MEDIATRICES AND CONNECTIVITY

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Abstract

If (X, d) is a connected metric space and a and b are points in X, then the locus L of the points x where $d(x,a) - d(x,b) = 0$ is called a mediatrix. For instance if X is a geodesic space then the geodesics emanating from a and b (starting at the same time, and travelling with unit speed) are said to focus at L.

In an earlier paper ([18]), Brillouin Spaces were defined. These are spaces in which mediatrices have desirable properties. Most importantly: they are minimally separating. This means that for every proper subset L' of L in X, $X - L$ is disconnected, but $X - L'$ is connected.

The purpose of this note is twofold. First, we give a very simple characterization of Brillouin Spaces, which shows that, for example, compact, connected Riemannian manifolds are Brillouin Spaces. Second, we give a description in terms of homology of these mediatrices. This leads to a complete topological classification of mediatrices in 2-dimensional, compact, connected, Riemannian manifolds.
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1 Introduction

Suppose that $x_1(t)$ and $x_2(t)$ are two distinct solutions of a second order differential equation with $x_1(0) = x_2(0)$ and there is some $T \neq 0$ so that

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$x_1(T) = x_2(T)$. Then the trajectories x_1 and x_2 are said to focus at time T. One can ask how the number of trajectories which focus varies with the endpoint $x(T)$— this gives rise to the concept of a focal decomposition. This concept (originally called sigma decomposition) was first proposed in [13]. In [14], several explicit examples were worked out, in particular the fundamental example of the pendulum $\ddot{x} = -\sin x$.

Later, in [9], the idea of focal decomposition was approached in the context of geodesics of a Riemannian manifold M (in addition to a reformulation of the main theorem of [14]). Here, one takes a basepoint x_0 in the manifold M: two geodesics γ_1 and γ_2 focus at some point $y \in M$ if $\gamma_1(T) = y = \gamma_2(T)$. This gives rise to a decomposition of the tangent space of M at x into regions where the same number of geodesics focus. In this setting there is also a connection with number theory (see [15]).

The flavor of these papers, and especially their connections with various other branches of mathematics, seemed to invite a separate treatment of this circle of ideas. We now present some of the results ([18]) that this gave rise to. In this note we will not go into details about the relation these results bear to the original problem. (However, see [18] for a discussion of this.)

Let S be a discrete set in a path-connected, metric space X with distance function $d(.,.)$. Assume further that the function $d(x,.)$ is proper, so that closed balls of finite radius are compact (see [5] where it proved that such a space is locally compact and complete). We will call such a space a proper (path-connected, metric) space. By a discrete set $S = \{x_i\}_{i\in I}$ we mean that any compact subset of X contains at most finitely many points of S. Note that if $\liminf_{a,b\in S} d(a,b) > 0$, then S is discrete. Choose a preferred point x_0 (the origin) in S. The most important notion in [18] is the formal definition of the n-th Brillouin zone $B_n(x_0)$ with base point x_0. We use $N_r(x)$ for an open ball of radius r and centered in x and $C_r(x)$ for the boundary of the closed ball (the circumference). The following notion is discussed in greater detail in [18] and several figures are included there).

Definition 1.1 *Let $x \in X$, let n be a positive integer, $n \leq \#(S)$, and let $r = d(x, x_0)$. Then define the sets $b_n(x_0)$ and $B_n(x_0)$ as follows.*

- $x \in b_n(x_0) \iff \#(N_r(x) \cap S) = n - 1$ *and* $C_r(x) \cap S = \{x_0\}$ *(one point only).*
- $x \in B_n(x_0) \iff \#(N_r(x) \cap S) = m$ *and* $\#(C_r(x) \cap S) = \ell \geq 1$, *where $l, m \in \mathbb{Z}^+$ with $m + 1 \leq n \leq m + \ell$.*

Remark: The notion of Brillouin Zone was introduced by Brillouin in the 1930's, and plays an important role in solid state physics (see [4] or [2]).

Remark: In geometry $b_1(x_0)$ coincides with the Dirichlet domain associated with x_0. In computational geometry, they are sometimes called Voronoi cells ([16]. There also is an intimate connection between this boundary and the cut-locus in Riemannian geometry (see [6] and [8]).

In the second part of the definition, if $m = n - 1$ and $\ell = 1$, then $x \in b_n(x_0)$. So $b_n(x_0) \subseteq B_n(x_0)$. Note also that $b_n(x_0)$ is open and that $B_n(x_0)$ is closed. Finally, observe that for fixed x_0 the sets $b_n(x_0)$ are disjoint, but the sets $B_n(x_0)$ are not: their "boundaries" overlap. Here by "boundaries" we really mean the complement of $b_n(x_0)$ in $B_n(x_0)$. These consist of subsets of mediatrices:

Definition 1.2 *For a and b distinct points in S, the associated mediatrix (sometimes called equidistant set or bisector) L_{ab} is the set given by:*

$$L_{ab} = \{x \in X \mid d(x, a) - d(x, b) = 0\} \quad .$$

The fact that $B_n(x_0) - b_n(x_0)$ consists entirely of pieces of mediatrices does not imply that $B_n(x_0)$ is the closure of $b_n(x_0)$. And in fact, this is not a very "natural" property. However, a slightly weaker result, Theorem 1.6, is true in many common spaces. To formulate it, we need some definitions first.

Definition 1.3 *The space X is called metrically consistent if, for all x in X, all $R > r > 0$ in $\mathbb{R}$ with r sufficiently small, and for each $a \in C_R(x)$, there is a $z \in C_r(x)$ satisfying $N_{d(z,a)}(z) \subseteq N_R(x)$ and $C_{d(z,a)}(z) \cap C_R(x) = \{a\}$.*

Definition 1.4 *A proper subset L of a connected set X is separating if $X - L$ consists of more than one component. A set L in X is minimally separating if it is separating and no proper subset of L is separating.*

Definition 1.5 *A proper, path connected metric space X is Brillouin if it is metrically consistent and if for all a, b in X, the mediatrices L_{ab} are minimally separating sets.*

For Brillouin spaces we have the following result.

Theorem 1.6 *(See [18].) If X is Brillouin , then $b_n(x_i)$ equals the interior of $B_n(x_i)$ (itself a closed set) and $B_n(x_0)$ is contained in the closure of $b_1(x_0) \cup \cdots b_n(x_0)$.*

In section 2 we give a very simple example (example 2.9) where $B_3(x_0)$ consists of a set with no interior (and therefore is not the closure of $b_3(x_0)$). While it is possible to give general conditions on (X,d) for which $B_n(x_0)$ is the closure of $b_n(x_0)$, we would have to exclude many common spaces (we will not pursue this here). One of the aims of this note is to prove that Brillouin spaces are "natural" in the sense that "common" spaces such as the n-sphere, hyperbolic n-space, and $\mathbb{R}^n$ belong to that category. All of these thus satisfy the conclusion of Theorem 1.6. This will be done in section 2.

Clearly, mediatrices play an important role in this discussion. The fact that they are minimally separating sets leads to a topological characterization of them. Let us suppose that X is a smooth, compact, connected n-dimensional manifold, and $L \subset X$ is a mediatrix. Denote by $i : L \to X$ the inclusion. In section 3, we assume a "triangulability" condition and prove that i induces a homomorphism $i_{n-1} : H_{n-1}(L;\mathbb{Z}_2) \to H_{n-1}(X;\mathbb{Z}_2)$ on the $n-1$-st homology groups with coefficients $\mathbb{Z}_2$ with the property that its Kernel has exactly one generator. Since for compact smooth n-manifolds the homology is finitely generated, this means that $H_{n-1}(L;\mathbb{Z}_2)$ has at most one generator more than $H_{n-1}(X;\mathbb{Z}_2)$. While this seems abstract, in the case where X is a smooth, compact, connected n-dimensional manifold, one can give a topological classification of mediatrices. This is done in section 5.

The discussion in section 3 depends upon the assumption that the n-manifold is triangulated and that the subset L is triangulated in such a way that both triangulations have a common refinement. We prove in section 4 that if X is a smooth 2-dimensional manifold, and $L \subset X$ is a mediatrix, then L is the topological image of a finite graph (a finite number of edges glued together in a finite number of vertices). Thus L admits a natural triangulation — namely the one inherited from the graph. We defer the proof that these triangulations have a common refinement to forthcoming work.

For completeness we mention that the two basic properties from which ultimately the physical and mathematical interest of Brillouin zones is derived.

Theorem 1.7 *(See [18].) If X is a proper, path connected metric space these Brillouin zones tile the underlying space X in two ways:*

$$\cup_i B_i(x_n) = X \quad \text{and} \quad b_i(x_n) \cap b_j(x_n) = \emptyset \quad \text{if } i \neq j.$$

$$\cup_i B_n(x_i) = X \quad \text{and} \quad b_n(x_i) \cap b_n(x_j) = \emptyset \quad \text{if } i \neq j.$$

In fact, it follows that if S is the image of x_0 under a discrete group of isometries then the Brillouin zones are fundamental set for the group (see [18]). Indeed, in the case of a lattice in $\mathbb{R}^2$, both results in this Theorem were proved by Bieberbach (see [3]).

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2 Brillouin Spaces

The aim of this section is to prove that a large class of metric spaces has the Brillouin property. That class is described in the next theorem and includes the connected Riemannian manifolds. All of these therefore obey the conclusion of Theorem 1.6.

Theorem 2.1 *Let (X,d) be a path connected metric space satisfying the following conditions*

1: Closed balls are compact.

2: Between any two points a and b there is always a shortest path whose length equals $d(a,b)$.

3: If two shortest paths, one from a to b, and the other from a to $c \neq b$, have a common segment, then one of the paths is a subset of the other.

Then (X,d) is Brillouin.

Suppose γ in X is a (connected) subset of a shortest path c, is longer than γ', a path with the same endpoints. Then c is longer than the concatenation of γ' and $c-\gamma$ (contradicting the assumption that c is shortest). Therefore, we have

Lemma 2.2 *Let X be a space satisfying the above requirements. Any path in X that is a subset of a shortest path, is itself a shortest path.*

Example 2.3 The figure Y with the usual shortest path distance does not satisfy the last of these requirements. It also is not Brillouin, as can be seen by choosing two points a and b, one in each of the upper branches and at the same "height". The mediatrix consists of the entire lower branch of the figure.

The central object of study are the mediatrices L_{ab} associated with two points a and b in X. We will also denote this set by L when no confusion is possible. We will also use the following conventions. Let (X, d) be a metric space satisfying the requirements of Theorem 2.1 and let a and b be two distinct points in X.

$$\begin{aligned} L_{+} &= \{x \in X \,|\, d(a,x) - d(b,x) > 0\} \quad . \\ L_{-} &= \{x \in X \,|\, d(a,x) - d(b,x) < 0\} \quad . \end{aligned}$$

We will speak of the mediatrix associated to a and b. For ease of notation, a and b will now be fixed. The function $f(x) = d(a,x) - d(b,x)$ is a Lipschitz function (with Lipschitz constant 2) from X to $\mathbb{R}$. For a curve γ, we denote by $\mathring{\gamma}$ the set $\gamma - \{\gamma(0) \cup \gamma(1)\}$.

Lemma 2.4 *Suppose $d(a,x) \leq d(b,x)$, and let γ be the minimizing path connecting the points x and a. Then $\mathring{\gamma} \subset L_{-}$ (the component containing a).*

Proof: Assume that the lemma is false. So there is a shortest path γ from $x \in (L \cup L^{-})$ to $a \in L^{-}$ such that $\mathring{\gamma} \cap (L \cup L_{+})$ contains a point y. Then

$$d(a,x) \leq d(b,x) \qquad \text{and} \qquad d(a,y) \geq d(b,y) \quad . \tag{2.1}$$

In the following, γ is a minimizing path from a to x.

$$\begin{aligned} d(a,x) &= d(a,y) + d(y,x) && (y \text{ lies on } \gamma) && (2.2) \\ &\geq d(b,y) + d(y,x) && (y \text{ lies in the set } L \cup L_{+}) && (2.3) \\ &\geq d(b,x) && (\text{triangle inequality}) \quad . && (2.4) \end{aligned}$$

It thus follows that the first inequality in (2.1) and the one in (2.4) must be equalities, and so:

$$d(a,x) = d(b,x) \quad .$$

It then follows that the inequality in (2.3) must also be an equality. That implies that

$$d(a,y) = d(b,y) \quad .$$

Finally, the equality in (2.4) implies that

$$d(x, y) + d(b, y) = d(b, x) \quad .$$

Thus we have that the path (x, y, b) is a shortest path. Since this is by equation (2.2) also true for (x, y, a), this violates the last condition of Theorem 2.1. ■

Corollary 2.5 *Let X have the properties in Theorem 2.1 and L be a mediatrix in X. Then $X - L$ has exactly 2 components, and each of these is path connected.*

Proof: Suppose there are at least 3 components. One of these contains neither a nor b. Let γ be a shortest path from a point x in this component to a. Then $\gamma(0)$ and $\gamma(1)$ are not in L and its endpoint lie in distinct components. This contradicts the previous lemma. To see that both components are path connected, note that if x and y in L_+ then there are paths from x to a and from y to a (and both do not intersect L). ■

For future reference we prove a slightly stronger version of Lemma 2.4 .

Lemma 2.6 *Suppose x and y are (not necessarily distinct) points in a mediatrix L. Let γ be a minimizing path connecting the points x and a or b and η a minimizing path connecting y to either a or b. Then $\mathring{\gamma} \cap \mathring{\eta} = \emptyset$.*

Proof: First assume that γ lands in a and η lands in b (or vice versa). Lemma 2.4 implies that $\mathring{\gamma}$ and $\mathring{\eta}$ lie in distinct components of $X - L$. Hence their intersection is empty.

Now assume that γ and η land in the same point, say a. Assume that the lemma is false. So there is a point $z \in \mathring{\gamma} \cap \mathring{\eta}$. By Lemma 2.2, the subpath along γ from x to z, and the one from z to a are shortest paths with respective lengths $d(x, z)$ and $d(z, a)$. The path along η from z to a is also a shortest path of length $d(z, a)$. Thus from x to a there are two distinct shortest paths coinciding in a sub path, contradicting Condition 3. ■

Corollary 2.7 *The set L is a closed, minimally separating set.*

Proof: The $g(x) = d(a,x)$ is continuous and therefore so is $f(x) = d(a,x) - d(b,x)$. Thus L_-, the set of x such that $f(x) < 0$, is open and non-empty (it contains the point a). The same holds for L_+. This proves in the first place that $X - L$ is not connected, and furthermore, that the complement L of $L_- \cup L_+$ is closed.

To prove minimality, suppose that for some $x \in L$, the set $L - x$ also separates X into two components, one containing a, the other containing b. Then x is in one of the components relative to $L - x$. Consider the minimizing segment from x to a and from x to b. At least one of them must pass through $L - x$, contradicting Lemma 2.4. ■

To prove the main result of the section, it is now sufficient to prove the following lemma.

Lemma 2.8 *A space X as defined in Theorem 2.1 is metrically consistent.*

Proof: Let $R > r > 0$ and suppose that $C_R(x)$ contains at least 2 points (otherwise we are done). Choose $a \in C_R(x)$, let γ be the minimizing path from x to a. Choose $z = \gamma \cap C_r(x)$ and let $y \neq a$ in $C_R(x)$. Now we have the strict inequality

$$d(z,a) < d(z,y) \quad .$$

This can be seen as follows. By the triangle inequality:

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) \\ \Rightarrow d(z,y) &\geq d(x,y) - d(x,z) = R - r = d(z,a) \quad . \end{aligned}$$

Suppose this were an actual equality. Then the segments from x to z and then from z to y would form a minimizing path from x to y, contradicting Lemma 2.2. ■

From the previous it would seem that for Brillouin spaces $B_n(x_0)$ is the closure of $b_n(x_0)$. There are conditions under which it does hold. We give a simple example to show that this is not always the case. As the example shows these conditions will have to exclude many “natural” spaces. We will not pursue this here.

Example 2.9 X is the flat cylinder obtained by identifying opposite sides of the strip $\{z \in \mathbb{C} \text{ such that } -1 \leq \Re z \leq 1\}$ in the usual way. We will denote points in the cylinder by a corresponding complex number. Let $x_0 = 1$, $x_1 = i$, and $x_2 = -i$. Each of the mediatrices $L_{0,2}$ and $L_{0,3}$ consists of a circle with a Lipschitz singularity at the origin. It is easy to see that in this example $B_3(x_0)$ consists of only the origin. Note also that B_3 is contained in B_2 and B_1, and also in the closures of b_1 and b_2.

3 Structure Theorems

The aim of this section is to give a topological classification of mediatrices. We assume that X is a compact smooth n-manifold. Then X admits a triangulation (see [11]). We also assume that (X, L) is a triangulated pair, which means that X admits some triangulation of which L is a subcomplex. We will be using results from [12]. For the record, we mention that the class of compact triangulated relative n-homology manifolds (X, L) (in terms of which Munkres states his results, see [12], section 63) includes the pairs (X, L) stated above. We also follow Munkres' convention concerning n-cells.

Corollary 3.1 *The set L is the boundary of the open set L_-. (L is also the boundary of the open set L_+.)*

Proof: It suffices to prove the statement for L_-. From the proof of Corollary 2.7, we conclude that L_- is open and $L \cup L_-$ is closed. From Lemma 2.4 it follows that each point in L is a limit point of L_-. ∎

Proposition 3.2 *Let (X, A) be a compact triangulated relative n-homology manifold such that $X - A$ is connected and A contains no n-cells. Then the inclusion homomorphism $i_{n-1} : H_{n-1}(A; \mathbb{Z}_2) \to H_{n-1}(X; \mathbb{Z}_2)$ is a monomorphism.*

Proof: We know that $H^0(|X| - |A|; \mathbb{Z}_2) = \mathbb{Z}_2$ ([12], Theorem 42.1). By Lefschetz duality, we have that $H^0(|X| - |A|; \mathbb{Z}_2) = H_n(X, A; \mathbb{Z}_2) = \mathbb{Z}_2$ ([12], Theorem 70.2). Because X is an n-manifold, we have that $H_n(X; \mathbb{Z}_2) = \mathbb{Z}_2$. Part of the relative homology sequence is :

$$\begin{aligned} \cdots \longrightarrow H_n(A; \mathbb{Z}_2) \quad &\xrightarrow{i_n} H_n(X; \mathbb{Z}_2) \quad \xrightarrow{\pi_n} H_n(X, A; \mathbb{Z}_2) \\ &\xrightarrow{\partial_n} \; H_{n-1}(A; \mathbb{Z}_2) \xrightarrow{i_{n-1}} \; H_{n-1}(X; \mathbb{Z}_2) \longrightarrow \cdots \end{aligned}$$

By assumption $H_n(A; \mathbb{Z}_2) = 0$ (since A contains no n-cells). It then follows that π_n is an isomorphism. Thus $Ker(\partial_n) = H_n(X, A; \mathbb{Z}_2)$. Hence $Ker(i_{n-1}) = 0$ and thus i_{n-1} is a monomorphism. ∎

Proposition 3.3 *Let (X, L) be a compact triangulated relative n-homology manifold such that $X - L$ has 2 components and L contains no n-cells. Then the inclusion homomorphism $i_{n-1} : H_{n-1}(L; \mathbb{Z}_2) \to H_{n-1}(X; \mathbb{Z}_2)$ is not a monomorphism (its kernel is $\mathbb{Z}_2$).*

Proof: The same as above but now the Lefschetz duality implies that $H_n(X, L; \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$, because Corollary 2.5 implies that $|X| - |L|$ has exactly 2 components. ∎

The two propositions together imply that the inclusion homomorphism of a minimally separating set must have a kernel equal to $\mathbb{Z}_2$. More precisely:

Theorem 3.4 *Let L be a minimally separating set in X and suppose that (X, L) is a compact triangulated relative n-homology manifold. Then the kernel of inclusion homomorphism $i_{n-1} : H_{n-1}(L; \mathbb{Z}_2) \to H_{n-1}(X; \mathbb{Z}_2)$ has exactly one generator. Thus $H_{n-1}(L; \mathbb{Z}_2)$ has at most one generator more than $H_{n-1}(X; \mathbb{Z}_2)$.*

We wish to remark here that the above result restricted to $n = 2$ states that the maximum number of "circular" cuts which do not disconnect the the surface is equal to the number of generators of the first homology group with coefficients $\mathbb{Z}_2$. For this reason this number is sometimes called the "connectivity number". This formulation – and the result – appears in the book by Seifert and Threlfall [17] (section 41). However, their reasoning is different.

We shall prove, in the next section, that, under suitable conditions, L is actually a homeomorphic image of a finite closed graph (a graph without boundary). For such objects, $H_1(L; \mathbb{Z}_2)$ is a free group with d generators over $\mathbb{Z}_2$, where d is determined as follows (as was pointed out to us by R. Beyl). Suppose L consists of several disjoint subgraphs L_i. The set of generators is the union of the generators of all of the L_i. To determine the number of generators in a connected graph, we must find the maximal tree it contains. By homotopy invariance, we may collapse this tree to a point. All remaining edges will then be circles glued together in a point. They are the generators of $H_1(L_i)$. For reference see [7] (section 2.5).

Example 3.5 : A mediatrix in a Riemannian 2-torus must be homeomorphic to a closed graph (see next section) whose first homology group has at most 3 generators, for instance 3 circles glued together in a point, or 2 circles glued together in 2 distinct points.

4 Regularity Theorems

We will assume that X is a compact, connected 2-dimensional manifold with a smooth Riemannian metric giving rise to a distance d. In this section we establish that L is the homeomorphic image of a finite graph. Thus L comes with a natural triangulation, namely the one inherited from the graph. We claim that the triangulation of the underlying manifold X and that of L have a common refinement (or (X, L) is a triangulated pair, see previous section). We will take this issue up in a forthcoming work.

Note that X is complete as a metric space, and thus the Hopf-Rinow Theorem implies that X is geodesically complete and furthermore satisfies Condition 2 in Theorem 2.1 (see for example [8]). The third condition in that theorem follows again from existence and uniqueness of the solutions of differential equations. This establishes that X is a Brillouin space.

In this section we will prove the following result:

Theorem 4.1 *If X is a compact 2 dimensional manifold with a smooth Riemannian metric, then any mediatrix L is a homeomorphic image of a finite graph.*

To prove this, we need some notation and intermediate results. The exponential at a point x in X will be denoted by $\exp_x$. Since X is compact, ρ_m, the minimum of the radii of injectivity, is greater than 0. Thus for $\rho < \rho_m$ the map

$$\exp_x : N_\rho(0) \subset T_x X \to X$$

is a diffeomorphism onto its image. However, the map $\exp_x$ is smooth on *any* bounded domain of $T_x X$. Suppose L is the mediatrix associated with the points a and b in X. Let $x \in L$ and ρ small enough – as specified before. First, apply $\exp_x^{-1}$, a smooth coordinate transformation, to $\exp_x(N_\rho(0)) \subset X$. In the disk thus obtained, we use geodesic polar coordinates r and θ. Thus (r, θ) is a point in $T_x X$ and (tr, θ) is a point on line through the origin and (r, θ) in $T_x X$. Note that x is mapped to 0. Define

$$\begin{aligned} \Theta_a &= \{\theta \in S^1 \,|\, t \to \exp_x(tr, \theta) \text{ is a minimal geodesic from } x \text{ to } a\} \\ \Theta_b &= \{\theta \in S^1 \,|\, t \to \exp_x(tr, \theta) \text{ is a minimal geodesic from } x \text{ to } b\} \end{aligned}$$

Define a wedge to be a subset of $N_\rho(0)$ with the property that

$$\begin{aligned} W = \; & \{(r, \theta) \in N_\rho(0) \mid \theta \in [\theta_-, \theta_+] \,,\, \theta_- \in \Theta_a \,,\, \theta_+ \in \Theta_b \\ & \text{and } \Theta_a \cap (\theta_-, \theta_+) = \Theta_b \cap (\theta_-, \theta_+) = \emptyset\} \end{aligned}$$

or the same definition with a and b interchanged. The proof of the main theorem of this section now proceeds in a number of steps here given as lemmas.

Lemma 4.2 *The sets Θ_a and Θ_b are disjoint compact subsets of the circle.*

Proof: The disjointness is clear from the that $d(x,a) = d(x,b)$, so that if the geodesics are not disjoint, then $a = b$.

Let $\{c_i\}_{i\in\mathbb{N}}$ be a sequence of minimal geodesics between $x_i \in X$ and $a \in X$ with initial velocities v_i (and $|v_i| = 1$). Furthermore, suppose they have the property that $\lim_i x_i = x$ and $\lim_i v_i = v$. By geodesic completeness, the initial point x with initial (unit) velocity v defines a geodesic.

Now let $t \to \phi(t,x,v)$ be the geodesic at time t, starting at x with initial velocity v. Then ϕ is the solution of a second order differential equation with a smooth right hand side. It follows that $\phi(t,x,v)$ is also smooth (therefore continuous) (see [1], section 32). Thus since $\lim_i \phi(t_i,x_i,v_i) = a$, we also get $\phi(t,x,v) = a$. Finally, the continuity of the distance function implies that $t = \lim_i t_i = \lim_i d(x_i,a) = d(x,a)$. And so the limiting geodesic is minimal.

Therefore, fixing $x_i = x$, the set of directions at x in which minimal geodesics to a leave, is closed. ∎

Lemma 4.3 *There are finitely many wedges W_i.*

Proof: If the number of wedges were infinite, then there would be a sequence in Θ_a accumulating to a point in Θ_b (and vice versa). This contradicts the previous lemma. ∎

Lemma 4.4 *Each wedge W_i is separated by L', and L' does not intersect the sides of the wedge.*

Proof: Consider the connected component of $\exp_x^{-1}(L) \cap N_\rho(0)$ which contains the point $0 = \exp_x^{-1}(x)$. For brevity we will call this set L' from now on. L separates X into a component containing a and another one containing b. Thus the geodesics parting in a direction in Θ_a must be separated by L from the geodesics parting in a direction Θ_b. Since L does not meet 0, they must be separated by L'. On the other hand, by Lemma 2.6, L' cannot intersect these geodesics (except at x). ∎

Let us call the set $W_i \cap L' = L_i$ a spoke based in x.

Lemma 4.5 *For r small enough, each spoke L_i is diffeomorphic to $\{(x, f(x)) \mid x \in [0, r)\}$ in $\mathbb{R}^2$, for some Lipschitz function f.*

Proof: Given a spoke L_i in a wedge $W_i = \{(r, \theta) \mid r \in [0, \rho] \text{ and } \theta \in [\theta_-, \theta_+]\}$. We know that the spoke L_i meets 0. Let $y \in L_i$ be distinct from 0. Suppose θ_1 is a direction in which a minimal geodesic from y to a departs, and θ_2 one in which a geodesic to b departs. By Lemma 2.6, $\theta_1 \in [\theta_- - \epsilon, \theta_+ + \epsilon]$, for some $\epsilon > 0$, otherwise the geodesics will cross. The same holds for θ_2. Notice by the same lemma that L_i cannot cross these rays leaving from y. Thus L_i preserves a cone field $\{\theta_- - \epsilon\} \cup \{\theta_+ + \epsilon\}$. Now perform a linear coordinate change on $N_\rho(0)$ that maps $\theta = \theta_- - 2\epsilon$ to $\theta = 0$ and $\theta = \theta_+ + 2\epsilon$ to $\theta = \pi/2$. Then L_i is a figure in the positive quadrant of $\mathbb{R}^2$ leaving invariant the cone field $\{\epsilon\} \cup \{\pi/2 - \epsilon\}$ which implies the result.

The wedges do not necessarily partition the disk. The remaining pieces (if any) are wedge shaped and have the form

$$S = \{(r, \theta) \in N_\rho(0) \,|\, \theta \in [\theta_-, \theta_+] \;,\; \theta_+ \,,\, \theta_- \in \Theta_a \;\text{ and }\; \Theta_b \cap (\theta_-, \theta_+) = \emptyset\}$$

(or the same definition with a and b interchanged). Suppose that S contained a spoke ℓ connected to x. Then at each point y of the spoke ℓ there would be a minimal geodesic departing in the direction $\theta_1(y)$ to a. As before $\theta_1(y) \in [\theta_- - \epsilon, \theta_+ + \epsilon]$. Since this would be true for y arbitrarily close to x, by the reasoning in the proof of Lemma 4.2, there would have to be a minimal geodesic from x to a departing in the direction $\theta_1(x) \in [\theta_-, \theta_+]$. This contradicts the definition of S. ■

From this proof we conclude the following. Each of the wedges contains exactly one spoke, and each spoke is homeomorphic to a radial segment. The complement of the wedges does not contains any points of L'. This implies that there are finitely many spokes associates with x, and that if x has n spokes associated with it, then there must be at least n distinct minimal geodesics connecting x with a or b (namely the boundaries of the wedges). we are now in a position to perform the last step of the proof of the main theorem. Denote a point in L with more than 2 spokes associated with it as a vertex.

Lemma 4.6 *L contains a finite number of vertices.*

Proof: By compactness, if the lemma is false then there is an accumulation point x_0 of vertices $\{x_i\}$. Transform again to the disk $N_\rho(0)$ by applying

$\exp_{x_0}^{-1}$. Possibly by thinning out the sequence $\exp_{x_0}(x_i)$, we obtain a sequence $\{y_i\}$ that lies entirely in one of the wedges associated with x_0 or in the complement of the wedges. In the first case, the x_i lie in a single wedge associated with 0. For the same reason as in the previous lemma, the directions θ in which the minimal geodesics from x_i to a or b leave, must be contained in the wedge. Since each x_i is a vertex, there must be at least 3 minimal geodesics leaving to a or b from each y_i. By continuity (see the proof of Lemma 4.2), this is also true for 0. But this contradicts the assumption that the x_i lie in a single wedge associated with 0. When x_i lies in the complement of the wedges, a similar contradiction can be derived, because now there must be geodesics to both a and b parting from x_i. ■

5 Further Remarks

The topological classification of mediatrices L in a compact Riemannian 2-manifold X is now an easy exercise. One calculates the rank of $H_1(X; \mathbb{Z}_2)$ and call this number q_1 (the first connectivity number, see [17]). Then the mediatrix is topological graph whose first homology group has rank less than or equal to $q_1 + 1$ (the graph has less than $q_1 + 1$ independent cycles). For the torus this means that the mediatrix consists of three or less topological circles forming one, two, or three connected components. Curiously, it appears that the case where $H_1(L; \mathbb{Z}_2)$ has three generators cannot occur. The non-orientable manifolds were not excluded. It is fun to draw mediatrices of a projective plane. According to section 3, a mediatrix of the projective plane has at most two cycles.

It is possible that one has more information. For instance a mediatrix in an orientable, connected, smooth compact Riemannian 2-manifold of genus g with no vertices consists of at most g disjoint "circles" (this was pointed out to us by J. Milnor). The reasoning is as follows: Let X be a surface with g handles. A "circular cut" either disconnects the surface or else opens up one of the handles. Since vertices are forbidden, a new cut cannot go through a previous cut. So fill the obtained holes by disk, which increases the genus by 2. Keep going until you disconnect the surface or until the genus is zero. In the last case the Jordan Curve Theorem now implies that with the next cut we separate the surface.

We end by giving an intuitive example of a mediatrix in a 3-dimensional manifold.

Example 5.1 : In $\mathbb{R}^3$, we can obtain a variety of two-dimensional shapes by choosing an appropriate metric. Start with the flat metric and consider two concentric spheres of almost the same radius. In the region between them (a "fat" sphere), we make the metric very big, in such a way that it is harder to cross the fat sphere than to travel the circumference of the outer sphere. Let a be a point inside the sphere, and b a point outside, but less than one radius away from the outer sphere. Clearly, the set $L_{1,0}$ is a topological sphere. In a similar way we can obtain a torus for $L_{1,0}$. If we vary the metric smoothly to go from one to the other, at the transition, we will have a branching at the center of the figure.

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