_S \;=\; \mathcal{F}_R\, \mathcal{F}_D\, \phi_{S_4} \;. \qquad (47)$$

Now ϕ_{S_4} is known to be the normalized gaussian. Using the implementation formulas of Proposition 8, it is straightforward to calculate the gaussian integrals giving (31). The Mehler kernel $\mathcal{M}_{S_4}(t;x,y)$ for $\mathfrak{H}_{S_4} = (P^2 + Q^2)/2$ is well-known (and can be read of (27) at imaginary time). Using (47) and the definition (10),

$$\mathcal{M}_S(t;x,y) \;=\; \int_{\mathbb{R}} dx' \int_{\mathbb{R}} dy'\; \langle x|\, \mathcal{F}_R\, \mathcal{F}_D\, |x'\rangle\; \mathcal{M}_{S_4}(t;x',y')\; \langle y'|\, \mathcal{F}_D^{-1}\, \mathcal{F}_R^{-1}\, |y\rangle \;.$$

The gaussian integrals herein give rise to (32). $\Box$

Let us conclude with the proof of the complementary result given in Section 2.

Proof of Proposition 5: The commutant $\mathcal{B}$ of the abelian C*-algebra generated by H_W contains the commutant of $\pi_W(\mathcal{A}_\theta)$, that is the von Neumann algebra $\overline{\pi_W(\mathcal{A}_{\theta'})}$ generated by $\pi_W(\mathcal{A}_{\theta'})$. As $\overline{\pi_W(\mathcal{A}_{\theta'})}$ is of type II$_1$ [Sak], there exist $*$-endomorphisms $\eta_q : \mathrm{Mat}_{q\times q} \to \mathcal{B}$ for every $q \in \mathbb{N}$ (here $\mathrm{Mat}_{q\times q}$ denotes the complex $q \times q$ matrices).

According to the spectral theorem, $\mathcal{H}$ decomposes according to the multiplicity of $\pi_W(H)$: $\mathcal{H} = \oplus_{n\geq 1} L^2(X_n, \mu_n) \otimes \mathbb{C}^n \oplus L^2(X_\infty, \mu_\infty) \otimes \ell^2(\mathbb{N})$ where the μ_n's are positive measures with pairwise disjoint supports $X_n \subset \mathbb{R}$. In this representation, $\pi_W(H) = \oplus_{n\geq 1}\mathrm{Mult}(E) \otimes \mathbf{1}_n \oplus \mathrm{Mult}(E) \otimes \mathbf{1}_\infty$ (here $\mathrm{Mult}(E)$ denotes the multiplication by the identity on $\mathbb{R}$) and $\mathcal{B} = \oplus_{n\geq 1} L^\infty(X_n, \mu_n) \otimes \mathrm{Mat}_{n\times n} \oplus L^\infty(X_\infty, \mu_\infty) \otimes \mathcal{B}(\ell^2(\mathbb{N}))$. Let P_n be the projection on $L^2(X_n, \mu_n) \otimes \mathbb{C}^n$. Then $P_n \mathcal{B} P_n = L^\infty(X_n, \mu_n) \otimes \mathrm{Mat}_{n\times n}$. Moreover $\phi_{n,x}(B) = P_n B P_n(x)$ defines a $*$-endomorphism from $\mathcal{B}$ to $\mathrm{Mat}_{n\times n}$ for μ_n-almost all $x \in X_n$. Combining with η_q, one gets $*$-endomorphisms $\phi_{n,x} \circ \eta_q : \mathrm{Mat}_{q\times q} \to \mathrm{Mat}_{n\times n}$ for any q satisfying $\phi_{n,x} \circ \eta_q(\mathbf{1}_q) = \mathbf{1}_n$. This is impossible for any $q > n$ so that $X_n = \emptyset$ for all $n \geq 1$.

If H_W had a cyclic vector, its spectrum would be simple. $\Box$

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