On the existence of WKB-type asymptotics for the generalized eigenvectors of discrete string operators

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ABSTRACT. Let J be a Jacobi real symmetric matrix on l_2 with zero diagonal and non-diagonal entries of the form $\{1+p_n\}$. If $p_{n-1} \pm p_n = O(n^{-\alpha})$ with some $\alpha > 2/3$, then we prove the existance of bounded solutions of $Ju = \lambda u$ for a.e. $\lambda \in (-2, 2)$ with the WKB-type asymptotic behavior.

1. Introduction

The main object of this note is a symmetric Jacobi matrix J of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix} \tag{1.1}$$

with $a_n > 0$ and $b_n \in \mathbb{R}$. Under certain conditions on a_n's, b_n's expression (1.1) represents a self-adjoint operator on $l_2(\mathbb{N})$. We are especially interested in J's close to the free matrix J_0 defined by (1.1) with $a_n = 1$ and $b_n = 0$ (the discrete free Schrödinger operator). If $\lim a_n = 1$ and $\lim b_n = 0, n \to \infty$, then J is a compact perturbation of J_0. Under the additional condition

$$\sum_{n\geq 1} (a_n - 1)^2 + \sum_{n \geq 1} b_n^2 < \infty, \tag{1.2}$$

the difference $J - J_0$ becomes a Hilbert-Schmidt operator on l_2. A complete description of the spectra of all such Jacobi matrices was recently found by Killip-Simon in [6]. In particular the absolutely continuous spectrum fills $(-2, 2)$.

In the present note we deal with discrete string operators of the form

$$J = J_0 + \begin{pmatrix} 0 & p_1 & 0 & 0 & \cdots \\ p_1 & 0 & p_2 & 0 & \cdots \\ 0 & p_2 & 0 & p_3 & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}. \tag{1.3}$$

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It was proven by Janas-Naboko in [5] that under conditions

$$p_{n-1} \pm p_n = O\left(n^{-\alpha_\pm}\right), \alpha_+ > 3/4, \alpha_- > 2/3, \alpha_+ + \alpha_- > 3/2 \tag{1.4}$$

the equation ($u := \{u_n\}_{n\geq 1}$)

$$Ju = \lambda u \tag{1.5}$$

has two linearly independent bounded solutions u and $\overline{u}$ with the WKB-type asymptotic behavior ($\varepsilon_n := \frac{1}{2}\frac{p_{n-1}-p_n}{1+p_n}$)

$$u_n(\theta) \sim e^{in\theta} \exp i \sum_{k=2}^{n} \left\{ p_k \cot\theta - \varepsilon_k \sum_{m\geq 0} \varepsilon_{m+k} \sin 2m\theta \right\}, n \to \infty, \tag{1.6}$$

for almost all $\lambda := 2\cos\theta \in (-2,2)$.

By the Gilbert-Pearson subordinacy theory [4], the existence of asymptotics (1.6) implies that the absolutely continuous spectrum of J is $(-2,2)$. Note that [5] belongs to a recent sequence of papers on conservation of absolutely continuous spectra of Schrödinger operators and Jacobi matrices under slowly decaying perturbations which began with Kiselev [7], [8], Christ-Kiselev [1], and Remling [11]. The original approach was based upon a subtle analysis of WKB-type asymptotics. It was discovered soon by Deift and Killip [3] that the final solution to the question on the support of absolutely continuous spectra of Schrödinger operators and Jacobi matrices with L_2 (respectively l_2) potentials follows from an elegant idea based on certain trace formulas. This idea was immediately employed by Molchanov-Novitskii-Vainberg [10] (the Schrödinger operator case) and Laptev-Naboko-Safronov [9] (the Jacobi matrix case) to come to a number of important conclusions on the absolutely continuous spectrum. The culminating paper in this series is the above mentioned preprint [6] by Killip-Simon. The Gilbert-Pearson theory and the WKB-analysis hence do not appear to be necessary tools in this circle of questions. However the existence of generalized eigenfunctions of absolutely continuous spectrum with the WKB asymptotic behavior has become a separate issue interesting in its own right (e.g. it is a principal ingredient in establishing existence of wave operators [2]). The main result of this paper is a further step in improving conditions (1.4) and asymptotics (1.6). Namely, we prove the following theorem.

THEOREM 1. *Let J be a symmetric Jacobi matrix of form* (1.3) *subject to the condition $p_{n-1} \pm p_n = O(n^{-\alpha})$ with some $\alpha > 2/3$. Then, for almost all $\lambda = 2\cos\theta \in (-2,2)$, there exist two linearly independent bounded solutions u and $\overline{u}$ of equation* (1.5) *with the WKB-type asymptotics*

$$u_n(\theta) \sim e^{in\theta} \exp\left\{ i \cot\theta \sum_{k=1}^{n} p_k \right\}, \ n \to \infty. \tag{1.7}$$

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2. Notation and Preliminaries

We will follow standard notation with a few exceptions. Given sequence of matrices $\{M_n\}_{n\geq 1}$, we write

$$\prod_{k=1}^{n} M_k := M_n M_{n-1}\cdots M_1.$$

We also agree to write $M_n = \tilde{O}(n^{-\alpha})$ if $\|M_n\| = O(n^{-\alpha-\epsilon}), \epsilon > 0, n \to \infty$. Through the paper, we denote by C (M) any scalar (matrix-valued) quantities (with variables or without) which specific values are not of importance to us.

We reserve A and B for the following two matrices

$$A(a) = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix},\; B(a) = \begin{pmatrix} 0 & \overline{a} \\ a & 0 \end{pmatrix},\; a \in \mathbf{C}, \tag{2.1}$$

which will be basic blocks in our construction. We are going to systematically use their elementary properties:

$$\begin{aligned} A(a)A(b) &= A(ab);\;\; A^n(a) = A(a^n), n \in \mathbf{Z}; \\ B(a)B(b) &= A(\overline{a}b);\;\; (I + B(a))^{-1} = (1-|a|^2)^{-1}(I - B(a)); \\ A(a)B(b) &= B(\overline{a}b);\;\; B(a)A(b) = B(ab). \end{aligned}$$

The following lemma will play a crucial role in our arguments.

Lemma 1. *Let $A_n = A\left(a_n\right)$, where $\{a_n\}$ is a complex sequence and $\inf\{|a_n|\} > 0$. If $\{M_n\}$ is a sequences of 2×2 matrices and $M_n = \tilde{O}(n^{-1})$ then*

$$\prod_{k=1}^{n}(A_k + M_k) = A\left(\prod_{k=1}^{n} a_k\right)\cdot\left\{M + \tilde{O}(1)\right\}, n \to \infty, \tag{2.2}$$

where M is some 2×2 matrix. If $\sup\limits_{n\geq 1}|a_n|^{-1}\|M_n\| < 1/4$ then M is invertible.

Proof. All matrices A_k are obviously invertible and

$$A_k + M_k = A_k(I + {A_k}^{-1}M_k).$$

It immediately follows by induction from

$$(I + {A_k}^{-1}M_k)A_{k-1} = A_{k-1}\left(I + {A_{k-1}}^{-1}{A_k}^{-1}M_k A_{k-1}\right)$$

that

$$\prod_{k=1}^{n}(A_k + M_k) = \prod_{k=1}^{n} A_k \cdot \prod_{k=1}^{n}\left\{I + \left(\prod_{m=1}^{k} A_m\right)^{-1} B_k \left(\prod_{m=1}^{k-1} A_m\right)\right\}. \tag{2.3}$$

Due to the specific form (2.1) of our matrices A_k we have

$$\prod_{k=1}^{n} A_k = A(\alpha_n),\;\text{where}\;\alpha_n := \prod_{k=1}^{n} a_k,$$

and (2.3) then reads

$$\prod_{k=1}^{n}(A_k+M_k)=A(\alpha_n)\cdot\prod_{k=1}^{n}\left\{I+A(\alpha_k^{-1})M_kA(\alpha_{k-1})\right\}. \tag{2.4}$$

Since $||A(a)||\le 2|a|$ one has

$$\begin{aligned}\left\|A(\alpha_k^{-1})M_kA(\alpha_{k-1})\right\| &\le 4\left|\alpha_k^{-1}\right||\alpha_{k-1}|\,\|M_k\|=4|a_k|^{-1}|\alpha_{k-1}|^{-1}|\alpha_{k-1}|\,\|M_k\|\\ &= 4|a_k|^{-1}\|M_k\|\le C\|M_k\|=\tilde O(k^{-1}),\end{aligned}$$

and hence the product on the right hand side of (2.4) converges to some invertible M. □

The following lemma can be extracted from some more general results of [1].

LEMMA 2. *Let K be a linear operation defined on finite vectors $x=\{x_n\}$ by*

$$(Kx)(\theta)=\varphi(\theta)\sum_{n\ge1}\xi_n(\theta)x_n, \tag{2.5}$$

where φ is a smooth indicator of a fixed compact interval I, and $|\xi_n(\theta)|=1$. If

$$||Kx||_{L_2(I)}\le C||x||_{l_2},\forall x\in l_2, \tag{2.6}$$

then the series $\sum_{n\ge1}\xi_n(\theta)x_n$ and $\sum_{n\ge m}\xi_n(\theta)\overline{\xi}_m(\theta)x_ny_m$ converge for a.e. $\theta\in I$ and every $x,y\in l_2$.

The proposition below can easily be assembled from the relevant statements of [1]. For the reader's convenience, we present here its streamlined proof.

PROPOSITION 1. *Let $x_n,y_n,z_n=\tilde O(n^{-2/3})$ and real. Define $\xi_n(\theta):=\exp i(2n\theta+\cot\theta\sum_{k=1}^{n}z_k)$, $0<\theta<\pi$. Then the series*

$$\sum_{n\ge1}\xi_n(\theta)x_n,\quad \sum_{n\ge m}\xi_n(\theta)\overline{\xi}_m(\theta)x_ny_m \tag{2.7}$$

converge for a.e. $\theta\in(0,\pi)$ and for those θ :

$$\sum_{k\ge n}\xi_k(\theta)x_k=O(n^{-1/6}). \tag{2.8}$$

PROOF. Consider a linear operator K defined on $x=\{x_n\}\in l_2$ by (2.5) with an interval $I\subset(0,\pi)$ and $\varphi(0)=0=\varphi(\pi)$. We show that estimate (2.6) holds. To this end, consider

(2.9)
$$||Kx||^2_{L_2(0,\pi)}=\int_0^\pi\left(\sum_{n,m\ge1}\xi_n(\theta)\overline{\xi_m}(\theta)x_nx_m\right)\varphi^2(\theta)d\theta=\sum_{n,m\ge1}K(n,m)x_nx_m,$$

where

$$K(n,m):=\int_0^\pi e^{2i(n-m)\theta}F_{nm}(\theta)d\theta,$$

$$F_{nm}(\theta):=e^{iZ_{nm}\cot\theta}\varphi^2(\theta),Z_{nm}:=\sum_{k=1}^{n}z_k-\sum_{k=1}^{m}z_k.$$

Let us estimate $K(n,m)$. Assume $n \neq m$. Twice by parts yields

$$K(n,m) = \left(\frac{1}{2i(n-m)}\right)^2 \int_0^\pi e^{2i(n-m)\theta} F''_{nm}(\theta)d\theta. \tag{2.10}$$

Note, that the integrated terms vanish in virtue of $\varphi(0) = 0 = \varphi(\pi)$. By a direct computation,

$$F''_{nm}(\theta) = e^{iZ_{nm}\cot\theta}\left\{2\left((\varphi')^2 + \varphi\varphi''\right) + 2iZ_{nm}\frac{\varphi^2\cot\theta - 2\varphi\varphi'}{\sin^2\theta} - Z^2_{nm}\frac{\varphi^2}{sin^4\theta}\right\},$$

and one easily derives

$$|F''_{nm}(\theta)| \leq C\left(1 + |Z_{nm}| + \left|Z^2_{nm}\right|\right), \quad C = C(\varphi, I). \tag{2.11}$$

By condition, $z_n = \tilde{O}(k^{-2/3})$ and hence $\{z_n\} \in l_{3/2}$. By Hölder's inequality it follows from (2.11) that ($z = \{z_n\}$)

$$|F''_{nm}(\theta)| \leq C\left(1 + ||z||_{l_{3/2}} \cdot |n-m|^{\frac{1}{3}} + ||z||^2_{l_{3/2}} \cdot |n-m|^{\frac{2}{3}}\right) \leq C\left(1 + |n-m|^{\frac{2}{3}}\right),$$

and hence (2.10) implies

$$|K(n,m)| \leq C|n-m|^{-2}|n-m|^{\frac{2}{3}} = C|n-m|^{-\frac{4}{3}}, \; n \neq m \tag{2.12}$$

If $n = m$, then

$$K(n,m) = \int_0^\pi \varphi^2(\theta)d\theta < \pi \tag{2.13}$$

and it follows from (2.9), (2.12), and (2.13) that

$$||Kx||^2_{L_2(0,\pi)} \leq C\left(||x||_{l_2} + \sum_{n\neq m\geq 1} |n-m|^{-\frac{4}{3}}|x_n||x_m|\right). \tag{2.14}$$

For the sum in (2.14), by Hölder's inequality, one has

$$\sum_{n\neq m} |n-m|^{-\frac{4}{3}}|x_n||x_m| = \sum_{n\neq m} |n-m|^{-\frac{2}{3}}|x_n||n-m|^{-\frac{2}{3}}|x_m|$$

$$\leq \sum_{n\geq 1}\left(\sum_{m\neq n}|n-m|^{-\frac{4}{3}}\right)x_n^2 \leq C\sum_{n\neq m}|x_n|^2 = C||x||^2_{l_2}$$

and estimate (2.6) is established. So, our operator K is subject to the conditions of Lemma 2 and both series (2.7) converge for a.e. $\theta \in I$ and hence for a.e. $\theta \in (0,\pi)$ since I is an arbitrary subinterval of $(0,\pi)$.

Now we prove (2.8). Introduce a new vector $\tilde{x}$ by

$$\tilde{x}_n := n^{1/6}x_n = n^{1/6}\tilde{O}(n^{-2/3}) = \tilde{O}(n^{-1/2}).$$

Since $\tilde{x} \in l_2$, by Lemma 2, the sequence

$$G_n(\theta) := \sum_{k=1}^n \xi_k(\theta)\tilde{x}_k$$

converges a.e. on $(0,\pi)$. Performing Abel's transformation:

$$\sum_{k\geq n}\xi_k(\theta)x_k=\sum_{k\geq n}k^{-1/6}\xi_k(\theta)\tilde{x}_k$$

$$=\sum_{k\geq n}k^{-1/6}G_k(\theta)=-n^{-1/6}G_{n-1}(\theta)+\sum_{k\geq n}\left(k^{-1/6}-(k+1)^{-1/6}\right)G_k(\theta)$$

$$=O(n^{-1/6}),\ \text{a.e. }\theta\in(0,\pi),$$

we arrive at (2.8). □

3. Proof of Theorem 1

Observe first that due to the asymptotic nature of the theorem we are free to drop in matrix (1.3) a finite number of rows and columns. Rewriting equation (1.5) in the form

$$(1+p_{n-1})u_{n-1}+(1+p_n)u_{n+1}=\lambda u_n,\ n=2,3,...,$$

or in the matrix form

$$\begin{pmatrix} u_n \\ u_{n+1}\end{pmatrix}=\begin{pmatrix} 0, & 1 \\ -\frac{1+p_{n-1}}{1+p_n}, & \frac{\lambda}{1+p_n}\end{pmatrix}\begin{pmatrix} u_{n-1} \\ u_n\end{pmatrix}, \qquad n=2,3...,$$

immediately implies

$$\begin{pmatrix} u_n \\ u_{n+1}\end{pmatrix}=\prod_{k=2}^{n}\begin{pmatrix} 0, & 1 \\ -\frac{1+p_{k-1}}{1+p_k}, & \frac{\lambda}{1+p_k}\end{pmatrix}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}. \tag{3.1}$$

Set $\lambda:=2cos\theta=\xi+\overline{\xi}$, where $\xi:=e^{i\theta},\theta\in(0,\pi)$, and

$$U:=\begin{pmatrix} 1 & \frac{1}{\xi} \\ \xi & \overline{\xi}\end{pmatrix}.$$

Introducing the sequence of matrices

$$T_n:=U^{-1}\begin{pmatrix} 0, & 1 \\ -\frac{1+p_{n-1}}{1+p_n}, & \frac{\lambda}{1+p_n}\end{pmatrix}U \tag{3.2}$$

transforms (3.1) into

$$\begin{pmatrix} u_n \\ u_{n+1}\end{pmatrix}=U\left(\prod_{k=2}^{n}T_k\right)U^{-1}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}=:T(n)\begin{pmatrix} u_1 \\ u_2\end{pmatrix}. \tag{3.3}$$

By involved but a straightforward computation one verifies that

$$T_n=\{I+A(\tilde{z_n})-B(\tilde{z_n})\}\cdot A(\xi), \tag{3.4}$$

where A and B are defined by (2.1) and

$$\tilde{z}_n:=\frac{z_n}{1+p_n},\quad z_n:=\frac{p_{n-1}-p_n}{2}+i\frac{p_{n-1}+p_n}{2}\cot\theta. \tag{3.5}$$

To derive the assertions of the theorem one needs to analyze the asymptotic behavior of the matrix-valued sequence

$$T(n)=U\left(\prod_{k=2}^{n}T_k\right)U^{-1}. \tag{3.6}$$

To this end we introduce a sequence of invertible matrices $\{U_k\}_{k\geq 2}$ to be chosen later and rewrite (3.6) as follows

(3.7)
$$T(n) = (UU_{n+1}) \cdot \left\{ \prod_{k=2}^{n} (U_{k+1}^{-1} T_k U_k) \right\} \cdot U_2^{-1} U^{-1} = (UU_{n+1}) \cdot \left(\prod_{k=2}^{n} \widetilde{T}_k \right) \cdot (UU_2)^{-1} ,$$

where $\widetilde{T}_n := U_{n+1}^{-1} T_n U_n$. With Lemma 1 in mind, we want to represent each $\widetilde{T}_k$ as a diagonal matrix plus a small remainder. We show now that the sequence

$$U_n = A(\xi^n) A(\alpha_n) \{I + B(\beta_n)\} , \tag{3.8}$$

with proper α_n and β_n will do it. Indeed, by (3.4) and (3.8) one has

$$\widetilde{T}_n = U_{n+1}^{-1} T_n U_n = \{I + B(\beta_{n+1})\}^{-1} \Omega_n \{I + B(\beta_n)\} , \tag{3.9}$$

where

$$\Omega_n := A^{-1}(\alpha_{n+1}) A^{-1}(\xi^{n+1}) \{I + A(\tilde{z_n}) - B(\tilde{z_n})\} A(\xi) A(\xi^n) A(\alpha_n).$$

We first treat Ω_n. Using the properties of matrices A and B, we get

$$\begin{aligned}
\Omega_n &= A(\xi^{-n-1} \alpha_{n+1}^{-1}) \cdot \{I + A(\tilde{z_n}) - B(\tilde{z_n})\} \cdot A(\xi^{n+1} \alpha_n) \\
&= A\left(\frac{\alpha_n}{\alpha_{n+1}} \right) + A(\xi^{-n-1} \alpha_{n+1}^{-1}) A(\tilde{z_n}) A(\xi^{n+1} \alpha_n) \\
&\quad - A(\xi^{-n-1} \alpha_{n+1}^{-1}) B(\tilde{z_n}) A(\xi^{n+1} \alpha_n) \\
&= A\left(\frac{\alpha_n}{\alpha_{n+1}} \right) + A\left(\frac{\alpha_n}{\alpha_{n+1}} \widetilde{z}_n \right) - B\left(\xi^{n+1} \overline{\alpha}_{n+1}^{-1} \widetilde{z}_n \right) A(\xi^{n+1} \alpha_n) \\
&= A\left(\frac{\alpha_n}{\alpha_{n+1}} (1 + \widetilde{z}_n) \right) - B\left(\xi^{2(n+1)} \frac{\alpha_n}{\overline{\alpha}_{n+1}} \widetilde{z}_n \right) . \qquad (3.10)
\end{aligned}$$

Choose now

$$\alpha_n(\theta) = \exp \sum_{k=2}^{n-1} z_k(\theta) .$$

Due to the conditions on p_n, we have $\dfrac{\alpha_n}{\alpha_{n+1}} = \exp(-z_n) = 1 - z_n + \tilde{O}(n^{-1})$, $\widetilde{z}_n = (1 + p_n)^{-1} z_n = z_n + \tilde{O}(n^{-1})$, and hence the first term on the right in (3.10) reads

$$A\left(\frac{\alpha_n}{\alpha_{n+1}} (1 + \widetilde{z}_n) \right) = I + \tilde{O}(n^{-1}). \tag{3.11}$$

Next, since $\dfrac{\overline{\alpha}_n}{\overline{\alpha}_{n+1}} = 1 - \overline{z}_n + \tilde{O}(n^{-1})$,

$$\begin{aligned}
\xi^{2(n+1)} \frac{\alpha_n}{\overline{\alpha}_{n+1}} \widetilde{z}_n &= \widetilde{z}_n \frac{\overline{\alpha}_n}{\overline{\alpha}_{n+1}} \xi_n = \left(z_n + \tilde{O}(n^{-1}) \right) \left(1 - \overline{z}_n + \tilde{O}(n^{-1}) \right) \xi_n \\
&= z_n \xi_n + \tilde{O}(n^{-1}),
\end{aligned}$$

where

$$\xi_n = \xi_n(\theta) := \exp 2i \left((n+1)\theta + Im \sum_{k=2}^{n-1} z_k(\theta) \right) , \tag{3.12}$$

and the second term on the right side of (3.10) becomes

$$B\left(\xi^{2(n+1)}\frac{\alpha_n}{\overline{\alpha}_{n+1}}\widetilde{z}_n\right) = B\left(z_n\xi_n\right) + \tilde{O}(n^{-1}). \tag{3.13}$$

Plugging (3.11) and (3.13) in (3.10) yields

$$\Omega_n = I - B\left(z_n\xi_n\right) + \tilde{O}(n^{-1}), \tag{3.14}$$

and combining (3.14) with (3.9), we get

$$\begin{aligned}\widetilde{T}_n &= \{I + B(\beta_{n+1})\}^{-1}\cdot\left\{I - B\left(z_n\xi_n\right) + \tilde{O}(n^{-1})\right\}\cdot\{I + B(\beta_n)\}\\ &= (1-|\beta_{n+1}|^2)^{-1}\cdot\left\{\Xi_n + \tilde{O}(n^{-1})\right\},\end{aligned} \tag{3.15}$$

where

$$\Xi_n := \{I - B(\beta_{n+1})\}\cdot\{I - B\left(z_n\xi_n\right)\}\cdot\{I + B(\beta_n)\}. \tag{3.16}$$

Multiplying (3.16) out and using the properties of A and B, after tedious but direct computations one obtains

$$\begin{aligned}\Xi_n &= A\left(I + \overline{\beta}_{n+1}z_n\xi_n - \overline{\beta}_{n+1}\beta_n - \overline{z_n\xi_n}\beta_n\right)\\ &\quad + B\left(\beta_n - \beta_{n+1} - z_n\xi_n + \beta_{n+1}\beta_n\overline{z_n\xi_n}\right)\end{aligned} \tag{3.17}$$

By condition, the sequences $\{p_{n-1} \pm p_n\}$ and $\{\xi_n(\theta)\}$ are subject to the conditions of Proposition 1 and hence the series

$$\sum_{n\geq 1} z_n\xi_n(\theta) = \frac{1}{2}\sum_{n\geq 1}\xi_n(\theta)\left(p_{n-1} - p_n\right) + \frac{i}{2}\cot\theta\sum_{n\geq 1}\xi_n(\theta)\left(p_{n-1} + p_n\right)$$

converges for almost all $\theta \in (0,\pi)$. Choose now β_n to satisfy

$$\beta_n = \beta_n(\theta) = -\sum_{k\geq n} z_k\xi_k(\theta). \tag{3.18}$$

The sequence β_n is well-defined and due to (2.8)

$$\beta_n = O(n^{-\frac{1}{6}}), \quad \text{a. e.} \quad \theta \in (0,\pi). \tag{3.19}$$

Since $\beta_n - \beta_{n+1} = z_n\xi_n$ and $\beta_{n+1}\beta_n\overline{z_n\xi_n} = \tilde{O}(n^{-1})$, (3.17) reduces to

$$\Xi_n = A\left(I - |\beta_{n+1}|^2 - \overline{z_n\xi_n}\beta_n\right) + \tilde{O}(n^{-1}),$$

and it follows from (3.15) that for a. e. $\theta \in (0,\pi)$

$$\widetilde{T}_n = A\left(1 + a_n\right) + \tilde{O}(n^{-1}), \quad \text{where} \quad a_n := -\overline{z_n\xi_n}\beta_n. \tag{3.20}$$

Since $a_n = \tilde{O}(n^{-2/3})O(n^{-1/6}) = \tilde{O}(n^{-5/6})$, the limit

$$\lim_{n\to\infty}\left\{\ln\prod_{k=2}^{n}\left(1 + a_k\right) - \sum_{k=2}^{n} a_k\right\} =: C,$$

clearly exists, and applying Lemma 1 to $\left\{\widetilde{T}_n\right\}$ defined by (3.20) yields

$$\begin{aligned}\prod_{k=2}^{n}\widetilde{T}_k &= A\left(\prod_{k=2}^{n}(1+a_k)\right)\left(M+\tilde{O}(1)\right)\\ &= A\left(\exp\sum_{k=2}^{n}a_k\right)\left(M+\tilde{O}(1)\right),\end{aligned}\tag{3.21}$$

where M is some matrix. Observe now that for U_{n+1} defined by (3.8) we have:

$$U_{n+1}=A\left(\xi^{n+1}\alpha_{n+1}\right)+B\left(\beta_{n+1}\overline{\xi^{n+1}\alpha_{n+1}}\right)=A\left(\xi^{n+1}\alpha_{n+1}\right)+\tilde{O}(1),\tag{3.22}$$

for a.e. $\theta\in(0,\pi)$. Combining (3.7), (3.21), and (3.22) we have

$$\begin{aligned}T(n) &= UA\left(\xi^{n+1}\alpha_{n+1}\right)A\left(\exp\sum_{k=2}^{n}a_k\right)M\left(UU_2\right)^{-1}+\tilde{O}(1)\\ &= U\cdot A\left(\xi^{n+1}\alpha_{n+1}\exp\sum_{k=2}^{n}a_k\right)\cdot M+\tilde{O}(1)\\ &= U\cdot A\left(\exp\left\{i(n+1)\theta+\sum_{k=2}^{n}z_k-\sum_{k=2}^{n}\overline{z_k\xi_k}\beta_k\right\}\right)\cdot M+\tilde{O}(1).\end{aligned}\tag{3.23}$$

Let us now treat the sums on the right in (3.23):

$$\begin{aligned}\sum_{k=2}^{n}z_k &= \sum_{k=2}^{n}\frac{p_{k-1}-p_k}{2}+i\cot\theta\sum_{k=2}^{n}\frac{p_{k-1}+p_k}{2}\\ &= \frac{p_1-p_n}{2}+i\cot\theta\sum_{k=1}^{n}p_k-i\frac{p_1+p_n}{2}\cot\theta\\ &= i\cot\theta\cdot\sum_{k=1}^{n}p_k+C(\theta)+\tilde{O}(1).\end{aligned}\tag{3.24}$$

We show that the other sum on the right side of (3.23) actually converges. Indeed, by (3.18) we have

$$\begin{aligned}-\sum_{k=2}^{n}\overline{z_k\xi_k}\beta_k &= \sum_{k=2}^{n}\overline{z_k\xi_k}\sum_{m=k}^{n}z_m\xi_m+\sum_{k=2}^{n}\overline{z_k\xi_k}\cdot\sum_{k\geq n}z_k\xi_k\\ &= \sum_{m\geq k\geq 2}^{n}\xi_m\overline{\xi}_k\,\overline{z}_kz_m-\beta_n\sum_{k=2}^{n}\overline{z_k\xi_k}=\sum_{m\geq k\geq 2}^{n}\xi_m\overline{\xi}_k\overline{z}_kz_m+\tilde{O}\left(n^{-1/6}\right).\end{aligned}\tag{3.25}$$

Since $z_n=x_n^{(-)}+ix_n^{(+)}\cot\theta$, where $x_n^{(\pm)}=p_{n-1}\pm p_n$, the sum on the right side of (3.25) breaks down into four series

$$\sum_{m\geq k\geq 2}^{n}\xi_m(\theta)\overline{\xi}_k(\theta)x_m^{(\pm)}x_k^{(\pm)}$$

which by Proposition 1 converge for a.e. $\theta\in(0,\pi)$ and (3.25) takes form

$$-\sum_{k=2}^{n}\overline{z_k\xi_k}\beta_k=C(\theta)+\tilde{O}\left(n^{-1/6}\right),\ \text{a.e. }\theta\in(0,\pi).$$

Combing this with (3.24) and inserting into (3.23), we finally obtain that for a.e. $\theta \in (0,\pi)$

$$T(n) = U \cdot A\left(\exp i\left\{n\theta + \cot\theta \cdot \sum_{k=1}^{n} p_k\right\}\right) \cdot M(\theta) + \tilde{O}(1). \tag{3.26}$$

It is not hard to see that the matrix M is invertible (after, if needed, dropping a finite number of rows and columns in the original matrix (1.3)). This means that by choosing initial conditions (u_1, u_2) we can always attain

$$M\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and it follows from (3.3) and (3.26) that for a.e. $\theta \in (0,\pi)$

$$\begin{aligned}\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} &= U \cdot A\left(\exp i\left\{n\theta + \cot\theta \cdot \sum_{k=1}^{n} p_k\right\}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{O}(1) \\ &= U\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \exp i\left\{n\theta + \cot\theta \cdot \sum_{k=1}^{n} p_k\right\} + \tilde{O}(1), n \to \infty.\end{aligned}$$

The latter immediately implies (1.7) and the theorem is proven.

Corollary 1. *Under the conditions of Theorem 1, the absolutely continuous spectrum of J fills the interval* $(-2,2)$.

Remark 1. *It is not difficult to prove Theorem 1 under conditions* $p_{n-1} \pm p_n = o\left(n^{-2/3}\right)$. *Moreover* [1] *makes one believe that the theorem still holds if* $p_{n-1} \pm p_n = \tilde{O}\left(n^{-1/2}\right)$. *As it was discussed in Introduction, the conclusion of Corollary 1 remains valid* [6] *under the only condition* $\{p_n\} \in l_2$ *(see (1.2)) and one may cautiously hope that so does asymptotics (1.7). It seems to be an open problem even in the case of Schrödinger operators.*

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