Long-periodic orbits and invariant tori in a singularly perturbed Hamiltonian system

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Abstract. In this paper we study a singularly perturbed two-degrees-of-freedom Hamiltonian system with a normally elliptic slow manifold. We prove that the slow manifold persists but can have a large number ($\sim \varepsilon^{-1}$) of exponentially small ($\leq \mathrm{e}^{-c/\varepsilon}$) gaps. We demonstrate the existence of KAM tori in a neighborhood of the slow manifold. In addition we investigate a bifurcation which describes the creation of a gap in the slow manifold and derive its normal form.

1 Introduction

In a generic dynamical system persistent manifolds are normally hyperbolic [6]. The situation is essentially different in the Hamiltonian perturbation theory. For example, Kolmogorov-Arnold-Moser (KAM) tori are not normally hyperbolic, see e.g [1]. On the other hand, not all the tori survive under a small perturbation. Some of them disappear due to resonances between different degrees of freedom. In this paper we show that a similar situation arises in the study of a singularly perturbed Hamiltonian system.

We consider a two-degrees-of-freedom Hamiltonian system with an analytic Hamilton function $H(x,y,u,v;\varepsilon)$ and an ε-dependent symplectic form $\Omega = \varepsilon \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}v \wedge \mathrm{d}u$. The corresponding dynamics is defined by a system of 4 differential equations

$$\begin{aligned}\varepsilon\dot{x} &= \frac{\partial H}{\partial y}, & \dot{u} &= \frac{\partial H}{\partial v},\\ \varepsilon\dot{y} &= -\frac{\partial H}{\partial x}, & \dot{v} &= -\frac{\partial H}{\partial u}.\end{aligned} \tag{1}$$

As usually, this slow-fast system defines two limit (slow and fast) systems.

The slow system is obtained in the following way. First we let $\varepsilon = 0$ and solve the equations

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial x} = 0 \tag{2}$$

with respect to the fast variables (x, y):

$$x = f_0(u, v), \qquad y = g_0(u, v). \tag{3}$$

Throughout this paper we assume that f_0 and g_0 are single-valued functions in a domain $D_0 \subset \mathbf{R}^2$; (3) defines a two dimensional unperturbed slow manifold W^0_{slow}.

Then we exclude x and y from the second pair of the Hamiltonian system (1) and obtain the slow system. As is easily seen, it is Hamiltonian with a Hamilton function

$$H_{slow}(u, v) = H(f_0(u, v), g_0(u, v), u, v, 0). \tag{4}$$

We also assume, as in [5], the slow manifold to be normally elliptic. The normal ellipticity is defined in terms of the fast system:

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}, \tag{5}$$

where (u, v) variables are considered as parameters. It is easy to see that (3) provides equilibria for the fast system. We call W^0_{slow} normally elliptic if all these equilibria are elliptic for $(u, v) \in D_0$. Equivalently, we can require the matrix of second derivatives of H with respect to the variables (x, y) (the relative Hessian) to be positive definite on W^0_{slow}.

The slow system has one degree of freedom only and its dynamics can be completely understood. In a typical situation any invariant compact region contains a finite number of elliptic and hyperbolic equilibria and a finite number of domains filled with one-parameter families of periodic orbits bounded by separatrix trajectories.

The main problem of singular perturbation theory is to connect the dynamics of a system at the singular limit $\varepsilon = 0$, that is, of slow and fast systems, with the dynamics of the system for small positive ε. It is clear that the persistence (or non-persistence) of periodic orbits existing at $\varepsilon = 0$ on the slow manifold is one of the principal problems here. This is the main topic of this paper. Another theme is KAM dynamics near the slow manifold.

In [5] our main result (a similar result was proved in [3]) is that the system (1) with n slow degrees of freedom (u, v are n-dimensional vectors) has an almost invariant normally elliptic $2n$-dimensional symplectic submanifold W^ε_{slow} in an ε-neighborhood of W^0_{slow}. Its almost invariance means that the normal component of the restriction of the vector field to W^ε_{slow} is exponentially small in ε. Therefore, an orbit starting on W^ε_{slow} remains exponentially close to it after exponentially long time. Moreover, in a neighborhood of this submanifold the system (1) has an adiabatic invariant — an action variable of the fast system.

In the case $n = 2$ the fast-slow system is almost completely integrable in a neighborhood of W^0_{slow} due to the presence of the adiabatic invariant. This allows us to study the dynamics in this neighborhood by methods of the Hamiltonian perturbation theory.

In this paper we first prove that the words "almost invariant" can be replaced by just "invariant" for a large part of the slow submanifold. We prove that exponentially close to W^ε_{slow} the system (1) has families of true periodic orbits, which form a true invariant manifold W_ε. Nevertheless, this manifold is in fact destroyed, since it has exponentially small annular gaps with stripe-like structure. The number of gaps grows linearly in ε^{-1} but their width vanishes exponentially fast as $\varepsilon \to 0$. Thus, we can loosely speak about persistence of the slow manifold for $\varepsilon > 0$. We derive a normal form for the system near a gap in Section 5 and provide numerical pictures for the slow manifold in Section 6. We also discuss the KAM tori in Sections 3 and 4.

2 Invariant slow manifold

The persistence (or non-persistence) of the slow manifold depends on the dynamics of the slow system described by the Hamiltonian H_{slow}. This system has one degree of freedom, so it is natural to assume that it has an annulus $K_0 \subset D_0$ filled with periodic trajectories. Each trajectory belongs to a level $H_{slow} = E$ and its frequency is a function $\omega_0(E)$. We say that K_0 is filled with a non-degenerate family of periodic orbits if there are positive constants c_1 and c_2 such that

1) $|\omega_0(E)| > c_1$;

2) $|\omega_0'(E)| > c_2$.

Since ω_0 is defined on a connected set K_0, the inequalities mean that each of the functions $\omega_0(E)$ and $\omega_0'(E)$ is either strictly positive or strictly negative. All four combinations of signs are possible.

Our main analytical result can be formulated in the form of the following theorem. We assume henceforth that $0 < \varepsilon < \varepsilon_0$, where ε_0 is a sufficiently small positive constant.

Theorem 1 *Let the system* (1) *have a slow manifold* W^0_{slow} *defined by* (3) *and assume the Hamilton function* H *satisfies the following properties:*

[H1] *Analyticity:* $H(x, y, u, v, \varepsilon)$ *is analytic in all its variables in a neighborhood of* W^0_{slow}*;*

[H2] *Normal ellipticity:* $\det(\partial^2_{x,y} H) > c_3 > 0$ *at* $(x, y, u, v) \in W^0_{slow}$*;*

[H3] *the slow system has an annulus $K_0 \subset D_0$ filled with a non-degenerate family of periodic orbits.*

Then there is an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the system (1) has an invariant manifold W_ε filled with periodic orbits; W_ε is diffeomorphic $K_\varepsilon = K_0 \setminus R_\varepsilon$, where R_ε is a union of n_ε disjoint annuli; W_ε can be represented as a graph

$$\begin{aligned} x &= f(u,v,\varepsilon) = f_0(u,v) + O(\varepsilon), \\ y &= g(u,v,\varepsilon) = g_0(u,v) + O(\varepsilon), \qquad (u,v) \in K_\varepsilon. \end{aligned}$$

The Lebesgue measure of R_ε is exponentially small in ε, n_ε grows at most linearly in $1/\varepsilon$.

Proof. Canonical changes are those, which preserve the form of the Hamiltonian equations and the symplectic form Ω. In an ε-independent neighborhood of W^0_{slow} there is a canonical change of variables [5], which transforms the original Hamiltonian to the following form:

$$H = h(I, u, v; \epsilon) + r(x, y, u, v; \epsilon), \tag{6}$$

where $I = \frac{1}{2}(x^2 + y^2)$, $\partial_I h > 0$ for $I = 0$, and

$$r = O\left(e^{-\frac{c}{\epsilon}}\right). \tag{7}$$

The new Hamiltonian is analytic in (x, y, u, v) and C^∞-smooth in ε. The (u, v) component of the coordinate change is ε-close to identity.

Let us study the system (1) with the Hamiltonian function (6). It is convenient to rewrite H in the form

$$H = h(0, u, v; \epsilon) + I h_1(I, u, v; \epsilon) + r(x, y, u, v; \epsilon).$$

The equation $x = y = 0$ defines the almost invariant slow manifold W^ε_{slow}. We see that I is almost an integral of the system: $I' = O\left(e^{-\frac{c}{\epsilon}}\right)$. It means that in a neighborhood of W^ε_{slow} the system is exponentially close to an integrable system with a true invariant manifold.

The annulus K_0 is filled with a non-degenerate family of periodic trajectories of the slow system with the Hamiltonian $H_{slow}(u, v)$. The (u, v) component of the coordinate change is ε-close to identity. Consequently, $h(0, u, v, \varepsilon) = H_{slow}(u, v) + O(\varepsilon)$, so the new slow system has an annulus ε-close to K_0 which is filled with a nondegenerate family of periodic orbits. We denote this annulus with the same letter. We can make a canonical change of coordinates $(u, v) \mapsto (s, \rho)$ introducing the action-angle variables for the system with Hamiltonian $h(0, u, v, \varepsilon)$. Let us denote the new Hamiltonian $h_0(\rho, \varepsilon)$. Due to [H3] this function satisfy the following inequalities:

$$\partial_\rho h_0(\rho, \varepsilon) > \tilde{c}_1, \qquad |\partial^2_\rho h_0(\rho, \varepsilon)| > \tilde{c}_2 \qquad \rho \in (\rho_1, \rho_2).$$

It is clear, that the assumption of positivity of $\partial_\rho h_0(\rho, \varepsilon)$ does not restrict the generality, since if this derivative is negative, one can change $t \mapsto -t$ in order to make it positive.

Applying the transformation to action-angle variables (being identical in (x, y)) to the full system we come to the Hamiltonian defined in a region $D^2 \times K$, $K = \mathbf{T}^1 \times [\rho_1, \rho_2]$. The new Hamiltonian is $2\pi-$periodic in the angle variable s. It is convenient to use the same letters for functions after the change of variables, since later on we use only this form of the Hamiltonian. In the new variables the Hamiltonian takes the form:

$$H = h_0(\rho, \varepsilon) + I h_1(I, s, \rho, \varepsilon) + r(x, y, s, \rho; \varepsilon) \tag{8}$$

and the system (1) is transformed to the following

$$\begin{aligned} \varepsilon\dot{x} &= (h_1 + I\partial_I h_1)\, y + \partial_y r, & \dot{s} &= \partial_\rho h_0 + I\partial_\rho h_1 + \partial_\rho r, \\ \varepsilon\dot{y} &= -(h_1 + I\partial_I h_1)\, x - \partial_x r, & \dot{\rho} &= -I\partial_s h_1 - \partial_s r. \end{aligned} \tag{9}$$

The last system has an important property: the section $\Pi_0 = \{\, s = 0 \,\}$ is transversal to the flow provided $|I| < \delta$ for some $\delta > 0$. We choose Π_0 to be our Poincaré section, and study the properties of the corresponding first-return map.

It is not too difficult to see that a trajectory, which starts in $\Pi_0 \cap \{\, |I| < \delta \,\}$, returns to the section with $|I| \le \delta + \varepsilon$ after completing one rotation in s variable.

The section Π_0 is 3-dimensional disk, it is foliated by levels $H = E$. A position of a point on the section is uniquely defined by its coordinates (x, y) and E, due to Implicit Function Theorem. Indeed, the equation $H(x, y, s, \rho, \varepsilon) = E$ with the Hamiltonian function (8) has a solution

$$\rho = U(x, y, s, ; \varepsilon, E), \tag{10}$$

since we assumed $\partial_\rho h_0 \neq 0$ in the annulus K. Observe that the function U has a representation

$$U(x, y, s, \varepsilon, E) = U_0(I, s, \varepsilon, E) + U_1(x, y, s, \varepsilon, E), \tag{11}$$

where U_0 is a solution to the equation

$$h_0(U, \varepsilon) + I h_1(I, s, U, \varepsilon) = E,$$

and $U_1 := U - U_0$. Similar to r, the function U_1 has an exponential upper bound

$$U_1 = O\left(e^{-\frac{c}{\varepsilon}}\right).$$

Differentiating with respect to I, we obtain

$$\partial_I U_0 = -\frac{h_1(I, s, U_0, \varepsilon) + I\partial_I h_1}{\partial_\rho h_0 + I\partial_\rho h_1}.$$

This implies

$$\partial_I U_0 \neq 0 \tag{12}$$

for I small enough.

Thus, invariant 3-dimensional submanifold $H = E$ is represented as the graph of the function U.

It is convenient to use (x, y, E) as local coordinates on Π_0. Since E is preserved by the flow it can be considered as a parameter. So we obtain a family $F_{E,\varepsilon}$ of area preserving Poincaré maps parameterized by E and ε.

Lemma 2 *The Poincaré map $F_{E,\varepsilon} : \underline{x} \mapsto \underline{x}_1$ is defined in a small ε-independent neighborhood of the origin. This area preserving map has the following form:*

$$\underline{x}_1 = R(2\pi\varepsilon^{-1}A_0(I,E,\varepsilon))\,\underline{x} + P(\underline{x},E,\varepsilon),$$

where $\underline{x} = (x,y)^T$, $I = \frac{1}{2}(x^2+y^2)$; $R(\alpha)$ is the matrix of rotation by an angle α,

$$|A_0(0,E,\varepsilon)| > c > 0, \qquad |\partial_E A_0(0,E,\varepsilon)| > c > 0,$$

and $P(\underline{x},E,\varepsilon) = O(\mathrm{e}^{-c/\varepsilon})$.

Proof of the lemma. The main contribution to the map $F_{E,\varepsilon}$ comes from the integrable part of the Hamiltonian which corresponds to the Hamiltonian (8) with r removed. Let us write down the related integrable map explicitly.

Let us use the angle variable s as the new time. In this way the restriction of system (9) on the submanifold $H = E$ can be rewritten as a singularly perturbed non-autonomous one-degree-of-freedom 2π-periodic in s system:

$$\varepsilon\frac{dx}{ds} = \frac{(h_1 + I\partial_I h_1)y + \varepsilon\partial_y r}{\partial_\rho h_0 + I\partial_\rho h_1 + \varepsilon\partial_\rho r} = -\frac{\partial U}{\partial y}, \quad \varepsilon\frac{dy}{ds} = -\frac{(h_1 + I\partial_I h_1)x + \varepsilon\partial_x r}{\partial_\rho h_0 + I\partial_\rho h_1 + \varepsilon\partial_\rho r} = \frac{\partial U}{\partial x}. \qquad (13)$$

The integrable part of the vector field (13) corresponds to $U_1 \equiv 0$ (or, equivalently, $r \equiv 0$). Let us consider the system (13) with U replaced by U_0. We introduce a complex variable $z = x + iy$ and rewrite the system in the complex form

$$\varepsilon\dot{z} = iz\partial_I U_0(I,s,\varepsilon,E) = izA(I,s,\varepsilon,E). \qquad (14)$$

Since I remains constant along a trajectory of this equation, it can be integrated explicitly:

$$z(s) = z_0 \exp\left(\frac{\mathrm{i}}{\varepsilon}\int_0^s \partial_I U_0(I,\tau,\varepsilon,E)\,\mathrm{d}\tau\right). \qquad (15)$$

The integrable part of the Poincaré map $F^0_{\varepsilon,E}$ is obtained if we set $s = 2\pi$:

$$z_0 \mapsto \mathrm{e}^{2\pi\mathrm{i}A_0(I,E,\varepsilon)/\varepsilon} z_0, \qquad (16)$$

where A_0 is the average value of the function $\partial_I U_0$ over a period in the variable s. As it should have been expected, $F^0_{\varepsilon,E}$ is a rotation on the angle, which depends on the amplitude, since $I = |z_0|^2/2$. Additionally the rotation angle depends on E and ε.

The origin is a fixed point of $F^0_{\varepsilon,E}$ with the multiplicator $\lambda_{\varepsilon,E}$,

$$|\lambda_{\varepsilon,E}| = 1, \qquad \arg\lambda_{\varepsilon,E} = 2\pi A_0(0,E,\varepsilon)\varepsilon^{-1}. \qquad (17)$$

The inequality (12) implies $A_0(0,E,\varepsilon) > c_0 > 0$. As ε goes to zero $\lambda_{\varepsilon,E}$ rapidly rotates on the unit circle going countably many times through the value 1. Outside of a small neighborhood of $\lambda_{\varepsilon,E} = 1$ we have to expect persistence of the fixed point under a small perturbation due to Implicit Function Theorem.

Returning to the real variables $\underline{x} = (x, y)^T$ we come to the following form of the map

$$\underline{x}_1 = R(2\pi A_0/\varepsilon)\underline{x}, \tag{18}$$

where $R(\alpha)$ is the matrix of rotation by an angle α.

This form of the integrable part of the map implies that the true Poincaré map has the form

$$\underline{x}_1 = R(2\pi A_0/\varepsilon)\underline{x} + P(\underline{x}, E, \varepsilon), \tag{19}$$

with P analytic in X, E and being exponentially small in ε due to exponential smallness of the perturbation r. $\square$

Now we can complete the proof of Theorem 1. The slow manifold W_ε is to be filled by periodic trajectories, so $W_\varepsilon \cap \Pi_0$ should consists of fixed points of the Poincaré map.

According to Lemma 2, the fixed point equation $\underline{x} = F_{E,\varepsilon}(\underline{x})$ can be rewritten in the form

$$\underline{x} = (E - R(2\pi A_0/\varepsilon))^{-1} P(\underline{x}, E, \varepsilon), \tag{20}$$

Denote the right hand side in (20) as $G(\underline{x})$, for brevity.

We solve this equation by contraction map principle. First we choose $\delta > 0$ and consider a disk $D_\delta = \{\, |I| < \delta \,\}$ on the $\underline{x}$-plane.

Now we need a bound of the inverse matrix in the right hand side of (20). Direct calculation gives

$$\|(E - R(2\pi A_0/\varepsilon))^{-1}\| = \frac{1}{2|\sin(\pi A_0/\varepsilon)|}.$$

Thus, we get

$$\|G(\underline{x})\| \le \frac{C_0 \mathrm{e}^{-c/\varepsilon}}{2|\sin(\pi A_0/\varepsilon)|},$$

where we used the upper bound for P from Lemma 2. By Cauchy estimates we obtain:

$$\|\partial_{\underline{x}} G(\underline{x})\| \le \frac{C_1 \mathrm{e}^{-c/\varepsilon}}{2|\sin(\pi A_0/\varepsilon)|}.$$

If $\|G(\underline{x})\| < \delta$ and $\|\partial_{\underline{x}} G(\underline{x})\| < 1/2$, the map G is obviously a contraction of the disk D_δ and has a unique fixed point inside D_δ. These two conditions are fulfilled provided

$$|\sin(\pi A_0/\varepsilon)| > \mathrm{e}^{-c/\varepsilon} \max\left\{ C_0\delta^{-1},\ C_1 \right\}.$$

We can choose $\delta = \mathrm{e}^{-c/2\varepsilon}$. Then the Poincaré map $F_{E,\varepsilon}$ has a unique fixed point in the disk D_δ provided

$$|\sin(\pi A_0(0, E, \varepsilon)/\varepsilon)| > C_2 \mathrm{e}^{-c/2\varepsilon}.$$

Since $\partial_E A_0(0, E, \varepsilon)$ does not vanish, the last inequality is fulfilled at all points of the interval (E_1, E_2) except exponentially small neighborhoods of the points, where $A_0(0, E, \varepsilon) = n\varepsilon$ with an integer n. This defines the strip-like structure of the slow manifold.

The other statements of the theorem follow by Implicit Function Theorem. In order to reconstruct the whole disk it is necessary to consider the family of section $\Pi_\alpha = \{s = \alpha\}$. On each of these section we have similar estimates for the Poincaré map, which allows us to show that W_ε is actually a graph of a function, which is exponentially close to W^ε_{slow}. After coming back to the original coordinates we obtain the invariant manifold W_ε, which is ε-close to W^0_{slow}

Thus, the persistence theorem is proved. □

3 Invariant tori

In this section we discuss the existence of invariant tori filled with quasi-periodic orbits. First we study the region outside of an exponentially thin neighborhood of the almost invariant elliptic manifold. We show that results of [1] (see a review in [2], chapter 5, Sec. 5.4) can be applied to prove the existence of invariant tori carrying quasiperiodic orbits in any level $H = E$ where the representation (11) is valid and I is small enough (but finite!).

To this end, let us return to the non-autonomous system (13) and introduce a new independent variable $\tau = s/\varepsilon$. Then we get the system that can be considered as that with slow varying parameter $s = \varepsilon\tau$. Transform the system to action-angle variables $(x, y) \mapsto (I, \theta)$, $x = \sqrt{2I}\cos\theta,\ y = \sqrt{2I}\sin\theta$. In these variables the system reads

$$\dot{I} = -\frac{\partial \hat{U}_1}{\partial \theta}, \qquad \dot{\theta} = \frac{\partial U_0}{\partial I} + \frac{\partial \hat{U}_1}{\partial I}, \tag{21}$$

where $\hat{U}_1 = U_1(\sqrt{2I}\cos\theta, \sqrt{2I}\sin\theta, E, \varepsilon)$. Both derivatives in the right hand sides of (21) are analytic in (I, θ) and bounded outside of some neighborhood of $I = 0$ since they depend analytically on $\sqrt{I}$, and $I^{-1/2}$ enters as a factor in front of $\partial\hat{U}_1/\partial I$. The derivatives of $\hat{U}_1$ are exponentially small, therefore one can take exponentially thin in ε neighborhood of $I = 0$, $I > C_2\exp[-c_1/\varepsilon]$, $c_1 \le c$, with some positive constant C_2, such that for $C_2\exp[-c_1/\varepsilon] < I < I_0$ functions in the right hand sides of (21) are analytic in (I, θ), periodic in θ and uniformly bounded. The range of E is defined by an interval where the annulus K_0 was given, say, if $E_0 < E < E_1$, then for $E^0 > E_0$, $E^1 < E_1$, $E^0 < E^1$, there are ε_1 and δ_1 such that for $0 < \varepsilon < \varepsilon_1$, $I < \delta_1$ all our considerations hold, in particular, to the system (21) the following theorem from [1] is applicable.

Recall that $A_0(I, E, \varepsilon)$ is the mean value over the period of $\partial U_0/\partial I$ and $A_0(I, E, \varepsilon) \ne 0$ in accordance with (12). This inequality implies

Theorem 3 *Let for system (21) with one degree of freedom $A_0(I, E, \varepsilon) \ne 0$. Then there are positive constants ε_1, δ_1 and E^0, E^1 such that for all $0 < \varepsilon < \varepsilon_1$, $\exp[-c_1/\varepsilon] < I < I_0$, $E^0 < E < E^1$ the action variable I is the eternal adiabatic invariant. The greater part of the the phase space $D^2 \times T^1$ is filled with invariant tori being close to tori $I = const$. These tori are filled with quasi-periodic solutions with Diophantine rotation numbers.*

Now let us show that for majority of periodic orbits on W_ε found in Sec. 2 there is a neighborhood of a periodic orbit such that the region of existence of invariant tori can be drawn till this periodic orbit, that is, for any small enough neighborhood of the periodic orbit there are invariant tori in this neighborhood.

We take a periodic orbit from W_ε such that its multiplicator $\exp[2\pi i\lambda]$ satisfies the condition of absence of strong resonances, i.e., $\lambda \neq 1, 1/2, 1/3, 1/4$. Let us observe that, in accordance with (18), the exponent λ of such the orbit has the following form

$$\lambda = \lambda_0 + O(\exp[-c/\varepsilon]), \quad \lambda_0 = A_0(0, E, \varepsilon)\varepsilon^{-1}.$$

To prove the existence of invariant tori we investigate the Poincaré map (19) near the related fixed point and apply KAM theorem [8] to this point. Invariant tori do exist if some Birkhoff coefficient of the related normal form of the map at this point does not vanish.

For our case the problem is facilitated by the exponential closeness of the map to the integrable one being just in the Birkhoff form. Since the position of the fixed point corresponding to the periodic orbit under consideration is of $O(\exp[-c/\varepsilon])$, then we see, shifting the origin to this point, that any given Birkhoff coefficient B_n is exponentially close to the same for the integrable approximation, i.e.,

$$B_n = \frac{1}{n!}\partial^n A_0/\partial I^n|_{I=0}.$$

Thus we get

Proposition 4 *Suppose for a given value of E the related periodic orbit exists and its multiplicator $\exp[2\pi i\lambda]$ satisfies the inequalities $\lambda \neq 1, 1/2, 1/3, 1/4$, and*

$$\left.\frac{\partial A_0}{\partial I}\right|_{I=0} \neq 0.$$

Then there is a neighborhood of this periodic orbit in the level $H = E$ such that almost full measure points of this neighborhood belong to invariant KAM tori, filled with quasiperiodic orbits.

Remark. The size of a neighborhood where Proposition works can depend on E and shrinks to zero where E approaches to the value where the related periodic orbit is at the strong resonance. But it is easy to verify that for any ε one can take exponentially small in ε neighborhoods of strong resonances such that the size of the neighborhood where KAM theorem works would be uniformly bounded and finite. For these E neighborhoods found in Theorem 3 and Proposition 4 will overlap themselves covering a whole neighborhood of the periodic orbit in the level $H = E$.

4 In a neighborhood of an elliptic point

In this section we study periodic and quasi-periodic orbits of system (1) in a neighborhood of its singular point that exists for $\varepsilon > 0$ near a center singular point on slow manifold. Let us assume that slow system has a singular point (u_0, v_0) of the center type. It means first that differential of the Hamiltonian $h_0(u,v) = h(0,u,v,0) = H(f(u,v), g(u,v), u, v, 0)$ is equal to zero at (u_0, v_0) which we assume, without loss of generality, to be $(0,0)$. The point will be a center, if at $(0,0)$

$$\det \begin{pmatrix} \dfrac{\partial^2 h_0}{\partial u^2} & \dfrac{\partial^2 h_0}{\partial u \partial v} \\ \\ \dfrac{\partial^2 h_0}{\partial u \partial v} & \dfrac{\partial^2 h_0}{\partial v^2} \end{pmatrix} > 0. \tag{22}$$

For $(u,v) = (0,0)$ fast system (5) has a center at $(x,y) = (f_0(0,0), g_0(0,0))$, due to the assumption [H2] of Theorem 1. In this Section we will study the system (1) introducing fast time $\tau = t/\varepsilon$. After this one gets the system in the form

$$\begin{aligned} x' &= \frac{\partial H}{\partial y}, & u' &= \varepsilon \frac{\partial H}{\partial v}, \\ y' &= -\frac{\partial H}{\partial x}, & v' &= -\varepsilon \frac{\partial H}{\partial u}, \end{aligned} \tag{23}$$

where prime means differentiation with respect to fast time τ. This system is Hamiltonian with respect to singular symplectic form

$$\Omega_1 = \mathrm{d}y \wedge \mathrm{d}x + \varepsilon \mathrm{d}v \wedge \mathrm{d}u. \tag{24}$$

For this system the following evident fact is valid

Proposition 5 *For $\varepsilon > 0$ small enough system (23) possesses a singular point O of elliptic type being close to $(f_0(0,0), g_0(0,0), 0, 0)$. Its eigenvalues are two imaginary pairs $\pm i\Omega$ and $\pm \varepsilon i \omega$. The former pair is close to the eigenvalues of the center of fast system for $(u,v) = (0,0)$, the pair $\pm i\omega$ is close to eigenvalues of the center of the slow system at $(u,v) = (0,0)$.*

Proof. The conclusions of the proposition follow directly from the Implicit Function Theorem due to assumptions about matrices of second derivatives in (x,y) and (u,v) (see, (22), and [H2]). □

Later on, without loss of generality, we suppose the singular point, found in Proposition, to locate at $(0,0,0,0)$. It is easily seen that linearization matrix of the system at $(0,0,0,0)$ has two symplectic subspaces corresponding eigenvalues $\pm i\Omega$ and $\pm \varepsilon i\omega$, respectively. The planes are ε-close to two dimensional invariant plane of linearized system for (23) at the singular point O: the plane corresponding to eigenvalues $i\Omega(0)$, and a tangent plane to the slow manifold at the point $(u,v) = (0,0)$.

Taking these subspaces as coordinate planes and making two linear symplectic transformation in each plane we transform the quadratic part of our Hamiltonian H to the form

$$H = \frac{\Omega}{2}(x^2 + y^2) + \frac{\omega}{2}(u^2 + v^2) + H_3,$$

where H_3 denotes terms of the third order and higher. Due to the form (24) of symplectic 2-form, eigenvalues of the related Hamiltonian system are $\pm i\Omega$ and $\pm i\varepsilon\omega$, respectively. Note, that the ratio $\varepsilon\omega/\Omega$ is small enough, so resonances of lower orders are absent here, in particular, those of order 3 and 4. Recall [2], that their absence is necessary to prove KAM theorem in a neighborhood of such an elliptic point.

To begin let us discuss the existence of periodic orbits near O. The Lyapunov center theorem (see, for instance, [2]) says that there exists an analytic two-dimensional symplectic submanifold through O being tangent to the plane corresponding eigenvalues $\pm i\Omega$ (i.e., being transversal to slow manifold), filled up with periodic orbits. When tending O period of such the periodic orbit approaches $2\pi/\Omega$. The existence of another family adjacent to equilibrium, corresponding to eigenvalues $\pm i\varepsilon\omega$, depends strongly on the value of ε. Indeed, the ratio $\Omega/\varepsilon\omega$, as $\varepsilon \to 0$, countably many time passes through resonant values $\varepsilon_n = \Omega/n\omega$ when the Lyapunov center theorem does not work. The structure of the system near these values is described well by integrable resonant normal form, see [2]. Creation of new families of periodic orbits when passing through these values of ε is the common feature.

To utilize the KAM theorem for proving invariant tori carrying quasi-periodic orbits, we need to check the condition for the point to be of the general elliptic type [1, 2]. But the condition for this is presented for a Hamiltonian and a related system given in standard Darboux coordinates. To transform the system and Hamiltonian to this form we make a scaling of variables $u = \sqrt{\varepsilon}u_1, v = \sqrt{\varepsilon}v_1$. Then our singular 2-form becomes standard one, and Hamiltonian takes the form

$$\hat{H}(x, y, u_1, v_1) = H(x, y, \sqrt{\varepsilon}u_1, \sqrt{\varepsilon}v_1),$$

and system (23) becomes standard Hamiltonian system with this Hamiltonian. Further, we will transform this Hamiltonian to the normal form. In order do not complicate notations we hold the same letters for new Hamiltonian and variables.

We need to transform Hamiltonian to the form

$$H = \frac{\Omega}{2}(x^2+y^2)+\frac{\varepsilon\omega}{2}(u^2+v^2)+\frac{1}{2}(A(x^2+y^2)^2+2\varepsilon B(x^2+y^2)(u^2+v^2)+\varepsilon^2 C(u^2+v^2)^2)+\cdots. \tag{25}$$

The condition of general ellipticity is expressed in one of two forms [1, 2]:

$$\begin{array}{lll} 1. & \varepsilon^2(B^2 - AC) \neq 0 & \text{(nondegenericity)} \\ 2. & \varepsilon^2 \det \begin{pmatrix} A & B & \Omega \\ B & C & \omega \\ \Omega & \omega & 0 \end{pmatrix} \neq 0 & \text{(isoenergetical nondegenericity).} \end{array} \tag{26}$$

To lighten carrying out the normalization procedure, we introduce complex symplectic variables:

$$x = \frac{i+1}{2}(z_1 + w_1),\ y = \frac{i-1}{2}(z - w),\ u = \frac{i+1}{2}(z_2 + w_2),\ v = \frac{i-1}{2}(z_2 - w_2).$$

Then our Hamiltonian takes the form

$$\begin{aligned}
&\hat{H} = i\Omega z_1 w_1 + i\varepsilon\omega z_2 w_2 + \hat{H}_3, \quad \hat{H}_3(z_1, w_1, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}w_2) = \\
&H_3(\frac{i+1}{2}(z_1+w_1), \frac{i-1}{2}(z_1-w_1), \sqrt{\varepsilon}\frac{i+1}{2}(z_2+w_2), \sqrt{\varepsilon}\frac{i-1}{2}(z_2-w_2)).
\end{aligned} \tag{27}$$

The reality of the initial Hamiltonian leads to the following reality relation

$$\overline{\hat{H}_3(z_1, -i\bar{z}_1, z_2, -i\bar{z}_2)} = \hat{H}_3(z_1, -i\bar{z}_1, z_2, -i\bar{z}_2),$$

where overbar means complex conjugation. The corresponding relations for coefficients of power series expansion are the following

$$\overline{\hat{H}}_{nmqp} = (-i)^{m+n+p+q}\hat{H}_{mnpq}.$$

In particular, for terms of the third and fourth orders one derives

$$\overline{\hat{H}}_{nmqp} = (-i)\hat{H}_{mnpq} \quad (m+n+p+q=3), \overline{\hat{H}}_{nmqp} = \hat{H}_{mnpq} \quad (m+n+p+q=4). \tag{28}$$

To find normalized Hamiltonian up to the fourth order terms, one needs only to eliminate third order terms in Hamiltonian by means of a canonical transformation determined by a generating function $S = z_1W_1 + z_2W_2 + s_3(z_1, W_1, z_2, W_2, \varepsilon)$, where $s_3(z_1, W_1, z_2, W_2, \varepsilon)$ is a homogeneous polynomial of the third order in (z_1, W_1, z_2, W_2). We will choose s_3 in such a way that all third order terms in the transformed Hamiltonian vanish. Then we calculate coefficients in front of the only unremovable fourth order terms $(Z_1W_1)^2, Z_1W_1Z_2W_2, (Z_2W_2)^2$. As is well known, they are not changed under the further normalization. The related coefficients, due to (28), are real ones, and therefore, they are exactly coefficients A, εB, and $\varepsilon^2 C$ we are seeking.

Formulas, determining a canonical change of variables, are the following

$$Z_k = \frac{\partial S}{\partial W_k}, \quad w_k = \frac{\partial S}{\partial z_k}, \quad k = 1, 2. \tag{29}$$

The new (transformed) Hamiltonian is merely the old one, in which new variables have been plugged. Expanding the initial and transformed Hamiltonians and s_3 in power series in phase variables and inserting relations (29) into initial and transformed Hamiltonians, we get the following equations for finding s_3

$$\begin{aligned}
&i\Omega(z_1\frac{\partial s_3}{\partial z_1} - W_1\frac{\partial s_3}{\partial W_1}) + i\varepsilon\omega(z_2\frac{\partial s_3}{\partial z_2} - W_2\frac{\partial s_3}{\partial W_2}) = \\
&\hat{H}_3^{(new)}(z_1, W_1, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}W_2) - \hat{H}_3^{(old)}(z_1, W_1, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}W_2),
\end{aligned}$$

here we denote $\hat{H}_j$ terms of the order j in the corresponding Hamiltonian.

Equating coefficients in front of the same monomials, we obtain

$$s_{mnpq} = \varepsilon^{(p+q)/2}\frac{\hat{H}^{(new)}_{mnpq} - \hat{H}^{(old)}_{mnpq}}{i[\Omega(m-n) + \varepsilon\omega(p-q)]}, \tag{30}$$

here we assume that variables z_2, w_2, Z_2, W_2 enter into Hamiltonians only in combinations with the multiplicator $\sqrt{\varepsilon}$. From these formulas we see that s_3 consists of two parts, one is regular with respect to $\sqrt{\varepsilon}$, and another one is singular, where ε enters as $\varepsilon^{-1/2}$. These latter terms are s_{1110} and s_{1101}. Polynomial s_3 becomes uniquely defined, if we put $\hat{H}^{(new)}_{mnpq} = 0$, that is supposed to be fulfilled further.

Now we can write coefficients of unremovable terms in the transformed Hamiltonian. They are deduced from the relations for the fourth order terms in the initial and transformed Hamiltonians. We need only coefficients which enter to the normal form.

The identity, which defines coefficients of the fourth order in new Hamiltonian, has the form

$$\begin{aligned}&\hat{H}^{(old)}_3(z_1, W_1 + \frac{\partial s_3}{\partial z_1}, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}(W_2 + \frac{\partial s_3}{\partial z_2})) - \hat{H}^{(old)}_3(z_1, W_1, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}W_2) =\\ &\hat{H}^{(new)}_4(z_1, W_1, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}W_2) - \hat{H}^{(old)}_4(z_1, W_1, \sqrt{\varepsilon}z_2, \sqrt{\varepsilon}W_2),\end{aligned}$$

We have to write down coefficients $\hat{H}^{(new)}_{2200}$, $\hat{H}^{(new)}_{1111}$, $\hat{H}^{(new)}_{0022}$. It is not hard to verify that contributions to the left hand side in the preceding relations can come only from monomials of two types:

$$n z_1^m W_1^{n-1} \frac{\partial s_3}{\partial z_1} z_2 W_2, \quad \text{and} \quad q z_1^m W_1^n z_2^p W_2^{q-1} \frac{\partial s_3}{\partial z_2}$$

Equating coefficients in front of the related fourth order monomials gives the relations

$$\begin{aligned}&\hat{H}^{(new)}_{2200} = \hat{H}^{(old)}_{2200} + \hat{H}^{(old)}_{2100} s_{1200} + 4\hat{H}^{(old)}_{1200} s_{2100} + 9\hat{H}^{(old)}_{0300} s_{3000} +\\ &\sqrt{\varepsilon}\hat{H}^{(old)}_{2001} s_{0210} + \sqrt{\varepsilon}\hat{H}^{(old)}_{0201} s_{2010} + \sqrt{\varepsilon}\hat{H}^{(old)}_{1101} s_{1110},\\ \\ &\varepsilon\hat{H}^{(new)}_{1111} = \varepsilon\hat{H}^{(old)}_{1111} + 2\hat{H}^{(old)}_{1200} s_{1011} + 4\sqrt{\varepsilon}\hat{H}^{(old)}_{0210} s_{2001} + 4\sqrt{\varepsilon}\hat{H}^{(old)}_{0201} s_{2010} +\\ &\sqrt{\varepsilon}\hat{H}^{(old)}_{1110} s_{1101} + \sqrt{\varepsilon}\hat{H}^{(old)}_{1101} s_{1110} + 2\sqrt{\varepsilon}\hat{H}^{(old)}_{1101} s_{0021} + \varepsilon\hat{H}^{(old)}_{1011} s_{0111} + \varepsilon\hat{H}^{(old)}_{0111} s_{1011} +\\ &2\varepsilon\hat{H}^{(old)}_{0111} s_{2100} + 4\varepsilon\hat{H}^{(old)}_{1002} s_{0120} +\\ &4\varepsilon\hat{H}^{(old)}_{0102} s_{1020} + 2\varepsilon^{3/2}\hat{H}^{(old)}_{0012} s_{1110},\\ \\ &\varepsilon^2\hat{H}^{(new)}_{0022} = \varepsilon^2\hat{H}^{(old)}_{0022} + \varepsilon\hat{H}^{(old)}_{0120} s_{1002} + \varepsilon\hat{H}^{(old)}_{0111} s_{1011} + \varepsilon\hat{H}^{(old)}_{0102} s_{1020} +\\ &\varepsilon^{3/2}\hat{H}^{(old)}_{0021} s_{0012} + 4\sqrt{\varepsilon}\hat{H}^{(old)}_{0012} s_{0021} + 9\varepsilon^{3/2}\hat{H}^{(old)}_{0003} s_{0030}.\end{aligned} \tag{31}$$

Inserting expressions from (30) we get

$$\hat{H}_{2200}^{(new)} = \hat{H}_{2200}^{(old)} + \hat{H}_{2100}^{(old)}(\hat{H}_{1200}^{(old)}/i\Omega) - 4\hat{H}_{1200}^{(old)}(\hat{H}_{2100}^{(old)}/i\Omega) - 3\hat{H}_{0300}^{(old)}(\hat{H}_{3000}^{(old)}/i\Omega) +$$
$$\varepsilon\hat{H}_{2001}^{(old)}(\hat{H}_{0210}^{(old)}/i[2\Omega - \varepsilon\omega]) - \varepsilon\hat{H}_{0201}^{(old)}(\hat{H}_{2010}^{(old)}/i[2\Omega + \varepsilon\omega]) - \hat{H}_{1101}^{(old)}(\hat{H}_{1110}^{(old)}/i\omega),$$

$$\varepsilon\hat{H}_{1111}^{(new)} = \varepsilon\hat{H}_{1111}^{(old)} - 2\varepsilon\hat{H}_{1200}^{(old)}\hat{H}_{1011}^{(old)}/i\Omega - 4\varepsilon\hat{H}_{0210}^{(old)}\hat{H}_{2001}^{(old)}/i[2\Omega - \varepsilon\omega] -$$
$$4\varepsilon\hat{H}_{0201}^{(old)}\hat{H}_{2010}^{(old)}/i[2\Omega + \varepsilon\omega] + \varepsilon\hat{H}_{1110}^{(old)}(\hat{H}_{1101}^{(old)}/i\varepsilon\omega) - \varepsilon\hat{H}_{1101}^{(old)}\hat{H}_{1110}^{(old)}/i\varepsilon\omega -$$
$$2\varepsilon\hat{H}_{0111}^{(old)}\hat{H}_{2100}^{(old)}/i\Omega + 4\varepsilon^2\hat{H}_{1002}^{(old)}\hat{H}_{0120}^{(old)}/i[\Omega - 2\varepsilon\omega] - 4\varepsilon^2\hat{H}_{0102}^{(old)}\hat{H}_{1020}^{(old)}/i[\Omega + 2\varepsilon\omega] -$$
$$2\varepsilon^2\hat{H}_{0012}^{(old)}\hat{H}_{1110}^{(old)}/i\varepsilon\omega - 2\varepsilon^2\hat{H}_{1101}^{(old)}\hat{H}_{0021}^{(old)}/i\varepsilon\omega + \varepsilon^2\hat{H}_{1011}^{(old)}\hat{H}_{0111}^{(old)}/i\Omega - \varepsilon^2\hat{H}_{0111}^{(old)}\hat{H}_{1011}^{(old)}/i\Omega$$

$$\varepsilon^2\hat{H}_{0022}^{(new)} = \varepsilon^2\hat{H}_{0022}^{(old)} + \varepsilon^2\hat{H}_{0120}^{(old)}(-\hat{H}_{1002}^{(old)}/i[\Omega - 2\varepsilon\omega]) + \varepsilon^2\hat{H}_{0102}^{(old)}(-\hat{H}_{1020}^{(old)}/i[\Omega + 2\varepsilon\omega]) +$$
$$\varepsilon^2\hat{H}_{0111}^{(old)}(-\hat{H}_{1011}^{(old)}/i\Omega) + \varepsilon^3\hat{H}_{0021}^{(old)}(\hat{H}_{0012}^{(old)}/i\varepsilon\omega) +$$
$$4\varepsilon^3\hat{H}_{0012}^{(old)}(-\hat{H}_{0021}^{(old)}/i\varepsilon\omega) + 3\varepsilon^3\hat{H}_{0003}^{(old)}(-\hat{H}_{0030}^{(old)}/i\varepsilon\omega).$$

Taking into account reality conditions (28), coefficients takes the form

$$\begin{aligned}
&A = \hat{H}_{2200}^{(new)} = \hat{H}_{2200}^{(old)} + 3\left(|\hat{H}_{1200}^{(old)}|^2 + |\hat{H}_{3000}^{(old)}|^2\right)/\Omega + |\hat{H}_{1110}^{(old)}|^2/\omega + \\
&\varepsilon\Big(\frac{|\hat{H}_{2010}^{(old)}|^2}{2\Omega + \varepsilon\omega} - \frac{|\hat{H}_{2001}^{(old)}|^2}{2\Omega - \varepsilon\omega}\Big), \\
&\varepsilon B = \varepsilon\hat{H}_{1111}^{(new)} = \varepsilon\hat{H}_{1111}^{(old)} - 2\varepsilon(\hat{H}_{1200}^{(old)}\hat{H}_{1011}^{(old)} - \overline{\hat{H}_{1200}^{(old)}\hat{H}_{1011}^{(old)}})/i\Omega - \\
&2\varepsilon(\hat{H}_{0012}^{(old)}\hat{H}_{1110}^{(old)} - \overline{\hat{H}_{0012}^{(old)}\hat{H}_{1110}^{(old)}})/i\omega + \\
&4\varepsilon\Big(\frac{|\hat{H}_{2001}^{(old)}|^2}{2\Omega - \varepsilon\omega} + \frac{|\hat{H}_{2010}^{(old)}|^2}{2\Omega + \varepsilon\omega}\Big) + 4\varepsilon^2\Big(\frac{|\hat{H}_{1020}^{(old)}|^2}{\Omega + 2\varepsilon\omega} - \frac{|\hat{H}_{1002}^{(old)}|^2}{\Omega - 2\varepsilon\omega}\Big), \\
&\varepsilon^2 C = \varepsilon^2\hat{H}_{0022}^{(new)} = \varepsilon^2\hat{H}_{0022}^{(old)} + \varepsilon^2\Big[|\hat{H}_{1002}^{(old)}|^2/(\Omega - 2\varepsilon\omega) + |\hat{H}_{1020}^{(old)}|^2/(\Omega + 2\varepsilon\omega) + \\
&|\hat{H}_{0111}^{(old)}|^2/\Omega + 3|\hat{H}_{0012}^{(old)}|^2/\omega + 3|\hat{H}_{0003}^{(old)}|^2/\omega\Big].
\end{aligned} \tag{32}$$

Theorem 6 *Let the inequality $B^2 - AC$ does not vanish at $\varepsilon = 0$. Then O is the point of the general elliptic type, and there is $O(\varepsilon)-$neighborhood of this point where KAM theorem is valid.*

This theorem ensures the existence of almost full measure points lying on invariant Lagrangian tori.

5 Gaps in slow manifold

Is there really any gap in the invariant slow manifold W_ε as it is stated in Theorem 1, or their existence is an artifact of our proof?

The manifold W_ε is filled with periodic orbits of the system (1), which belong to an exponentially small neighborhood of the almost invariant slow manifold W_{slow}^ε. The periodic orbits were constructed as fixed points of the Poincaré map $F_{\varepsilon,E}$, which acts on $\Pi_{0,E}$, the intersection of the energy surface $H = E$ by the Poincaré section Π_0 (see Section 2). Actually $F_{\varepsilon,E}$ has more periodic orbits than it can be established by the perturbation technique of Section 2. Indeed, according to Section 3 the system (1) has a lot of Cantor families of KAM tori. An intersection of an invariant torus with the section $\Pi_{0,E}$ is a closed invariant curve for the map $F_{\varepsilon,E}$. The interior of such a curve is an invariant disk of the map. A closed invariant disk always has at least one fixed point inside.

On each section $\Pi_{0,E}$ *the Poincaré map* $F_{\varepsilon,E}$ *has at least one fixed point.*

Some of the fixed points belong to an exponentially small neighborhood of W^ε_{slow} but other ones may exist. The fixed points can form families parameterized by E, these families may branch, coalesce, etc. Fixed points found in Section 2 do lie in an exponentially small neighborhood of W^ε_{slow}, and they form a smooth family in E till E reaches an exponentially small neighborhood of the resonance value E_0 defined by $A_0(0, E_0, \varepsilon)/\varepsilon = n$ ($1 : n$ resonance between slow and fast subsystems). Only at a resonance of this form the family may break forming a gap.

Is it possible that the periodic trajectories of the system (1) form an invariant slow manifold without any gap? This happens, in particular, if the Hamiltonian H can be canonically transformed to the form (6) with $r \equiv 0$. Thus, an existence proof for gaps implies that the exponentially averaging residue r cannot be eliminated completely. A lower bound for an exponentially small residue is usually a very difficult analytical problem. In this paper we will not try to give this lower bound. Instead of this we suppose that r does not vanish, and compare the results of our analysis with numerical experiments in Section 6.

So we will assume that the effect of the residue is exponentially small but non-vanishing. Using this assumption we derive an approximate form of the map near the gap.

According to Lemma 2 the Poincaré map is a small perturbation of the rotation:

$$\underline{x}_1 = R(2\pi A_0(I, E, \varepsilon)/\varepsilon)\underline{x} + P(\underline{x}, E, \varepsilon),$$

If $P \equiv 0$ the origin is a fixed point of $F_{\varepsilon,E}$.

Let us fix ε small and consider the energy E as a governing parameter. Let $E = E_0$ correspond to $1 : n$ resonance between the slow and fast subsystems,

$$A_0(0, E_0, \varepsilon) = n\varepsilon$$

for some $n \in \mathbf{N}$. The multiplicator of the integrable approximation is equal to 1. Since we expect to find a parabolic fixed point, we make a usual in such cases scaling of variables and governing parameter:

$$x = \mu\sqrt{\varepsilon}X, \qquad y = \mu\sqrt{\varepsilon}Y, \qquad E = E_0 + \mu^2\varepsilon\delta.$$

The auxiliary rescaling parameter μ is to be chosen later, it will describe the scale at which we look on the gap. The parameter δ describes the passage through the resonance. Denote

$J = \frac{1}{2}(X^2 + Y^2)$, then $I = \mu^2 \varepsilon J$. Expanding A_0 in the Taylor series near $I = 0, E = E_0$ we obtain

$$A_0(I, E, \varepsilon) = n\varepsilon + \mu^2 \varepsilon a_1(\varepsilon)\delta + \mu^2 \varepsilon a_2(\varepsilon) J + \mu^4 \varepsilon^2 B_0(J, \delta, \varepsilon),$$

where B_0 is bounded. Then $F_{\varepsilon,E}$ takes the form

$$\underline{X}_1 = R(2\pi\mu^2(a_1\delta + a_2 J + \mu^2 \varepsilon B_0))\,\underline{X} + \mu^{-1}\varepsilon^{-1/2} P(\mu\varepsilon^{1/2}\underline{X}, E, \varepsilon).$$

We expand it in series in μ:

$$\underline{X}_1 = \underline{X} + 2\pi\mu^2(a_1\delta + a_2 J) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{X} + \mu^{-1}\varepsilon^{-1/2} P(0, E_0, \varepsilon) + O(\mu) + O(\|P\|). \quad (33)$$

Keeping in mind smallness of terms $O(\mu) + O(\|P\|)$, we consider this map as being close to that given by remaining terms in (33) which, in turn, is close to the identical map. A close to identity map can be interpolated by an one-degrees of freedom Hamiltonian system [4]. In the analytic case the interpolation can be done up to an error exponentially small in μ. If we chose μ itself to be exponentially small in ε, the Poincaré map will be super-exponentially close ($\exp(-c_1 \exp(c_2 \varepsilon^{-1}))$ for some positive c_1 and c_2) to an integrable system. A super-exponential approximation is a good starting point for studying exponentially small phenomena. We do not have to know the super-exponentially interpolating Hamiltonian exactly. For our purposes it is sufficient to construct its first order approximation.

We choose the time step to be $\tau = 2\pi\mu^2$. Let $\Phi_{\delta,\varepsilon,\mu}$ be a time τ map of a Hamiltonian system

$$H_{res} = \delta a_1(\varepsilon) J + a_2(\varepsilon) \frac{J^2}{2} + p_1(\varepsilon, \mu) X + p_2(\varepsilon, \mu) Y,$$

where

$$(p_1, -p_2)^T = \frac{1}{2\pi\mu^3\sqrt{\varepsilon}} P(0, E_0, \varepsilon).$$

It is easy to see that

$$\|F_{\varepsilon,E} - \Phi_{\delta,\varepsilon,\mu}\| \le c(\mu + \|P\|).$$

Here we arrive to the central part of our analysis: we have an exponentially small upper bound on P, but we do not have any lower bounds for $P(0, E_0, \varepsilon)$. If $P(0, E, \varepsilon) \equiv 0$ the origin is a fixed point of $F_{\varepsilon,E}$ for all E and ε; and the slow manifold has nor gaps. In order to justify that the following bifurcation analysis correctly reflects the situation we conjecture that $\|P(0, E_0, \varepsilon)\|\mu^3\varepsilon^{-1/2}$ is bounded away from zero. Then we can make our choice μ^3 to be of the order of $\varepsilon^{-1/2}\|P(0, E_0, \varepsilon)\|$. We will see that this assumption leads to a qualitatively correct bifurcation picture.

We study the bifurcation of the equilibria of the system described by H_{res}. We recall that ε is fixed and δ is our bifurcation parameter. To simplify H_{res} we rotate the (X, Y) plane to eliminate p_2, and rescale the variables and the parameter normalizing to unity as many coefficients of H_{res} as possible. Of course, one assumes that a_1 and a_2 do not

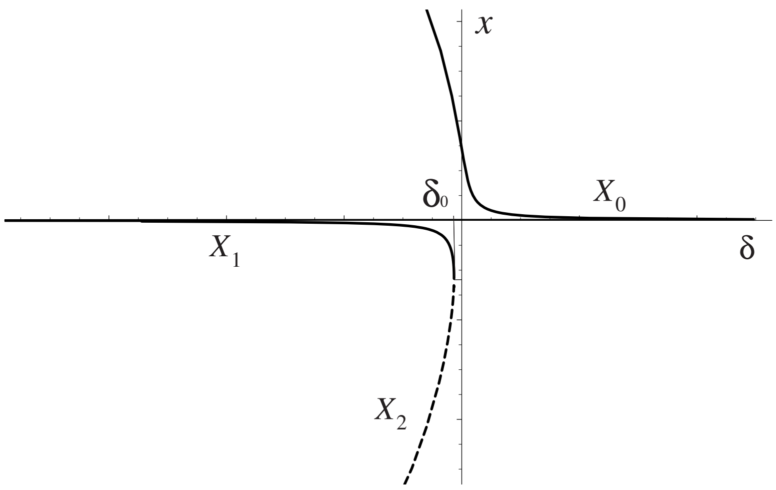

Figure 1: Bifurcation diagram for the equilibria of the Hamiltonian (34).

vanish due to non-degeneracy conditions. In this way we obtain the first order resonant form

$$H_{res} = \delta J + \frac{J^2}{2} + pX, \qquad J = \tfrac{1}{2}(X^2 + Y^2). \tag{34}$$

The corresponding Hamiltonian equation have the form

$$\begin{aligned} \dot{X} &= \delta Y + JY, \\ \dot{Y} &= -\delta X - JX - p. \end{aligned}$$

If $p = 0$, the system is rotationally symmetric, the origin is an equilibrium for all δ; when δ becomes negative, a circle of equilibria $J = -\delta$ bifurcates from the origin.

The constant term p breaks the rotational symmetry of the system, and we see an asymmetric pitchfork bifurcation. If $p > 0$, all equilibria of the system belong to the X-axis. The bifurcation diagram on the (δ, X) plane is shown on Fig. 1. There are 3 families of fixed points: $X_0(\delta)$ and $X_1(\delta)$ are elliptic and $X_2(\delta)$ is hyperbolic. For each of these families X-coordinate is a monotone function of δ. The first family $X_0(\delta)$ is defined for all $\delta \in \mathbf{R}$; $X_0(-\infty) = +\infty$ and $X_0(+\infty) = 0$. The families $X_1(\delta)$ and $X_2(\delta)$ are defined for $\delta \in (-\infty, \delta_0)$ with $\delta_0 = -\frac{3}{2}p^{2/3}$;

$$X_1(\delta_0) = X_2(\delta_0) = -p^{1/3}, \qquad X_1(-\infty) = 0, \qquad X_2(-\infty) = -\infty.$$

What does this bifurcation picture imply for the original singularly perturbed system (1)? First, the periodic trajectories which remain in an exponentially thin neighborhood of the almost invariant manifold W^{ε}_{slow} do not form a continuous family. Thus the exact slow manifold W_ε has gaps at the resonances $1 : n$. In an exponentially small neighborhood of the resonance this manifold can be considered as a graph of a two-valued function with an exponentially small jump between the two branches. As the energy E deviates from the resonant value E_0 one of the branches recedes from W^ε_{slow} and can be continued at least till $E - E_0 \sim \varepsilon^2$. The corresponding bifurcation diagram looks like a finite sequence of asymmetric pitchfork bifurcations with an exponentially small gaps in their pseudo-intersections.

In the next section we will provide results of simulations which show that the bifurcation picture described here is qualitatively correct.

6 Numerical construction of slow manifold

In this section we numerically study the destruction of the slow manifold due to resonances between the slow and fast degrees of freedom. As an example we choose the Hamiltonian function

$$H(x, y, u, v; \varepsilon) = h_1(x, y) + h_2(u, v) + \varepsilon x u,$$

where

$$h_1(x, y) = \frac{x^2 + y^2}{2} + \frac{(x^2 + y^2)^2}{2}, \qquad h_2(u, v) = \frac{u^2 + v^2}{2} - \frac{u^3}{3}.$$

The slow system is governed by the Hamiltonian $h_2(u, v)$. It has a region around the origin filled with periodic orbits. According to Theorem 1 the exact slow manifold W_ε is also foliated into periodic orbits, and can be represented as a graph:

$$x = f(u, v, \varepsilon), \qquad y = g(u, v, \varepsilon).$$

It is a two-dimensional surface in the four dimensional space. We apply different techniques to visualize it. We chose $\varepsilon = 0.15$ for all numerical simulations reported in the present section.

In order to plot the slow manifold we consider a cross-section $\Pi_0 = \{ v = 0 \}$. Next we note that our system is reversible with respect to the involution $(x, y, u, v) \mapsto (x, -y, u, -v)$. The periodic orbits on the slow manifold are symmetric, thus $g(u, 0, \varepsilon) \equiv 0$, and the intersection $W_\varepsilon \cap \Pi_0$ can be described by a single function $x = f(u, 0, \varepsilon)$. Each point on the graph of this function gives initial conditions for a periodic orbit of the original system (1). On the other hand, it corresponds to a fixed point of the Poincaré map.

Fixed points of the Poincaré map are shown on Fig. 2. The straight line represents $W_\varepsilon \cap \Pi_0$. The branching points correspond to the resonances. We see that at each resonance two families of fixed points branch off the slow manifold. The families marked by dashed lines consist of saddle points, and the families of elliptic fixed points are drawn by solid lines.

We choose one crossing near $u = 0.616..$ and show its magnification on Fig. 3. We immediately see that the intersections of lines in Fig. refFig:1 are actually pseudo-intersections. The diagram looks very similar the normal form diagram shown on Fig. 1. This crossing corresponds to the resonance 1 : 8. It is clearly seen that the slow manifold has a gap which is so small, that we do not see it on Fig. 2. The picture is very similar to

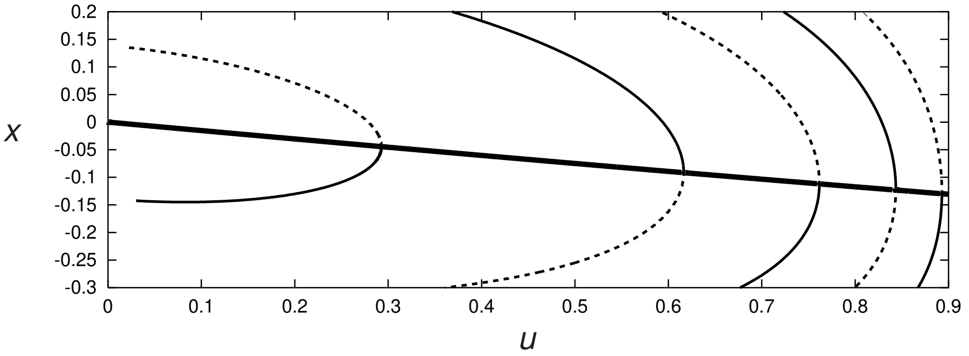

Figure 2: Intersection of the slow manifold W_ε with Π_0 (bold line) and bifurcating periodic orbits

the bifurcation diagram for a non-symmetric pitchfork bifurcation of (34) shown on Fig. 1: as u increases a saddle collides with a center and disappears. Another center comes from outside and occupies the place of the first one. The bifurcation region is very small but all participating trajectories can be continued to an essentially larger region.

We can also visualize W_ε by plotting the graph of the function $x = f(u, v, \varepsilon)$ for a fixed ε. The plot of $y = g(u, v, \varepsilon)$ looks quite similar, so we omit it. The manifold W_ε is a union of annuli filled with periodic trajectories. The boundaries of the annuli corresponds to $1 : n$ and $1 : (n + 1)$ resonances. We choose the annulus between the $1 : 7$ and $1 : 8$ resonances and plot the corresponding piece of the graph $x = f(u, v, \varepsilon)$ on Fig. 4. Each line on this plot represents a numerically computed periodic trajectory of the system (1). We started with a single periodic trajectory and use a method of continuation in order to compute other trajectories. In this way we obtained not only the trajectories on W_ε but also some branching off trajectories.

The periodic trajectories, which belong to W_ε, look special: they do not oscillate in rapid directions. This is clearly seen on Fig. 4. The flat annulus corresponds to W_ε. We see also some of oscillating trajectories, which are attached to the flat annulus from inside and outside. The trajectories outside the inner boundary are elliptic; and the trajectories outside the outer boundary are hyperbolic.

Finally we study the Poincaré map near the periodic orbits on W^0_{slow}. The section Π_0 is foliated by two dimensional energy surfaces $H = E$. We choose the variables (x, y) to be local coordinates on the intersection $\Pi_0 \cap \{ H = E \}$. For most values of E the Poincaré map $F_{\varepsilon,E}$ looks like a rotation. Near the resonances the dynamics is more interesting.

Similar to the previous section we consider E as a parameter and study the bifurcation of $F_{\varepsilon,E}$, when E crosses a resonant values. As an example, we consider a neighborhood of the resonance $1 : 8$. On Fig. 5 we plotted phase portraits of the map $F_{\varepsilon,E}$ for different values of E. We see their remarkable similarity to the phase portraits of the Hamiltonian system (34).

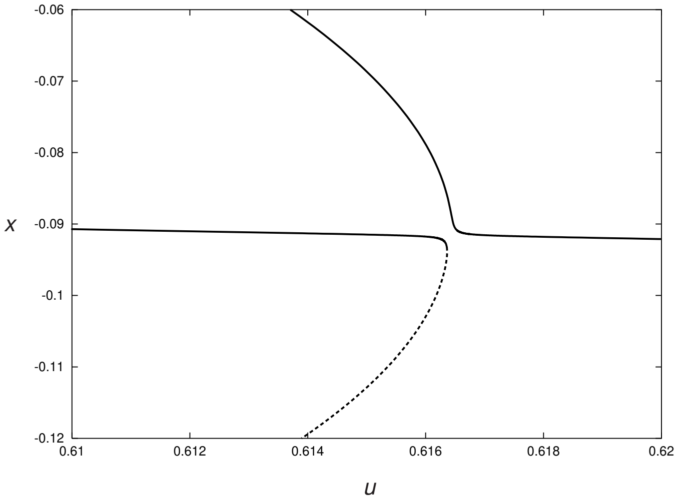

Figure 3: Magnification of a neighborhood of the pseudo-intersection in Fig. 2.

Note, that in this region the Poincaré map is exponentially close to identity and super exponentially close to an autonomous Hamiltonian flow. For that reason the trajectories of the map look like continuous lines.

Conclusions

In this paper we studied the orbit behavior of a singularly perturbed system near its elliptic slow manifold. First we considered a piece of the slow manifold, which is an annulus K_0 filled with periodic trajectories of the slow system. Then we showed that the whole system has an invariant slow manifold filled with periodic trajectories. It can be represented in the form

$$W_\varepsilon = \{\, x = f(u,v,\varepsilon),\ y = g(u,v,\varepsilon),\ (u,v) \in K \,\}.$$

The set $K = \cup_m K_{m,\varepsilon}$ is the union of $n(\varepsilon) \sim c\varepsilon^{-1}$ smaller annuli K_m. A gap between $K_{m+1,\varepsilon}$ and $K_{m,\varepsilon}$ is exponentially thin for each m. The functions f and g are analytic for $(u,v) \in K_{m,\varepsilon}$, but they may have breaks between adjacent annuli. The breaks are exponentially small. Thus W_ε can have a lot of exponentially small gaps.

The gaps are due to $1:n$ resonances between the slow and fast degrees of freedom, so n is of the order of ε^{-1}. We derived the resonant normal form (34), which describes the crossing of a single resonance. We also proved that in an exponentially small neighborhood of a gap the singularly perturbed system (1) is super-exponentially close to an integrable system, which can be considered as a perturbation of (34).

Finally, we constructed KAM tori near the slow manifold including those in neighborhoods of elliptic points of the initial system existing near center points of slow manifold.

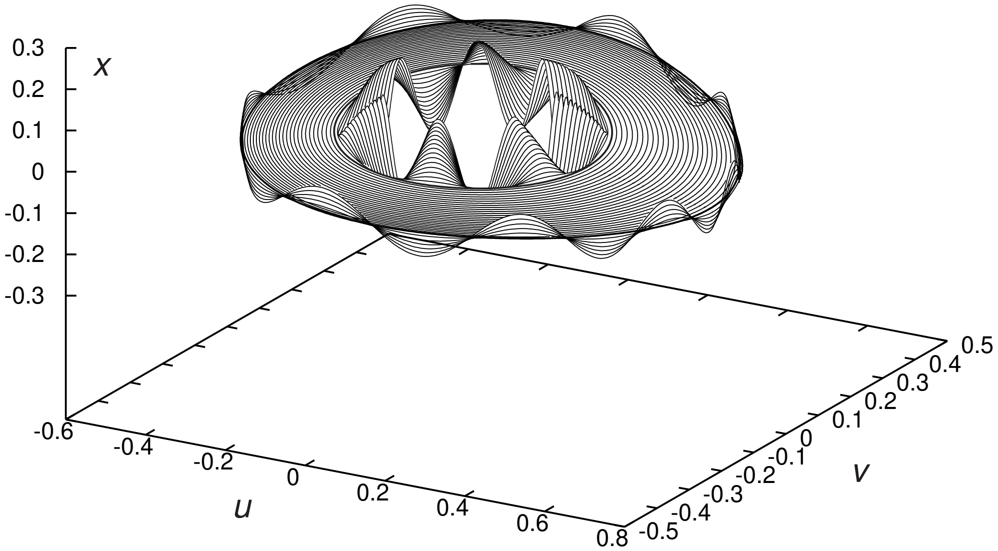

Figure 4: A piece of the slow manifold W_ε between the resonances $1:7$ (inside) and $1:8$ (outside).

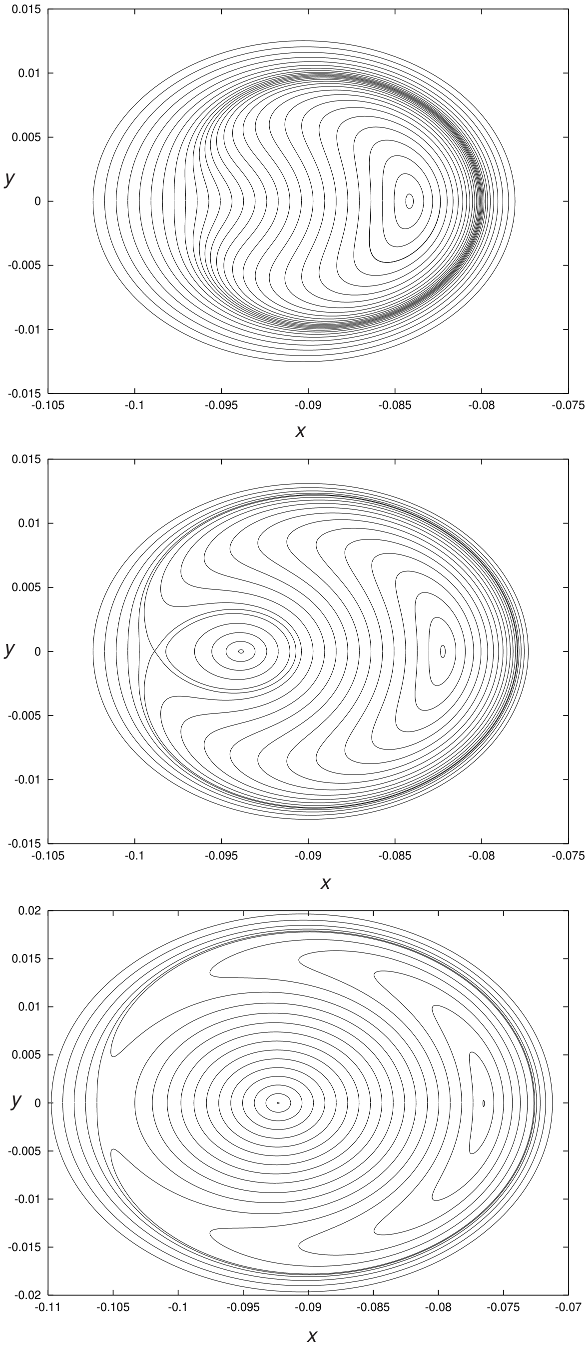

Figure 5: Poincaré sections $\Pi_0 \cap \{ H = \text{const} \}$ near the $1:8$ resonance for different values of E

Most of the presented picture was supported by analytical proofs. Numerical experiments showed that the gaps do really occur, and support the resonant normal form theory developed in Section 5.

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