$\mathbf{SL_2}(\mathbb{R})$, EXPONENTIAL HERGLOTZ REPRESENTATIONS, AND SPECTRAL AVERAGING

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Abstract. We revisit the concept of spectral averaging and point out its origin in connection with one-parameter subgroups of $SL_2(\mathbb{R})$ and the corresponding Möbius transformations. In particular, we identify exponential Herglotz representations as the basic ingredient for the absolute continuity of average spectral measures with respect to Lebesgue measure and the associated spectral shift function as the corresponding density for the averaged measure. As a by-product of our investigations we unify the treatment of rank-one perturbations of self-adjoint operators and that of self-adjoint extensions of symmetric operators with deficiency indices $(1,1)$. Moreover, we derive separate averaging results for absolutely continuous, singularly continuous, and pure point measures and conclude with an averaging result of the κ-continuous part (with respect to the κ-dimensional Hausdorff measure) of singularly continuous measures.

1. Introduction

Spectral averaging is usually concerned with integrating the spectral measure of a one-parameter family of self-adjoint operators with respect to a parameter, typically a coupling constant or a boundary condition parameter. One then proceeds to proving the absolute continuity of the integrated (averaged) spectral measure with respect to Lebesgue measure. Actually, one is usually more ambitious and tries to establish the universality of spectral averaging, provided that averaging is carried out over the whole parameter space. That is, one intends to prove that the averaged measure does not depend upon the concrete choice of the one-parameter family of operators and that it is mutually equivalent to Lebesgue measure.

In this paper we revisit this circle of ideas and present a discussion of the following topics:

• The intimate connection between spectral averaging, $SL_2(\mathbb{R})$, and Möbius transformations.

• The exponential Herglotz representation theorem is shown to be the underlying reason for absolute continuity of averaged spectral measures with respect to Lebesgue measure. In particular, this identifies the spectral shift function as the density in the absolutely continuous averaged spectral measure.

• Various existing results on (universality of) spectral averaging are extended. In particular, we don't assume the existence of a spectral gap (or boundedness from below) in the associated self-adjoint operators.

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• Conditions for (non)universality of spectral averaging to hold are identified.
• A unified treatment of self-adjoint rank-one perturbations of a self-adjoint operator and self-adjoint extensions of a densely defined closed symmetric operator with deficiency indices $(1, 1)$ is presented.
• Separate averaging for point spectra, absolutely continuous, and singularly continuous spectra are discussed.
• A partial result for averaging the κ-continuous part (with respect to the κ-dimensional Hausdorff measure) of singularly continuous measures is derived.

We next illustrate these ideas in two canonical cases: the case of rank-one perturbation theory and that of the theory of self-adjoint extensions of symmetric operators with deficiency indices $(1, 1)$.

Let A be a self-adjoint operator in a separable complex Hilbert space $\mathcal{H}$ and P an orthogonal rank-one projection in $\mathcal{H}$. We introduce two Herglotz functions, M and N, associated with the pair (A, P)

$$M(z) = \operatorname{tr}(P(A-z)^{-1}P), \quad z \in \mathbb{C}_+ \tag{1.1}$$

and

$$N(z) = \operatorname{tr}(P(zA+I)(A-z)^{-1}P), \quad z \in \mathbb{C}_+, \tag{1.2}$$

with $\mathbb{C}_+$ the open upper complex half-plane. One then has the Herglotz representations,

$$M(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+, \tag{1.3}$$

with μ a probability measure on $\mathbb{R}$, $\mu(\mathbb{R}) = 1$, and

$$N(z) = B + \int_{\mathbb{R}} d\nu(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}_+, \tag{1.4}$$

with $B \in \mathbb{R}$ and ν a Borel measure satisfying

$$\int_{\mathbb{R}} \frac{d\nu(\lambda)}{1+\lambda^2} < \infty. \tag{1.5}$$

Actually, a short computation reveals that

$$N(z) = z + (1+z^2)M(z) \tag{1.6}$$

and

$$d\nu(\lambda) = (1+\lambda^2)d\mu(\lambda), \quad B = \operatorname{Re}(N(i)) = 0. \tag{1.7}$$

Thus, (1.4) simplifies to

$$N(z) = \int_{\mathbb{R}} (1+\lambda^2) d\mu(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}_+. \tag{1.8}$$

If A is unbounded and $\operatorname{ran}(P) \cap \operatorname{dom}(A) = \{0\}$, then the measure ν is infinite.

Lemma 1.1. *Consider the one-parameter family of self-adjoint operators,*

$$A_t = A + tP, \quad t \in \mathbb{R}, \tag{1.9}$$

with resolvents

$$(A_t - z)^{-1} = (A-z)^{-1} - \frac{1}{M(z) + (1/t)} (A-z)^{-1} P (A-z)^{-1}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+, \tag{1.10}$$

and M given by (1.1). *Moreover, introduce*

$$M_t(z) = \frac{M(z)}{tM(z)+1}, \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+. \tag{1.11}$$

Then M_t is the corresponding M-function associated with the pair (A_t, P) (cf. (1.1)*). Denote by μ_t the measure in* (1.3) *associated with M_t and by Δ a bounded Borel set on $\mathbb{R}$. Then averaging μ_t yields an absolutely continuous measure with respect to Lebesgue measure,*

$$\int_{t_1}^{t_2} dt\, \mu_t(\Delta) = \int_\Delta d\lambda\, [\xi(\lambda; A_{t_2}, A) - \xi(\lambda; A_{t_1}, A)], \tag{1.12}$$

where $\xi(\cdot, B, A)$ is the spectral shift function associated with the pair (B, A) of self-adjoint operators. Moreover, spectral averaging is universal in the sense that

$$\int_{-\infty}^{\infty} dt\, \mu_t(\Delta) = |\Delta|, \tag{1.13}$$

with $|\cdot|$ denoting Lebesgue measure on $\mathbb{R}$.

Remark 1.2. The proof of the lemma is well-known and can be found in [25] and [42]. In fact, (1.12) is a particular case of the Birman–Solomyak spectral averaging formula [6] proven in the mid-seventies.

Lemma 1.3. *Assume that A is an unbounded self-adjoint operator and*

$$\operatorname{ran}(P) \cap \operatorname{dom}(A) = 0. \tag{1.14}$$

Then the one-parameter family of operator-valued functions

$$R_t(z) = (A-z)^{-1} - \frac{1}{N(z) + (1/t)}(A-i)(A-z)^{-1}P(A+i)(A-z)^{-1}, \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+, \tag{1.15}$$

with N given by (1.2)*, are the resolvents of a one-parameter family of self-adjoint operators $\{A_t\}_{t\in\mathbb{R}}$. Introducing,*

$$N_t(z) = \frac{N(z) - t}{tN(z)+1}, \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+, \tag{1.16}$$

then N_t is the N-function (in the sense of (1.15)*) of the pair (A_t, P) (cf.* (1.8)*). The family $\{A_t\}_{t\in\mathbb{R}}$ is a one-parameter family of self-adjoint extensions of a closed symmetric densely defined operator $\dot{A}$ with deficiency indices $(1,1)$,*

$$\dot{A} = A\big|_{\operatorname{dom}(\dot{A})}, \quad \operatorname{dom}(\dot{A}) = \bigcap_{t\in\mathbb{R}} \operatorname{dom}(A_t). \tag{1.17}$$

In particular, $\lim_{t\to 0} A_t = A$ in the strong resolvent sense. Denote by ν_t the measure in (1.8) *associated with N_t and by Δ a bounded Borel set on $\mathbb{R}$. Then averaging ν_t yields an absolutely continuous measure with respect to Lebesgue measure,*

$$\frac{1}{\pi}\int_{t_1}^{t_2} \frac{dt}{1+t^2}\, \nu_t(\Delta) = \int_\Delta d\lambda\, [\xi(\lambda; A_{t_2}, A) - \xi(\lambda; A_{t_1}, A)]. \tag{1.18}$$

Moreover, spectral averaging is universal in the sense that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2}\, \nu_t(\Delta) = |\Delta|. \tag{1.19}$$

Remark 1.4. The resolvent formula (1.15) is due to Krein [34] and Naimark [37]. The proof of the transformation law (1.16) can be found, for instance, in [21]. The spectral averaging formula (1.18) in the case of boundary condition dependence for a semibounded Schrödinger operator is due to Javrjan [29]. Javrjan's method can easily be adapted to the case of arbitrary self-adjoint operators A having a spectral gap. The treatment of the general case of A with $\operatorname{spec}(A) = \mathbb{R}$ needs some additional information on the spectral shift function theory in the case of relatively trace class perturbation. In this case the spectral shift function should be viewed as a path-dependent homotopy invariant characteristics of the perturbation (see, [47, Ch. 8, Sect. 8]) and the proof of (1.18) requires minor additional efforts.

In the case of perturbation theory the transformation (1.11) can be represented in the form

$$M_t(z) = g_t(M(z)), \tag{1.20}$$

where $\{g_t\}_{t\in\mathbb{R}}$ is a one-parameter group of automorphisms of the open upper half-plane $\mathbb{C}_+$

$$g_t \circ g_s = g_{t+s}, \quad s,t \in \mathbb{R}, \tag{1.21}$$

where

$$g_t(z) = \frac{z}{tz+1}, \quad (t,z) \in \mathbb{R}\times\mathbb{C}_+. \tag{1.22}$$

In the case of self-adjoint extension theory the transformation (1.16) can be written as

$$N_t(z) = f_t(N(z)), \tag{1.23}$$

where $\{f_t\}_{t\in\mathbb{R}}$ is a one-parameter family of automorphisms of $\mathbb{C}_+$

$$f_t(z) = \frac{z-t}{tz+1}, \quad (t,z) \in \mathbb{R}\times\mathbb{C}_+. \tag{1.24}$$

The family of transformations $\{f_t\}_{t\in\mathbb{R}}$ is not a one-parameter subgroup of $SL_2(\mathbb{R})$. However, by a change of parametrization $t \mapsto \tan(t)$, the group law (1.21) can be restored with

$$g_t(z) = f_{\arctan(t)}(z), \quad (t,z) \in \mathbb{R}\times\mathbb{C}_+. \tag{1.25}$$

In either case, the one-parameter family $\{g_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathbb{C}_+$ gives rise to a dynamical system on a certain "phase space" of measures as discussed in Section 3.

We continue with an intuitive explanation of how exponential Herglotz representations, and hence spectral shift functions, naturally enter the averaging process (1.12), (1.18). In both cases, Lemma 1.1 and 1.3, M_t, respectively, N_t (the latter after reparametrizing $t \mapsto \tan(t)$) are of the type,

$$M_t(z) = \frac{a_t M_0(z) + b_t}{c_t M_0(z) + d_t} = \frac{d}{dt}\mathrm{Ln}(c_t M_0(z) + d_t), \quad (t,z) \in \mathbb{R}\times\mathbb{C}_+. \tag{1.26}$$

Here M_0 represents M and N in Lemmas 1.1 and 1.3, respectively, $\mathrm{Ln}(\cdot)$ denotes the logarithm on the standard infinitely sheeted Riemann surface branched at zero and infinity (and some care taking appropriate sheets must be exercised), and the coefficients a_t, b_t, c_t, d_t are all real-valued satisfying

$$\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}\bigg|_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } a_t d_t - b_t c_t = 1,\ t \in \mathbb{R}. \tag{1.27}$$

Since M_0 is a Herglotz function, so is M_t for each $t \in \mathbb{R}$. Similarly, $c_t M_0 + d_t$ is a Herglotz or anti-Herglotz function and thus M_t and $c_t M_0 + d_t$ admit Herglotz and exponential Herglotz representations of the type,

$$M_t(z) = B_t + \int_{\mathbb{R}} d\,\omega_t(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \tag{1.28}$$

$$\mathrm{Ln}(c_t M_0(z) + d_t)) = C_t + \int_{\mathbb{R}} d\lambda\, \xi_t(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \tag{1.29}$$

where $B_t, C_t \in \mathbb{R}$,

$$\omega_t((\lambda_1, \lambda_2]) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, \mathrm{Im}(M_t(\lambda + i\varepsilon)), \tag{1.30}$$

$$\xi_t(\lambda) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\big(\mathrm{Ln}(c_t M_0(\lambda + i\varepsilon) + d_t)\big) \text{ for a.e. } \lambda \in \mathbb{R}, \tag{1.31}$$

and

$$\int_{\mathbb{R}} \frac{d\omega_t(\lambda)}{1+\lambda^2} < \infty, \quad \xi_t(\cdot) \in L^\infty(\mathbb{R}), \quad t \in \mathbb{R}. \tag{1.32}$$

Thus, one formally obtains for any bounded Borel set $\Delta \subset \mathbb{R}$,

$$\begin{aligned}\int_{t_1}^{t_2} dt\, \omega_t(\Delta) &= \frac{1}{\pi}\int_\Delta d\lambda \int_{t_1}^{t_2} dt\, \lim_{\varepsilon\downarrow 0}\frac{d}{dt}\mathrm{Im}\big(\mathrm{Ln}(c_t M_0(\lambda + i\varepsilon) + d_t)\big)\\ &= \int_\Delta d\lambda \int_{t_1}^{t_2} dt\, \frac{d}{dt}\xi_t(\lambda)\\ &= \int_\Delta d\lambda\, [\xi_{t_2}(\lambda) - \xi_{t_1}(\lambda)],\end{aligned} \tag{1.33}$$

freely interchanging integrals, limits, and differentiation. Once rigorously established, (1.33) proves that averaging ω_t over the interval $[t_1, t_2]$ yields a measure absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ and density related to the spectral shift function $\xi = \xi_{t_2} - \xi_{t_1}$. Moreover, in the case of perturbations discussed in Lemma 1.1, one can show that

$$\xi_{t_2}(\lambda) - \xi_{t_1}(\lambda) \to 1 \text{ as } t_1 \downarrow -\infty \text{ and } t_2 \uparrow \infty \tag{1.34}$$

and hence the universal behavior (1.13)

$$\int_{-\infty}^{\infty} dt\, \omega_t(\Delta) = |\Delta| \tag{1.35}$$

emerges. The case of self-adjoint extensions discussed in Lemma 1.3 requires some additional periodicity considerations with respect to t but in the end also yields the universality in (1.19). However, a third case of one-parameter subgroups of $SL_2(\mathbb{R})$ considered in the following sections shows that universality cannot be taken for granted and may in fact fail. The material in Sections 2 and 3 will justify the formal procedures in (1.33).

Before describing the contents of each section we briefly review the historical development of this subject, which appears to be less well-known. To the best of our knowledge, the credit for the first paper on spectral averaging belongs to Javrjan [28] (see also the subsequent [29]), who considered half-line Schrödinger operators on $(0, \infty)$ and averaged over the boundary condition parameter at $x = 0$ as early as 1966. The next step is due to Birman and Solomyak [6] in 1975. They considered trace class perturbations of self-adjoint operators and averaged over the coupling constant parameter (using the differentiation formula for operator-valued functions by Daleckiĭ and S. Kreĭn [14]). Aleksandrov [1] appears to be the first to consider spectral averaging of a measure and separately averaging of its singular part in connection with the boundary behavior of inner functions in the unit disk in 1987. More recent treatments of spectral averaging can be found in Birman and Pushnitski [5], Gesztesy and Makarov [23], Gesztesy Makarov, and Naboko [22] (the latter references discuss an operator-valued version of the Birman–Solomyak averaging formula), Gesztesy, Makarov, and Motovilov [24], and Simon [42], [43].

The concept of spectral averaging became an important tool in investigations of disordered systems, in particular, in connection with random Schrödinger and Jacobi operators in the early eighties. In 1983, Carmona [8] (see also [9]), apparently unaware of previous results by Javrjan and Birman and Solomyak, used spectral averaging over boundary condition parameters to prove the existence of an absolutely continuous (a.c.) component in random and deterministic Schrödinger operators (he also proved that the rest of the spectrum consists of eigenvalues dense in certain intervals with exponentially localized eigenfunctions in some random cases). Kotani also used this approach to link the existence of pure point spectrum and exponentially decaying eigenfunctions with the positivity of the Lyapunov exponent in 1984 [32] (published in 1986). Kotani's work inspired new proofs of exponential localization by Delyon, Lévy, and Souillard [18], [19], Simon and Wolff [44], Simon [41], Delyon, Simon and Souillard [20], and Kotani and Simon [33] for one- and quasi-one-dimensional as well as multi-dimensional Anderson models (the latter for large disorder or sufficiently high energy) and one-dimensional random Schrödinger operators. In all these references spectral averaging over coupling constants plays a crucial role. This is especially transparent in the paper by Simon and Wolff [44], which uses results by Aronszajn [2] and Donoghue [21] as their point of departure to study the variation of singular spectra under rank-one perturbations of self-adjoint operators. This is also discussed in Simon's review [42]. (For textbook presentations of spectral averaging in this context we refer to [10, Sect. VIII.2], [38, Sect. 13].) Subsequently, Gordon [26], [27] used spectral averaging in his studies of eigenvalues embedded in the essential spectrum. Spectral averaging has also been used to prove exponential localization for the one-dimensional Poisson model by Stolz [46]. A more general approach, involving two-parameter spectral averaging, has recently been employed to prove exponential localization in the Poisson and random displacement models in one dimension by Buschmann and Stolz [7]. The latter approach was again used by Sims and Stolz [45] in their discussion of exponential localization of the one-dimensional random displacement model and in a one-dimensional model of wave propagation in a random medium. Combes and Hislop [11] use averaging of spectral families to prove a Wegner-type estimate for a family of Anderson and Poisson-like multi-dimensional random Hamiltonians. Moreover, Combes, Hislop, and Mourre [12] in their discussion of perturbations of singular spectra and exponential localization for certain multi-dimensional random Schrödinger operators, and Combes, Hislop, Klopp, and Nakamura [13] in their study of the Wegner estimate and the integrated density of states, discuss spectral averaging in the spirit of Birman and Solomyak.

In Section 2 we collect basic facts on $SL_2(\mathbb{R})$, Möbius transformations, and the infinitely sheeted Riemann surface of the logarithm, as needed in the subsequent sections. Section 3, the principal section of this paper, then develops spectral averaging for spectral measures as well as for the associated absolutely continuous, singularly contionuous, and pure point parts (with respect to Lebesgue measure). Finally, Section 4 obtains a partial result concerning spectral averaging of the κ-continuous part (with respect to the κ-dimensional Hausdorff measure) of the singularly continuous part of measures.

2. Preliminaries on $SL_2(\mathbb{R})$ and on Möbius transformations

$SL_2(\mathbb{R})$ denotes the group of 2×2 real matrices with determinant equal to 1. By definition, its Lie algebra, $sl_2(\mathbb{R})$, consists of those matrices X such that $e^{tX} \in SL_2(\mathbb{R})$ for all $t \in \mathbb{R}$ (cf., e.g., [35, Ch. VI]). Therefore, $sl_2(\mathbb{R})$ consists of all 2×2 real matrices X with zero trace, $\operatorname{tr}(X) = 0$. The following three matrices then form a basis for $sl_2(\mathbb{R})$

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

and one verifies the following commutation relations

$$[X_2, X_1] = 2X_1, \quad [X_1, X_3] = X_2, \quad [X_3, X_2] = 2X_3. \tag{2.2}$$

If $X \in sl_2(\mathbb{R})$ then the map $t \mapsto e^{tX}$, $t \in \mathbb{R}$ is a one-parameter subgroup of $SL_2(\mathbb{R})$ and all one-parameter subgroups can be obtained in that way.

For future reference we recall the notion of automorphisms of the open complex upper half-plane $\mathbb{C}_+$, denoted by $\operatorname{Aut}(\mathbb{C}_+)$:

$$\operatorname{Aut}(\mathbb{C}_+) = \{g \colon \mathbb{C}_+ \to \mathbb{C}_+ \mid g \text{ is biholomorphic (i.e., a conformal self-map of } \mathbb{C}_+)\}. \tag{2.3}$$

$\operatorname{Aut}(\mathbb{C}_+)$ becomes a group with respect to compositions of maps. For simplicity, this group is denoted by the same symbol.

To fix the notational setup we now introduce the following hypothesis.

Hypothesis 2.1. *Given $\alpha, \beta, \gamma \in \mathbb{R}$, represent an element $X = X(\alpha, \beta, \gamma) \in sl_2(\mathbb{R})$ as*

$$X = \alpha X_1 + \beta X_2 + \gamma X_3 = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix} \tag{2.4}$$

and denote by

$$\begin{aligned} g_t(z) &= \frac{a_t z + b_t}{c_t z + d_t}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+, \\ g_0(z) &= z, \quad z \in \mathbb{C}_+ \end{aligned} \tag{2.5}$$

the corresponding one-parameter group of automorphisms of the open upper-half plane $\mathbb{C}_+$ such that

$$\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = e^{tX} \in SL_2(\mathbb{R}), \quad t \in \mathbb{R}. \tag{2.6}$$

We briefly recall a few facts in connection with Möbius (i.e., linear fractional) transformations (2.5). Let M be a Möbius transformation of the type

$$M(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C} \cup \{\infty\},\ a, b, c, d \in \mathbb{C},\ ad - bc \neq 0. \tag{2.7}$$

Then,
(i) M maps $\mathbb{R} \cup \{\infty\}$ onto itself if and only if M admits a representation where $a, b, c, d \in \mathbb{R}$ and $|ad - bc| = 1$.
(ii) M maps $\mathbb{C}_+$ onto itself if and only if M admits a representation where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
(iii) $\mathrm{Aut}(\mathbb{C}_+)$ is isomorphic to $SL_2(\mathbb{R})/\{I_2, -I_2\}$ (I_2 the identity matrix in $\mathbb{R}^2$).
(iv) Assuming $\det(M) = ad - bc = 1$ in (2.7), one uses $\mathrm{tr}(M) = (a + d)$ to classify M as
 elliptic, if $(a + d) \in \mathbb{R}$ and $|a + d| < 2$
 parabolic, if $(a + d) = \pm 2$
 hyperbolic, if $(a + d) \in \mathbb{R}$ and $|a + d| > 2$
 loxodromic, if $(a + d) \in \mathbb{C} \backslash \mathbb{R}$.

On the other hand, assuming $\left(\begin{smallmatrix} a_t & b_t \\ c_t & d_t \end{smallmatrix}\right) = e^{tX}$, $t \in \mathbb{R}$, with $\mathrm{tr}(X) = 0$ and $X = \left(\begin{smallmatrix} \beta & \alpha \\ \gamma & -\beta \end{smallmatrix}\right)$, one can use $\det(X) = -\alpha\gamma - \beta^2$ to classify the one-parameter subgroups of Möbius transformations in (2.5) and distinguish three cases:
 Case I: $\det(X) > 0$ (cyclic subgroup)
 Case II: $\det(X) = 0$
 Case III: $\det(X) < 0$ (hyperbolic subgroup).

Lemma 2.2. *Assume Hypothesis* 2.1 *and let* $(t, z) \in \mathbb{R} \times \mathbb{C}_+$.
(*i*) *If* $\det(X) > 0$, *then*

$$g_t(z) = \frac{\big(\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)\big)z + \frac{\alpha}{\omega}\sin(\omega t)}{\big(\frac{\gamma}{\omega}\sin(\omega t)\big)z + \cos(\omega t) - \frac{\beta}{\omega}\sin(\omega t)}, \tag{2.8}$$

where $\omega = \sqrt{\det(X)} > 0$.
(*ii*) *If* $\det(X) = 0$, *then*

$$g_t(z) = \frac{\big(1 + \beta t\big)z + \alpha t}{\gamma t z + (1 - \beta t)}. \tag{2.9}$$

(*iii*) *If* $\det(X) < 0$, *then*

$$g_t(z) = \frac{\big(\cosh(\omega t) + \frac{\beta}{\omega}\sinh(\omega t)\big)z + \frac{\alpha}{\omega}\sinh(\omega t)}{\big(\frac{\gamma}{\omega}\sinh(\omega t)\big)z + \cosh(\omega t) - \frac{\beta}{\omega}\sinh(\omega t)}, \tag{2.10}$$

where $\omega = \sqrt{|\det(X)|} > 0$.

Proof. Since $\mathrm{tr}(X) = 0$, every entry a_t, b_t, c_t, and d_t of the matrix e^{tX} in (2.6) is a solution of the initial value problem

$$\ddot{y} + \det(X)y = 0, \tag{2.11}$$

$$y(0) = 1, \quad \dot{y}(0) = \beta \quad \text{ for } y(t) = a_t, \tag{2.12}$$

$$y(0) = 0, \quad \dot{y}(0) = \alpha \quad \text{ for } y(t) = b_t, \tag{2.13}$$

$$y(0) = 0, \quad \dot{y}(0) = \gamma \quad \text{ for } y(t) = c_t, \tag{2.14}$$

$$y(0) = 1, \quad \dot{y}(0) = -\beta \quad \text{ for } y(t) = d_t, \tag{2.15}$$

where the dot $\cdot$ denotes d/dt. Solving the initial value problems (2.11), (2.12)–(2.15) proves (2.8)–(2.10). ∎

Remark 2.3. If $\gamma = 0$ in (2.4) the subgroup g_t is a group of linear transformations of $\mathbb{C}_+$. If $\gamma \in \mathbb{R}\backslash\{0\}$, the subgroup g_t corresponds to the case of linear fractional transformations of $\mathbb{C}_+$. If $\gamma \in \mathbb{R}\backslash\{0\}$, the automorphism $g_t(z)$ is a linear function in z if and only if $t \in (\pi/\omega)\mathbb{Z}$ in case I and $t = 0$ in cases II and III, respectively. In other words,

$$\gamma \in \mathbb{R}\backslash\{0\} \text{ if and only if } c_t \neq 0 \text{ for } \begin{cases} t \in \mathbb{R}\backslash\{(\pi/\omega)\mathbb{Z}\} & \text{in case I,} \\ t \in \mathbb{R}\backslash\{0\} & \text{in cases II, III.} \end{cases} \tag{2.16}$$

Moreover, suppose that $t \in \mathbb{R}\backslash\{(\pi/\omega)\mathbb{Z}\}$, that is, g_t is not the identity transformation ($g_t(z) \neq z$). Then case I consists of elliptic Möbius transformations. Case II always corresponds to parabolic Möbius transformations, and as long as $t \neq 0$, case III corresponds to hyperbolic Möbius transformations.

Remark 2.4. The case of self-adjoint rank-one perturbations tP of self-adjoint operators A discussed in Lemma 1.1, corresponds to the case $\det(X) = 0$ with $\alpha = \beta = 0$, $\gamma = 1$ as one readily verifies upon comparison with (1.11). Similarly, the case of self-adjoint extensions of a closed symmetric densely defined operator $\dot{A}$ with deficiency indices $(1,1)$ discussed in Lemma 1.3 corresponds to the case $\det(X) = 1$, $\omega = 1$ with $\alpha = -1$, $\beta = 0$, $\gamma = 1$ upon comparison with (1.16) and the change of parametrization $t \mapsto \tan(t)$ in (1.25).

Remark 2.5. The geometry of the trajectories $\bigcup_{t\in\mathbb{R}}\{g_t(z)\}$, $z \in \mathbb{C}_+$ of the one-parameter groups of automorphisms (2.8)–(2.10) can be understood in terms of the trajectories $\bigcup_{t\in\mathbb{R}}\{F_t(z)\}$, $z \in \mathbb{C}_+$, of the map F_t given by

$$F_t(z) = \frac{(1+\beta t)z + \alpha t}{\gamma t z + (1-\beta t)}, \quad (\alpha, \beta, \gamma) \in \mathbb{R}^3. \tag{2.17}$$

In fact, one has the following representations ($X = X(\alpha, \beta, \gamma)$, cf. (2.4))

$$g_t(z) = \begin{cases} F_{\tan(\sqrt{\det(X)}t)/\sqrt{\det(X)}}(z) & \text{if } \det(X) = -\alpha\gamma - \beta^2 > 0, \\ F_t(z) & \text{if } \det(X) = -\alpha\gamma - \beta^2 = 0, \\ F_{\tanh(\sqrt{|\det(X)|}t)/\sqrt{|\det(X)|}}(z) & \text{if } \det(X) = -\alpha\gamma - \beta^2 < 0. \end{cases} \tag{2.18}$$

Therefore, the trajectories of the groups (2.8)–(2.10) can be described by

$$\bigcup_{t\in\mathbb{R}}\{g_t(z)\} = \bigcup_{t\in\mathbb{R}}\{F_t(z)\} \quad \text{in cases I and II} \tag{2.19}$$

and

$$\bigcup_{t\in\mathbb{R}}\{g_t(z)\} = \bigcup_{|t|<|\det(X)|^{-1/2}}\{F_t(z)\} \subsetneqq \bigcup_{t\in\mathbb{R}}\{F_t(z)\} \quad \text{in case III.} \tag{2.20}$$

One observes that F_t is a one-parameter group of transformations of $\mathbb{C}_+$ with respect to t, that is, $F_{t+s} = F_t \circ F_s$ for all $s, t \in \mathbb{R}$, if and only if $\alpha\gamma + \beta^2 = 0$.

Next, we denote by $\log(\cdot)$ the branch of the logarithm on the cut plane $\Pi = \mathbb{C}\backslash[0,\infty)$ assuming

$$0 < \arg(\log(z)) < 2\pi \text{ for } z \in \Pi, \tag{2.21}$$

extending $\log(\cdot)$ to the upper rim, $\partial_+\Pi$, of Π by

$$\lim_{\varepsilon\downarrow 0}\log(x + i\varepsilon) \in \mathbb{R}, \quad x > 0 \tag{2.22}$$

and hence

$$\mathrm{Im}(\log(x)) = \pi, \quad x < 0. \tag{2.23}$$

Analytic continuation of the the branch $\log(\cdot)$ defined above then leads to the infinitely sheeted Riemann surface $\mathcal{R}$ of the logarithm with branch points of infinite order at zero and infinity. We denote the resulting analytic function on $\mathcal{R}$ by $\mathrm{Ln}(\cdot)$. For future reference we also introduce the nth sheet, $\mathcal{S}_n$, of $\mathcal{R}$. We use the convention $\mathcal{S}_0 = \Pi \cup \partial_+\Pi$. $\mathrm{Ln}\colon v \mapsto w = \mathrm{Ln}(v)$ then maps the interior, $\mathrm{int}(\mathcal{S}_n)$, of each sheet $\mathcal{S}_n$ biholomorphically onto the strip $2\pi n < \mathrm{Im}(z) < 2\pi(n+1)$ and

$$v \in \mathcal{S}_n \text{ if and only if } 2\pi n \leq \arg(w) < 2\pi(n+1), \quad n \in \mathbb{Z}. \tag{2.24}$$

Assuming Hypothesis 2.1 with $\gamma \neq 0$, we will in the following

$$\begin{aligned} &\text{denote by } \overleftrightarrow{c_t z + d_t} \text{ the lift of the trajectory } t \mapsto c_t z + d_t \text{ to } \mathcal{R}, \\ &\text{with } \overleftrightarrow{c_0 z + d_0} = d_0 = 1 \in \partial\mathcal{S}_0, \quad z \in \mathbb{C}_+. \end{aligned} \tag{2.25}$$

Lemma 2.6. *Assume Hypothesis* 2.1 *with* $\gamma \in \mathbb{R}\backslash\{0\}$ (*cf.* (2.16)), *let* $-\infty < t_1 < t_2 < \infty$, $z \in \mathbb{C}_+$, *and recall our convention* (2.25). *Then,*

$$\int_{t_1}^{t_2} dt\, \mathrm{Im}(g_t(z)) = \frac{1}{\gamma}\mathrm{Im}\big(\mathrm{Ln}(\overleftrightarrow{c_{t_2} z + d_{t_2}}) - \mathrm{Ln}(\overleftrightarrow{c_{t_1} z + d_{t_1}})\big). \tag{2.26}$$

Proof. Since the entries of the matrix (2.6) solve the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix}\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}, \quad t \in \mathbb{R}, \tag{2.27}$$

the following relations hold

$$\dot c_t = \gamma a_t - \beta c_t, \quad \dot d_t = \gamma b_t - \beta d_t, \tag{2.28}$$

implying

$$a_t = \frac{\dot c_t + \beta c_t}{\gamma} \quad \text{and} \quad b_t = \frac{\dot d_t + \beta d_t}{\gamma}. \tag{2.29}$$

Thus,

$$a_t z + b_t = \frac{\beta}{\gamma}(c_t z + d_t) + \frac{1}{\gamma}(\dot c_t z + \dot d_t), \tag{2.30}$$

and hence

$$g_t(z) = \frac{a_t z + b_t}{c_t z + d_t} = \frac{\beta}{\gamma} + \frac{1}{\gamma}\frac{\dot c_t z + \dot d_t}{c_t z + d_t} = \frac{\beta}{\gamma} + \frac{1}{\gamma}\frac{d}{dt}\mathrm{Ln}(\overleftrightarrow{c_t z + d_t}). \tag{2.31}$$

Integrating (2.31) from t_1 to t_2 and taking imaginary parts of the resulting expression proves (2.26). ☐

3. Dynamical systems on a space of measures

As shown below, each one-parameter subgroup $\{e^{tX}\}_{t\in\mathbb{R}}$ of $SL_2(\mathbb{R})$, or, what is the same, each one-parameter group $\{g_t\}_{t\in\mathbb{R}}$ of automorphisms of the open upper-half plane $\mathbb{C}_+$, generates a dynamical system $\{g_t^*\}_{t\in\mathbb{R}}$ on the (phase) space $\mathcal{M} = [0,\infty) \times \mathbb{R} \times \Omega$. Here Ω denotes the space of Borel measures μ on $\mathbb{R}$ with the property

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2} < \infty. \tag{3.1}$$

Let $\{g_t\}_{t\in\mathbb{R}}$ be a one-parameter subgroup of $\mathrm{Aut}(\mathbb{C}_+)$, the group of automorphisms of $\mathbb{C}_+$,

$$g_t(z) = \frac{a_t z + b_t}{c_t z + d_t}, \quad (t,z) \in \mathbb{R}\times\mathbb{C}_+. \tag{3.2}$$

Given a point $(A_0, B_0, \mu_0) \in \mathcal{M}$, introduce the Herglotz function

$$M_0(z) = A_0 z + B_0 + \int_{\mathbb{R}} d\mu_0(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \quad z \in \mathbb{C}_+, \tag{3.3}$$

where

$$\mu_0((\lambda_1,\lambda_2)) + \frac{1}{2}\mu_0(\{\lambda_1\}) + \frac{1}{2}\mu_0(\{\lambda_2\}) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1}^{\lambda_2} d\lambda\, \mathrm{Im}(M_0(\lambda+i\varepsilon)). \tag{3.4}$$

Since for each $t\in\mathbb{R}$, $g_t \in \mathrm{Aut}(\mathbb{C}_+)$, the one-parameter family of functions

$$M_t(z) = g_t(M_0(z)), \quad (t,z)\in\mathbb{R}\times\mathbb{C}_+ \tag{3.5}$$

is a one-parameter family of Herglotz functions. Therefore, M_t admits the representation

$$M_t(z) = A_t z + B_t + \int_{\mathbb{R}} d\mu_t(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \quad (t,z)\in\mathbb{R}\times\mathbb{C}_+ \tag{3.6}$$

for a unique triple $(A_t, B_t, \mu_t) \in \mathcal{M}$. Define the map

$$g_t^* : \mathcal{M} \to \mathcal{M}, \quad (A_0, B_0, \mu_0) \mapsto (A_t, B_t, \mu_t), \quad t\in\mathbb{R}. \tag{3.7}$$

Then,

$$g_{t+s}^* = g_t^* \circ g_s^*, \quad s,t\in\mathbb{R}. \tag{3.8}$$

That is, $\{g_t^*\}_{t\in\mathbb{R}}$ defines a dynamical system on $\mathcal{M}$ as claimed.

We note that

$$M_t(i) = A_t i + B_t + i\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} \tag{3.9}$$

and thus,

$$A_t = \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} - \mathrm{Im}(M_t(i)), \quad B_t = \mathrm{Re}(M_t(i)), \quad t\in\mathbb{R}. \tag{3.10}$$

Moreover, if $c_t \neq 0$ in (3.2), then $A_t = 0$, and hence

$$\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} = \mathrm{Im}(M_t(i)) \text{ if } c_t \neq 0. \tag{3.11}$$

For the remainder of this section it is convenient to introduce the following assumptions.

Hypothesis 3.1. *Assume Hypothesis* 2.1 *and*

$$\gamma \in \mathbb{R}\backslash\{0\}, \textit{ or equivalently, } c_t \neq 0 \textit{ for } \begin{cases} t\in\mathbb{R}\backslash\{(\pi/\omega)\mathbb{Z}\} & \textit{in case I,} \\ t\in\mathbb{R}\backslash\{0\} & \textit{in cases II, III.} \end{cases} \tag{3.12}$$

The following statement is a variant of the exponential Herglotz representation theorem due to Aronszajn-Donoghue [3] (see also [4]).

Lemma 3.2. *Assume Hypothesis* 3.1, *let* $(z,t) \in \mathbb{C}_+ \times \mathbb{R}$*, and recall our convention* (2.25)*. Given a Herglotz function* M_0 *with* $M_0(i) \neq 0$*, introduce the function*

$$N_t(z) = \mathrm{Ln}(\overleftrightarrow{c_t M_0(z) + d_t}). \tag{3.13}$$

Then $N_t(\cdot)$ *is analytic on* $\mathbb{C}_+$ *and the following representation holds*

$$N_t(z) = \mathrm{Re}(N_t(i)) + \int_{\mathbb{R}} d\lambda\, \xi_t(\lambda)\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg), \tag{3.14}$$

where

$$\xi_t(\lambda) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\big(\mathrm{Ln}(\overleftrightarrow{c_t M_0(\lambda + i\varepsilon) + d_t})\big) \ \textit{for a.e.}\lambda \in \mathbb{R}. \tag{3.15}$$

Proof. Since $M_0(i) \neq 0$, the expression $c_t M_0(z) + d_t$ never vanishes, and hence the lift $\overleftrightarrow{c_t M_0(z) + d_t}$ is well-defined as a point on $\mathcal{R}$. To set the stage, we assume that $\overleftrightarrow{c_t M_0(z) + d_t}$ is a point on the nth sheet $\mathcal{S}_n$ of $\mathcal{R}$ for some (and hence for all) $z \in \mathbb{C}_+$, that is,

$$2\pi n \leq \arg(\overleftrightarrow{c_t M_0(z) + d_t}) < 2\pi(n+1), \quad n \in \mathbb{Z}. \tag{3.16}$$

Then, by the definition of $\mathrm{Ln}(\cdot)$ on $\mathcal{R}$, one obtains

$$N_t(z) = \log(c_t M_0(z) + d_t) + 2\pi i n, \tag{3.17}$$

where $\log(\cdot)$ denotes the branch (2.21), (2.22) on $\mathcal{S}_0 = \Pi \cup \partial_+\Pi$.

Given $t \in \mathbb{R}$, there are three possible outcomes for N_t depending on whether $c_t > 0$, $c_t < 0$, and $c_t = 0$. If $c_t > 0$, the function $c_t M_0(z) + d_t$ is a Herglotz function and thus,

$$N_t(z) = \mathrm{Re}(N_t(i)) + \int_{\mathbb{R}} d\lambda\, \eta_t(\lambda)\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg) + 2\pi i n, \tag{3.18}$$

where

$$\eta_t(\lambda) = \frac{1}{\pi}\mathrm{Im}\big(\log(c_t m_0(\lambda) + d_t)\big) \text{ for a.e.}\lambda \in \mathbb{R} \tag{3.19}$$

and

$$m_0(\lambda) = \lim_{\varepsilon\downarrow 0} M_0(\lambda + i\varepsilon) \text{ for a.e.}\lambda \in \mathbb{R}. \tag{3.20}$$

Since

$$\frac{1}{\pi}\int_{\mathbb{R}} d\lambda\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) = i, \quad z \in \mathbb{C}_+, \tag{3.21}$$

one can rewrite (3.18) in the form (3.14) with

$$\begin{aligned} \xi_t(\lambda) &= \eta_t(\lambda) + 2n \\ &= \frac{1}{\pi}\mathrm{Im}\big(\log(c_t m_0(\lambda) + d_t) + 2\pi n i\big) \\ &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0} \mathrm{Im}\big(\mathrm{Ln}(\overleftrightarrow{c_t M_0(\lambda + i\varepsilon) + d_t})\big), \end{aligned} \tag{3.22}$$

proving (3.14), (3.15) in the case $c_t > 0$.

If $c_t < 0$ one obtains

$$\begin{aligned} N_t(z) &= \log(c_t M_0(z) + d_t) + 2\pi i n \\ &= \log(|c_t| M_0(z) - d_t) + 2\pi i n + \pi i. \end{aligned} \tag{3.23}$$

Using the Herglotz representation theorem for $|c_t|M_0(z)-d_t$ one arrives at

$$N_t(z)=\mathrm{Re}(N_t(i))+\int_{\mathbb{R}} d\lambda\,\eta_t(\lambda)\Big(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\Big)+2\pi i n+\pi i, \tag{3.24}$$

where

$$\eta_t(\lambda)=\frac{1}{\pi}\mathrm{Im}(\log(|c_t|m_0(\lambda)-d_t)) \text{ for a.e.}\lambda\in\mathbb{R} \tag{3.25}$$

and (3.20) results again. Thus, (3.14) holds with

$$\begin{aligned}\xi_t(\lambda)&=\eta_t(\lambda)+2n-1=\\ &=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\big(\log(c_tM_0(\lambda+i\varepsilon)+d_t)+2\pi n i\big)\\ &=\frac{1}{\pi}\mathrm{Im}\big(\mathrm{Ln}(\overleftrightarrow{c_tm_0(\lambda)+d_t})\big).\end{aligned} \tag{3.26}$$

Finally, if $c_t=0$, $N_t(z)$ is a constant with respect to z

$$\begin{aligned}N_t(z)&=\log(d_t)+2\pi i(n-1)\\ &=\log|d_t|+i(2\pi(n-1)+\arg(d_t)),\quad \mathrm{Im}(z)>0,\end{aligned} \tag{3.27}$$

which proves (3.14) with

$$\xi_t(\lambda)=2(n-1)+\pi^{-1}\arg(d_t)\in\mathbb{Z}, \tag{3.28}$$

a λ-independent integer constant. □

Now we can prove the absolute continuity of the measure associated with the Herglotz representation of the integrated (averaged) Herglotz function

$$M_{t_1,t_2}(z)=\int_{t_1}^{t_2} dt\,M_t(z),\quad \mathrm{Im}(z)>0,\quad t_1,t_2\in\mathbb{R},\ t_1<t_2. \tag{3.29}$$

Theorem 3.3. *Assume Hypothesis* 3.1*, let* $z\in\mathbb{C}_+$*,* $t_j\in\mathbb{R}$*,* $j=1,2$*,* $t_1<t_2$*, and recall our convention* (2.25)*. Then the integrated Herglotz function*

$$M_{t_1,t_2}(z)=\int_{t_1}^{t_2} dt\,M_t(z) \tag{3.30}$$

admits the Herglotz representation

$$M_{t_1,t_2}(z)=B_{t_1,t_2}+\int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\Big(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\Big). \tag{3.31}$$

Here $B_{t_1,t_2}\in\mathbb{R}$ *and the measure* μ_{t_1,t_2} *is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}$ *with Radon–Nikodym derivative* (*density*) *a bounded function* $\frac{d\mu_{t_1,t_2}}{d\lambda}=\xi_{t_1,t_2}\in L^\infty(\mathbb{R})$*. In fact,*

$$\xi_{t_1,t_2}(\lambda)=\frac{1}{\gamma}\big(\xi_{t_2}(\lambda)-\xi_{t_1}(\lambda)\big), \tag{3.32}$$

where

$$\xi_t(\lambda)=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}\big(\mathrm{Ln}(\overleftrightarrow{c_tM_0(\lambda+i\varepsilon)+d_t})\big),\quad t\in\mathbb{R}. \tag{3.33}$$

Proof. By Lemma 2.6, $\operatorname{Im}(M_{t_1,t_2})$ admits the representation

$$\begin{aligned}\operatorname{Im}(M_{t_1,t_2}(z)) &= \int_{t_1}^{t_2} dt\,\operatorname{Im}(g_t(M_t(z))) \\ &= \frac{1}{\gamma}\operatorname{Im}\big(\operatorname{Ln}(\overleftrightarrow{c_{t_2}M_0(z)+d_{t_2}}) - \operatorname{Ln}(\overleftrightarrow{c_{t_1}M_0(z)+d_{t_1}})\big). \end{aligned} \tag{3.34}$$

Hence, $\operatorname{Im}(M_{t_1,t_2})$ is uniformly bounded on $\mathbb{C}_+$, which proves that M_{t_1,t_2} has no linear term in its Herglotz representation. Moreover, by Fatou's theorem, the boundedness of $\operatorname{Im}(M_{t_1,t_2})$ on $\mathbb{C}_+$ ensures the absolute continuity of the measure μ_{t_1,t_2} in (3.31) with respect to Lebesgue measure on $\mathbb{R}$. Hence, (3.32) is a consequence of (3.34). ☐

Corollary 3.4. *Assume in addition to the hypotheses of Theorem* 3.3 *that* $\gamma > 0$*, and* $t_1 < 0 < t_2$*. If* $\det(X) > 0$*, assume in addition that*

$$-\frac{\pi}{2\sqrt{\det(X)}} < t_1 < 0 < t_2 < \frac{\pi}{2\sqrt{\det(X)}}. \tag{3.35}$$

Then the density (3.32) *has the form*

$$\xi_{t_1,t_2}(\lambda) = \frac{1}{\gamma} + \frac{1}{\gamma\pi}\operatorname{Im}\bigg(\log\bigg(\frac{\Theta(t_2)m_0(\lambda)+1}{-\Theta(t_1)m_0(\lambda)-1}\bigg)\bigg) \text{ for a.e. } \lambda\in\mathbb{R}, \tag{3.36}$$

where

$$m_0(\lambda) = \lim_{\varepsilon\downarrow 0}(\gamma M_0(\lambda+i\varepsilon)-\beta) \text{ for a.e.} \lambda\in\mathbb{R}, \tag{3.37}$$

and

$$\Theta(t) = \lim_{s\to\sqrt{\det(X)}}\frac{\tan(st)}{s} = \begin{cases} \frac{\tan(\sqrt{\det(X)}t)}{\sqrt{\det(X)}} & \text{if } \det(X)>0, \\ t & \text{if } \det(X)=0, \\ \frac{\tanh(\sqrt{|\det(X)|}t)}{\sqrt{|\det(X)|}} & \text{if } \det(X)<0, \end{cases} \quad t\in\mathbb{R}. \tag{3.38}$$

Remark 3.5. Define

$$T_1(X) = \begin{cases} -\frac{\pi}{2\sqrt{\det(X)}}, & \det(X)>0, \\ -\infty, & \det(X)\le 0, \end{cases}, \quad T_2(X) = \begin{cases} \frac{\pi}{2\sqrt{\det(X)}}, & \det(X)>0, \\ \infty, & \det(X)\le 0, \end{cases}, \tag{3.39}$$

then the density (3.32) has the form

$$\xi_{T_1(X),T_2(X)}(\lambda) = \frac{1}{\gamma} + \begin{cases} 0 & \text{if } \det(X)\ge 0, \\ \frac{1}{\gamma\pi}\operatorname{Im}\bigg(\log\bigg(\frac{\pi m_0(\lambda)+2\sqrt{|\det(X)|}}{\pi m_0(\lambda)-2\sqrt{|\det(X)|}}\bigg)\bigg) & \text{if } \det(X)<0. \end{cases} \tag{3.40}$$

Next, we discuss the following technical result.

Lemma 3.6. *Assume Hypothesis* 3.1*, let* $t_j\in\mathbb{R}\cup\{-\infty,\infty\}$*,* $j=1,2$*,* $t_1<t_2$*, and denote by* μ_{t_1,t_2} *the Borel measure in the Herglotz representation* (3.31) *of the integrated Herglotz function* (3.30)*. Then for any bounded Borel set* $\Delta\subset\mathbb{R}$ *the function* $t\mapsto\mu_t(\Delta)$ *is measurable and one has*

$$\int_{t_1}^{t_2} dt\,\mu_t(\Delta) = \mu_{t_1,t_2}(\Delta). \tag{3.41}$$

Proof. The proof is based on the following representation

$$\int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\, f(\lambda) = \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, f(\lambda), \tag{3.42}$$

which holds for a wide function class of f to be specified below. We split the proof into four steps. First, we establish (3.42) for functions f of the following type

$$f(\lambda) = \phi_\varepsilon(\lambda - \delta), \quad \varepsilon > 0, \quad \delta \in \mathbb{R}, \tag{3.43}$$

where $\phi_\varepsilon(\lambda)$ is an approximate identity,

$$\phi_\varepsilon(\lambda) = \varepsilon^{-1}\phi(\varepsilon^{-1}\lambda), \text{ with } \phi(\lambda) = \frac{1}{\pi}\frac{1}{1+\lambda^2}, \quad \lambda \in \mathbb{R}. \tag{3.44}$$

Second, we prove (3.42) for the functions f that can be represented as a convolution of ϕ_ε with a C_0^∞-function k,

$$f(\lambda) = (\phi_\varepsilon * k)(\lambda), \quad k \in C_0^\infty(\mathbb{R}), \quad \varepsilon > 0. \tag{3.45}$$

Third, we prove the validity of representation (3.42) for $f \in C_0^\infty(\mathbb{R})$. Finally, we establish (3.42) for characteristic functions of finite intervals, implying assertion (3.41).

Step I. Let $z = \delta + i\varepsilon \in \mathbb{C}_+$. By representation (3.6)

$$\operatorname{Im}\big(g_t(M_0(\delta + i\varepsilon))\big) = A_t\varepsilon + \varepsilon \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{(\lambda-\delta)^2 + \varepsilon^2}. \tag{3.46}$$

Since $\gamma \neq 0$, $A_t = 0$ for almost all $t \in \mathbb{R}$ by (2.16) and hence

$$\int_{t_1}^{t_2} dt \operatorname{Im}\big(g_t(M_0(\delta + i\varepsilon))\big) = \varepsilon \int_{t_1}^{t_2} dt \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{(\lambda-\delta)^2 + \varepsilon^2}. \tag{3.47}$$

On the other hand, by Theorem 3.3 one infers

$$\int_{t_1}^{t_2} dt \operatorname{Im}\big(g_t(M_0(\delta + i\varepsilon))\big) = \varepsilon \int_{\mathbb{R}} \frac{d\mu_{t_1,t_2}(\lambda)}{(\lambda-\delta)^2 + \varepsilon^2}. \tag{3.48}$$

Comparing (3.47) and (3.48) proves (3.42) for the functions of the type (3.43), (3.44).

Step II. Let $k \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp}(k) \subset [\delta_1, \delta_2]$ for some $-\infty < \delta_1 < \delta_2 < \infty$.

We start with two observations. Given $\varepsilon > 0$, the function $k_\varepsilon(\lambda, \delta) = \phi_\varepsilon(\lambda - \delta)k(\delta)$, $(\lambda, \delta) \in \mathbb{R} \times [\delta_1, \delta_2]$, is summable with respect to the product measures $d\mu_t \times d\delta$, $t \in [t_1, t_2]$ as well as with respect to the product measure $d\mu \times d\delta$, that is,

$$k_\varepsilon \in L^1(\mathbb{R} \times [\delta_1, \delta_2]; d\mu_t \times d\delta), \quad t \in [t_1, t_2] \tag{3.49}$$

and

$$k_\varepsilon \in L^1(\mathbb{R} \times [\delta_1, \delta_2]; d\mu \times d\delta). \tag{3.50}$$

Moreover, we claim that the function F_ε (cf. (3.45)),

$$F_\varepsilon(t, \delta) = \int_{\mathbb{R}} d\mu_t(\lambda)\, \phi_\varepsilon(\lambda - \delta) f(\delta), \quad t \in [t_1, t_2], \tag{3.51}$$

is summable on $[t_1, t_2] \times [\delta_1, \delta_2]$, that is,

$$F_\varepsilon \in L^1([t_1, t_2] \times [\delta_1, \delta_2]; dt \times d\delta). \tag{3.52}$$

In order to prove (3.52), one notes that by representation (3.6), (3.46) holds again. Thus, given $\varepsilon > 0$, the function h,

$$h(t,\delta) = A_t\varepsilon + \varepsilon \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{(\lambda-\delta)^2+\varepsilon^2}, \tag{3.53}$$

is continuous on $[t_1,t_2]\times[\delta_1,\delta_2]$. Hence $h(t,\delta)f(\delta)$ is also continuous on $[t_1,t_2]\times[\delta_1,\delta_2]$ and thus bounded. Since $A_t = 0$ a.e., $F_\varepsilon(t,\delta)$ is measurable and essentially bounded on $[t_1,t_2]\times[\delta_1,\delta_2]$. This proves (3.52).

Now the validity of (3.42) for the function class (3.45) follows from the following chain of equalities

$$\begin{aligned}
&\int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t\,(\lambda)\,(\phi_\varepsilon * k)(\lambda) \\
&= \int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda) \int_{\delta_1}^{\delta_2} d\delta\, \phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{since } \operatorname{supp}(k)\subset[\delta_1,\delta_2]) \\
&= \int_{t_1}^{t_2} dt \int_{\delta_1}^{\delta_2} d\delta \int_{\mathbb{R}} d\mu_t(\lambda)\, \phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by (3.49) using Fubini's theorem}) \\
&= \int_{\delta_1}^{\delta_2} d\delta \int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\, \phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by (3.52) using Fubini's theorem}) \\
&= \int_{\delta_1}^{\delta_2} d\delta \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, \phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by step I}) \\
&= \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda) \int_{\delta_1}^{\delta_2} d\delta\, \phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by (3.50) using Fubini's theorem}) \\
&= \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\,(\phi_\varepsilon * k)(\lambda) \quad (\text{since } \operatorname{supp}(k)\subset[\delta_1,\delta_2]).
\end{aligned} \tag{3.54}$$

Step III. Let $f\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(f)\subset[\delta_1,\delta_2]$. One infers

$$\lim_{\varepsilon\downarrow 0}(\phi_\varepsilon * f)(\lambda) = \lim_{\varepsilon\downarrow 0}\int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)f(\delta) = f(\lambda), \tag{3.55}$$

uniformly with respect to λ as long as λ varies in a compact set $\Lambda\subset\mathbb{R}$. With $\Lambda = [\delta_1-1,\delta_2+1]$, one obtains the estimate

$$\left|\int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)f(\delta)\right| \le \max_{\delta\in\operatorname{supp}(f)}|f(\delta)|(\delta_2-\delta_1)\frac{1}{\pi}\frac{\varepsilon}{\operatorname{dist}^2(\lambda,[\delta_1,\delta_2])}, \quad \lambda\in\mathbb{R}\backslash\Lambda. \tag{3.56}$$

Thus, there exists a constant $C = C(\delta_1,\delta_2)$ such that

$$|(\phi_\varepsilon * f)(\lambda)| = \left|\int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)f(\delta)\right| \le C\frac{\varepsilon}{1+\lambda^2}, \quad \lambda\in\mathbb{R}\backslash\Lambda. \tag{3.57}$$

Taking into account that

$$\sup_{t\in[t_1,t_2]}\int_{\mathbb{R}}\frac{d\mu_t(\lambda)}{1+\lambda^2} < \infty, \tag{3.58}$$

the uniform convergence (3.55) combined with the estimate (3.57) and the result of Step II proves (3.42) for $f\in C_0^\infty(\mathbb{R})$.

Step IV. Let Δ be a finite interval and $f_1(\lambda) \geq f_2(\lambda) \geq \dots$ a monotone sequence of non-negative functions, $f_n \in C_0^\infty(\mathbb{R})$, $n \in \mathbb{N}$ converging pointwise to the characteristic function of the interval Δ as n approaches infinity, that is

$$\lim_{n\to\infty} f_n(\lambda) = \chi_\Delta(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.59}$$

By the dominated convergence theorem one then obtains

$$\lim_{n\to\infty} \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, f_n(\lambda) = \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, \chi_\Delta(\lambda) = \mu_{t_1,t_2}(\Delta) \tag{3.60}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) = \int_{\mathbb{R}} d\mu_t(\lambda)\, \chi_\Delta(\lambda) = \mu_t(\Delta), \quad t \in [t_1, t_2]. \tag{3.61}$$

Since

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) \leq \int_{\mathbb{R}} d\mu_t(\lambda)\, f_1(\lambda) \\ &\leq \max_{s\in\operatorname{supp}(f_1)} \big((1+s^2) f_1(s)\big) \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} \\ &\leq \max_{s\in\operatorname{supp}(f_1)} \big((1+s^2) f_1(s)\big) \sup_{t\in\mathbb{R}} \bigg(\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} \bigg), \end{aligned} \tag{3.62}$$

one obtains

$$\lim_{n\to\infty} \int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) = \int_{t_1}^{t_2} dt \lim_{n\to\infty} \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) = \int_{t_1}^{t_2} dt\, \mu_t(\Delta), \tag{3.63}$$

using the dominated convergence theorem again. By Step III and by taking into account (3.60), this proves (3.42) for $f(\lambda) = \chi_\Delta(\lambda)$.

The extension from the case of bounded intervals Δ to the case of bounded Borel sets Δ is now straightforward, completing the proof. $\square$

Given a general Herglotz function M of the type

$$M(z) = Az + B + \int_{\mathbb{R}} d\mu(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \bigg), \quad A \geq 0,\ B \in \mathbb{R},\ z \in \mathbb{C}_+, \tag{3.64}$$

we next introduce the following subsets of $\mathbb{R}$,

$$\mathcal{A}(M) = \Big\{ \lambda \in \mathbb{R} \,\Big|\, \lim_{\varepsilon\downarrow 0} M(\lambda + i\varepsilon) \in \mathbb{C}_+ \Big\}, \tag{3.65}$$

$$\begin{aligned} \mathcal{P}(M) = &\Big\{ \lambda \in \mathbb{R} \Big| \lim_{\varepsilon\downarrow 0} \varepsilon \operatorname{Im}(M(\lambda + i\varepsilon)) \in (0,\infty) \Big\} \\ &\bigcup \Big\{ \lambda \in \mathbb{R} \Big| \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \operatorname{Im}(M(\lambda + i\varepsilon)) \in (0,\infty) \Big\}, \end{aligned} \tag{3.66}$$

$$\mathcal{S}(M) = \mathbb{R} \backslash \{\mathcal{A}(M) \cup \mathcal{P}(M)\}. \tag{3.67}$$

By results of Aronszajn [2], Donoghue [21], and Simon and Wolff [44], the subsets (3.65)–(3.67) are invariant with respect to the whole family $\{g \circ M\}_{g\in\operatorname{Aut}(\mathbb{C}_+)}$ of Herglotz functions, that is,

$$\mathcal{A}(M) = \mathcal{A}(g\circ M), \quad \mathcal{P}(M) = \mathcal{P}(g\circ M), \quad \mathcal{S}(M) = \mathcal{S}(g\circ M), \quad g \in \operatorname{Aut}(\mathbb{C}_+). \tag{3.68}$$

Strictly speaking, these results were obtained for Herglotz functions being the Stieltjes transforms of finite Borel measures. For the sake of completeness we prove this invariance in the case of general Herglotz functions. The invariance of the set $\mathcal{A}(M)$

is obvious from (2.8)–(2.10). The invariance of the set $\mathcal{S}(M)$ is then a corollary of the one of $\mathcal{P}(M)$. In order to prove the invariance of the set $\mathcal{P}(M)$ one needs some additional considerations.

We start by recalling the following well-known result.

Lemma 3.7 (see, e.g., [2], [3], [44]). *Let M be a Herglotz function with representation* (3.64). *Then, for any $\lambda_0 \in \mathbb{R}$,*

$$\lim_{\varepsilon\downarrow 0}(-i\varepsilon)M(\lambda_0+i\varepsilon)=\mu(\{\lambda_0\}), \tag{3.69}$$

in particular,

$$\lim_{\varepsilon\downarrow 0}\varepsilon \mathrm{Im}(M(\lambda_0+i\varepsilon))=\mu(\{\lambda_0\}) \tag{3.70}$$

and

$$\lim_{\varepsilon\downarrow 0}\varepsilon \mathrm{Re}(M(\lambda_0+i\varepsilon))=0. \tag{3.71}$$

Definition 3.8. A Herglotz function M of the type (3.64) is said to have a normal derivative at the point $\lambda \in \mathbb{R}$ if the following two limits exist (finitely).
(*i*) $M(\lambda)=\lim_{\varepsilon\downarrow 0} M(\lambda+i\varepsilon)\in\mathbb{C}$.
(*ii*) $M'(\lambda)=\lim_{\varepsilon\downarrow 0}(M(\lambda+i\varepsilon)-M(\lambda))/(i\varepsilon)\in\mathbb{C}$.

Lemma 3.9. *Let M be a Herglotz function with representation* (3.64). *Assume, in addition, that*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}(M(\lambda_0+i\varepsilon))\in(0,\infty) \tag{3.72}$$

for some point $\lambda_0\in\mathbb{R}$. Then M has a real normal boundary value at λ_0 and M has a strictly positive normal derivative at λ_0, that is,

$$M(\lambda_0)=\lim_{\varepsilon\downarrow 0}M(\lambda_0+i\varepsilon)\in\mathbb{R}, \tag{3.73}$$

$$M'(\lambda_0)=\lim_{\varepsilon\downarrow 0}\frac{M(\lambda_0+i\varepsilon)-M(\lambda_0)}{i\varepsilon}\in(0,\infty). \tag{3.74}$$

Proof. Let $\mathcal{I}$ be a finite open interval containing λ_0 and decompose M as $M = M_1+M_2$, where

$$M_1(z)=Az+B+\int_{\mathbb{R}\backslash\mathcal{I}}d\mu(\lambda)\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)-\int_{\mathcal{I}}d\mu(\lambda)\frac{\lambda}{1+\lambda^2}, \tag{3.75}$$

$$M_2(z)=\int_{\mathcal{I}}\frac{d\mu(\lambda)}{\lambda-z}. \tag{3.76}$$

Clearly,

$$M_1(\lambda_0)=\lim_{\varepsilon\downarrow 0}M_1(\lambda_0+i\varepsilon)\in\mathbb{R} \tag{3.77}$$

and

$$M_1'(\lambda_0)=\begin{cases}\lim_{\varepsilon\downarrow 0}\frac{M_1(\lambda_0+i\varepsilon)-M_1(\lambda_0)}{i\varepsilon}>0 & \text{if } A\neq 0 \text{ or } \mu(\mathbb{R}\backslash\mathcal{I})\neq 0,\\ 0 & \text{if } A=0 \text{ and } \mu(\mathbb{R}\backslash\mathcal{I})=0.\end{cases} \tag{3.78}$$

Hypothesis (3.72) and (3.76) imply

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}(M_1(\lambda_0+i\varepsilon))=0 \tag{3.79}$$

and hence

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}(M(\lambda_0+i\varepsilon)) = \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}(M_2(\lambda_0+i\varepsilon)) = \lim_{\varepsilon\downarrow 0}\int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0)^2+\varepsilon^2}$$
$$= \int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0)^2} \in [0,\infty), \tag{3.80}$$

using the monotone convergence theorem in the last step. Since $\mathcal{I}$ is a finite interval and $\int_{\mathcal{I}} d\mu(\lambda)(\lambda-\lambda_0)^{-2} < \infty$ by (3.72) and (3.80), applying the dominated convergence theorem yields

$$\lim_{\varepsilon\downarrow 0}\mathrm{Re}(M_2(\lambda_0+i\varepsilon)) = \lim_{\varepsilon\downarrow 0}\int_{\mathcal{I}} d\mu(\lambda)\frac{(\lambda-\lambda_0)}{(\lambda-\lambda_0)^2+\varepsilon^2} = \int_{\mathcal{I}}\frac{d\mu(\lambda)}{\lambda-\lambda_0} \in \mathbb{R}. \tag{3.81}$$

Thus,

$$M_2(\lambda_0) = \lim_{\varepsilon\downarrow 0} M_2(\lambda_0+i\varepsilon) \in \mathbb{R}, \tag{3.82}$$

and combining (3.77) and (3.82) then proves (3.73). Applying the dominated convergence theorem again yields

$$M_2'(\lambda_0) = \lim_{\varepsilon\downarrow 0}\frac{M_2(\lambda_0+i\varepsilon)-M_2(\lambda_0)}{i\varepsilon} = \lim_{\varepsilon\downarrow 0}\int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0-i\varepsilon)(\lambda-\lambda_0)}$$
$$= \begin{cases}\int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0)^2} > 0 & \text{if } \mu(\mathcal{I}) \neq 0,\\ 0 & \text{if } \mu(\mathcal{I}) = 0.\end{cases} \tag{3.83}$$

Taking into account that by hypothesis (3.72), either $A > 0$ or $\mu(\mathbb{R}) \neq 0$ in the Herglotz representation (3.64) of M, combining (3.78) and (3.83) proves (3.74). ☐

Lemma 3.10. *Let M be a Herglotz function of the type* (3.64). *Then*

$$\mathcal{P}(M) = \mathcal{P}(g\circ M), \quad g \in \mathrm{Aut}(\mathbb{C}_+). \tag{3.84}$$

Proof. It sufficies to prove the inclusion

$$\mathcal{P}(M) \subset \mathcal{P}(g\circ M), \quad g \in \mathrm{Aut}(\mathbb{C}_+). \tag{3.85}$$

Moreover, any automorphism $g \in \mathrm{Aut}(\mathbb{C}_+)$ admits the representation

$$g = g_1 \circ f \circ g_2, \tag{3.86}$$

where $g_j \in \mathrm{Aut}(\mathbb{C}_+)$, $j = 1, 2$ are linear transformations of the upper half-plane, and

$$f(z) = -\frac{1}{z}, \quad z \in \mathbb{C}_+. \tag{3.87}$$

Since $\mathcal{P}(M)$ is obviously invariant for linear transformations of $\mathbb{C}_+$, it suffices to establish the inclusion

$$\mathcal{P}(M) \subset \mathcal{P}(f\circ M). \tag{3.88}$$

Let $\lambda \in \mathcal{P}(M)$. By definition of $\mathcal{P}(M)$ either

$$\lim_{\varepsilon\downarrow 0}\varepsilon\mathrm{Im}(M(\lambda+i\varepsilon)) \in (0,\infty) \tag{3.89}$$

or

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}(M(\lambda+i\varepsilon)) \in (0,\infty). \tag{3.90}$$

If (3.89) holds, then

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}(f\circ M(\lambda+i\varepsilon))=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\frac{\mathrm{Im}(M(\lambda+i\varepsilon))}{|M(\lambda+i\varepsilon)|^2}\in(0,\infty), \tag{3.91}$$

using (3.69)–(3.71). Therefore, $\lambda\in\mathcal{P}(f\circ M)$. Next, assume that (3.90) holds. By Lemma 3.9 $M(\lambda)=\lim_{\varepsilon\downarrow 0}M(\lambda+i\varepsilon)\in\mathbb{R}$ and $M(z)$ has a positive normal derivative at the point λ. If $M(\lambda)\neq 0$, then

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}((f\circ M)(\lambda+i\varepsilon))=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathrm{Im}\bigg(\frac{1}{M(\lambda)}-\frac{1}{M(\lambda+i\varepsilon)}\bigg)=\frac{M'(\lambda)}{(M(\lambda))^2}>0. \tag{3.92}$$

If $M(\lambda)=0$, then

$$\lim_{\varepsilon\downarrow 0}\varepsilon\mathrm{Im}((f\circ M)(\lambda+i\varepsilon))=\lim_{\varepsilon\downarrow 0}\varepsilon\mathrm{Im}\bigg(-\frac{1}{M(\lambda+i\varepsilon)}\bigg)=\frac{1}{M'(\lambda)}>0, \tag{3.93}$$

that is, (3.90) implies $\lambda\in\mathcal{P}(f\circ M)$. Therefore, in both cases (3.89) and (3.90), $\lambda\in\mathcal{P}(f\circ M)$, which proves (3.88) and hence (3.84). □

The following result provides a spectral characterization of the invariant sets $\mathcal{A}(M)$, $\mathcal{S}(M)$, and $\mathcal{P}(M)$ (see [44] for a strategy of the proof). We recall that a measure μ on $\mathbb{R}$ is supported by the set $\mathcal{T}\subseteq\mathbb{R}$ if $\mu(\mathbb{R}\backslash\mathcal{T})=0$.

Lemma 3.11. *Let M be a Herglotz function of the type* (3.64), $g\in\mathrm{Aut}(\mathbb{C}_+)$, μ_g *the measure in the Herglotz representation of $g\circ M$, and*

$$\mu_g=\mu_g^{ac}+\mu_g^{sc}+\mu_g^{pp},\quad g\in\mathrm{Aut}(\mathbb{C}_+) \tag{3.94}$$

the Lebesgue decomposition of μ_g into its absolute continuous, singularly continuous, and pure point parts, respectively. Then μ_g^{ac}, μ_g^{sc}, and μ_g^{pp} are supported by $\mathcal{A}(M)$, $\mathcal{S}(M)$ and $\mathcal{P}(M)$, respectively.
Moreover, for any point $\lambda\in\mathcal{P}(M)$ there exists an automorphism $g\in\mathrm{Aut}(\mathbb{C}_+)$ such that

$$\mu_g^{pp}(\{\lambda\})>0. \tag{3.95}$$

Remark 3.12. Originally, the set $\mathcal{P}(M)$ has been introduced in the context of rank one perturbations in [44] by

$$\mathcal{P}(M)=\Big\{\lambda\in\mathbb{R}\,\Big|\,\mu(\{\lambda\})>0\Big\}\bigcup\Big\{\lambda\in\mathbb{R}\,\Big|\,\int_{\mathbb{R}}\frac{d\mu(s)}{(s-\lambda)^2}<\infty\Big\}. \tag{3.96}$$

Naively one might think that the set $\mathbb{R}\backslash\mathcal{A}(M)$ coincides (modulo Lebesgue null sets) with the complement of the support of the absolutely continuous component μ^{ac} of the measure μ associated with the Herglotz function M. Thus, one might erroneously conclude that

$$|\mathrm{supp}(\mu^{\mathrm{ac}})\cap(\mathbb{R}\backslash\mathcal{A}(M))|=0. \tag{3.97}$$

The following counterexample illustrates the situation.

Example 3.13. Let $K\subset[0,1]$ be a closed nowhere dense set of a positive Lebesgue measure and let

$$M(z)=\int_{\mathbb{R}}\frac{d\mu(\lambda)}{\lambda-z},\quad d\mu(\lambda)=\chi_{[0,1]\backslash K}(\lambda)\,d\lambda, \tag{3.98}$$

where χ_Λ denotes the characteristic set of a set $\Lambda \subset \mathbb{R}$. Then

$$\operatorname{supp}(\mu) = \operatorname{supp}(\mu^{\mathrm{ac}}) = [0,1], \tag{3.99}$$

but

$$|\operatorname{supp}(\mu^{\mathrm{ac}}) \cap (\mathbb{R}\backslash\mathcal{A}(M))| = |K| > 0. \tag{3.100}$$

Thus, (3.97) does not hold in general.

Combining the results of Corollary 3.4, Remark 3.5, and Lemma 3.6, we can now formulate the following spectral averaging theorems.

Theorem 3.14. *Assume Hypothesis* 3.1 *and let $t_j \in \mathbb{R}\cup\{-\infty,\infty\}$, $j=1,2$, $t_1 < t_2$. Suppose M_0 is a Herglotz function of the type* (3.64) *and M_t, $t\in\mathbb{R}$, is the one-parameter family of Herglotz functions given by* (3.5) *and* (3.6). *Denote by μ_t^{ac}, μ_t^{sc}, and μ_t^{pp} the absolutely continuous, singularly continuous, and pure point parts in the Lebesgue decomposition of μ_t in* (3.6),

$$\mu_t = \mu_t^{ac} + \mu_t^{sc} + \mu_t^{pp}, \quad t\in\mathbb{R}. \tag{3.101}$$

Then the following averaged measures

$$\int_{t_1}^{t_2} dt\, d\mu_t, \quad \int_{t_1}^{t_2} dt\, d\mu_t^{ac}, \quad \int_{t_1}^{t_2} dt\, d\mu_t^{sc}, \quad \int_{t_1}^{t_2} dt\, d\mu_t^{pp} \tag{3.102}$$

are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. More precisely, given a bounded Borel set $\Delta\subset\mathbb{R}$, the functions $t\mapsto\mu_t(\Delta)$, $t\mapsto\mu_t^{ac}(\Delta)$, $t\mapsto\mu_t^{sc}(\Delta)$, and $t\mapsto\mu_t^{pp}(\Delta)$ are measurable and

$$\int_{t_1}^{t_2} dt \begin{cases} \mu_t(\Delta) \\ \mu_t^{ac}(\Delta) \\ \mu_t^{sc}(\Delta) \\ \mu_t^{pp}(\Delta) \end{cases} = \begin{cases} \mu_{t_1,t_2}(\Delta) \\ \mu_{t_1,t_2}(\Delta\cap\mathcal{A}) \\ \mu_{t_1,t_2}(\Delta\cap\mathcal{S}) \\ \mu_{t_1,t_2}(\Delta\cap\mathcal{P}) \end{cases} \tag{3.103}$$

where μ_{t_1,t_2} is the absolutely continuous measure in the Herglotz representation (3.31) *of the integrated Herglotz function* (3.30) *in Theorem* 3.3 *and $\mathcal{A}(M_0)$, $\mathcal{P}(M_0)$, and $\mathcal{S}(M_0)$ are the invariant sets* (3.65)–(3.67) *associated with the Herglotz function M_0. In particular,*

$$|\{t\in\mathbb{R} \mid \mu_t^{sc}(\mathbb{R})\neq 0\}| = 0 \quad \text{if } |\mathcal{S}(M_0)| = 0, \tag{3.104}$$

$$|\{t\in\mathbb{R} \mid \mu_t^{pp}(\mathbb{R})\neq 0\}| = 0 \quad \text{if } |\mathcal{P}(M_0)| = 0. \tag{3.105}$$

Proof. Equation (3.41) implies the result (3.103) since

$$\begin{aligned} \mu_t(\Delta\cap\mathcal{A}(M_0)) &= \mu_t^{ac}(\Delta\cap\mathcal{A}(M_0)), \\ \mu_t(\Delta\cap\mathcal{S}(M_0)) &= \mu_t^{sc}(\Delta\cap\mathcal{S}(M_0)), \\ \mu_t(\Delta\cap\mathcal{P}(M_0)) &= \mu_t^{pp}(\Delta\cap\mathcal{P}(M_0)), \end{aligned} \tag{3.106}$$

and $\mathcal{A}(M_0)$, $\mathcal{S}(M_0)$, and $\mathcal{P}(M_0)$ are known to be Borel sets. ☐

Remark 3.15. The "life-time" $|\{t\in\mathbb{R} \mid \mu_t^{\mathrm{sing}}(\mathbb{R}\backslash\mathcal{A}(M_0))\neq 0\}|$ is never zero whenever $|\mathcal{S}(M_0)\cup\mathcal{P}(M_0)|\neq 0$. Here

$$\mu_t^{sing} = \mu_t^{sc} + \mu_t^{pp}. \tag{3.107}$$

As concrete examples show (cf. [15]), it may be finite or infinite depending upon the choice of the Herglotz function M_0.

Remark 3.16. Example 3.13 shows that the sets $\operatorname{supp}(\mu_t^{\mathrm{ac}})$ and $\mathbb{R}\backslash\mathcal{A}$ may have nontrivial intersection of positive Lebesgue measure and that

$$|\{t \in \mathbb{R} \mid \mu_t^{\mathrm{sing}}(\operatorname{supp}(\mu_t^{\mathrm{ac}})) \neq 0\}| \neq 0 \tag{3.108}$$

in general.

As a corollary of the previous theorem we get the following global result.

Theorem 3.17. *Assume the hypotheses of Theorem* 3.14 *and let* $\mathcal{A}(M_0)$, $\mathcal{S}(M_0)$, *and* $\mathcal{P}(M_0)$ *be the invariant sets associated with the Herglotz function* M_0. *Then for any bounded Borel set* $\Delta \subset \mathbb{R}$ *the following holds.*
(*i*) *If* $\det(X) > 0$ *then,*

$$|\gamma| \int_{-\pi/\left(2\sqrt{\det(X)}\right)}^{\pi/\left(2\sqrt{\det(X)}\right)} dt \begin{cases} \mu_t(\Delta) \\ \mu_t^{ac}(\Delta) \\ \mu_t^{sc}(\Delta) \\ \mu_t^{pp}(\Delta) \end{cases} = \begin{cases} |\Delta| \\ |\Delta \cap \mathcal{A}(M_0)| \\ |\Delta \cap \mathcal{S}(M_0)| \\ |\Delta \cap \mathcal{P}(M_0)| \end{cases} \tag{3.109}$$

(*ii*) *If* $\det(X) = 0$ *then,*

$$|\gamma| \int_{-\infty}^{\infty} dt \begin{cases} \mu_t(\Delta) \\ \mu_t^{ac}(\Delta) \\ \mu_t^{sc}(\Delta) \\ \mu_t^{pp}(\Delta) \end{cases} = \begin{cases} |\Delta| \\ |\Delta \cap \mathcal{A}(M_0)| \\ |\Delta \cap \mathcal{S}(M_0)| \\ |\Delta \cap \mathcal{P}(M_0)| \end{cases} \tag{3.110}$$

(*iii*) *If* $\det(X) < 0$ *then,*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t(\Delta) = \int_{\Delta} d\lambda\, \xi(\lambda), \tag{3.111}$$

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{ac}(\Delta) = \int_{\Delta\cap\mathcal{A}} d\lambda\, \xi(\lambda), \tag{3.112}$$

with

$$\xi(\lambda) = 1 + \frac{1}{\pi}\mathrm{Im}\bigg(\log\bigg(\frac{\pi m_0(\lambda) + 2\sqrt{|\det(X)|}}{\pi m_0(\lambda) - 2\sqrt{|\det(X)|}}\bigg)\bigg) \text{ for a.e. } \lambda \in \mathbb{R} \tag{3.113}$$

and m_0 *given by* (3.37). *Moreover,*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{sc}(\Delta) = |\Delta \cap \mathcal{S}(M_0) \cap \mathcal{R}(M_0)| \tag{3.114}$$

and

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{pp}(\Delta) = |\Delta \cap \mathcal{P}(M_0) \cap \mathcal{R}(M_0)|, \tag{3.115}$$

where

$$\mathcal{R}(M_0) = \Big\{\lambda \in \mathbb{R} \;\Big|\; \lim_{\varepsilon\downarrow 0} M_0(\lambda + i\varepsilon) \in \mathbb{R} \text{ and } |m_0(\lambda)| < (2/\pi)\sqrt{|\det(X)|}\Big\}. \tag{3.116}$$

Remark 3.18. Theorem 3.14 shows, in particular, the universality of the averaging on the whole parameter space in the case of cyclic groups g_t (associated with the one-parameter subgroups e^{tX} of $SL_2(\mathbb{R})$), where $\det(X) > 0$, and in their limiting cases corresponding to $\det(X) = 0$. However, the theorem also shows that averaging in the case of hyperbolic one-parameter subgroups g_t, with $\det(X) < 0$, depends on the initial Herglotz function M_0.

Remark 3.19. An analogous result concerning the decomposition of Lebesgue measure $|\cdot|$ restricted to $\mathcal{A}$ and $|\cdot|$ restricted to $\mathbb{R}\backslash\mathcal{A}$ into integrals of the measures μ_t^{ac} and $\mu_t^{sing} = \mu_t^{sc} + \mu_t^{sing}$ on the unit circle first appeared in [1]. In the case of self-adjoint rank-one perturbations of self-adjoint operators (which is a special case of (3.110) as observed in Remark 2.4), (3.110) appeared in [17].

4. Spectral averaging and Hausdorff measures

Lebesgue's decomposition of measures (3.101) is a particular case of a more general result in the theory of decomposing measures with respect to Hausdorff measures. This result states, in particular, that for each $\kappa \in [0,1]$, a Borel measure μ can be decomposed uniquely as

$$\mu = \mu^{\kappa\text{-}c} + \mu^{\kappa\text{-}s}, \tag{4.1}$$

where $\mu^{\kappa\text{-}c}$ is κ-continuous with respect to the κ-dimensional Hausdorff measure h^κ (i.e., $\mu^{\kappa\text{-}c}$ gives zero weight to sets with zero κ-dimensional Hausdorff measure h^κ) and $\mu^{\kappa\text{-}s}$ is κ-singular with respect to the κ-dimensional Hausdorff measure (i.e., $\mu^{\kappa\text{-}s}$ is supported on a set with of zero κ-dimensional Hausdorff measure h^κ). For more details on the decomposition (4.1) we refer to [36], [39], [40].

We recall that the κ-dimensional Hausdorff (outer) measure h^κ, $\kappa \in [0,1]$ of a set $S \subset \mathbb{R}$ is defined as

$$h^\kappa(S) = \lim_{\delta\downarrow 0} \inf_{\delta-\text{covers}} \sum_{n\in\mathbb{N}} |I_n(\delta)|^\kappa, \tag{4.2}$$

where the infimum is taken over countable collections of intervals $\{I_n(\delta)\}_{n\in\mathbb{N}}$, the δ-covers, such that

$$S \subset \bigcup_{n\in\mathbb{N}} I_n(\delta) \quad \text{ and } |I_n(\delta)| < \delta \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

We also recall that the Hausdorff dimension of a set S is defined by

$$\dim_H(S) = \inf\{\kappa \in [0,1] \mid h^\kappa(S) = 0\}. \tag{4.4}$$

The goal of this section is to obtain partial results concerning spectral averaging of the κ-continuous part $\mu_t^{\text{sc},\kappa\text{-}c}$ with respect to h^κ, $\kappa \in (0,1)$ of the singular continuous part μ_t^{sc} (with respect to Lebesgue measure) (3.101) of the measure μ_t associated with the family of Herglotz functions $M_t = g_t(M_0)$,

$$\mu_t^{sc} = \mu_t^{\text{sc},\kappa\text{-}c} + \mu_t^{\text{sc},\kappa\text{-}s}, \quad t \in \mathbb{R}, \tag{4.5}$$

where g_t is a one-parameter group of automorphisms of $\mathrm{Aut}(\mathbb{C}_+)$.

We introduce the following hypothesis.

Hypothesis 4.1. *Let M_0 be a Herglotz function of the type* (3.64)*, $\kappa \in (0,1)$,*

$$\mathcal{S}_\kappa(M_0) = \Big\{\lambda \in \mathbb{R} \;\Big|\; \liminf_{\varepsilon\downarrow 0} \varepsilon^{\kappa-1}\mathrm{Im}(M_0(\lambda + i\varepsilon)) \in (0,\infty)\Big\}, \tag{4.6}$$

and assume that the set $\mathcal{A}_\kappa(M_0)$, defined by

$$\mathcal{A}_\kappa(M_0) = \bigcup_{\kappa'\in[\kappa,1)} \mathcal{S}_{\kappa'}(M_0), \tag{4.7}$$

is a Borel set of positive Lebesgue measure.

We note that by Hypothesis 4.1,

$$\mathcal{A}_\kappa(M_0) \subseteq \mathcal{S}(M_0), \tag{4.8}$$

where $\mathcal{S}(M_0)$ is the invariant set (3.67) associated with the Herglotz function M_0.

Lemma 4.2. *Assume Hypothesis* 4.1 *and the hypotheses of Theorem* 3.14. *Let*

$$\mu_t^{sc} = \mu_t^{sc,\kappa\text{-}c} + \mu_t^{sc,\kappa\text{-}s}, \quad t \in \mathbb{R} \tag{4.9}$$

be the decomposition of the measure μ_t^{sc} (3.101) *such that* $\mu^{sc,\kappa\text{-}c}$ *is* κ*-continuous and* $\mu^{sc,\kappa\text{-}s}$ *is* κ*-singular (with respect to the* κ*-dimensional Hausdorff measure* h^κ*). Then, for any bounded Borel set* $\Delta \subset \mathbb{R}$ *and* $0 \neq |t| < \pi/\big(2\sqrt{\det(X)}\big)$ *in case I, and* $0 \neq t \in \mathbb{R}$ *in cases II and III,*

$$\mu_t^{sc}(\Delta \cap \mathcal{A}_\kappa(M_0)) = \mu_t^{sc,\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)). \tag{4.10}$$

Proof. We note that

$$0 < \liminf_{\varepsilon \downarrow 0} \varepsilon^{\kappa-1} \mathrm{Im}(M_0(\lambda + i\varepsilon)) \text{ (possibly equal to } +\infty) \text{ for } \lambda \in \mathcal{A}_\kappa(M_0). \tag{4.11}$$

Using the estimate

$$\mathrm{Im}(M_t(z)) = \frac{\mathrm{Im}(M_0(z))}{|c_t M_0(z) + d_t|^2} \leq \frac{1}{c_t^2 \mathrm{Im}(M_0(z))}, \quad z \in \mathbb{C}_+ \tag{4.12}$$

(we recall that $c_t \neq 0$ and $d_t \in \mathbb{R}$ by hypothesis), one infers

$$0 \leq \limsup_{\varepsilon \downarrow 0} \varepsilon^{1-\kappa} \mathrm{Im}(M_t(\lambda + i\varepsilon)) < \infty, \quad \lambda \in \mathcal{A}_\kappa(M_0). \tag{4.13}$$

It is known (cf. [16, Lemma 3.2]) that (4.13) implies

$$0 \leq \limsup_{\delta \downarrow 0} \frac{\mu_t(\lambda - \delta, \lambda + \delta)}{\delta^\kappa} < \infty, \quad \lambda \in \mathcal{A}_\kappa(M_0). \tag{4.14}$$

Hence, by a result of Rogers-Taylor [39], [40] (also see [16, Theorem 2.1]) the measure $\mu_t \restriction \mathcal{A}_\kappa(M_0)$ is a κ-continuous measure, which proves (4.10), since

$$\mu_t \restriction \mathcal{A}_\kappa(M_0) = \mu_t^{sc} \restriction \mathcal{A}_\kappa(M_0) \tag{4.15}$$

by (4.8). □

Remark 4.3. In general, we can neither state that $\mathcal{A}_\kappa(M_0)$ is a Borel set (cf. Hypothesis 4.1), nor that

$$\mu_t^{sc}(\Delta) = \mu_t^{\mathrm{sc},\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)), \quad t \neq 0. \tag{4.16}$$

It was pointed out to us by Barry Simon that a different but not unrelated discussion of singular continuous measures for continuous and discrete half-line Schrödinger operators, based on asymptotic behavior of solutions, was recently provided in [31] (following a previous result in [30]).

As a corollary we get the following result.

Corollary 4.4. *Assume the hypotheses of Lemma* 4.2. *Then for any bounded Borel set* $\Delta \subset \mathbb{R}$ *the following hold.*
(*i*) *If* $\det(X) > 0$ *then,*

$$|\gamma| \int_{-\pi/\left(2\sqrt{\det(X)}\right)}^{\pi/\left(2\sqrt{\det(X)}\right)} dt\, \mu_t^{sc,\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)) = |\Delta \cap \mathcal{A}_\kappa(M_0)|. \tag{4.17}$$

(ii) If $\det(X) = 0$ *then,*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{sc,\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)) = |\Delta \cap \mathcal{A}_\kappa(M_0)|. \tag{4.18}$$

(iii) If $\det(X) < 0$ *then,*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{sc,\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)) = |\Delta \cap \mathcal{A}_\kappa(M_0) \cap \mathcal{R}(M_0)|, \tag{4.19}$$

where

$$\begin{aligned} \mathcal{R}(M_0) = \Big\{\lambda \in \mathbb{R} \,\Big|\, & \lim_{\varepsilon \downarrow 0} M_0(\lambda + i\varepsilon) \in \mathbb{R} \text{ and} \\ & \big|\lim_{\varepsilon \downarrow 0}(\gamma M_0(\lambda + i\varepsilon) - \beta)\big| < (2/\pi)\sqrt{|\det(X)|}\Big\}. \end{aligned} \tag{4.20}$$

Even though Corollary 4.4 appears to be a new result, it cannot be considered a complete analog of (3.103), since first of all we have no results for the singular part $\mu_t^{\mathrm{sc},\kappa-\mathrm{s}}$, and secondly, we were not able to remove the set $\mathcal{A}_\kappa(M_0)$ on the left-hand sides of (4.17)–(4.19). We hope our present attempt will encourage future work in this direction.

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