Scattering Theory for the Spin Fermion Model.

Zied Ammari

Fachbereich Mathematik (17)
Johannes Gutenberg-Universität
55099 Mainz, Germany
E-mail: zied@mathematik.uni-mainz.de

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The scattering theory for the interacting spin fermion model is considered. We give a proof on the asymptotic completeness of the dynamics in the case of massive fermions.

1. INTRODUCTION

We study in this note scattering theory for a spin system interacting with a massive fermions. The Hamiltonian of the spin system is given by a bounded self-adjoint operator K acting on a finite dimensional Hilbert space $\mathcal{K}$. The most common physical example for this formulation is the Pauli matrix $K := \sigma_z$. Fermions are described by the second quantization operator $\mathrm{d}\Gamma(\omega(k))$ of the dispersion relation $\mathbb{R}^d \ni k \to \omega(k) \in \mathbb{R}_+$ with $m := \inf_{k\in\mathbb{R}^d} \omega(k) > 0$, and acting on the fermionic Fock space $\Lambda(L^2(\mathbb{R}^d))$. The interaction is taken to be an infinite sum of bounded even wick polynomials constructed from a family of kernels $w_{p,q} \in S'(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))$. Under some hypothesis which will be detailed in the Sect. 3, concerning the spin system, the free fermion energy and the kernels $w_{p,q}$, we prove the existence of the wave operator and it is completeness. Namely, we show that the wave operator whom its existence derives from a Cook argument and which is given by

$$\begin{aligned}\Omega_\pm \ : \ & \mathcal{H} \otimes \Lambda(L^2(\mathbb{R}^d)) \to \mathcal{H}, \\ & \psi \otimes \prod_{i=1}^n b^*(f_i)\Omega \to \prod_{i=1}^n b^{*,\pm}(f_i)\psi,\end{aligned}$$

constructed using asymptotic creation operators $b^{*,\pm}$, Subsect. 5.1, satisfies the property

$$\mathrm{Ran}(\Omega_{\pm|\mathcal{H}_{\mathrm{bd}}(H)\otimes\Lambda(L^2(\mathbb{R}^d))}) = \mathcal{H},$$

where $\mathcal{H}_{\mathrm{bd}}(H)$ is the space of bound states of the perturbed Hamiltonian. For this we heavily use the method initiated in the work [7]. In fact, asymptotic completeness is expected to hold for the spin fermion system since it is proved for more complicated models such as spin boson [7], self-interacting boson $P(\varphi)_2$ [8], ultraviolet removed Nelson model [1]. Nevertheless we are unaware in the literature of any precise result concerning the scattering theory of the considered system. Moreover, it has features such that it is an interesting example in the study of stability of KMS states and where asymptotic completeness could has application as well as for the scattering theory for automorphism groups of $\mathbb{C}^*-$algebras. Also, it serves as a non trivial counter example where the property of L^1-asymptotic abelianness does not hold and therefore pointing out a gap in the non-stationary approach on the stability of KMS states [2].

The section below is an introductory one where we define objects used later. We state the hypothesis and the main theorem in Sect. 3 and we begin by establishing spectral results on Sect. 4. The last section Sect. 5 is devoted to the proof of asymptotic completeness.

2. BASIC NOTATIONS

In this section we introduce the main notations which will be used in the sequel. We recall the definitions of the most important operations in the fermionic Fock space, especially those are related to the study of the scattering theory. Among this operations we cite the *scattering identification operator*, introduced in [16], [7] and which play a fundamental role in the proof of asymptotic completeness. In Subsect. 2.2, we follow the formalism in [8] defining Wick polynomials in the fermionic case. We derive some commutation relations and estimates satisfied by this Wick polynomials.

2.1. Fermionic Fock spaces

Let $\mathfrak{h}$ be a complex Hilbert space. Let $\Lambda^n(\mathfrak{h})$ denotes the anti-symmetric n-fold tensor product of $\mathfrak{h}$. The *fermionic Fock space* over $\mathfrak{h}$ is

$$\Lambda(\mathfrak{h}) := \bigoplus_{n\geq 0} \Lambda^n(\mathfrak{h}),$$

where $\Lambda^0(\mathfrak{h}) := \mathbb{C}$. The *vacuum vector* (1,0,...) will be denoted by Ω. We define the finite particle space $\Lambda^{\text{fin}}(\mathfrak{h})$ as the subspace of finite sums of vectors in $\Lambda^n(\mathfrak{h}), n \in \mathbb{N}$.
We also define Λ_n the orthogonal projection from the usual tensor product $\otimes^n \mathfrak{h}$ into the subspace $\Lambda^n(\mathfrak{h})$. It acts in a pure tensor product state as follows:

$$\Lambda_n(f_1 \otimes ... \otimes f_n) := \frac{1}{n!} \sum_{\pi\in S_n} \epsilon(\pi) f_{\pi_1} \otimes ... \otimes f_{\pi_n}, \text{ where } f_i \in \mathfrak{h}, \text{ for } i = 1, ..., n.$$

The above sum is taken over all the permutations of n elements and $\epsilon(\pi)$ denotes the signature of those permutations. Let $\psi \in \Lambda^n(\mathfrak{h}), \phi \in \Lambda^m(\mathfrak{h})$, we denote by $\psi \wedge \phi$ the anti-symmetric vector $\Lambda_{n+m}\psi \otimes \phi$. We will also use the notation $\Lambda_+ := \oplus_{n\geq 0} \Lambda_n$.

creation-annihilation. We recall the annihilation and creation distributions in the case where $\mathfrak{h} := L^2(\mathbb{R}^d)$, they are operator-valued tempered distributions. Let $S(\mathbb{R}^d)$ be the Schwartz space. Let $\Psi \in \Lambda^{\text{fin}}(\mathfrak{h})$:

$$(b(k)\Psi)^{(n)}(k_1, ..., k_n) := (n+1)^{\frac{1}{2}}\, \Psi^{(n+1)}(k, k_1, ..., k_n),$$

$$(b^*(k)\,\Psi)^{(n)}(k_1, ..., k_n) := n^{-\frac{1}{2}} \sum_{j=1}^{n} (-1)^{j-1}\, \delta(k - k_j)\, \Psi^{(n-1)}(k_1, ..., \hat{k}_j, ..., k_n),$$

where $\hat{k}_j$ means that k_j is omitted and δ is the Dirac function. The following canonical anti-commutation relations hold in the sense of sesquilinear form-valued tempered distribution:

$$\{b(k), b(k')\} = \{b^*(k), b^*(k')\} = 0,$$
$$\{b(k), b^*(k')\} = \delta(k - k').$$

We can retrieve the annihilation and creation operators $b(h), b^*(h)$ for $h \in \mathfrak{h}$, using the following relations:

$$b(h) := \int b(k)\, \overline{h(k)}\, dk,$$
$$b^*(h) := \int b^*(k)\, h(k)\, dk.$$

We use the notation $b^\sharp(.)$ to simplify the writing of statements which hold for both $b(.)$ and $b^*(.)$. As creation annihilation distributions $b(h)$ and $b^*(h)$ satisfy canonical anti-commutation relations.

$$\{b^\sharp(h), b^\sharp(g)\} = 0, \tag{2.1}$$

$$\{b(h), b^*(g)\} = (h|g)\mathbb{1}. \tag{2.2}$$

A consequence of the relation (2.1) is that $b^\sharp(h)^2 = 0$. This means that a fermion system can note has the same state for more then one-particle. This is the so-called *Pauli principle* and it leads with (2.2) to the boundedness of creation-annihilation operators $||b^\sharp(h)|| = ||h||$.

Further operators. We define the second quantization of a one particle operator A denoted by $\mathrm{d}\Gamma(A)$ as

$$\mathrm{d}\Gamma(A)_{|\Lambda^n(\mathfrak{h})} = \sum_{j=1}^{n} A_j,$$

where A_j is the operator acting as A in the j^{th} component and as the identity in the others. An example is the number operator N which we obtain putting $A = \mathbb{1}$. We mention that in general $\mathrm{d}\Gamma(A)$ are unbounded operators even if A are bounded and they are bounded if and only if A are trace class operators. If ω is a multiplication operator on $\mathfrak{h}$ then $\mathrm{d}\Gamma(\omega)$ can be expressed as

$$\mathrm{d}\Gamma(\omega) \;=\; \int b^*(k)\, \omega(k)\, b(k) dk$$

Let B be an operator acting on $\mathfrak{h}$. The functor operator $\Gamma(B)$ is defined by

$$\Gamma(B) : \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}),$$
$$\Gamma(B)_{|\Lambda^n(\mathfrak{h})} := B^{\otimes(n)},$$

with the notation $B^{\otimes(n)} := B \otimes ... \otimes B$, n times.
Let A, B be two self-adjoint operators on $\mathfrak{h}$. We define

$$\mathrm{d}\Gamma(A, B) : \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}),$$
$$\mathrm{d}\Gamma(A, B)_{|\Lambda^n(\mathfrak{h})} := \sum_{j=1}^{n} A^{\otimes(j-1)} \otimes B \otimes A^{\otimes(n-j)}.$$

Let $\mathcal{K}$ be a complex Hilbert space describing another system of particles. Let $\mathcal{H}$ be the Hilbert space of the joint system $\mathcal{K} \otimes \Gamma(\mathfrak{h})$. We can extend $b^\sharp(h)$ to an operator acting on the whole space $\mathcal{H}$ and satisfying anti-commutation relations. Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, we define

$$b^*(v)\phi \otimes \psi := \sqrt{N}\;\; v\phi \,\wedge\, \psi,$$
$$b(v)\phi \otimes \psi := \sqrt{N+1}\;\; \Lambda_+ \; v^*\phi \otimes \psi, \quad \psi \in \Lambda(\mathfrak{h}),\; \phi \in \mathcal{K}.$$

We notice that in the definition of $b(v)$, v^* acts only in the first component $\mathcal{K}\otimes\mathfrak{h}$ and in the rest as the identity. In the case where $\mathfrak{h} := L^2(\mathbb{R}^d)$, we can represent $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ as a function $\mathbb{R}^d \ni k \to v(k) \in \mathcal{B}(\mathcal{K})$

defined almost every where and satisfying

$$v(k)\,\phi = (v\phi)(k), \text{ for all } \phi \in \mathcal{K}, \text{ and } k \text{ a.e. in } \mathbb{R}^d,$$
$$\mathcal{K}\times\mathcal{K} \ni (\phi_1,\phi_2) \to \int (v(k)^*\, v(k)\phi_1|\phi_2)_{\mathcal{K}}\, dk = (v\phi_1|v\phi_2)_{\mathcal{K}\otimes\mathfrak{h}},$$

is a continuous quadratic form. We have the following anti-commutation relations for all $v, w \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ such that $[v^*(k), w(k')] = 0$ for all k, k' a.e. in $\mathbb{R}^d$:

$$\{b^\sharp(v), b^\sharp(w)\} = 0,$$
$$\{b(v), b^*(w)\} = v^*w \otimes 1\!\!1_{\Lambda(\mathfrak{h})}.$$

The Fermionic exponential law. Let $\mathfrak{h}_1, \mathfrak{h}_2$ two Hilbert spaces. There exists an unitary transformation U mapping the space $\Lambda(\mathfrak{h}_1\oplus\mathfrak{h}_2)$ into $\Lambda(\mathfrak{h}_1)\otimes\Lambda(\mathfrak{h}_2)$ and satisfying

$$U\,\Omega := \Omega\otimes\Omega, \tag{2.3}$$
$$U\, b^\sharp(h_1\oplus h_2) := (b^\sharp(h_1)\otimes 1\!\!1 + (-1\!\!1)^N \otimes b^\sharp(h_2))\, U. \tag{2.4}$$

Since the vector space generated by the family $\{\prod_{i\in I} b^*(h_i)\Omega, \text{for all finite sets } I\}$ is dense in the Fock space, the identities (2.3)-(2.3) are enough to define U. We notice that the factor $(-1\!\!1)^N$ in (2.4) is coherent with the Pauli principle. We can also write the action of U in a sector of n particle subspace:

$$U_{|\Lambda^n(\mathfrak{h}_1\oplus\mathfrak{h}_2)} = \sum_{m=0}^{n} \sqrt{\frac{n!}{(n-m)!m!}}\ p_1^{\otimes(m)}\otimes p_2^{\otimes(n-m)},$$

where p_i is the projection from $\mathfrak{h}_1\oplus\mathfrak{h}_2$ into $\mathfrak{h}_i$, for $i = 1, 2$. One of the feature of this map is the decoupling propriety when it acts on second quantization operator. We understand more when we see the following formula:

$$U\, \mathrm{d}\Gamma\left(\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\right) = (\mathrm{d}\Gamma(b_1)\otimes 1\!\!1 + 1\!\!1\otimes \mathrm{d}\Gamma(b_2))\, U.$$

Scattering identification operator. Let i be the injection defined by

$$i:\ \mathfrak{h}\oplus\mathfrak{h} \to \mathfrak{h},$$
$$(h_0, h_\infty) \to h_0 + h_\infty.$$

We define the scattering identification operator I:

$$I : \Lambda(\mathfrak{h})\otimes\Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}),$$
$$I := \Gamma(i)U^*\mathcal{J},$$
$$\mathcal{J} := (-1\!\!1)^N \otimes 1\!\!1, \text{ acting on } \Lambda(\mathfrak{h})\otimes\Lambda(\mathfrak{h}).$$

We have the following formula

$$I \prod_{i=1}^{n} b^*(h_i)\Omega \prod_{j=1}^{p} b^*(g_j)\Omega = \prod_{j=1}^{p} b^*(g_j) \prod_{i=1}^{n} b^*(h_i)\Omega.$$

We notice that I is an unbounded operator since it contains an operator $\Gamma(i)$ and $||i|| = \sqrt{2}$. Another way to express this operator is using an integral formula. Let $\psi \in \Lambda^n_{\text{alg}}(\mathfrak{h})$ the space of finite sums of pure tensor states in $\Lambda^n(\mathfrak{h})$

$$Iu \otimes \psi = \frac{1}{\sqrt{n!}} \int \psi(k_1, ..., k_n)\ b^*(k_1)...b^*(k_n)u\ dk, \ \ u \in \Lambda(\mathfrak{h}).$$

The map $I : u \to Iu \otimes \psi$ is continuous on $\Lambda(\mathfrak{h})$ for all $\psi \in \Lambda^n_{\text{alg}}(\mathfrak{h})$. Moreover $I(N+1)^{-\frac{n}{2}} \otimes \mathbb{1}$ is bounded on $\Lambda(\mathfrak{h}) \otimes \Lambda^n(\mathfrak{h})$. Let $\tilde{i} := (i_0, i_\infty)$ be a pair of maps such that $i_0, i_\infty : \mathfrak{h} \to \mathfrak{h}$. Let $I(\tilde{i})$ be the map:

$$I(\tilde{i}) : \Lambda^{\text{fin}}(\mathfrak{h}) \otimes \Lambda^{\text{fin}}(\mathfrak{h}) \to \Lambda^{\text{fin}}(\mathfrak{h}),$$
$$I(\tilde{i}) := I\Gamma(i_0) \otimes \Gamma(i_\infty).$$

Let $\tilde{i} = (i_0, i_\infty), \tilde{j} = (j_0, j_\infty)$ be tow pairs of maps from $\mathfrak{h}$ into $\mathfrak{h}$. We set $\mathrm{d}I(\tilde{i}, \tilde{j})$ as

$$\mathrm{d}I(\tilde{i}, \tilde{j}) : \Lambda^{\text{fin}}(\mathfrak{h}) \otimes \Lambda^{\text{fin}}(\mathfrak{h}) \to \Lambda^{\text{fin}}(\mathfrak{h}),$$
$$\mathrm{d}I(\tilde{i}, \tilde{j}) := I\big(\mathrm{d}\Gamma(i_0, i_\infty) \otimes \Gamma(i_\infty) + \Gamma(i_0) \otimes \mathrm{d}\Gamma(i_0, j_\infty)\big).$$

We construct a right inverse for the scattering identification operator. For this, we define first an intermediate operator $\check{\Gamma}(j)$ then under certain restriction on j we get a right inverse operator for I. Let $j := (j_0, j_\infty)$ a map such that

$$\begin{aligned} j\ :\ & \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}, \\ & h \to (j_0 h, j_\infty h). \end{aligned}$$

Set $N_\infty := \mathbb{1} \otimes N$ and $N_0 := N \otimes \mathbb{1}$ acting on $\Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h})$. We define the operator $\check{\Gamma}(j)$ by

$$\begin{aligned} \check{\Gamma}(j) : \quad & \Lambda(\mathfrak{h}) \to \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h}), \\ & \check{\Gamma}(j) := U\Gamma(j). \end{aligned}$$

We have

$$\check{\Gamma}(j) \prod_{i=1}^n b^*(h_i)\Omega = \prod_{i=1}^n \big(b^*(j_0 h_i) \otimes \mathbb{1} + (-\mathbb{1})^N \otimes b^*(j_\infty h_i)\big)\Omega \otimes \Omega,$$

and

$$\begin{aligned} \mathbb{1}_{\{k\}}(N_\infty)\check{\Gamma}(j)_{|\Lambda^n(\mathfrak{h})} &= \mathbb{1}_{\{k\}}(N_\infty) \sum_{m=0}^n \sqrt{\frac{n!}{(n-m)!m!}}\ p_1^{\otimes(m)} \otimes p_2^{\otimes(n-m)} \Gamma(j)_{|\Lambda^n(\mathfrak{h})} \\ &= \sqrt{\frac{n!}{(n-k)!k!}}\ j_0^{\otimes(n-k)} \otimes j_\infty^{\otimes(k)}. \end{aligned}$$

Consider now j such that $0 \leq j_0 \leq 1$, $0 \leq j_\infty$ and $j_0 + j_\infty = 1$, then we have

$$I\mathcal{J}\ \ \check{\Gamma}(j) = \mathbb{1}.$$

Furthermore

$$I\mathcal{J}\mathbb{1}_{\{1,\dots,k\}}(N_\infty)\ \ \check{\Gamma}(j)_{|\Lambda^n(\mathfrak{h})} = \sum_{\sharp\{\varepsilon_i = \infty\} \leq k} j_{\varepsilon_1} \otimes ... \otimes j_{\varepsilon_n}.$$

2.2. Wick Polynomials

Let $w \in \mathcal{B}(\Lambda^p(\mathfrak{h}), \Lambda^q(\mathfrak{h}))$, we define the wick monomial with the symbol w to be the operator $Wick(w)$ acting as following:

$$\begin{aligned} &Wick(w) : \Lambda^{\text{fin}}(\mathfrak{h}) \to \Lambda^{\text{fin}}(\mathfrak{h}), \\ &Wick(w)_{|\Lambda^n(\mathfrak{h})} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\; w \wedge 1\!\!1^{\otimes(n-p)}, \text{ if } n \geq p, \end{aligned}$$

otherwise it is equal to zero. We call Wick polynomial a finite sum of Wick monomials with symbols belonging to $\mathcal{B}(\Lambda^{p_i}(\mathfrak{h}), \Lambda^{q_i}(\mathfrak{h})), i \in I$ a finite family of index. We mention that a Wick monomial can be defined even if the symbol is unbounded operator, but we restrict our selves to bounded symbols.
The following lemma should enlighten the motivation behind the above definition.

Lemma 2.1. *We have*

$$\begin{aligned} &i)\ \lambda \in \mathbb{C},\ Wick(\lambda) = \lambda 1\!\!1, \\ &ii)\ w \in \mathcal{B}(\mathfrak{h}),\ Wick(w) = \mathrm{d}\Gamma(w), \\ &iii)\ h \in \mathfrak{h},\ Wick(\langle h|) = b(h),\ Wick(|h\rangle) = b^*(h). \\ &iv)\ h_1, ..., h_n \in \mathfrak{h},\ Wick(|h_1 \wedge ... \wedge h_n\rangle) = b^*(h_1)...b^*(h_n), \\ &\qquad\qquad Wick(\langle h_1 \wedge ... \wedge h_n|) = b(h_n)...b(h_1). \end{aligned}$$

Let $\mathcal{K}$ be an auxiliary complex Hilbert space. We extend the definition of Wick polynomials to be observables taking in account the degrees of freedom of $\mathcal{K}$. Hence, for the symbol $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$, we define Wick monomial as follows:

$$\begin{aligned} &Wick(w) : \mathcal{K} \otimes \Lambda^{\text{fin}}(\mathfrak{h}) \to \mathcal{K} \otimes \Lambda^{\text{fin}}(\mathfrak{h}), \\ &Wick(w)_{|\mathcal{K}\otimes\Lambda^n(\mathfrak{h})} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\; 1\!\!1_{\mathcal{K}} \otimes \Lambda_{n+q-p} \quad w \otimes 1\!\!1^{\otimes(n-p)}. \end{aligned}$$

Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, we have the basic cases

$$Wick(v) = b^*(v), \quad Wick(v^*) = b(v),$$

where $b^\sharp$ are the extended creation-annihilation operators in Subsect.2.1. Let summarize properties of this Wick polynomials in the following lemma.

Lemma 2.2. *Let* $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$, $u \in \Lambda^n(\mathfrak{h})$, $v \in \Lambda^m(\mathfrak{h})$. *We have*

$$\begin{aligned} &i)\ Wick(w)^* = Wick(w^*). \\ &ii)\ Wick(|u\rangle \wedge w \wedge \langle v|) = Wick(|u\rangle)\, Wick(w)\, Wick(\langle v|). \end{aligned}$$

In the case $\mathfrak{h} := L^2(\mathbb{R}^d)$. Using the kernel theorem (see e.g. [20]), we know that

$$L(S(\mathbb{R}^{dp}), S'(\mathbb{R}^{dq}, \mathcal{B}(\mathcal{K}))) \cong S'(\mathbb{R}^{dp}) \hat{\otimes} S'(\mathbb{R}^{dq}, \mathcal{B}(\mathcal{K})) \cong S'(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))$$

Then $w \in \mathcal{B}(\mathcal{K}\otimes\Lambda^p(\mathfrak{h}),\mathcal{K}\otimes\Lambda^q(\mathfrak{h}))$ has a distribution kernel $w(k_1,...,k_p,k_1^{'},...,k_q^{'}) \in S^{'}(\mathbb{R}^{d(p+q)},\mathcal{B}(\mathcal{K}))$. Hence, Wick monomials have the following weak integral representation:

$$Wick(w) = \int w(k_1,...,k_p,k_1^{'},...,k_q^{'})\ b^*(k_1^{'})...b^*(k_q^{'})b(k_1)...b(k_p)\ dkdk^{'}.$$

The above expression is equivalent to the one given in the beginning of this subsection. Another way to express Wick polynomials is to decompose the symbol in an orthonormal basis.
Let $e_0(s) := \pi^{-\frac{1}{4}}\mathrm{e}^{-\frac{s^2}{2}}$ and $e_j(s) := -\frac{1}{\sqrt{2^j j!}}P_j(s)e_0(s)$, $j\in\mathbb{N}$, where P_j are the Hermite polynomials satisfying $\frac{d^j}{ds^j}\mathrm{e}^{-\frac{s^2}{2}} = (-1)^j P_j(s)\mathrm{e}^{-\frac{s^2}{2}}$. The family of Hermite functions

$$\{e_J := \prod_{i=1}^{d} e_{j_i},\ J=(j_i)_{1..d}\in\mathbb{N}^d\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$. Let $w \in \mathcal{B}(\mathcal{K}\otimes\Lambda^p(\mathfrak{h}),\mathcal{K}\otimes\Lambda^q(\mathfrak{h}))$, then we have

$$Wick(w) = \sum_{I,J}\lambda_{I,J}\ b^*(e_{I_1})...b^*(e_{I_q})\times b(e_{J_p})...b(e_{J_1}), \tag{2.5}$$

$$\lambda_{I,J} = \langle w, e_{I_1}\wedge...\wedge e_{I_q}\otimes e_{J_1}\wedge...\wedge e_{J_p}\rangle \in \mathcal{B}(\mathcal{K}), \tag{2.6}$$

where the sum runs over all $I=(I_i)_{1..q}$, $J=(J_i)_{1..p}$, $I_i, J_i\in\mathbb{N}^d$. The r.h.s. in (2.5) is weakly convergent on $\mathcal{K}\otimes\Lambda^{\mathrm{fin}}(\mathfrak{h})$.

Boundedness of Wick polynomials. Since we are considering fermions, bounded observables were for particular interest. We recall a well known estimate [11].

Theorem 2.3. *Let $w \in \mathcal{B}(\mathcal{K}\otimes\Lambda^p(\mathfrak{h}),\mathcal{K}\otimes\Lambda^q(\mathfrak{h}))$. We have*

$$\left\|Wick(w)\right\| \le c\ \left\|w(k_1,...,k_p,k_1^{'},...,k_q^{'})\right\|_{S_2(\mathbb{R}^{d(p+q)},\mathcal{B}(\mathcal{K}))},$$

where the norm in the r.h.s. of the estimate is a Schwartz norm

$$\sum_{|\alpha|+|\beta|\le 2d(p+q)}\sup_{k,k^{'}}\left\|(k,k^{'})^\alpha D^\beta_{k,k^{'}} w(k,k^{'})\right\|_{\mathcal{B}(\mathcal{K})},$$

defining the Banach space $S_2(\mathbb{R}^{d(p+q)},\mathcal{B}(\mathcal{K}))$ of elements of $C^2(\mathbb{R}^{d(p+q)},\mathcal{B}(\mathcal{K}))$ with the above norm finite.

Proof. Using the decomposition (2.5), we see that $||Wick(w)|| \le \sum_{I,J}||\lambda_{I,J}||_{\mathcal{B}(\mathcal{K})}$. Since the basis $\{e_J\}$ is formed of Hermite functions which are eigenvectors of the harmonic oscillator $\Theta_h := \frac{1}{2}\sum_{i=1}^d -\frac{d^2}{ds_i^2}+s_i^2-1$ and using Hölder inequality, we can write

$$\begin{aligned}
\sum_{I,J}||\lambda_{I,J}||_{\mathcal{B}(\mathcal{K})} &\le \sum_{I,J}||\lambda_{I,J}||_{\mathcal{B}(\mathcal{K})}\prod_{k=1}^{d(p+q)}(1+i_k)\prod_{k=1}^{d(p+q)}(1+i_k)^{-1}\\
&\le \left(\sum_{I,J}||\lambda_{I,J}||^2_{\mathcal{B}(\mathcal{K})}\prod_{k=1}^{d(p+q)}(1+i_k)^2\right)^{\frac{1}{2}}\left(\sum_{I,J}\prod_{k=1}^{d(p+q)}(1+i_k)^{-2}\right)^{\frac{1}{2}}\\
&\le c||\prod_{k=1}^{p+q}(1+\Theta_h)w||_{L^2(\mathbb{R}^{d(p+q)},\mathcal{B}(\mathcal{K}))}.
\end{aligned}$$

The norm appearing in the last inequality is equivalent to the Schwartz norm claimed in the theorem (see e.g. [18, exemple 7]). ∎

Hence, we conclude that the map $S_2(\mathbb{R}^{d(p+q)}, B(\mathcal{K})) \ni w \to Wick(w) \in \mathcal{B}(\mathcal{K} \otimes \Lambda(\mathfrak{h}))$ is continuous, which also equivalent to the continuity of the map which corresponds to a sequences $\lambda_{I,J}$ with values in $\mathcal{B}(\mathcal{K})$ satisfying $\sum_{I,J} ||\lambda_{I,J}||^2_{\mathcal{B}(\mathcal{K})} \prod_{k=1}^{d(p+q)}(1+i_k)^2 < \infty$, a Wick monomial with respect to the decomposition (2.5).

Wick tensor product. The idea of adding a copy of Fock space interpreting asymptotic free particles had already proved its depth. In this way, we need to construct observables with an another Fock copy.

Let $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^{p_1}(\mathfrak{h}_1) \otimes \Lambda^{p_2}(\mathfrak{h}_2), \mathcal{K} \otimes \Lambda^{q_1}(\mathfrak{h}_1) \otimes \Lambda^{q_2}(\mathfrak{h}_2))$, we define a tensor Wick monomial with symbol w as following

$$\begin{aligned}
&Wick^{\otimes}(w) : \mathcal{K} \otimes \Lambda^{\text{fin}}(\mathfrak{h}_1) \otimes \Lambda^{\text{fin}}(\mathfrak{h}_2) \to \mathcal{K} \otimes \Lambda^{\text{fin}}(\mathfrak{h}_1) \otimes \Lambda^{\text{fin}}(\mathfrak{h}_2),\\
&Wick^{\otimes}(w)_{|\mathcal{K}\otimes\Lambda^n(\mathfrak{h}_1)\otimes\Lambda^m(\mathfrak{h}_2)} := \frac{\sqrt{n!(n-p_1+q_1)!}}{(n-p_1)!} \frac{\sqrt{m!(m-p_2+q_2)!}}{(m-p_2)!}\\
&\mathbb{1}_{\mathcal{K}} \otimes \Lambda_{n-p_1+q_1} \otimes \Lambda_{(m-p_2+q_2)} \; (T_n^{p_1})^{-1} \; (-\mathbb{1})^{\alpha_{q_1 \cdot q_2}(N)} \otimes \mathbb{1}_{\Lambda(\mathfrak{h}_2)}\\
&\mathbb{1}_{\otimes^{n-p_1}\mathfrak{h}_1} \otimes w \otimes \mathbb{1}_{\otimes^{n-p_2}\mathfrak{h}_2} \; T_n^{p_1} \; (-\mathbb{1})^{\alpha_{p_1 \cdot p_2}(N)} \otimes \mathbb{1}_{\Lambda(\mathfrak{h}_2)},
\end{aligned}$$

where T_n^p is the map:

$$\begin{aligned}
T_n^p :\; &\mathcal{K} \otimes \otimes^n \mathfrak{h}_1 \to \otimes^{n-p}\mathfrak{h}_1 \otimes \mathcal{K} \otimes \otimes^p \mathfrak{h}_1,\\
&g \otimes f_1 ... \otimes f_n \to f_{p+1} \otimes .... \otimes f_n \otimes g \otimes f_1 ... \otimes f_p,
\end{aligned}$$

and $\alpha_{q_1,q_2}(N) := Nq_2 + q_2 q_1,\ \alpha_{p_1,p_2}(N) := Np_2 + p_2 p_1$.
When $\mathfrak{h}_1 = \mathfrak{h}_2 = L^2(\mathbb{R}^d)$, we have the following integral representation for tensor Wick monomial with symbol $w \in S'(\mathbb{R}^{d(p_1+q_1+p_2+q_2)}, \mathcal{B}(\mathcal{K}))$:

$$\begin{aligned}
Wick^{\otimes}(w) \; = \int w(k_{1,1}, .., k'_{1,1}, .., k_{2,1}, .., k'_{2,1}, ..) \; &\prod_{i=1}^{q_1} b^*(k_{1,i}) \otimes \mathbb{1} \prod_{i=1}^{q_2} (-\mathbb{1})^N \otimes b^*(k_{2,i})\\
&\times \prod_{i=1}^{p_2} (-\mathbb{1})^N \otimes b(k'_{2,i}) \prod_{i=1}^{p_1} b(k'_{1,i}) \otimes \mathbb{1} \; dk dk', \qquad (2.7)
\end{aligned}$$

where we integrate over all variables $k_{i,j}, k'_{k,l}$.
We collect some properties of tensor Wick monomials in the following lemma.

Lemma 2.4. *i) Let $w_1 \in \mathcal{B}(\mathcal{K} \otimes \Lambda^{p_1}(\mathfrak{h}_1), \mathcal{K} \otimes \Lambda^{q_1}(\mathfrak{h}_1))$, and $w_2 \in \mathcal{B}(\Lambda^{p_2}(\mathfrak{h}_2), \Lambda^{q_2}(\mathfrak{h}_2))$. We have*

$$Wick^{\otimes}(w_1 \otimes w_2) = (-\mathbb{1})^{\alpha_{q_1 \cdot q_2}(N)} Wick(w_1)(-\mathbb{1})^{\alpha_{p_1 \cdot p_2}(N)} \otimes Wick(w_2).$$

ii) Let $w = | \wedge_{i=1}^{q_1} h_{i,1}\rangle\langle \wedge_{i=1}^{p_1} g_{i,1}| \otimes | \wedge_{i=1}^{q_2} h_{i,2}\rangle\langle \wedge_{i=1}^{p_2} g_{i,2}|$. Then

$$Wick^{\otimes}(w) = \prod_1^{q_1} b^*(h_{1,i}) \otimes \mathbb{1} \prod_1^{q_2} (-\mathbb{1})^N \otimes b^*(h_{2,i}) \times \prod_1^{p_2} (-\mathbb{1})^N \otimes b(g'_{2,i}) \prod_1^{p_1} b(g'_{1,i}) \otimes \mathbb{1}.$$

iii) Let $j_i, i = 1,2$ be two maps acting on $\mathfrak{h}_i \to \mathfrak{h}_i$. Then

$$\Gamma(j_1) \otimes \Gamma(j_2)\, Wick^{\otimes}(w\, \Gamma(j_1) \otimes \Gamma(j_2)) = Wick^{\otimes}(w)\, \Gamma(j_1) \otimes \Gamma(j_2).$$

iv) $Wick^{\otimes}(w^*) = Wick^{\otimes}(w)^*$.

v) The following map is continuous

$$S_2(\mathbb{R}^{d(p_1+q_1+p_2+q_2)}, \mathcal{B}(\mathcal{K})) \ni w \to Wick^{\otimes}(w) \in \mathcal{B}(\mathcal{K} \otimes \Lambda(\mathfrak{h}) \otimes \Lambda(\mathfrak{h})).$$

Proof. $i) - iv)$ follows by simple computation in one sector $\mathcal{K} \otimes \Lambda^n(\mathfrak{h}_1) \otimes \Lambda^m(\mathfrak{h}_2)$ and even one can easily checked this points for the case $\mathfrak{h}_1 = \mathfrak{h}_2 = L^2(\mathbb{R}^d)$, using the integral representation. We can decompose a symbol of a tensor Wick monomial using Hermite functions. Then (2.6) can appear as sum of elements of the form *ii)*. Now applying the same argument as in the proof of Thm. 2.3 we obtain *v)*. ▌

The properties of tensor Wick monomials definition are mentioned in the following proposition.

Proposition 2.5. *The following assertions hold:*
i) Let $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}_1 \oplus \mathfrak{h}_2), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}_1 \oplus \mathfrak{h}_2))$.

$$\begin{aligned}
&U\, Wick(w)\, U^* = Wick^{\otimes}(\tilde{U} w \tilde{U}^*), \\
&\textit{where}\ \ \tilde{U}^* u \otimes v := \frac{(p+q)!}{p!q!} \Gamma(i_1) u \wedge \Gamma(i_2) v,\ u \in \Lambda^p(\mathfrak{h}_1), v \in \Lambda^q(\mathfrak{h}_2), \\
&\textit{and}\ \ \tilde{U}_{|\Lambda^n(\mathfrak{h}_1 \oplus \mathfrak{h}_2)} := \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} P_1^{\otimes k} \otimes P_2^{\otimes (n-k)}.
\end{aligned}$$

ii) Let $w \in B(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$.

$$\begin{aligned}
&Wick(w) I = I Wick^{\otimes}(P w \tilde{I}), \\
&\tilde{I} u \otimes v = (-1)^p \frac{(p+q)!}{p!q!} u \wedge v,\ u \in \Lambda^p(\mathfrak{h}), v \in \Lambda^q(\mathfrak{h}), \\
&\textit{and}\ \ P : \Lambda^{\rm fin}(\mathfrak{h}) \to \Lambda^{\rm fin}(\mathfrak{h}),\ Pu = u \otimes \Omega.
\end{aligned}$$

iii) Let $j = (j_0, j_\infty)$ *be a pair of maps acting on* $\mathfrak{h}$ *and* $w \in B(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$.

$$Wick(\Gamma(j_0) w) I(j) = I(j) Wick^{\otimes}(P w \tilde{I} \Gamma(j_0) \otimes \Gamma(j_\infty)).$$

Proof. To prove *i)* it is enough to show it only for symbols w having the form $|h_1 \oplus g_1 \wedge ... \wedge h_n \oplus g_n\rangle$. We have

$$\begin{aligned}
UWick(w)U^* &= \prod_{i=1}^{n} (b^*(h_i) \otimes 1\!\!1 + (-1\!\!1)^N \otimes b^*(g_i)) \\
&= \sum_{k=0}^{n} \sum_{I,J \in \mathfrak{S}} \varepsilon(I,J) \prod_{s=1}^{k} b^*(h_{i_s}) \otimes 1\!\!1 \times \prod_{s=1}^{n-k} (-1\!\!1)^N \otimes b^*(g_{j_s}) \\
&= Wick^{\otimes}(\sum_{k=0}^{n} \sum_{I,J \in \mathfrak{S}} \varepsilon(I,J) h_{i_1} \wedge ... \wedge h_{i_k} \otimes g_{j_1} \wedge ... \wedge g_{j_{n-k}}) \\
&= Wick^{\otimes}(\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} P_1^{\otimes (k)} \otimes P_2^{\otimes (n-k)} h_1 \oplus g_1 \wedge ... \wedge h_n \oplus g_n) \\
&= Wick^{\otimes}(\tilde{U} w).
\end{aligned}$$

As for $i)$ we have only to check $ii)$ for symbols of the from $|h_1 \wedge ... \wedge h_n\rangle$. We have:

$$\begin{aligned} Wick(w)I &= \prod_{i=1}^{n} b^*(h_i)I \\ &= (-1)^n I \prod_{i=1}^{n} b^*(h_i) \otimes 1\!\!1 \\ &= (-1)^n I\, Wick^{\otimes}(|h_1 \wedge ... \wedge h_n\rangle \otimes 1\!\!1_{\Omega}). \end{aligned}$$

We obtain $iii)$ using $ii)$ and $iii)$ of Lemma 2.4. We have

$$\begin{aligned} Wick(\Gamma(j_0)w)I(j) &= Wick(\Gamma(j_0)w)I\Gamma(j_0) \otimes \Gamma(j_\infty) \\ &= IWick^{\otimes}(P\Gamma(j_0)w\tilde{I})\Gamma(j_0) \otimes \Gamma(j_\infty) \\ &= IWick^{\otimes}(\Gamma(j_0) \otimes \Gamma(j^\infty)_{|\mathbb{C}\Omega} Pw\tilde{I})\Gamma(j_0) \otimes \Gamma(j_\infty) \\ &= I(j)Wick^{\otimes}(Pw\tilde{I}\Gamma(j_0) \otimes \Gamma(j_\infty)). \end{aligned}$$

∎

We give now some commutation relations satisfied by the Wick and tensor Wick polynomials.

Proposition 2.6.
i) Let $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$ and $a \in \mathcal{B}(\mathfrak{h})$.

$$[\mathrm{d}\Gamma(a), Wick(w)] = Wick([\mathrm{d}\Gamma(a), w]).$$

ii) Let $q \in \mathcal{B}(\mathfrak{h})$,

$$\begin{aligned} &\Gamma(q)Wick(w\Gamma(q)) = Wick(\Gamma(q)w)\Gamma(q). \\ &\Gamma(q)Wick(w) = Wick(\Gamma(q)w\Gamma(q^*))\Gamma(q), \ \textit{for } q \textit{ isometric }, \\ &\Gamma(q)Wick(w)\Gamma(q^{-1}) = Wick(\Gamma(q)w\Gamma(q^{-1})) \ \textit{for } q \textit{ unitary}, \\ &[\Gamma(q), Wick(w)] = \Gamma(q)Wick(w(1-\Gamma(q))) + Wick((\Gamma(q)-1)w)\Gamma(q). \end{aligned}$$

iv) Let $h \in \mathfrak{h}$ and $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$, $p+q$ even.

$$\begin{aligned} &[Wick(w), b^*(h)] = p \times Wick(w|h\rangle), \\ &[Wick(w), b(h)] = q \times Wick(\langle h|w). \end{aligned}$$

v) Let $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^{p_1}(\mathfrak{h}) \otimes \Lambda^{p_2}(\mathfrak{h}), \mathcal{K} \otimes \Lambda^{q_1}(\mathfrak{h}) \otimes \Lambda^{q_2}(\mathfrak{h}))$ and $j = (j_0, j_\infty) : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$.

$$\begin{aligned} I^*(j)Wick(w) &= Wick(w) \otimes 1\!\!1\, I^*(j) \\ &\quad + I^*(j)Wick(w(1-\Gamma(j_0^*))) \\ &\quad + Wick^{\otimes}\big((\Gamma(j_0^*)-1) \otimes \Gamma(j_\infty)\tilde{I}^*(j_\infty^*)wP^*\big)I^*(j) \\ &\quad + Wick^{\otimes}\big(1\!\!1 - (\Gamma(j_\infty^*) - |\Omega\rangle\langle\Omega|)\tilde{I}^*wP^*\big)I^*(j). \end{aligned}$$

The proof follows from simple computation, we leave it for the reader.
We finish this subsection with some commutator estimates.

Theorem 2.7. *Let q be a bounded operator on $\mathfrak{h}$ such that $||q|| \leq 1$ and $w \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h}))$. The following inequalities hold:*

$$||[\Gamma(q), Wick(w)]|| \leq \sum_i ||1\!\!1_{\otimes_{\mathfrak{h}}^{i-1}} \otimes (1-q) \otimes 1\!\!1_{\otimes_{\mathfrak{h}}^{p+q-i}} w||_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))},$$

$$\begin{aligned}
&||I^*(j)Wick(w) - Wick^{\otimes}(w \otimes \Omega)I^*(j)|| \leq \\
&c\Bigg(\sum_i ||1\!\!1_{\mathfrak{h}^{\otimes(i-1)}} \otimes (1-j_0) \otimes 1\!\!1_{\mathfrak{h}^{\otimes(p+q-i)}} w||_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))} \\
&+ \sum_i ||1\!\!1_{\mathfrak{h}^{\otimes(i-1)}} \otimes j_\infty \otimes 1\!\!1_{\mathfrak{h}^{\otimes(p+q-i)}} w||_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))}\Bigg).
\end{aligned}$$

Proof. Using Lemma 2.6 *ii*), we see that the commutator is equal to

$$[\Gamma(q), Wick(w)] = \Gamma(q)Wick(w(1-\Gamma(q))) + Wick((\Gamma(q)-1)w)\Gamma(q).$$

Assuming that $||q|| \leq 1$, we have only to estimate $Wick((\Gamma(q)-1)w)$ and $Wick(w(1-\Gamma(q)))$ using Thm. 2.3. Hence this leads to estimate the kernel $\sum_{i=1}^{p+q} 1\!\!1_{\mathfrak{h}^{\otimes(i-1)}} \otimes (1-q) \otimes 1\!\!1_{\mathfrak{h}^{\otimes(p+q-i)}} w(k,k')$. This proves *i*). The second point *ii*) is exactly similar to the above one. ▮

3. SPIN-FERMION MODEL AND MAIN RESULTS

The spin-fermion model considered here interpret a small system with finite number of degrees of freedom interacting with a fermionic reservoir at zero temperature. Free and perturbed time-evolution dynamics are one-parameter groups of $*$-automorphisms acting on the algebra of observables. In this notes we consider the zero temperature case described by a free and interacting Hamiltonians acting in the full Hilbert space $\mathcal{K} \otimes \mathcal{H}$ precised below.

3.1. The Model

The free dynamic. Let $\mathcal{K}$ be a complex Hilbert space (*the small system*). In all the sequel we assume

$$(H0) \quad \mathcal{K} \text{ finite dimension.}$$

We are concerned with the case $\mathfrak{h} := L^2(\mathbb{R}^d)$. We distinguish states considered as wave functions on the position variables from ones considered on the momentum variables. We denote by the italic letter k the one particle fermionic momentum and by the italic letter x the one particle fermionic position. The one particle space $\mathfrak{h}$ will be $L^2(\mathbb{R}^d, dk)$ considered in the momentum representation.

Let $\omega \in C(\mathbb{R}^d, \mathbb{R})$ be the dispersion relation for fermions. We assume that ω satisfies the following hypothesis.

$$(H1) \quad \begin{cases} \nabla\omega \in L^\infty(\mathbb{R}^d), \\ \nabla\omega(k) \neq 0, \text{ for } k \neq 0, \\ \lim\limits_{k\to\infty} \omega(k) = +\infty, \\ \inf\limits_k \omega(k) = \omega(0) =: m > 0. \end{cases}$$

Let $\mathcal{H}$ denotes the Hilbert space of the small system interacting with the reservoir i.e:

$$\mathcal{H} := \mathcal{K} \otimes \Lambda(\mathfrak{h}).$$

The free Hamiltonian is defined by:

$$H_0 := K \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega), \text{ acting on } \mathcal{H}.$$

We mention that H_0 is essentially self-adjoint on $\mathcal{D}(K) \otimes \Lambda^{\mathrm{fin}}(C_0^\infty(\mathbb{R}^d))$.

Perturbation. Let P be an element in $\mathcal{B}(\mathcal{H})$ such that $P^* = P$. Then we can construct the perturbed Hamiltonian H respectively to the interaction P by Dyson's series. For this one can use the general result stated in [5, Thm. 3.1.33]. Namely, We have that

$$H := H_0 + P, \text{ acting on } \mathcal{H},$$

is a well defined self-adjoint operator with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$.
We consider now the perturbation having the following form:

$$\begin{aligned} P &:= \sum_{p,q\in\Xi} Wick_\pi(w_{p,q}), \quad \text{where } \Xi := \{(p,q) \in \mathbb{N}^2 \mid p+q \text{ even } \} \\ &\text{and } w_{p,q} \in \mathcal{B}(\mathcal{K} \otimes \Lambda^p(\mathfrak{h}), \mathcal{K} \otimes \Lambda^q(\mathfrak{h})). \end{aligned}$$

One can assume

$$(I0) \quad \sum_{p,q\in\Xi} \sum_{I,J} \| \langle w_{p,q}, e_I \otimes e_J \rangle \|_{\mathcal{B}(\mathcal{K})} < \infty.$$

The hypothesis $(I0)$ ensure that P is bounded and therefore the perturbed dynamic is well defined by the Hamiltonian H which is a self-adjoint operator acting on $\mathcal{H}$. But, we will assume a slightly stronger condition for P which has the advantage to be more convenient to check since it involves kernels of the symbols $w_{p,q}$ instead of their operator norms. Namely,

$$(I0^{'}) \quad \sum_{p,q\in\Xi} \|w_{p,q}\|_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))} < \infty.$$

We introduce the following notations:

$$\begin{aligned} &\mathcal{H}^{\mathrm{ext}} := \mathcal{H} \otimes \Lambda(\mathfrak{h}), \\ &H_0^{\mathrm{ext}} := H_0 \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega), \quad H^{\mathrm{ext}} := H \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega), \text{ acting on } \mathcal{H}^{\mathrm{ext}}, \\ &N_0 := N \otimes 1\!\!1, \quad N_\infty := 1\!\!1 \otimes N, \text{ acting on } \mathcal{H}^{\mathrm{ext}}. \end{aligned}$$

We assume also an hypothesis concerning the Mourre estimate. Let conjugate operator a introduced in Subsect. 4.3.

$$(Mo) \quad \sum_{p,q\in\Xi} || \sum_{i=1}^{p} a_i w_{p,q} - \sum_{i=1}^{q} a_i w_{p,q} ||_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))} < \infty.$$

To prove asymptotic completeness we need more restrictive hypothesis on the interaction. We assume a decay of the perturbation of type *short range* , it appears also in the bosonic case [7].

$$\sum_{p,q\in\Xi} \sum_{i=1}^{p+q} (p+q) || 1\!\!1_{[R,\infty[}(|x_i|)\, w_{p,q} ||_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))} \in \begin{cases} O(R^\mu), \ \mu > 1, & (SR), \\ o(R^0), & (I1). \end{cases}$$

3.2. Result

We formulate the main result in the next theorem. We define the asymptotic *vacua* space

$$\mathcal{K}^{\pm} := \{\Psi \in \mathcal{H}, b^{\pm}(h)\Psi = 0, h \in \mathfrak{h}\},$$

where $b^{\pm}$ are asymptotic annihilation operators defined by

$$b^{\pm,\sharp}(h) := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}b^{\sharp}(h)\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}, h \in \mathfrak{h}.$$

Their existence will be proved in Subsect. 5.1. We set

$$\mathcal{H}_{\mathrm{bd}}(H) := \mathrm{Ran}\mathbb{1}_{\mathrm{pp}}(H)\mathcal{H}.$$

Theorem 3.8. *Assume hypothesis $(H0), (H1), (I0^{'}), (Mo)$ and (SR) satisfied. Then asymptotic completeness holds for the spin fermion model i.e:*

$$\mathcal{K}^{\pm} = \mathcal{H}_{\mathrm{bd}}(H).$$

The proof figure in Sect.5.4

4. SPECTRAL ANALYSIS

In this section we study the spectral properties of the spin-fermion Hamiltonian. We prove the existence of a ground state far from the essential spectrum with a gap equal to m, Thm. 4.11. Such property was been extensively studied specially in the massless case, see [4], [19], [10], [13]. Here, we follow the proof in [7].

We establish in Subsect. 4.3 a Mourre estimate which has the consequence that we have local finite pure point spectrum outside thresholds. This is an important steep in the proof of asymptotic completeness as it is known for the N-body Schrödinger operator [9].

We start in the following subsection by collecting some commutator estimate which will be often useful in the sequel.

4.1. Commutator estimates

In this subsection, we will often use the following functional calculus formula for a self-adjoint operator A and a function $\chi \in C_0^{\infty}(\mathbb{R})$.

$$\chi(A) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(z-A)^{-1}dz\wedge d\bar{z}, \tag{4.1}$$

where $\tilde{\chi}$ is an almost analytic extension of χ satisfying

$$\tilde{\chi}_{|\mathbb{R}} = \chi, \ \ |\partial_{\bar{z}}\tilde{\chi}(z)| \le c_n|\mathrm{Im}(z)|^n, n \in \mathbb{N}.$$

Let $q \in C_0^{\infty}(\mathbb{R}^d)$, satisfying $0 \le q \le 1$ and $q = 1$ near 0. We set $q^R := q(\frac{x}{R})$.

Lemma 4.9. *Let $\chi \in C_0^{\infty}(\mathbb{R})$. We have for R large*

$$[\chi(H), \Gamma(q^R)] \in \begin{cases} O(R^{-1}), & \textit{under } (SR), \\ o(R^0), & \textit{under } (I1). \end{cases}$$

Proof. We first, compute $[H, \Gamma(q^R)]$ as a quadratic form on $\mathcal{D}(H_0)$.

$$[H_0, \Gamma(q^R)] = \mathrm{d}\Gamma(q^R, [\omega, q^R]),$$
$$[P, \Gamma(q^R)] = \sum_{p,q\in\Xi} \Gamma(q^R) Wick(w_{p,q}(1-\Gamma(q^R))) + Wick((\Gamma(q^R)-1)w_{p,q})\Gamma(q^R).$$

We notice that $[H, \Gamma(q^R)]$ extends to a bounded operator on the domain of H_0.
Applying formula (4.1), we obtain

$$[\chi(H), \Gamma(q^R)] = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)\ (z-H)^{-1}[H, \Gamma(q^R)](z-H)^{-1}\ dz\wedge d\bar{z}.$$

Then, it is enough to estimate $(\mathrm{i}+H_0)^{-1}[H, \Gamma(q^R)](\mathrm{i}+H_0)^{-1}$. We have $[\omega, \Gamma(q^R)] \in O(R^{-1})$ and using Thm. 2.7 in the two cases $(SR), (I1)$, we obtain the claimed estimate. ▮

Let $j_0 \in C_0^\infty(\mathbb{R}^d), j_\infty \in C^\infty(\mathbb{R}^d)$, such that $0 \le j_0, j_\infty \le 1$, $j_0 = 1$ near 0 and $j_0^2 + j_\infty^2 \le 1$. We set $j_0^R := j_0(\frac{x}{R}), j_\infty^R := j_\infty(\frac{x}{R})$ and $j^R := (j_0^R, j_\infty^R)$.

Lemma 4.10. *Let $\chi \in C_0^\infty(\mathbb{R})$. We have for R large*

$$\chi(H^{\mathrm{ext}})I^*(j^R) - I^*(j^R)\chi(H) \in \begin{cases} O(R^{-1}), & \textit{under } (SR), \\ o(R^0), & \textit{under } (I1). \end{cases}$$

Proof. The proof is similar to the above one and it is based on commutation relation on Prop. 2.6 and estimate in Thm. 2.7. We have

$$\begin{aligned}&\chi(H^{\mathrm{ext}})I^*(j^R) - I^*(j^R)\chi(H) = \\ &\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)\ (z-H^{\mathrm{ext}})^{-1}\Big(H^{\mathrm{ext}}I^*(j^R) - I^*(j^R)H\Big)(z-H)^{-1} dz\wedge d\bar{z},\end{aligned} \tag{4.2}$$

Thm.2.7 gives us

$$\begin{aligned}&\|(\mathrm{i}+H_0^{\mathrm{ext}})^{-1}\Big(H^{\mathrm{ext}}I^*(j^R) - I^*(j^R)H\Big)(\mathrm{i}+H_0)^{-1}\| \le \\ &\|(\mathrm{i}+H_0^{\mathrm{ext}})^{-1}\mathrm{d}I^*(j^R, [j^R, \omega])(\mathrm{i}+H_0)^{-1}\| \\ &+ \sum_{p,q\in\Xi}\sum_i \|\mathbb{1}_{\mathfrak{h}^{\otimes(i-1)}} \otimes (1-j_0^R) \otimes \mathbb{1}_{\mathfrak{h}^{\otimes(p+q-i)}} w_{p,q}\|_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))} \\ &+ \sum_{p,q\in\Xi}\sum_i \|\mathbb{1}_{\mathfrak{h}^{\otimes(i-1)}} \otimes j_\infty^R \otimes \mathbb{1}_{\mathfrak{h}^{\otimes(p+q-i)}} w_{p,q}\|_{S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))}.\end{aligned}$$

The fact that $[j^R, \omega] \in O(R^{-1})$ and hypothesis (SR) and $(I1)$ imply the following

$$\|(\mathrm{i}+H_0^{\mathrm{ext}})^{-1}\Big(H^{\mathrm{ext}}I^*(j^R) - I^*(j^R)H\Big)(\mathrm{i}+H_0)^{-1}\| \in \begin{cases} O(R^{-1}) \text{ under } (SR), \\ o(R^0) \text{ under } (I1). \end{cases} \tag{4.3}$$

Now, combining (4.2) and (4.3) we prove the lemma. ▮

4.2. The nature of the spectrum

Next, we give a sort of HVZ theorem describing the qualitative aspect of the spectrum of H.

Theorem 4.11. *Assume $(H0)-(H1)$ and $(I1)$ satisfied, then*

$$\sigma_{\rm ess}(H) = [\inf \sigma(H) + m, +\infty[.$$

Moreover, $\inf \sigma(H)$ *is a discrete eigenvalue.*

Proof. Let $\chi \in C_0^\infty(]-\infty, \inf\sigma(H)+m[)$. Since $H^{\rm ext}1\!\!1_{[1,+\infty[}(N_\infty) \geq \inf\sigma(H)+m$, we conclude that we have

$$\chi(H^{\rm ext}) = \chi(H^{\rm ext})1\!\!1_{\{0\}}(N_\infty).$$

We use Lemma 4.10 to write

$$\begin{aligned}
\chi(H) &= \chi(H)I(j^R)I^*(j^R) \\
&= I(j^R)1\!\!1_{\{0\}}(N_\infty)\chi(H^{\rm ext})I^*(j^R) + o(R^0) \\
&= I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R)\chi(H) + o(R^0) \\
&= \Gamma((j_0^R)^2)\chi(H) + o(R^0).
\end{aligned}$$

One can prove that $\Gamma((j_0^R)^2)\chi(H)$ is compact operator. In fact, it suffices to see that $\prod_{i=1}^n j_0^2(\frac{x_i}{R})\ (K + \sum_{i=1}^n \omega(D_{x_i}) + {\rm i})^{-1}$ is compact for every $n \in \mathbb{N}$. Then using the norm limit

$$\Gamma((j_0^R)^2)1\!\!1_{[0,n[}(N) \to \Gamma((j_0^R)^2), n \to \infty,$$

we obtain compactness of $\Gamma((j_0^R)^2)\chi(H)$ and hence of $\chi(H)$. This shows

$$\sigma_{\rm ess}(H) \subset [\inf\sigma(H)+m, +\infty[.$$

Now, it is enough to find a Weyl sequence for every energy level $\lambda \in]\inf\sigma(H)+m, +\infty[$. Let Ψ be a ground state. The existence of Ψ is ensured by the first step of the proof. Let $h \in C_0^\infty(\mathbb{R}^d)$ with $||h|| = 1$ and k_0 such that $\omega(k_0) + \inf\sigma(H) = \lambda$. Let $R_j \in \mathbb{R}^d$ be a sequence satisfying $\lim_{j\to\infty} j^{-1}|R_j| = \infty$. We set

$$h_j(k) = j^{\frac{d}{2}} h(j(k-k_0)){\rm e}^{{\rm i}R_j.k}.$$

Then $w-\lim_{j\to\infty} h_j = 0$, $||h_j|| = 1$ and $\lim_{j\to\infty}(\omega(k)-\omega(k_0))h_j = 0$. Let $\Psi_j = b^*(h_j)\Psi$.
We have

$$\begin{aligned}
(H-\lambda)\Psi_j &= b^*(h_j)(H-\lambda)\Psi + b^*(\omega h_j)\Psi + [P, b^*(h_j)]\Psi \\
&= b^*((\omega(k)-\omega(k_0))h_j)\Psi + [P, b^*(h_j)]\Psi.
\end{aligned}$$

Since $\lim_{j\to\infty}[P, b^*(h_j)]\Psi = 0$, we conclude that $\lim_{j\to\infty}(H-\lambda)\Psi_j = 0$. Clearly Ψ_j is a Weyl sequence at level λ. ▌

4.3. Mourre Theory

Let a be the operator acting on $\mathfrak{h}$ defined by

$$a := \frac{1}{2}(\nabla\omega(k).D_k + D_k.\nabla\omega(k)), \text{ on } C_0^\infty(\mathbb{R}^d).$$

The closure of a is a self-adjoint operator with $C_0^\infty(\mathbb{R}^d)$ as a core [3, Prop. 4.2.3]. It is also the infinitesimal generator of the strong continuous group Φ_t associated to the vector field $\nabla\omega$, given by

$$\Phi_t F(k) = [\det \nabla\phi_{-t}(k)]^{\frac{1}{2}} F(\phi_{-t}(k)), \text{ for } F \in S'(\mathbb{R}^d),$$

where ϕ_t is the flow of the vector field $\nabla\omega$.
We denote the *thresholds* set by

$$\tau := \sigma_{\mathrm{pp}}(H) + m\mathbb{N}^\times,$$

where $\mathbb{N}^\times$ is the set of positive integers. We define the conjugate operator

$$A := \mathrm{d}\Gamma(a).$$

It is essentially self-adjoint on $\Lambda^{\mathrm{fin}}(C_0^\infty(\mathbb{R}^d))$.

Theorem 4.12. *The following three assertions hold:*
i) Let $\lambda \in \mathbb{R}\setminus\tau$. Then there exist $\epsilon > 0, C_0 > 0$ and compact operator K_0 such that

$$1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,\mathrm{i}A]1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq C_0\, 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) + K_0.$$

ii) For all $[\lambda_1,\lambda_2]$ such that $[\lambda_1,\lambda_2]\cap\tau = \emptyset$, one has

$$\dim 1\!\!1^{\mathrm{pp}}_{[\lambda_1,\lambda_2]}(H)\mathcal{H} < \infty.$$

Consequently $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, which is a closed countable set.
iii) Let $\lambda \in \mathbb{R}\setminus(\tau\cup\sigma_{\mathrm{pp}}(H))$. Then there exists $\epsilon > 0, C_0 > 0$ such that

$$1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,\mathrm{i}A]1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq C_0\, 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

Proof. We will follow the logic of the proof of Mourre estimate in the case of a Pauli-Fierz Hamiltonian for bosons [7].

We set

$$d(\lambda) := \inf\{\sum_{i=1}^n |\nabla\omega(k_i)|^2; \tau + \sum_{i=1}^n \omega(k_i) = \lambda,\ n = 1,2\ldots,\ \tau\in\sigma_{\mathrm{pp}}(H)\},$$

$$\tilde{d}(\lambda) := \inf\{\sum_{i=1}^n |\nabla\omega(k_i)|^2; \tau + \sum_{i=1}^n \omega(k_i) = \lambda,\ n = 0,1,\ldots,\ \tau\in\sigma_{\mathrm{pp}}(H)\}.$$

$$\Delta_\lambda^\mu := [\lambda-\mu,\lambda+\mu],\ \mu > 0,$$

$$d^\mu(\lambda) := \inf_{\nu\in\Delta_\lambda^\mu} d(\nu),$$

$$\tilde{d}^\mu(\lambda) := \inf_{\nu\in\Delta_\lambda^\mu} \tilde{d}(\nu),$$

$$E_0 := \inf \sigma(H).$$

We give statements, implying Thm. 4.12, that we will prove by induction in n:
$H_1(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists a compact operator K_0, an interval $\Delta \ni \lambda$ such that

$$1\!\!1_\Delta(H)[H, \mathrm{i}A]1\!\!1_\Delta(H) \geq (d(\lambda) - \epsilon)1\!\!1_\Delta(H) + K_0.$$

$H_2(n)$: Let $\epsilon > 0$ and $\lambda \in [E_0, E_0 + nm[$. There exists an interval $\Delta \ni \lambda$ such that

$$1\!\!1_\Delta(H)[H, \mathrm{i}A]1\!\!1_\Delta(H) \geq (\tilde{d}(\lambda) - \epsilon)1\!\!1_\Delta(H).$$

$H_3(n)$: Let $\mu > 0, \epsilon_0 > 0$. There exists $\delta > 0$ such that for all $\lambda \in [E_0, E_0 + nm - \epsilon_0]$, one has

$$1\!\!1_{\Delta_\lambda^\delta}(H)[H, \mathrm{i}A]1\!\!1_{\Delta_\lambda^\delta}(H) \geq (\tilde{d}^\mu(\lambda) - \epsilon)1\!\!1_{\Delta_\lambda^\delta}(H).$$

$S_1(n)$: τ is closed countable set in $[E_0, E_0 + nm]$.
$S_2(n)$: For all $\lambda_1 \leq \lambda_2 \leq E_0 + nm$ with $[\lambda_1, \lambda_2] \cap \tau = \emptyset$, one has

$$\dim 1\!\!1^{\mathrm{pp}}_{[\lambda_1, \lambda_2]}(H)\mathcal{H} < \infty.$$

The sketch of the proof is given by

$$\begin{aligned}
&S_2(n-1) \Rightarrow S_1(n),\\
&(S_1(n), H_3(n-1)) \Rightarrow H_1(n),\\
&H_1(n) \Rightarrow H_2(n),\\
&H_2(n) \Rightarrow H_3(n),\\
&H_1(n) \Rightarrow S_2(n).
\end{aligned}$$

Let check Mourre hypothesis:

$$\begin{aligned}
&(i)\, \mathrm{e}^{\mathrm{i}tA} \text{ preserves } \mathcal{D}(H_0),\\
&(ii)\, |(H\Psi, A\Psi) - (A\Psi, H\Psi)| \leq c(||H_0\Psi||^2 + ||\Psi||^2),\ \Psi \in \mathcal{D}(H_0) \cap \mathcal{D}(A).
\end{aligned}$$

It is enough show that $H_0 \mathrm{e}^{\mathrm{i}tA}(H_0 + \mathrm{i})^{-1}$ is bounded, to prove (i). We have

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}tA}\mathrm{d}\Gamma(\omega)\mathrm{e}^{\mathrm{i}tA} &= \mathrm{d}\Gamma(\mathrm{e}^{-\mathrm{i}ta}\omega\mathrm{e}^{\mathrm{i}ta})\\
&= \mathrm{d}\Gamma(\omega(\phi_{-t}(k))).
\end{aligned}$$

Using the fact that $|\sum_{i=1}^N \omega(\phi_{-t}(k_i)) - \omega(k_i)| \leq N \sup_k |\nabla\omega(k)|$, we obtain (i). We have as quadratic form on $\mathcal{D}(H_0) \cap \mathcal{D}(A)$

$$[H, \mathrm{i}A] = \mathrm{d}\Gamma(|\nabla\omega|) + \sum_{p,q \in \Xi} Wick([w_{p,q}, \mathrm{i}\, \mathrm{d}\Gamma(a)]).$$

Assuming the hypothesis (Mo) and using Lemma 2.7, we see that $[H, \mathrm{i}A]$ extends on bounded operator on $\mathcal{D}(H_0)$. This prove $ii)$.

We notice that $H(1)$ and $S(1)$ are immediate because the spectrum of H is discrete in $[E_0, E_0 + m[$. $S_2(n-1) \Rightarrow S_1(n)$ is obvious. Under Mourre hypothesis $i) - ii)$ satisfied, we have $H_1(n) \Rightarrow H_2(n)$, $H_2(n) \Rightarrow$

$H_3(n)$ following from standard arguments in [6],[17]. The implication $H_1(n) \Rightarrow S_2(n)$ is based in the Virial theorem, which hold also under Mourre hypothesis. Then, we have only to prove the implication $(S_1(n), H_3(n-1) \Rightarrow H_1(n))$.

Let $\chi \in C_0^\infty(\mathbb{R})$.

$$\begin{aligned}\chi(H) &= I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R)\chi(H) + I(j^R)1\!\!1_{[1,\infty[}(N_\infty)I^*(j^R)\chi(H) &(4.4)\\ &= \Gamma((j_0^R)^2)\chi(H) + I(j^R)1\!\!1_{[1,\infty[}(N_\infty)\chi(H^{\text{ext}})I^*(j^R) + o(R^0). &(4.5)\end{aligned}$$

(4.4) follows from the fact that $I(j^R)I^*(j^R) = 1\!\!1$ and $I(j^R)$ is bounded. (4.5) follows from the fact that one has

$$\Gamma((j_0^R)^2) = I(j^R)1\!\!1_{\{0\}}(N_\infty)I^*(j^R), \qquad I(j^R) = \Gamma(j^R)U,$$

and application of Lemma 4.10. We recall that the term $\Gamma((j_0^R)^2)\chi(H)$ is compact since the operator $\Gamma((j_0^R)^2)(H_0+1)^{-\frac{1}{2}}$ is compact.

Let $\lambda \in [E_0, E_0 + nm[$. Since $S_2(n-1) \Rightarrow S_1(n)$, the set τ is closed in $[E_0, E_0+nm]$, which gives $d(\lambda) = \sup_{\mu>0} d^\mu(\lambda)$. So we can choose μ such that $d^\mu(\lambda) \geq d(\lambda) - \frac{\epsilon}{3}$. $H_3(n-1)$ gives for $\lambda_1 < E_0 + (n-1)m$

$$1\!\!1_{\Delta^\delta_{\lambda_1}}(H)[H, \mathrm{i}A]1\!\!1_{\Delta^\delta_{\lambda_1}}(H) \geq (\tilde{d}^\mu(\lambda_1) - \frac{\epsilon}{3})1\!\!1_{\Delta^\delta_{\lambda_1}}(H).$$

Replacing λ_1 with $\lambda - \mathrm{d}\Gamma(\omega(k))$, we obtain

$$\begin{aligned}&1\!\!1_{\Delta^\delta_\lambda}(H + 1\!\!1\otimes \mathrm{d}\Gamma(\omega(k)))([H,\mathrm{i}A] + 1\!\!1\otimes\mathrm{d}\Gamma(|\nabla\omega|^2))\\ &\qquad\qquad \times 1\!\!1_{\Delta^\delta_\lambda}(H + 1\!\!1\otimes\mathrm{d}\Gamma(\omega(k)))1\!\!1_{[1,\infty[}(N_\infty)\\ &\geq 1\!\!1_{\Delta^\delta_\lambda}(H + 1\!\!1\otimes\mathrm{d}\Gamma(\omega(k)))\\ &\qquad \times(\tilde{d}^\mu(\lambda - 1\!\!1\otimes\mathrm{d}\Gamma(\omega(k))) + 1\!\!1\otimes\mathrm{d}\Gamma(|\nabla\omega|^2) - \frac{\epsilon}{3})1\!\!1_{[1,\infty[}(N_\infty)\\ &\geq (d^\mu(\lambda) - \frac{\epsilon}{3})1\!\!1_{\Delta^\delta_\lambda}(H + 1\!\!1\otimes\mathrm{d}\Gamma(\omega(k)))(H + 1\!\!1\otimes\mathrm{d}\Gamma(\omega))1\!\!1_{[1,\infty[}(N_\infty)\\ &\geq (d^\mu(\lambda) - \frac{2\epsilon}{3})1\!\!1_{\Delta^\delta_\lambda}(H + 1\!\!1\otimes\mathrm{d}\Gamma(\omega(k)))(H + 1\!\!1\otimes\mathrm{d}\Gamma(\omega))1\!\!1_{[1,\infty[}(N_\infty)\end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R}), \chi_1 \in C_0^\infty(\mathbb{R})$ such that $\chi_1\chi = \chi$. One has

$$\begin{aligned}&\chi(H)[H,\mathrm{i}A]\chi(H)\\ &= \Gamma((j_0^R)^2)\chi(H)[H,\mathrm{i}A]\chi(H)\\ &+ I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(H^{\text{ext}})I^*(j^R)\chi_1(H)[H,\mathrm{i}A]\chi(H) + o(R^0) &(4.6)\\ &= \Gamma((j_0^R)^2)\chi(H)[H,\mathrm{i}A]\chi(H)\\ &+ I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(H^{\text{ext}})I^*(j^R)[H,\mathrm{i}A]\chi(H) + o(R^0) &(4.7)\\ &= \Gamma((j_0^R)^2)\chi(H)[H,\mathrm{i}A]\chi(H)\\ &+ I^*(j^R)1\!\!1_{[1,+\infty[}(N_\infty)\chi(H^{\text{ext}})[H^{\text{ext}},\mathrm{i}A^{\text{ext}}]\chi(H^{\text{ext}})I^*(j^R) + o(R^0). &(4.8)\end{aligned}$$

(4.6) follows by (4.5). Lemma 4.10 gives (4.7). Since $[H, \mathrm{i}A]$ is similar to the Hamiltonian H, one can easily derive an equivalent estimate for $[H, \mathrm{i}A]$ as in Lemma 4.10. This will imply (4.8).
Since the operator $K_1^R := \Gamma((j_0^R)^2)\chi(H)[H, \mathrm{i}A]\chi(H)$ is a compact operator, this gives for χ such that $\mathrm{supp}\chi \subset [\lambda-\delta, \lambda+\delta[$

$$\chi(H)[H, \mathrm{i}A]\chi(H) \geq (d(\lambda) - \frac{2\epsilon}{3})\chi^2(H) + K_1(R) + o(R^0).$$

Choosing R large enough, we obtain $H_1(n)$. Properties $ii), iii)$ are standard consequences of $i)$. ▮

5. ASYMPTOTIC COMPLETENESS

In this section we prove asymptotic completeness for the spin-fermion Hamiltonian H. We first study the asymptotic limits of Wick polynomials, then we construct the wave operator and we show that it is an unitary map. In Subsect. 5.2, we prove a minimal velocity estimate using the Mourre estimate and some others propagation estimates. We construct the asymptotic velocity projection in Subsect. 5.3 and then we finish the proof of asymptotic completeness.

5.1. The Wave Operator

We prove existence of asymptotic limits of Wick polynomials, using a Cook method [14]. We are interested in both limits in time $\mp\infty$, but since their are equivalent we will limit proofs only to a one limit $+\infty$. Let $S_2(\mathbb{R}^{d(p+q)})$ be the space complex-valued functions defined by the similar norm defining $S_2(\mathbb{R}^{d(p+q)}, \mathcal{B}(\mathcal{K}))$.

Theorem 5.13. *i) The following norm limit*

$$b^{\sharp,\pm}(h) := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0} b(h)\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}, h \in \mathfrak{h}, \tag{5.1}$$

exists and is called asymptotic creation-annihilation operators.
ii) Furthermore, we have the norm limits

$$Wick^{\pm}(w) = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0} Wick(w)\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}, \ \textit{for} \ w \in S_2(\mathbb{R}^{d(p+q)}). \tag{5.2}$$

iii) The map

$$S_2(\mathbb{R}^{d(p+q)}) \ni w \mapsto Wick^{\pm}(w) \ \textit{is norm continuous.}$$

iv) The CAR relations hold:

$$\{b^{\pm}(g),\, b^{\pm^*}(h)\} = \langle g|h\rangle,$$
$$\{b^{\pm^\sharp}(g),\, b^{\pm^\sharp}(h)\} = 0$$

v) The Hamiltonian preserves the asymptotic Wick polynomials:

$$\mathrm{e}^{\mathrm{i}tH} Wick^{\pm}(w)\mathrm{e}^{-\mathrm{i}tH} = Wick^{\pm}(\mathrm{e}^{-\mathrm{i}tH_0} w \mathrm{e}^{\mathrm{i}tH_0}), \ \textit{for} \ w \in S_2(\mathbb{R}^{d(p+q)}).$$

Proof. We use a Cook argument to prove the existence of the limit 5.1. Let $\mathfrak{h}_0$ denotes the subspace $C_0^\infty(\mathbb{R}^d \backslash \{0\})$ and $h_t := \mathrm{e}^{\mathrm{i}t\omega}h, h \in \mathfrak{h}$. We have for $h \in \mathfrak{h}_0$,

$$\mathrm{e}^{\mathrm{i}tH} b^*(h_t)\mathrm{e}^{-\mathrm{i}tH} = b^*(h) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH}[P, b^*(h_s)]\mathrm{e}^{-\mathrm{i}sH} ds. \tag{5.3}$$

Now, using commutation relation in Prop. 2.6 and the short range condition (SR), we see that for $h \in \mathfrak{h}_0$ we have

$$\sum_{p,q \in \Xi} \sum_{i=1}^{p} p \langle w_{p,q}(k_i) | h(k_i) \rangle \in O(t^{-\mu}), \mu > 1.$$

We extend the existence of the limit using density argument and the fact that the norm of $b^{*,\pm}(h)$ is preserved. The case of $b(h)$ is similar and the norm limits in $ii)$ follows from $i)$ using for example the decomposition 2.5. The anti-commutation relations and the action of $\mathrm{e}^{\mathrm{i}tH}$ on the asymptotic creation annihilation operators are obvious. ▮

Proposition 5.14. *We have for* $h \in \mathfrak{h}$*,*

$$b^{\pm}(h) 1\!\!1_{]-\infty,\lambda]}(H)\mathcal{H} \subset 1\!\!1_{]-\infty,\lambda-m]}(H)\mathcal{H}.$$

Proof. The proof follows using a standard argument due to Høegh-Krohn [15]. ▮

We define the space of *asymptotic vacua* associated to the CAR representation given by the asymptotic creation-annihilation operators:

$$\mathcal{K}^{\pm} := \{\Psi \in \mathcal{H} \mid b^{\pm}(h)\Psi = 0, \, h \in \mathfrak{h}\}.$$

We denote by $\mathcal{H}^{\pm}$ the space $\mathcal{K}^{\pm} \otimes \Lambda(\mathfrak{h})$.

Proposition 5.15. *The following assertions hold:*
i) $\mathcal{K}^{\pm}$ *is closed* H*-invariant space.*
ii) One has

$$\mathrm{Ran}\, 1\!\!1_{\mathrm{pp}}(H) \subset \mathcal{K}^{\pm}.$$

Proof. The fact that $\mathcal{K}^{\pm}$ is $H-$invariant follows from Thm. 5.13. Let us prove now $ii)$. Let $u \in \mathcal{H}$ such that $Hu = Eu$, one has

$$\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} b(\mathrm{e}^{\mathrm{i}t\omega} h)\mathrm{e}^{-\mathrm{i}tH} u = 0,$$

since $s - \lim_{t \to \pm\infty} b(\mathrm{e}^{\mathrm{i}t\omega} h) = 0$ and $\mathrm{e}^{\mathrm{i}tH} b(\mathrm{e}^{\mathrm{i}t\omega} h)\mathrm{e}^{-\mathrm{i}tH} u = \mathrm{e}^{\mathrm{i}t(H-E)} b(\mathrm{e}^{\mathrm{i}t\omega} h)u$. This gives $b^{\pm}(h)u = 0$. ▮

We define

$$H^{\pm} := H_{|\mathcal{K}^{\pm}} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega),$$

and the *wave operator*

$$\Omega^{\pm} : \mathcal{H}^{\pm} \to \mathcal{H},$$
$$\Omega^{\pm}\psi \otimes \prod_{i=1}^{n} b^{*}(h_i)\Omega := \prod_{i=1}^{n} b^{\pm *}(h_i)\psi, \text{ for } \psi \in \mathcal{K}^{\pm}, \;\; h_i \in \mathfrak{h}.$$

Theorem 5.16. $\Omega^{\pm}$ *is a unitary map satisfying*

$$\begin{aligned}&Wick^{\pm}(w)\Omega^{\pm}=\Omega^{\pm}\,1\!\!1_{\mathcal{H}}\otimes Wick(w),\ \text{for}\, w\in S_2(\mathbb{R}^{d(p+q)}),\\ &H\Omega^{\pm}=\Omega^{\pm}H^{\pm}.\end{aligned}$$

Proof. The intertwining relations in Prop. 5.13 gives that $\Omega^{\pm}$ is isometric and satisfies properties announced in the theorem. Let prove that $\Omega^{\pm}$ is unitary. Let $\Psi\in(\mathrm{Ran}(\Omega^{\pm}))^{\perp}$. Since H preserves the later subspace one may assume that Ψ is localized in energy i.e: $\Psi=1\!\!1_{]-\infty,\lambda]}(H)\Psi$. Then, if $nm>\inf\sigma(H)$ and using Prop. 5.14, we obtain that

$$\prod_{i=1}^{n}b^{\pm}(h_i)\Psi=0$$

Therefore, using the orthogonality

$$0=(\Psi,\prod_{i=2}b^{*\pm}(h_i)\prod_{i=2}^{n}b^{\pm}(h_i)\Psi)=||\prod_{i=2}^{n}b^{\pm}(h_i)\Psi||.$$

This leads to the fact that $\Psi=0$. ∎

We define an *extended wave operator*

$$\begin{aligned}&\Omega^{\mathrm{ext},\pm}:\mathcal{H}^{\mathrm{ext}}\to\mathcal{H},\\ &\Omega^{\mathrm{ext},\pm}\psi\otimes\prod_{i=1}^{n}b^{*}(h_i)\Omega:=\prod_{i=1}^{n}b^{\pm *}(h_i)\psi.\end{aligned}$$

We notice that $\Omega^{\mathrm{ext},\pm}_{|\mathcal{H}^{\pm}}=\Omega^{\pm}$. This suggests to treat sometimes $\Omega^{\mathrm{ext},\pm}$ as a partial isometry. Another construction of the extended wave operator is given by the following theorem, see [7, Thm. 5.7].

Theorem 5.17. *i) Let* $u\in\mathcal{D}(\Omega^{\mathrm{ext},\pm})$. *Then one has*

$$\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}I\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}u=\Omega^{\mathrm{ext},\pm}u,$$

where I *is the scattering identification operator defined in the Subsect. 2.1.*
ii) Let $\chi\in C_0^{\infty}(\mathbb{R})$. *Then* $\mathrm{Ran}\,\chi(H^{\mathrm{ext}})\subset\mathcal{D}(\Omega^{\mathrm{ext},\pm})$ *and the operators* $I\,\chi(H^{\mathrm{ext}}),\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})$ *are bounded. Moreover*

$$\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}I\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}}\chi(H^{\mathrm{ext}})=\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}}).$$

5.2. Minimal Velocity Estimate

In this subsection we will derive a minimal velocity estimate, following the approach of Graf [12]. It is based on the Mourre estimate and it does not need a double commutator estimate. We first prove a maximal velocity propagation estimate Prop.5.19, then we state a phase space propagation estimates, which will be

used to show the minimal velocity estimate.
We will use the following notations for Heisenberg derivatives

$$\mathrm{d}_0 := \partial_t + \mathrm{i}[\omega(k), .],$$
$$\mathbf{D}_0 := \partial_t + \mathrm{i}[\mathrm{d}\Gamma(\omega), .],$$
$$\mathbf{D} := \partial_t + \mathrm{i}[H, .].$$

We recall a standard argument often used to prove propagation estimates, see e.g. [9, lemma B.4.1].

Lemma 5.18. *Let $\Phi(t)$ be a bounded family of self-adjoint operators. Assume that there exist $C_0 > 0$ and operator valued function $B(t)$ and $B_i(t), i = 1, ..., n$, such that*

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^{n} B_i^*(t)B_i(t),$$
$$\int_1^\infty ||B_i(t)\mathrm{e}^{-\mathrm{i}tH}u||^2 dt \leq C||u||^2, i = 1, ..., n.$$

Then there exists C_1 such that

$$\int_1^\infty ||B(t)\mathrm{e}^{-\mathrm{i}tH}u||^2 dt \leq C_1||u||^2.$$

Proposition 5.19. *Let $\chi \in C_0^\infty(\mathbb{R})$. For $R' > R > v_{\max} := \sup|\nabla\omega(k)|$, there exists c such that we have*

$$\int_1^\infty ||\mathrm{d}\Gamma(1\!\!1_{[R,R']}(\frac{|x|}{t}))^{\frac{1}{2}}\chi(H)\,\mathrm{e}^{-\mathrm{i}tH}u||^2\frac{dt}{t} \leq c\,||u||^2.$$

Proof. Let $F \in C^\infty(\mathbb{R})$ be a cutoff function equal to 1 near ∞, to 0 near the origin, with $F'(s) \geq 1\!\!1_{[R,R']}(s)$. We consider the observable

$$\Phi(t) := \chi(H)\mathrm{d}\Gamma(F(\frac{|x|}{t}))\chi(H).$$

We claim that

$$\mathbf{D}\Phi(t) \geq t^{-1}C_0\ \ \chi(H)\mathrm{d}\Gamma(F'(\frac{|x|}{t}))\,\chi(H) + O(t^{-\mu}), \mu > 1. \tag{5.4}$$

One has

$$\mathbf{D}\Phi(t) = \chi(H)\ \mathrm{d}\Gamma(\mathrm{d}_0 F(\frac{|x|}{t}))\ \chi(H) + \chi(H)[V, \mathrm{i}\,\mathrm{d}\Gamma(F(\frac{|x|}{t}))]\chi(H).$$

Using the fact that $\mathrm{d}_0 F(\frac{|x|}{t}) \geq \frac{c_0}{t}\ F'(\frac{|x|}{t}) + O(t^{-2})$, it is sufficient to show that the second term in the previous identity is $O(t^{-1-\mu}), \mu > 0$, to have (5.4).
We have

$$\begin{aligned}[P, \mathrm{d}\Gamma(F(\frac{|x|}{t}))] &= \sum_{p,q\in\Xi}[Wick(w_{p,q}), \mathrm{d}\Gamma(F(\frac{|x|}{t}))] \\ &= \sum_{p,q\in\Xi} Wick\big([w_{p,q}, \mathrm{d}\Gamma(F(\frac{|x|}{t}))]\big).\end{aligned} \tag{5.5}$$

Using Thm. 2.7, it is enough to estimate the kernel of the operator $[w_{p,q}, \mathrm{d}\Gamma(F(\frac{|x|}{t}))]$ which is equal to $\sum_{i=1}^{q} F(\frac{|x_i^{'}|}{t})w_{p,q}(k,k^{'}) - \sum_{i=1}^{q} F(\frac{|x_i|}{t})w_{p,q}(k,k^{'})$. Now the hypothesis (SR) gives us

$$||[P, \mathrm{d}\Gamma(F(\frac{|x|}{t}))]|| \in O(t^{-\mu}), \mu > 1.$$

This proves (5.4). Then using Lemma 5.18 and inequality (5.4), we obtain the claimed estimate announced in the proposition. ▌

Proposition 5.20. *Let* $\chi \in C_0^\infty(R)$, $0 < c_0 < c_1$, *and*

$$\Theta_{[c_0,c_1]}(t) := \mathrm{d}\Gamma(\langle \frac{x}{t} - \nabla\omega(k), 1\!\!1_{[c_0,c_1]}(\frac{|x|}{t})(\frac{x}{t} - \nabla\omega(k))\rangle).$$

One has

$$\int_1^\infty ||\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u||^2 \frac{dt}{t} \leq c||u||^2.$$

Proof. Let $R_0(x) \in C^\infty$ be a function such that:

$$R_0(x) = 0, \text{ for } |x| \leq \frac{c_0}{2},$$
$$R_0(x) = \frac{1}{2}x^2 + c, \text{ for } |x| \geq 2c_1,$$
$$\nabla_x^2 R_0 \geq 1\!\!1_{[c_0,c_1]}(|x|).$$

We choose $c_1 > 2$, $c_2 > c_1 + 1$ and we define the function

$$R(x) := F(|x|)R_0(x),$$

where $F(s) = 1$, if $s \leq c_1, F(s) = 0$, if $s \geq c_2$.
We set

$$b(t) := R(\frac{x}{t}) - \frac{1}{2}(\langle \nabla R(\frac{x}{t}), \frac{x}{t} - \nabla\omega(k)\rangle + hc).$$

We consider the observable

$$\Phi(t) := \chi(H)\mathrm{d}\Gamma(b(t))\chi(H).$$

Pseudo-differential calculus gives

$$\chi(H)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(H) \geq$$
$$\chi(H)(\frac{1}{t}\Theta_{[c_0,c_2]}(t) - \frac{1}{t}\mathrm{d}\Gamma(1\!\!1_{[v_{max}+1,c_2]}(\frac{|x|}{t})))\chi(H) + O(t^{-2}).$$

The first term will serves in the application of Lemma 5.18 and the second is integrable along the evolution using Prop. 5.19. To complete the proof of the proposition, it suffices to show that:

$$\chi(H)[P, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(H) \in O(t^{-\mu}), \mu > 1. \tag{5.6}$$

As in the Prop. 5.19, it is enough to see that $[P, \mathrm{d}\Gamma(b(t))]$ has the kernel $\sum_{p,q\in\Xi}\sum_{i=1}^{p} b(t)_i w_{p,q}(k,k^{'}) - \sum_{i=1}^{q} b(t)_i w_{p,q}(k,k^{'})$ which is using the (SR) condition is estimated by $O(t^{-\mu}), \mu > 1$. ▌

Proposition 5.21. *Let* $0 < c_0 < c_1$, $J \in C_0^\infty(\{c_0 < |x| < c_1\})$, $\chi \in C_0^\infty(\mathbb{R})$. *For* $1 \leq i \leq d$, *one has*

$$\int_1^\infty ||\mathrm{d}\Gamma(|\ J(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k)) + hc\ |)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u||^2\frac{dt}{t} \leq c||u||^2.$$

Proof. We set

$$\begin{aligned} A &:= (\frac{x}{t} - \nabla\omega(k))^2 + t^{-\delta}, \\ b(t) &:= J(\frac{x}{t})A^{\frac{1}{2}}J(\frac{x}{t}). \end{aligned}$$

Let $J_1 \in C_0^\infty(\{c_0 < |x| < c_1\})$, $0 \leq J \leq 1,\ J = 1$ near the support of J_1. We consider the observable

$$\Phi(t) := -\chi(H)\mathrm{d}\Gamma(b(t))\chi(H).$$

One has

$$\chi(H)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(H) = -\chi(H)\mathrm{d}\Gamma(\mathrm{d}_0 b(t))\chi(H),$$

and we have using [7, Lemma 6.4]

$$\begin{aligned} &\chi(H)\mathbb{D}_0\mathrm{d}\Gamma(b(t))\chi(H) \geq \\ &\frac{c_0}{t}\chi(H)\mathrm{d}\Gamma(|\ J_1(\frac{x}{t})(\frac{x_i}{t} - \partial_i\omega(k)) + hc\ |)\chi(H) \\ &-\tfrac{c}{t}\chi(H)\mathrm{d}\Gamma(\langle\tfrac{x}{t} - \nabla\omega, J_2(\tfrac{x}{t})(\tfrac{x}{t} - \nabla\omega)\rangle)\chi(H) + O(t^{-1-\mu}). \end{aligned}$$

The second term is integrable along the evolution by Prop. 5.20. Then, it is enough to show that

$$\chi(H)[P, \mathrm{i}\,\mathrm{d}\Gamma(b(t))]\chi(H) \in O(t^{-\mu}), \mu > 1.$$

This follows by using (5.5), the fact that $J(\frac{x}{t})A^{\frac{1}{2}} \in O(1)$. The proof ends by application of Lemma 5.18. ∎

Proposition 5.22. *Let* $\chi \in C_0^\infty(\mathbb{R})$, *supported in* $\mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(H))$. *There exist* $\epsilon > 0$, C *such that we have*

$$\int_1^\infty ||\Gamma(1\!\!1_{[0,\epsilon]}(\frac{|x|}{t}))\chi(H)\,\mathrm{e}^{-\mathrm{i}tH}u||^2\frac{dt}{t} \leq C||u||^2.$$

Proof. Let $\epsilon > 0$. Let $q \in C_0^\infty(|x| \leq 2\epsilon)$ such that $0 \leq q \leq 1, q(x) = 1$, if $|x| \leq \epsilon$.
We set

$$\Phi(t) := \chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\chi(H),$$

where A is the conjugate operator considered in Sect. 4.3.
The Heisenberg derivative of $\Phi(t)$ is

$$\begin{aligned}
\mathbb{D}\Phi(t) &= \chi(H)\mathrm{d}\Gamma(q^t, \mathrm{d}_0 q^t)\ \tfrac{A}{t}\Gamma(q^t)\chi(H) + hc \\
&\quad +\chi(H)[P, \mathrm{i}\Gamma(q^t)]\ \frac{A}{t}\Gamma(q^t)\chi(H) + hc \\
&\quad +t^{-1}\chi(H)\Gamma(q^t)[H, \mathrm{i}A]\Gamma(q^t)\chi(H) \\
&\quad -t^{-1}\chi(H)\Gamma(q^t)\frac{A}{t}\Gamma(q^t)\Gamma(q^t)\chi(H) \\
&=: R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

We claim that

$$R_2 \in O(t^{-\mu}),\ \mu > 1. \tag{5.7}$$

To prove (5.7) it suffices to use Lemma 4.9 and the fact that $\frac{A}{t}\Gamma(q^t)(N+1)^{-2} \in O(1)$.

We consider now R_1. We have:

$$\mathrm{d}_0 q^t = -\frac{1}{2t}\langle \frac{x}{t} - \nabla\omega(k), \nabla q(\frac{x}{t})\rangle + hc + r^t =: \frac{1}{t}g^t + r^t,$$

where $r^t \in O(t^{-2})$. We have

$$||\chi(H)\mathrm{d}\Gamma(q^t, r^t)\frac{A}{t}\Gamma(q^t)\chi(H)|| \in O(t^{-2}).$$

We set

$$B_1 := \chi(H)\mathrm{d}\Gamma(q^t, g^t)(N+1)^{-\frac{1}{2}},\ B_2 := (N+1)^{\frac{1}{2}}\frac{A}{t}\Gamma(q^t)\chi(H).$$

So we obtain the inequality

$$R_1 \geq -\epsilon_0^{-1}t^{-1}B_1B_1^* - \epsilon_0 t^{-1}B_2B_2^*.$$

Using arguments in [7, Prop. 6.5], we obtain

$$\begin{aligned}
-B_2B_2^* &\geq -C_1\chi(H)\Gamma(q^t)^2\chi(H) - Ct^{-1}, \\
\int_1^\infty &||B_1\mathrm{e}^{-\mathrm{i}tH}u||^2\frac{dt}{t} \leq C||u||^2.
\end{aligned}$$

Using Lemma 4.9 and Thm. 4.12 *iii*), we have

$$R_3 \geq C_0 t^{-1}\chi(H)\Gamma(q^t)^2\chi(H) - Ct^{-2}.$$

We have

$$-R_4 \leq C_2\frac{\epsilon}{t}\chi(H)\Gamma(q^t)^2\chi(H) + Ct^{-2}.$$

Collecting the four terms, we obtain

$$\begin{aligned}
\mathbb{D}\phi(t) &\geq -\epsilon_0 t^{-1} B_2 B_2^* - \varepsilon_0^{-1} t^{-1} B_1 B_1^* + R_2 + R_3 + R_4 \\
&\geq (C_0 - \epsilon_0 C_1 - \epsilon C_2) t^{-1} \chi(H) \Gamma(q^t)^2 \chi(H) + R(t) \\
&\geq \tilde{C}_0 \chi(H) t^{-1} \Gamma(q^t)^2 \chi(H) - R(t),
\end{aligned}$$

where $R(t)$ is integrable. By Lemma 5.18, we obtain the inequality announced in the proposition for χ supported near a one energy level λ. Then we complete the proof for an arbitrary χ using a standard argument, see e.g.[9, Prop. 4.4.7]. ▌

5.3. Asymptotic Velocity

In this subsection we construct asymptotic velocity projection and we give the proof for asymptotic completeness Thm. 3.8.

Theorem 5.23. *Let $q, \tilde{q} \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq q, \tilde{q} \leq 1$, $q, \tilde{q} = 1$ on a neighborhood of 0 and $q^t := q(\frac{x}{t})$*
i) The following limits exist

$$\Gamma^\pm(q) \ := \ s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \, \Gamma(q^t) \mathrm{e}^{-\mathrm{i}tH}.$$

ii) We have

$$\begin{aligned}
&\Gamma^\pm(q\tilde{q}) = \Gamma^\pm(q)\Gamma^\pm(\tilde{q}). \\
&0 \leq \Gamma^\pm(q) \leq \Gamma^\pm(\tilde{q}) \leq \mathbb{1}, \text{ if } 0 \leq q \leq \tilde{q}. \\
&[H, \Gamma^\pm(q)] = 0.
\end{aligned}$$

iii) We have

$$\mathrm{Ran}\Gamma^\pm(q) \subset \mathcal{K}^\pm.$$

Proof. It is sufficient using a density argument and Lemma 4.9 to show for $\chi \in C_0^\infty(\mathbb{R})$ the existence of the limit

$$s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \chi(H) \, \Gamma(q^t) \chi(H) \mathrm{e}^{-\mathrm{i}tH}. \tag{5.8}$$

In general, showing asymptotic limits amounts to bound Heisenberg derivatives. We have on $\mathcal{H}$:

$$\begin{aligned}
\partial_t(\mathrm{e}^{\mathrm{i}tH} \chi(H) \, \Gamma(q^t) \chi(H) \mathrm{e}^{-\mathrm{i}tH}) \ &= \ \mathrm{e}^{\mathrm{i}tH} \chi(H) \mathrm{d}\Gamma(q^t, \mathbb{d}_0 q^t) \chi(H) \mathrm{e}^{-\mathrm{i}tH} \\
&\quad + \mathrm{e}^{\mathrm{i}tH} \chi(H) [P, \mathrm{i}\Gamma(q^t)] \chi(H) \mathrm{e}^{-\mathrm{i}tH}.
\end{aligned} \tag{5.9}$$

We know from the proof of Lemma 4.9 that

$$\chi(H)[P, \mathrm{i}\Gamma(q^t)]\chi(H) \in O(t^{-\mu}), \mu > 1. \tag{5.10}$$

Let now analyze the r.h.s. in (5.9). One can write

$$\begin{aligned}
&\mathbb{d}_0 q^t = \tfrac{1}{t} g^t + r^t, \\
\text{where} \quad & g^t = -\tfrac{1}{2}((\tfrac{x}{t} - \partial\omega(k)) \, \partial q(\tfrac{x}{t}) + hc), \\
\text{and} \quad & r^t \in O(t^{-2}).
\end{aligned}$$

Then we estimate the part of r^t, using the inequality $||(N+1)^{-1}\Gamma(q^t,r^t)\Psi|| \leq ||\mathrm{d}\Gamma((r^t)^2)^{\frac{1}{2}}\Psi||$, we obtain

$$||\chi(H)\mathrm{d}\Gamma(q^t,r^t)\chi(H)|| \in O(t^{-2}). \tag{5.11}$$

The term coming from $\frac{1}{t}g^t$, will be estimated as follows:

$$|(\mathrm{e}^{-\mathrm{i}tH}u, \chi(H)\mathrm{d}\Gamma(q^t,g^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)| \leq ||\mathrm{d}\Gamma(|g^t|)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u||^2, u \in \mathcal{H}. \tag{5.12}$$

We notice that the r.h.s. in (5.12) recall the observable in the propagation estimate Prop. 5.21. Now, combining (5.10)-(5.12) and Prop. 5.21, we prove the integrability of the Heisenberg derivative 5.9 and thus we prove the existence of the limit. (5.8).

The first statement in *ii*) follows from the fact that

$$\Gamma(q^t\tilde{q}^t) = \Gamma(q^t)\Gamma(\tilde{q}^t).$$

The second statement follows by

$$0 \leq \Gamma(q^t) \leq \Gamma(\tilde{q}^t) \leq 1 \; if \; 0 \leq q \leq \tilde{q}.$$

The last statement is a consequence of *i*) and Lemma 4.9.

Let us prove *iii*). Let $u \in \mathcal{H}$, we have

$$\begin{aligned} b^{\pm}(h)\Gamma^{\pm}(q)u &= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}b(h_t)\mathrm{e}^{-\mathrm{i}tH}\Gamma^{\pm}(q)u \\ &= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}b(h_t)\Gamma(q^t)\mathrm{e}^{-\mathrm{i}tH}u \\ &= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(q^t)b(q^t h_t)\mathrm{e}^{-\mathrm{i}tH}u. \end{aligned}$$

If $h \in \Lambda^{\mathrm{fin}}(C_0^\infty(\mathbb{R}^d))$, by a stationary phase argument we have $q^t h_t \in o(1), t \to \pm\infty$. Using the fact that the map $h \to b(h_t)$ is continuous, we obtain $b^{\pm}(h)\Gamma^{\pm}(q)u = 0$ for all $h \in \mathfrak{h}$. ▮

Let state a corollary of the above theorem, giving the construction of the asymptotic velocity projection $P_0^{\pm}$.

Corollary 5.24. *Let $\{q_n\} \in C_0^\infty(\mathbb{R}^d)$ be a decreasing sequence of functions such that $0 \leq q_n \leq 1$, $q_n = 1$ on a neighborhood of 0 and $\cap_{n=1}^\infty \mathrm{supp}\, q_n = \{0\}$. Then the following limit exist and it does not depend in the choose of the sequence*

$$i)\; P_0^{\pm} := \lim_{n\to\infty} \Gamma^{\pm}(q_n),$$

$$ii)\; \mathrm{Ran} P_0^{\pm} \subset \mathcal{K}^{\pm}.$$

Moreover $P_0^{\pm}$ is an orthogonal projection.

Proof. The existence of the limit *i*) follows from Thm. 5.23 *ii*) and the argument that the limit of a decreasing sequence of self-adjoint operators satisfying $\Gamma^{\pm}(q_{n+1})\Gamma^{\pm}(q_n) = \Gamma^{\pm}(q_{n+1})$ exist in the strong limit. The independence from the choose of the sequence follows from the fact that there exists an index m_n such that $q_n \geq \tilde{q}_{m_n}, \tilde{q}_n \geq q_{m_n}; \lim_{n\to\infty} m_n = +\infty$ and Thm. 5.23 *ii*). The statement *ii*) is a consequence of Thm. 5.23 *iii*) and *i*). ▮

5.4. Proof of Thm. 3.8

Let $j_0 \in C_0^\infty(\mathbb{R}^3), 0 \le j_0, 0 \le j_\infty, j_0^2 + j_\infty^2 \le 1, j_0 = 1$ near 0. Set $j := (j_0, j_\infty)$ and $j^t := (j_0^t, j_\infty^t)$, where $j_0^t := j_0(\frac{x}{t}), j_\infty^t := j_\infty(\frac{x}{t})$.

Theorem 5.25. *i) The following limits exist*

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}} I^*(j^t)\mathrm{e}^{-\mathrm{i}tH} =: W^\pm(j),$$

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} I(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} = W^\pm(j)^*.$$

ii) For a bounded Borel function F, we have

$$W^\pm(j)\; F(H) = F(H^{\mathrm{ext}})W^\pm(j).$$

iii) Let $q_0, q_\infty \in C^\infty(\mathbb{R}^d)$, ∇q_0, $\nabla q_\infty \in C_0^\infty(\mathbb{R}^d)$, $0 \le q_0, q_\infty \le 1$, $q_0 = 1$ near 0. Set $\tilde{j} := (q_0 j_0, q_\infty j_\infty)$. Then

$$\Gamma^\pm(q_0) \otimes \Gamma(q_\infty(\nabla\omega(k)))W^\pm(j) = W^\pm(\tilde{j}).$$

iv) Let $q \in C^\infty(\mathbb{R}^d)$, $\nabla q \in C_0^\infty(\mathbb{R}^d)$, $0 \le q \le 1$, $q = 1$ near 0. Then

$$W^\pm(j)\Gamma^\pm(q) = W^\pm(qj), \;\; \text{where } qj = (qj_0, qj_\infty).$$

v) Let $\tilde{j} = (\tilde{j}_0, \tilde{j}_\infty)$ be another pair satisfying the conditions stated before the theorem. Then

$$W^\pm(\tilde{j})^* W^\pm(j) = \Gamma(\tilde{j}_0 j_0 + \tilde{j}_\infty j_\infty),$$

in particular if $j_0^2 + j_\infty^2 = 1$, then $W^\pm(j)$ is isometric.
vi) Let $j_0 + j_\infty = 1$. If $\chi \in C_0^\infty(\mathbb{R})$, then

$$\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^\pm(j) = \chi(H).$$

Proof. To prove *i)* we use the same arguments as in Thm. 5.23. It is enough to prove the existence of the following limit for some $\chi \in C_0^\infty(\mathbb{R})$.

$$s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}} \chi(H^{\mathrm{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}\chi(H), \tag{5.13}$$

We compute

$$\begin{aligned} &\partial_t(\mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}}\chi(H^{\mathrm{ext}})I^*(j^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}) = \\ &\mathrm{e}^{\mathrm{i}tH^{\mathrm{ext}}}(\chi(H^{\mathrm{ext}})\mathfrak{D}_0 I^*(j^t)\chi(H) \\ &+\mathrm{i}\chi(H^{\mathrm{ext}})(P\otimes 1\!\!1 I^*(j^t) - I^*(j^t)V)\chi(H))\mathrm{e}^{-\mathrm{i}tH}, \end{aligned}$$

where $\mathfrak{D}_0$ is the asymmetric Heisenberg derivative defined by $\partial_t + \mathrm{i}H_0^{\mathrm{ext}}. - .\mathrm{i}H_0$. We have $\mathfrak{D}_0 I^*(j^t) = \mathrm{d}I^*(j^t, \mathfrak{d}_0 j^t)$.
Pseudo-differential calculus gives

$$\begin{aligned} \mathfrak{d}_0 j^t &= \frac{1}{t}g^t + r^t, \\ g^t = (g_0^t, g_\infty^t), \;\; g_\epsilon^t &= -\frac{1}{2}((\frac{x}{t} - \partial\omega(k))\partial j_\epsilon(\frac{x}{t}) + hc), \epsilon = 0, \infty \end{aligned}$$

with $r^t \in O(t^{-2})$. We obtain

$$||\chi(H^{\text{ext}})\mathrm{d}I^*(j^t, r^t)\chi(H)|| \in O(t^{-2}). \tag{5.14}$$

Now with $u_i^t := \mathrm{e}^{\mathrm{i}tH}u_i$, one obtain

$$\begin{aligned}
&|(u_1^t|\chi(H^{\text{ext}})\mathrm{d}I^*(j^t, g^t)\chi(H)u_2^t)| \\
&\leq ||\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}} \otimes 1\!\!1\chi(H^{\text{ext}})u_2^t||\ ||\mathrm{d}\Gamma(|g_0^t|)^{\frac{1}{2}}\chi(H)u_1^t|| \\
&+||(1\!\!1 \otimes \mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}})\chi(H^{\text{ext}})u_2^t||\ ||\mathrm{d}\Gamma(|g_\infty^t|)^{\frac{1}{2}}\chi(H)u_1^t||.
\end{aligned}$$

Then the integrability of the term $\chi(H^{\text{ext}})\mathfrak{D}_0 I^*(j^t)\chi(H)$ follows using Prop. 5.19. Using Thm. (4.10) we obtain

$$\chi(H^{\text{ext}})(P \otimes 1\!\!1 I^*(j^t) - I^*(j^t)V)\chi(H) \in O(t^{-\mu}), \mu > 1.$$

Then, the existence of the limit in $i)$ follows.

$ii)$ follows by spectral theorem and the point $i)$. $iii)$ follows using the fact that

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}\Gamma(q^t)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)} = \Gamma(q(\nabla\omega)),$$

$$\Gamma(q_0^t) \otimes \Gamma(q_\infty^t)I^*(j^t) = I^*(\tilde{j}^t).$$

$iv)$ is true since

$$I^*(j^t)\Gamma(q^t) = I^*((jq)^t).$$

$v)$ is a consequence of the fact

$$I(\tilde{j}^t)I^*(j^t) = \Gamma(\tilde{j}_0^t j_0^t + \tilde{j}_\infty^t j_\infty^t).$$

$vi)$ One has

$$H^{\text{ext}} 1\!\!1_{[k,\infty[}(N_\infty) \geq mk + \inf\sigma(H).$$

Then for $\chi \in C_0^\infty(\mathbb{R})$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\chi(H^{\text{ext}})1\!\!1_{]n,\infty[}(N_\infty) = 0. \tag{5.15}$$

We have

$$\begin{aligned}
&\Omega^{\text{ext},\pm}\chi(H^{\text{ext}})W^\pm(j) \\
&= \Omega^{\text{ext},\pm}1\!\!1_{[0,n]}(N_\infty)\chi(H^{\text{ext}})W^\pm(j) &(5.16)\\
&= s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}I1\!\!1_{[0,n]}(N_\infty)\chi(H^{\text{ext}})I^*(j^t)\mathrm{e}^{-\mathrm{i}tH} &(5.17)\\
&= s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}I1\!\!1_{[0,n]}(N_\infty)I^*(j^t)\mathrm{e}^{-\mathrm{i}tH}\chi(H). &(5.18)
\end{aligned}$$

(5.15) follows from (5.15). (5.17) follows from the limit $i)$ and Thm. 5.17. Lemma 4.10 and the boundedness of the operator $I1\!\!1_{[0,n]}(N_\infty)(N_0+1)^{-\frac{n}{2}}$ gives (5.18). We use now an estimate proved in [8] and which extends also to our case.

$$||I1\!\!1_{]n,\infty[}(N_\infty)I^*(j^t)(N+1)^{-1}|| \leq (n+1)^{-1}. \tag{5.19}$$

Since $II^*(j^t) = 1\!\!1$, letting $n \to \infty$ we obtain $\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^\pm(j) = \chi(H)$. This completes the proof. ▮

Theorem 5.26. *Let $j_n = (j_{0,n}, j_{\infty,n})$ be a sequence satisfying the hypothesis stated in the beginning of Thm. 5.25 such that $j_0 + j_\infty = 1$ and for any $\epsilon > 0$ there exists $m, \forall n > m$, $\mathrm{supp} j_{0,n} \subset [-\epsilon, \epsilon]$. Then*

$$\Omega^{\pm *} = w - \lim_{n\to+\infty} \tilde{W}^\pm(j_n),$$
$$\mathcal{K}^\pm = \mathrm{Ran} P_0^\pm,$$

Proof. Let $q \in C_0^\infty(\mathbb{R}), 0 \leq q \leq 1$ and $q = 1$ in a neighborhood of zero such that $qj_{0,n} = j_{0,n}$ for n large enough. Using Thm. 5.25 *iii)* and Corollary 5.24 we obtain

$$\Gamma^\pm(q) \otimes 1\!\!1\, W^\pm(j_n) = W^\pm(j_n),$$

$$w - \lim_{n\to+\infty} P_0^\pm \otimes 1\!\!1\, W^\pm(j_n) - W^\pm(j_n) = 0. \tag{5.20}$$

Let $\chi \in C_0^\infty(\mathbb{R})$. We have

$$\begin{aligned}
\Omega^{\pm *}\chi(H) &= \Omega^{\pm *}\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})W^\pm(j_n) & (5.21)\\
&= w - \lim_{n\to\infty} \Omega^{\pm *}\Omega^{\mathrm{ext},\pm}\chi(H^{\mathrm{ext}})P_0^\pm \otimes 1\!\!1\, W^\pm(j_n) & (5.22)\\
&= w - \lim_{n\to\infty} P_0^\pm \otimes 1\!\!1\chi(H^{\mathrm{ext}})W^\pm(j_n) & (5.23)\\
&= w - \lim_{n\to\infty} \chi(H^{\mathrm{ext}})W^\pm(j_n) & (5.24)\\
&= w - \lim_{n\to\infty} W^\pm(j_n)\chi(H). & (5.25)
\end{aligned}$$

(5.21) follows from Thm. 5.25 *iv)*. (5.22) follows by (5.20). (5.23) is true since $P_0^\pm$ commutes with H^{ext} and that $\mathrm{Ran}P_0^\pm \subset \mathcal{K}^\pm, \Omega^{\mathrm{ext},\pm}1\!\!1_{\mathcal{K}^\pm} \otimes 1\!\!1 = \Omega^\pm$ and $\Omega^{\pm *}\Omega^\pm = 1\!\!1_{\mathcal{K}^\pm} \otimes 1\!\!1$. (5.24) follows from the fact that $P_0^\pm \otimes 1\!\!1$ commutes with H^{ext} and (5.20). (5.25) is Thm. 5.25 *ii)*. So we conclude by a density argument that

$$\Omega^{\pm *} = w - \lim_{n\to+\infty} W^\pm(j_n),$$

$$P_0^\pm \otimes 1\!\!1\, \Omega^{\pm *} = \Omega^{\pm *}.$$

So, we obtain

$$\mathrm{Ran}\Omega^{\pm *} = \mathcal{K}^\pm \otimes \Gamma(\mathfrak{h}) \subset \mathrm{Ran}P_0^\pm \otimes \Gamma(\mathfrak{h}) \subset \mathcal{K}^\pm \otimes \Gamma(\mathfrak{h}).$$

Thus, we conclude that

$$\mathcal{K}^{\pm} = \mathrm{Ran} P_0^{\pm}$$

▮

Proof of Thm. 3.8. It suffices to show that $\mathcal{K}^{\pm} \subset \mathrm{Ran}\mathbb{1}_{\mathrm{pp}}(H)$, to prove Thm.3.8. Prop. 5.22 gives the existence of $\epsilon > 0$ and $C > 0$ such that

$$\int_1^{+\infty} ||\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u||^2 \frac{\mathrm{d}t}{t} \leq C||u||^2,$$

where $\chi \in C_0^\infty(\mathbb{R} \setminus (\tau \cup \sigma_{\mathrm{pp}}(H)))$ and $q \in C_0^\infty([-\epsilon, \epsilon])$, $q = 1$ for $|x| < \frac{\epsilon}{2}$. Thm. 5.23 gives that

$$||\Gamma(q^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u|| \to ||\Gamma^{\pm}(q)\chi(H)u|| = 0, t \to \pm\infty.$$

Hence $\Gamma^{\pm}(q)\chi(H) = 0$. Therefore, we have $\mathrm{Ran}P_0^{\pm} \subset \mathrm{Ran}\mathbb{1}_{\tau\cup\sigma_{\mathrm{pp}}(H)}(H)$. Now, Thm. 4.3 gives that τ is a closed countable set and $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, so $\mathbb{1}_{\mathrm{pp}}(H) = \mathbb{1}_{\tau\cup\sigma_{\mathrm{pp}}(H)}(H)$. This proves

$$\mathcal{H}_{\mathrm{bd}}(H) = \mathcal{K}^{\pm}.$$

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