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A semiclassical Egorov theorem and quantum ergodicity for matrix valued operators

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Abstract

We study the semiclassical time evolution of observables given by matrix valued pseudodifferential operators and construct a decomposition of the Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ into a finite number of almost invariant subspaces. For a certain class of observables, that is preserved by the time evolution, we prove an Egorov theorem. We then associate with each almost invariant subspace of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ a classical system on a product phase space $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}$, where $\mathcal{O}$ is a compact symplectic manifold on which the classical counterpart of the matrix degrees of freedom is represented. For the projections of eigenvectors of the quantum Hamiltonian to the almost invariant subspaces we finally prove quantum ergodicity to hold, if the associated classical systems are ergodic.

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Introduction

The relation between dynamical properties of a quantum system and its classical limit is a central subject in the field of quantum chaos. In this context quantum ergodicity is a well-established concept [Zel87, CdV85, HMR87, Zel96]. It states for quantisations of ergodic classical systems that the phase space lifts of almost all eigenfunctions of the quantum Hamiltonian converge in the semiclassical limit to an equidistribution on the level surfaces of the classical Hamiltonian. The principal goal of this paper is to establish quantum ergodicity in systems whose degrees of freedom can be divided into two classes such that they are represented in the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. The semiclassical limit shall be performed in terms of a parameter $\hbar \to 0$ which is primarily linked to the (translational) degrees of freedom that are described by the infinite-dimensional factor $L^2(\mathbb{R}^d)$. The finite dimension n of the other factor is fixed. Examples for systems where this description can be applied are relativistic particles with spin 1/2 in slowly varying external fields governed by a Dirac-Hamiltonian, or adiabatic situations modelled with a Born-Oppenheimer Hamiltonian.

This setting leads to a representation of quantum mechanical observables as matrix valued pseudodifferential operators acting on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, whose symbols are suitable matrix valued functions on the phase space $\mathrm{T}^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ with an expansion in $\hbar$. In particular, the principal symbol H_0 of the selfadjoint quantum Hamiltonian $\mathcal{H}$ is a hermitian matrix valued function on $\mathrm{T}^*\mathbb{R}^d$. Its spectral resolution requires to introduce several classical dynamics on $\mathrm{T}^*\mathbb{R}^d$, each of them generated by one eigenvalue of H_0. Lifted to the quantum level, this structure results in a decomposition of the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ into almost invariant subspaces with respect to the dynamics generated by the quantum Hamiltonian $\mathcal{H}$ that is directly associated with the spectral resolution of H_0. Recently the case of matrix valued operators for certain quantum Hamiltonians with scalar principal symbol, such that on the classical side one still has to deal with a single system, has been considered in [BG00, BGK01]. Here we extend this approach to the general setting of matrix valued operators where one has to define suitable classical systems corresponding to each almost invariant subspace of the Hilbert space.

So far it appears that the semiclassical limit has only been performed with respect to one type of the degrees of freedom. For a complete (semi-)classical description of the quantum systems under consideration one would also require the second type of degrees of freedom, that are represented by the factor $\mathbb{C}^n$ of the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, to be transferred to a classical level. It turns out, however, that for this purpose no further semiclassical parameter is needed and the dimension n of the second factor can be held fixed. Indeed, a suitable Stratonovich-Weyl calculus [Str57] allows to map the principal symbols (with respect to the parameter $\hbar$) of observables and their dynamics in a one-to-one manner to genuinely classical systems associated with the decomposition of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ into almost invariant subspaces. On this classical level the hierarchy of the two types of degrees of freedom is reflected in the structure of the classical dynamics: these are skew-product flows built over the Hamiltonian dynamics generated by the eigenvalues of H_0.

Apart from classical ergodicity the proof of quantum ergodicity typically requires two essential inputs. The first one is a suitable version of an Egorov theorem [Ego69] that allows to express the time evolution of quantum observables in the semiclassical limit in terms of a classical dynamics of principal symbols. We achieve this in two steps: beginning with matrix valued principal symbols, we proceed to a completely classical level by exploiting the Stratonovich-Weyl calculus in the form developed in [FGV90]. It is in the last step where the skew-product flows become relevant. The second input is a Szegö-type limit formula that relates averaged expectation values of observables to classical phase space averages. This can be obtained by a straight-forward generalisation of previous results [HMR87, BG00].

Our main results are the Egorov theorem in section 3 and the quantum ergodicity theorem in section 6. In order to formulate the Egorov theorem we first identify a subalgebra in the class of bounded semiclassical pseudodifferential operators that is invariant under the time evolution. The operators in this subalgebra have to be block-diagonal with respect to the projections onto the almost invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. Theorem 3.2 then asserts that the (matrix valued) principal symbol of each block is evolved with the Hamiltonian flow associated with that block. In addition, it is conjugated with unitary transport matrices that describe the time evolution of the matrix degrees of freedom along the trajectories of the Hamiltonian flow.

We next identify the dynamics provided by the transport matrices with a coadjoint action of a certain Lie group. Kirillov's method of orbits [Kir76] then enables us to connect the apparently quantum mechanical dynamics with a genuinely classical dynamics on a certain coadjoint orbit $\mathcal{O}$, which is a symplectic manifold. This relation can be constructed explicitly with the help of the Stratonovich-Weyl calculus developed in [FGV90]. As a result we obtain that after a Stratonovich-Weyl transform the principal symbol of each block of an observable is evolved with a skew-product dynamics on the combined symplectic phase space $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}$. This observation restores the general picture behind Egorov-type theorems: in leading semiclassical order the quantum mechanical time evolution is determined by classical dynamics.

The decomposition of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ into almost invariant subspaces and the corresponding set of classical flows force quantum ergodicity to be concerned with projections of the eigenvectors of $\mathcal{H}$ to the subspaces, since only these are associated with unique classical systems. The projected eigenvectors, however, are no longer genuine eigenvectors of $\mathcal{H}$, but only provide approximate solutions to the eigenvalue problem and thus yield, after normalisation, quasimodes (see [Laz93]). For the latter we prove quantum ergodicity to hold in the usual sense. In this context the relevant version of the Egorov theorem introduces on the classical side the skew-product flow associated with the given subspace as described above. We show that if this flow is ergodic, the phase space lifts of almost all normalised projected eigenvectors converge to equidistribution on the product phase space.

1 Background on matrix valued pseudodifferential operators

In this section we recall some basic results of pseudodifferential calculus which are well known in the context of operators with scalar symbols. They carry over to the case of matrix valued symbols by only slight modifications of the results known for operators with scalar symbols which can, e.g., be found in [Rob87, DS99]; for the matrix valued case see also [BG00].

The quantities we are primarily concerned with are linear and continuous operators $\mathcal{B} : \mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n \to \mathscr{S}'(\mathbb{R}^d) \otimes \mathbb{C}^n$ with Schwartz kernels $K_{\mathcal{B}}$ taking values in the $n \times n$ matrices $\mathrm{M}_n(\mathbb{C})$. Instead of using a kernel $K_{\mathcal{B}} \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ an operator $\mathcal{B}$ can alternatively be represented by its (Weyl) symbol $B \in \mathscr{S}'(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ that is related to the Schwartz kernel through

$$K_{\mathcal{B}}(x,y) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi} B\Big(\frac{x+y}{2}, \xi\Big)\, \mathrm{d}\xi. \tag{1.1}$$

Here $\hbar \in (0, \hbar_0]$, with $\hbar_0 > 0$, serves as a semiclassical parameter and $\mathrm{T}^*\mathbb{R}^d := \mathbb{R}^d \times \mathbb{R}^d$ denotes the cotangent bundle of the configuration space $\mathbb{R}^d$, i.e., $\mathrm{T}^*\mathbb{R}^d$ is the phase space of the translational degrees of freedom. Below (see section 4) $\mathrm{T}^*\mathbb{R}^d$ will provide one component of a certain combined phase space, which also represents the degrees of freedom described by the matrix character of the symbol in terms of points on a suitable symplectic manifold.

According to the Schwartz kernel theorem every continuous linear map $\mathcal{B} : \mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n \to \mathscr{S}'(\mathbb{R}^d) \otimes \mathbb{C}^n$ can be viewed as an operator with kernel of the above form. However, operators with kernels in $\mathscr{S}'(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ are too general for many purposes; e.g., they can in general not be composed with each other. One therefore has to restrict to smaller sets of kernels and hence to smaller classes of symbols. To achieve this we make use of order functions $m : \mathrm{T}^*\mathbb{R}^d \to (1, \infty)$, which have to fulfill a certain growth property in the sense that there are positive constants C, N such that

$$m(x,\xi) \le C \left(1 + (x-y)^2 + (\xi-\eta)^2\right)^{N/2} m(y,\eta)$$

for all $(x,\xi), (y,\eta) \in \mathrm{T}^*\mathbb{R}^d$. A typical example for such an order function is given by

$$m(x,\xi) = \left(1 + x^2 + \xi^2\right)^M, \quad M \ge 0.$$

This notion allows us to define the symbol classes which we will employ in the subsequent discussions (see [DS99]).

Definition 1.1. Let $m : \mathrm{T}^*\mathbb{R}^d \to (1,\infty)$ be an order function. Then define the symbol class $\mathrm{S}(m) \subset C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ to be the set of $B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ such that for every $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$ and all $\alpha, \beta \in \mathbb{N}_0^d$ there exist constants $C_{\alpha,\beta} > 0$ with

$$\|\partial_\xi^\alpha \partial_x^\beta B(x,\xi)\|_{n\times n} \le C_{\alpha,\beta}\, m(x,\xi). \tag{1.2}$$

Here $\|\cdot\|_{n\times n}$ denotes an arbitrary (matrix) norm on $\mathrm{M}_n(\mathbb{C})$. If in addition the symbol $B(x,\xi;\hbar)$ depends on the parameter $\hbar\in(0,\hbar_0]$, we say that $B\in\mathrm{S}(m)$ if $B(\cdot,\cdot;\hbar)$ is uniformly bounded in $\mathrm{S}(m)$ when $\hbar$ varies in $(0,\hbar_0]$. In particular, for $q\in\mathbb{R}$ let $\mathrm{S}^q(m)$ consist of $B:\mathrm{T}^*\mathbb{R}^d\times(0,\hbar_0]\to\mathrm{M}_n(\mathbb{C})$ belonging to $\hbar^{-q}\mathrm{S}(m)$, i.e.,

$$\|\partial_\xi^\alpha\partial_x^\beta B(x,\xi;\hbar)\|_{n\times n}\leq C_{\alpha,\beta}\hbar^{-q}m(x,\xi)$$

for all $\alpha,\beta\in\mathbb{N}_0^d$, $(x,\xi)\in\mathrm{T}^*\mathbb{R}^d$, and $\hbar\in(0,\hbar_0]$.

An asymptotic expansion of $B\in\mathrm{S}^{q_0}(m)$ is defined by a sequence $\{B_j\in\mathrm{S}^{q_j}(m)\}_{j\in\mathbb{N}_0}$ of symbols, where q_j decreases monotonically to $-\infty$ and

$$B-\sum_{j=0}^{N}B_j\in\mathrm{S}^{q_{N+1}}(m)$$

for all $N\in\mathbb{N}_0$. In this case we write

$$B\sim\sum_{j=0}^{\infty}B_j.$$

In the following we will often use the class $\mathrm{S}^q_{\mathrm{cl}}(m)$ of classical symbols, whose elements have asymptotic expansions in integer powers of $\hbar$,

$$B\sim\sum_{j=0}^{\infty}\hbar^{-q+j}B_j,$$

where $B_j\in\mathrm{S}(m)$ is independent of $\hbar$. We also use the notation

$$\mathrm{S}^\infty(m):=\bigcup_{q\in\mathbb{R}}\mathrm{S}^q(m)\quad\text{and}\quad\mathrm{S}^{-\infty}(m):=\bigcap_{q\in\mathbb{R}}\mathrm{S}^q(m).$$

An operator with a kernel of the form (1.1) and symbol $B\in\mathrm{S}(m)$ clearly maps both $\mathscr{S}(\mathbb{R}^d)\otimes\mathbb{C}^n$ and $\mathscr{S}'(\mathbb{R}^d)\otimes\mathbb{C}^n$ into themselves, whereby according to (1.1) it acts on $\mathbb{C}^n$-valued functions $\psi\in\mathscr{S}(\mathbb{R}^d)\otimes\mathbb{C}^n$ as

$$(\mathcal{B}\psi)(x)=\big(\mathrm{op}^W[B]\psi\big)(x)=\frac{1}{(2\pi\hbar)^d}\iint_{\mathrm{T}^*\mathbb{R}^d}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi}B\Big(\frac{x+y}{2},\xi\Big)\psi(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

Operators $\mathcal{B}=\mathrm{op}^W[B]$ of this type are called Weyl operators, and $\mathrm{symb}^W[\mathcal{B}]=B$ denotes the Weyl symbol of $\mathcal{B}$. If the Weyl symbol of an operator is a classical symbol with asymptotic expansion $B\sim\sum_{j\in\mathbb{N}_0}\hbar^{-q+j}B_j$, we also call $\mathrm{op}^W[B]$ a semiclassical pseudodifferential operator. The leading order term $\mathrm{symb}^W_P[\mathcal{B}]=B_0$ is then referred to as the principal symbol of $\mathcal{B}$, and the subsequent term B_1 as the subprincipal symbol.

The set of Weyl operators with symbols from the classes $\mathrm{S}(m)$ is stable under operator multiplication, in the sense that the operator product is again a Weyl operator with symbol in a certain class:

Lemma 1.2. *Let m_1, m_2 be order functions. Then for $B_j \in \mathrm{S}(m_j)$, $j = 1, 2$, the product of the corresponding operators $\mathcal{B}_j = \mathrm{op}^W[B_j]$ is again a Weyl operator that can be expressed in terms of the symbols B_1, B_2 as*

$$\mathcal{B}_1\mathcal{B}_2 = \mathrm{op}^W[B_1]\,\mathrm{op}^W[B_2] = \mathrm{op}^W[B_1 \# B_2],$$

where the symbol product $(B_1, B_2) \mapsto B_1 \# B_2$ is continuous from $\mathrm{S}(m_1)\times\mathrm{S}(m_2)$ to $\mathrm{S}(m_1 m_2)$ in the topology generated by the seminorms associated with the estimate (1.2). In explicit terms the symbol product reads

$$(B_1 \# B_2)(x,\xi) = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}\sigma(\partial_x,\partial_\xi;\partial_y,\partial_\eta)} B_1(x,\xi) B_2(y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}},$$

where $\sigma(v_x, v_\xi; w_x, w_\xi) = v_x \cdot w_\xi - v_\xi \cdot w_x$ denotes the symplectic two-form on $\mathrm{T}^*\mathbb{R}^d$. Furthermore, $B_j \in \mathrm{S}^0_{\mathrm{cl}}(m_j)$ are mapped to $B_1 \# B_2 \in \mathrm{S}^0_{\mathrm{cl}}(m_1 m_2)$ with (classical) asymptotic expansion

$$(B_1 \# B_2)(x,\xi) \sim \sum_{k,j_1,j_2 \in \mathbb{N}_0} \frac{\hbar^{k+j_1+j_2}}{k!} \left(\frac{\mathrm{i}}{2}\sigma(\partial_x,\partial_\xi;\partial_y,\partial_\eta)\right)^k B_{1,j_1}(x,\xi) B_{2,j_2}(y,\eta)\Bigg|_{\substack{y=x\\ \eta=\xi}}.$$

The following result, which in its original version is due to Beals [Bea77], is useful in situations where one wishes to identify a given operator as a pseudodifferential operator.

Lemma 1.3. *Let $\mathcal{B}(\hbar) : \mathscr{S}(\mathbb{R}^d)\otimes\mathbb{C}^n \to \mathscr{S}'(\mathbb{R}^d)\otimes\mathbb{C}^n$ be a linear and continuous operator depending on the semiclassical parameter $\hbar \in (0, \hbar_0]$. The following statements are then equivalent:*

(i) $\mathcal{B}(\hbar) = \mathrm{op}^W[B]$ is a Weyl operator with symbol $B \in \mathrm{S}^0(1)$.

(ii) For every sequence $l_1(x,\xi), \dots, l_N(x,\xi)$, $N \in \mathbb{N}$, of linear forms on $\mathrm{T}^*\mathbb{R}^d$ the operator given by the multiple commutator $[\mathrm{op}^W[l_N], [\mathrm{op}^W[l_{N-1}], \cdots, [\mathrm{op}^W[l_1], \mathcal{B}]\cdots]$ is bounded as an operator on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ and its norm is of the order $\hbar^N$.

The direction (i)⇒(ii) is a simple consequence of the symbolic calculus outlined above. For the reverse direction see [HS88, DS99].

In the discussions below we will basically encounter two types of (Weyl) operators: quantum Hamiltonians $\mathcal{H} = \mathrm{op}^W[H]$ with symbols $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ generating the quantum mechanical time evolution, and observables $\mathcal{B} = \mathrm{op}^W[B]$. In typical cases a Hamiltonian $\mathcal{H}$ is given and one is interested in a suitable algebra of observables that allows to study dynamical properties of the quantum system. For this purpose it is often convenient to consider bounded operators. In the scalar case it is sufficient to know the boundedness of the symbols in order to obtain a bounded Weyl operator. This result, originally going back to Calderón and Vaillancourt [CV71], generalises to the context of pseudodifferential operators with matrix valued symbols without changes.

Proposition 1.4. *Let $B(\hbar) \in \mathrm{S}(1)$, then the Weyl quantisation* $\mathrm{op}^W[B(\hbar)]$ *of this symbol is continuous on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. Furthermore, for $\hbar \in (0, \hbar_0]$ there exists an upper bound for the operator norm of* $\mathrm{op}^W[B(\hbar)]$.

For a proof of this result in the context of semiclassical pseudodifferential operators (depending on a parameter $\hbar$) see [Rob87, HS88, DS99].

A quantum Hamiltonian is required to be (essentially) selfadjoint. Thus, in the case of a Weyl operator $\mathcal{H} = \mathrm{op}^W[H]$ one requires the symbol H to take values in the hermitian $n \times n$ matrices. In order to trace back spectral properties of $\mathcal{H}$ to properties of the principal symbol H_0 we will have to construct (asymptotic) inverses of $\mathcal{H} - z$ and relate them to $(H_0 - z)^{-1}$. In this context an operator $\mathcal{B} = \mathrm{op}^W[B]$ is called elliptic, if its symbol $B \in \mathrm{S}(m)$ is invertible, i.e., if the matrix inverse B^{-1} exists in $\mathrm{S}(m^{-1})$. In such a case one can construct a parametrix $Q \in \mathrm{S}(m^{-1})$ which is an asymptotic inverse of B in the sense of symbol products,

$$B\#Q \sim Q\#B \sim 1.$$

To see that such an inverse exists for elliptic operators, consider

$$\mathrm{op}^W[B]\,\mathrm{op}^W[B^{-1}] = 1 - \hbar\,\mathrm{op}^W[R],$$

with $R \in \mathrm{S}(m)$. For sufficiently small $\hbar$ the operator $1 - \hbar\,\mathrm{op}^W[R]$ possesses a bounded inverse and one can define a (left and right) inverse $\mathrm{op}^W[B^{-1}](1 - \hbar\,\mathrm{op}^W[R])^{-1}$ for $\mathrm{op}^W[B]$. Furthermore, the Beals characterisation of pseudodifferential operators (Lemma 1.3) implies that this inverse is again a bounded pseudodifferential operator, see also [DS99]. To obtain an asymptotic expansion for the parametrix Q one next defines the operator $\mathcal{Q}_N := \mathrm{op}^W[B^{-1}](1 + \hbar\mathcal{R} + \cdots + \hbar^N\mathcal{R}^N)$, with $\mathcal{R} = \mathrm{op}^W[R]$, which is equivalent to $\mathcal{Q} = \mathrm{op}^W[Q]$ modulo terms of order $\hbar^{N+1}$. One can hence write

$$Q \sim B^{-1} + \hbar(B^{-1}\#R) + \hbar^2(B^{-1}\#R\#R) + \cdots, \tag{1.3}$$

and finally observes:

Lemma 1.5. *Let $B \in \mathrm{S}(m)$ be elliptic in the sense that $B^{-1}(x, \xi)$ exists for all $(x, \xi) \in \mathrm{T}^*\mathbb{R}^d$ and is in the class $\mathrm{S}(m^{-1})$. Then there exists a parametrix $Q \in \mathrm{S}(m^{-1})$ with an asymptotic expansion of the form (1.3) such that*

$$B\#Q \sim Q\#B \sim 1.$$

From (1.3) one moreover observes that an elliptic operator with classical symbol has a parametrix that is a classical symbol.

Frequently it is very convenient to have a functional calculus of pseudodifferential operators available. In some places, e.g., we would like to apply the Helffer-Sjöstrand formula, which shows that a smooth and compactly supported function $f \in C_0^\infty(\mathbb{R})$ of an essentially selfadjoint operator $\mathcal{B}$ with symbol in $B \in \mathrm{S}(m)$ yields a pseudodifferential operator

$$f(\mathcal{B}) = -\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{f}(z)(\mathcal{B} - z)^{-1}\,\mathrm{d}z,$$

whose symbol is in $\mathrm{S}(m^{-N})$ for every $N \in \mathbb{N}$. Here $\tilde{f} \in C_0^\infty(\mathbb{C})$ is an almost-analytic extension of f, i.e., $\tilde{f}(z) = f(z)$ for $z \in \mathbb{R}$ and $|\overline{\partial}\tilde{f}(z)| \leq C_N |\mathrm{Im}\, z|^N$ for all $N \in \mathbb{N}_0$. In the scalar case these results were shown in [HS89] (see also [DS99]) and have been extended to the matrix valued situation in [Dim93, Dim98]. A criterion that guarantees the essential selfadjointness of $\mathcal{B}$ is that first its symbol $B \in \mathrm{S}(m)$ is hermitian and, second, that $B + \mathrm{i} \in \mathrm{S}(m)$ is elliptic in the sense described above. If $B \in \mathrm{S}^0_{\mathrm{cl}}(m)$ one can even write down a classical asymptotic expansion for the symbol of the operator $f(\mathcal{B})$ whose principal symbol reads $f(B_0)$, where B_0 denotes the principal symbol of $\mathcal{B}$, see [Rob87, DS99].

2 Semiclassical projections

We motivate the following construction of semiclassical projection operators by considering the time evolution generated by a quantum Hamiltonian $\mathcal{H}$, i.e., the Cauchy problem

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(t) = \mathcal{H}\psi(t) \tag{2.1}$$

for an essentially selfadjoint operator $\mathcal{H}$ defined on a dense domain $D(\mathcal{H})$ in the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. If one introduces the strongly continuous one-parameter group of unitary operators $\mathcal{U}(t) := \exp(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t)$, $t \in \mathbb{R}$, a solution of (2.1) can be obtained by defining $\psi(t) := \mathcal{U}(t)\psi_0$ for $\psi_0 \in D(\mathcal{H})$. Therefore the time evolution $\mathcal{B}(t) := \mathcal{U}(t)^*\mathcal{B}\mathcal{U}(t)$ of a bounded operator $\mathcal{B} \in B(L^2(\mathbb{R}^d) \otimes \mathbb{C}^n)$ in the Heisenberg picture has to fulfill the (Heisenberg) equation of motion

$$\frac{\partial}{\partial t}\mathcal{B}(t) = \frac{\mathrm{i}}{\hbar}[\mathcal{H}, \mathcal{B}(t)]. \tag{2.2}$$

If one assumes $\mathcal{B}$ and $\mathcal{H}$ to be semiclassical pseudodifferential operators with symbols in the classes $\mathrm{S}^q_{\mathrm{cl}}(1)$ and $\mathrm{S}^0_{\mathrm{cl}}(m)$, respectively, equation (2.2) yields in leading semiclassical order an equation for the principal symbols:

$$\frac{\partial}{\partial t}B_0(t) = \frac{\mathrm{i}}{\hbar}[H_0, B_0(t)] + O(\hbar^0), \quad \hbar \to 0.$$

If one now requires the time evolution to respect the filtration of the algebra $\mathrm{S}^\infty_{\mathrm{cl}}(1) := \bigcup_{q\in\mathbb{Z}} \mathrm{S}^q_{\mathrm{cl}}(1)$ then, in particular, the principal symbol $B_0(t)$ should stay in its class that derives from the associated grading $\mathrm{S}^q_{\mathrm{cl}}(1)/\mathrm{S}^{q-1}_{\mathrm{cl}}(1)$, $q \in \mathbb{Z}$. One thus has to restrict to operators whose principal parts B_0 commute with the principal symbol H_0 of the operator $\mathcal{H}$. This condition is equivalent to a block-diagonal form of B_0,

$$B_0(x,\xi) = \sum_{\mu=1}^{l} P_{\mu,0}(x,\xi)B_0(x,\xi)P_{\mu,0}(x,\xi), \tag{2.3}$$

with respect to the projection matrices $P_{\mu,0} : \mathrm{T}^*\mathbb{R}^d \to \mathrm{M}_n(\mathbb{C})$, $\mu = 1,\ldots,l$, onto the eigenspaces corresponding to the eigenvalue functions $\lambda_\mu : \mathrm{T}^*\mathbb{R}^d \to \mathbb{R}$ of the hermitian principal symbol matrix $H_0 : \mathrm{T}^*\mathbb{R}^d \to \mathrm{M}_n(\mathbb{C})$. Since (2.3) is the semiclassical limit of the symbol of the operator $\hbar^q \sum_{\mu=1}^{l} \mathrm{op}^W[P_{\mu,0}]\mathcal{B}\,\mathrm{op}^W[P_{\mu,0}]$, when $\mathcal{B}$ is a semiclassical Weyl operator with symbol $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$, one can ask how the symbols $P_{\mu,0}$, which are projectors onto the eigenspaces of H_0 in $\mathbb{C}^n$, are related to projection operators onto (almost) invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ with respect to $\mathcal{H} = \mathrm{op}^W[H]$. We are hence looking for quantisations $\tilde{\mathcal{P}}_\mu$ of symbols $P_\mu \in \mathrm{S}^0_{\mathrm{cl}}(1)$, with principal symbols $P_{\mu,0}$, which are (almost) orthogonal projections, i.e.,

$$\tilde{\mathcal{P}}_\mu \tilde{\mathcal{P}}_\mu = \tilde{\mathcal{P}}_\mu = \tilde{\mathcal{P}}_\mu^* \mod O(\hbar^\infty) \tag{2.4}$$

in the operator norm. Moreover, in order that these operators map to almost invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ with respect to the time evolution $\mathcal{U}(t) = \exp(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t)$ generated by $\mathcal{H}$, we require them to fulfill

$$\|[\mathcal{H}, \tilde{\mathcal{P}}_\mu]\|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^n} = 0 \mod O(\hbar^\infty). \tag{2.5}$$

As it will turn out, it is even possible to modify the operators $\tilde{\mathcal{P}}_\mu$ in such a way that they satisfy the relation (2.4) exactly, i.e., not only mod $O(\hbar^\infty)$.

The above requirements lead us to consider (formal) asymptotic expansions for the symbols P_μ,

$$P_\mu(x,\xi) \sim \sum_{j=0}^{\infty} \hbar^j P_{\mu,j}(x,\xi),$$

which satisfy (2.4) and (2.5) on a (formal) symbol level:

$$P_\mu \# P_\mu \sim P_\mu \sim P_\mu^*, \tag{2.6}$$

and

$$[P_\mu, H]_\# := P_\mu \# H - H \# P_\mu \sim 0. \tag{2.7}$$

The solutions of the above equations will be called semiclassical projections and can be constructed by two different methods. The first one is based on solving the recursive problem arising from (2.6) and (2.7) by employing asymptotic expansions of P_μ and H in $\mathrm{S}^0_{\mathrm{cl}}(1)$ and $\mathrm{S}^0_{\mathrm{cl}}(m)$, respectively, using the symbolic calculus outlined in section 1 and finally equating equal powers of the semiclassical parameter $\hbar$. For this procedure cf. [EW96, BN99]. The second method employs the Riesz projection formula in the context of pseudodifferential calculus [HS88, NS01]. In the following we will pursue the latter method.

To this end we consider the matrix valued hermitian principal symbol $H_0 \in \mathrm{S}(m)$ of the operator $\mathcal{H}$, and in the following we assume:

(H0) The (real) eigenvalues λ_μ, $\mu = 1,\dots,l$, of H_0 have constant multiplicities $k_1,\dots,k_l$ and fulfill the hyperbolicity condition

$$|\lambda_\nu(x,\xi) - \lambda_\mu(x,\xi)| \geq Cm(x,\xi) \quad \text{for} \quad \nu \neq \mu \quad \text{and} \quad |x| + |\xi| \geq c.$$

This requirement is analogous to a condition imposed in [Cor82] on the eigenvalues of the symbol of an operator in a strictly hyperbolic system, i.e., where the eigenvalues are non-degenerate. In particular, the problem of mode conversion that arises from points where multiplicities of eigenvalues change is avoided. Since the eigenvalues are solutions of the algebraic equation

$$\det\big(H_0(x,\xi)-\lambda\big)=\sum_{\nu=0}^{n}\eta_\nu(x,\xi)\lambda^\nu=0, \tag{2.8}$$

they are smooth functions on $\mathrm{T}^*\mathbb{R}^d$. Moreover, since H_0 is supposed to be hermitian, the eigenvalues are bounded by the matrix norm of H_0. Using the smoothness of the eigenvalues and the hyperbolicity condition (H0), one obtains:

Proposition 2.1. *Let $H\in\mathrm{S}^0_{\mathrm{cl}}(m)$ be hermitian and let the hyperbolicity condition (H0) be fulfilled. Then there exist symbols $P_\mu\in\mathrm{S}^0_{\mathrm{cl}}(1)$ with asymptotic expansions*

$$P_\mu\sim\sum_{j=0}^{\infty}\hbar^j P_{\mu,j},\quad \mu=1,\dots,l, \tag{2.9}$$

that fulfill the conditions (2.6) and (2.7). In particular, the coefficients $P_{\mu,j}$, $j\in\mathbb{N}_0$, are unique, i.e., the symbols P_μ are uniquely determined modulo $\mathrm{S}^{-\infty}(1)$.

Furthermore, the corresponding almost projection operators $\tilde{\mathcal{P}}_\mu=\mathrm{op}^W[P_\mu]$ provide a semiclassical resolution of the identity on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^d$,

$$\tilde{\mathcal{P}}_1+\dots+\tilde{\mathcal{P}}_l=\mathrm{id}_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^n}\mod O(\hbar^\infty).$$

Proof. We use the technique of [HS88, NS01] and consider the Riesz projections

$$P_\mu(x,\xi):=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\mu(x,\xi)}Q(x,\xi,z)\,\mathrm{d}z, \tag{2.10}$$

where $\Gamma_\mu(x,\xi)$ is a simply closed and positively oriented regular curve in the complex plane enclosing the, and only the, eigenvalue $\lambda_\mu(x,\xi)\in\mathbb{R}$ of $H_0(x,\xi)$. Moreover, $Q(x,\xi,z)$ denotes a parametrix for $H-z$, i.e., $(H-z)\#Q\sim Q\#(H-z)\sim 1$ that will be constructed below. For technical considerations one may choose the contour as $\Gamma_\mu(x,\xi)=\{\lambda_\mu(x,\xi)+\rho_\mu(x,\xi)\,\mathrm{e}^{\mathrm{i}\varphi},\ 0\le\varphi\le 2\pi\}$ with $0<c\le\rho_\mu<\frac{1}{2}\min_{\nu\neq\mu}\{|\lambda_\mu-\lambda_\nu|\}$. Since H_0 is hermitian with eigenvalues λ_ν, $\nu=1,\dots,l$, one can estimate the matrix norm of $(H_0-z)^{-1}$ for $z\in\Gamma_\mu(x,\xi)$ as

$$\|(H_0(x,\xi)-z)^{-1}\|_{n\times n}\le\frac{C}{\rho_\mu(x,\xi)}.$$

The condition (H0) then allows to choose $\rho_\mu(x,\xi)\ge cm(x,\xi)$, so that H_0-z is elliptic for $z\in\Gamma_\mu$. If then $\hbar$ is sufficiently small, also $H-z=H_0-z+O(\hbar)$ is elliptic and the relation

$$\big(H-z\big)\#\big(H_0-z\big)^{-1}=1-\hbar R$$

enables one to construct a parametrix $Q(x,\xi,z) \in \mathrm{S}^0_{\mathrm{cl}}(m^{-1})$ for $H - z$ with asymptotic expansion

$$Q(x,\xi,z) \sim \sum_{j=0}^{\infty} \hbar^j Q_j(x,\xi,z)$$

in the same manner as in (1.3), see also [Rob87, DS99]. Plugging this expansion into (2.10) one obtains

$$\begin{aligned} P_\mu(x,\xi) &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\mu(x,\xi)} Q(x,\xi,z)\ \mathrm{d}z \\ &\sim \sum_{j=0}^{\infty} \hbar^j \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\mu(x,\xi)} Q_j(x,\xi,z)\ \mathrm{d}z =: \sum_{j=0}^{\infty} \hbar^j P_{\mu,j}(x,\xi) \end{aligned} \tag{2.11}$$

by using the Borel construction to sum asymptotic series of symbols. According to the properties of the Riesz integral the symbols P_μ therefore fulfill (2.6) and (2.7). Since these equations have unique solutions modulo $O(\hbar^\infty)$ [EW96], the coefficients $P_{\mu,j}$ are unique.

We now consider more general $z \in \mathbb{C}$, and by inspecting the above construction notice that the parametrix $Q(z)$ is well-defined as long as z has a sufficiently large distance from the eigenvalues of H_0. According to equation (1.3) its asymptotic expansion then reads

$$Q(z) \sim (H_0 - z)^{-1} + \hbar (H_0 - z)^{-1} \# R(z) \# (\mathrm{id}_{\mathbb{C}^n} + \hbar R(z) + \hbar^2 R(z) \# R(z) + \cdots).$$

Since

$$R(z) = \frac{1}{\hbar}\left(1 - (H - z)\#(H_0 - z)^{-1}\right),$$

it follows according to the composition formula of Lemma 1.2 that $R(z)$ contains a factor $(H_0 - z)^{-1}$, and therefore the only singularities of $Q(z)$ are caused by the eigenvalues of H_0. Thus, according to the Cauchy formula the expression

$$P_1 + \cdots + P_l = \frac{1}{2\pi \mathrm{i}} \int_{\bigcup_{\mu=1}^{l} \Gamma_\mu} Q(z)\ \mathrm{d}z$$

can be replaced by

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma(r)} Q(z)\ \mathrm{d}z,$$

where $\Gamma(r)$ is a contour with minimal distance r from the origin in $\mathbb{C}$ that encloses all eigenvalues of H_0 while keeping a sufficient distance from them. The value of the above integral does not depend on the particular choice of $\Gamma(r)$ so that one can take the limit $r \to \infty$ and hence obtains

$$\lim_{r\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma(r)} Q(z)\ \mathrm{d}z = \lim_{r\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma(r)} (H_0 - z)^{-1}\ \mathrm{d}z = \mathrm{id}_{\mathbb{C}^n} \mod O(\hbar^\infty).$$

□

The so constructed symbols P_μ yield semiclassical almost projection operators

$$\tilde{\mathcal{P}}_\mu := \mathrm{op}^W[P_\mu]$$

which according to Proposition 1.4 are bounded and obviously satisfy the relations (2.4) and (2.5). Following [Nen99] one can even construct pseudodifferential operators $\mathcal{P}_\mu$ that are semiclassically equivalent to $\tilde{\mathcal{P}}_\mu$ in the sense that $\|\tilde{\mathcal{P}}_\mu - \mathcal{P}_\mu\| = O(\hbar^\infty)$, and which fulfill (2.4) exactly. To see this, consider the operator

$$\mathcal{P}_\mu := \frac{1}{2\pi \mathrm{i}} \int_{|z-1|=\frac{1}{2}} (\tilde{\mathcal{P}}_\mu - z)^{-1} \, \mathrm{d}z,$$

which is well-defined since the spectrum of $\tilde{\mathcal{P}}_\mu$ is concentrated near 0 and 1. Thus $\mathcal{P}_\mu$ is an orthogonal projector acting on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, with $\|[\mathcal{P}_\mu, \mathcal{H}]\| \leq c\|[\tilde{\mathcal{P}}_\mu, \mathcal{H}]\| = O(\hbar^\infty)$. Since $\mathcal{P}_\mu$ is close to $\tilde{\mathcal{P}}_\mu$ in operator norm, Beals' characterisation of pseudodifferential operators (see Lemma 1.3) yields that $\mathcal{P}_\mu$ is again a pseudodifferential operator with symbol in the class $\mathrm{S}^0(1)$. This has already been noticed in [NS01] and follows from the fact that $(\tilde{\mathcal{P}}_\mu - z)^{-1}$ for $|z-1| = 1/2$ is a pseudodifferential operator according to the parametrix construction of Lemma 1.5. Having projectors available, one can also construct (pseudo-differential) unitary transformations of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$ which convert $\mathcal{H}$ by conjugation in an almost block-diagonal form, see [Cor83b, LF91, BR99, NS01, PST02]. Such unitary transformations are obviously not unique, and since for most purposes it suffices to work with the projectors we hence refrain from using the unitary operators here.

In view of the fact that $\mathcal{P}_\mu$ is an orthogonal projector on the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, one can ask if it is possible to satisfy also the relation (2.5) exactly. In other words, to what extent can $\mathcal{P}_\mu$ be related to a spectral projection of $\mathcal{H}$? (See [HS88, Cor00, Cor01] for examples.) We want to illustrate this question in the case where the principal symbol H_0 of $\mathcal{H}$ possesses two well-separated eigenvalues $\lambda_\nu < \lambda_{\nu+1}$ with constant multiplicities k_ν and $k_{\nu+1}$, respectively, among the eigenvalues $\lambda_1, \dots, \lambda_l$. For $l = 2$ this is exactly the situation that occurs in the case of a Dirac-Hamiltonian that we will discuss in some detail elsewhere [BG]. We also assume that there exists $\lambda \in \mathbb{R}$ separated from the spectrum $\mathrm{spec}(\mathcal{H})$ of $\mathcal{H}$ along with a fixed compact subset $W \subset \mathrm{T}^*\mathbb{R}^d$ such that

$$\lambda - \lambda_\nu(x,\xi) > Cm(x,\xi) \quad \text{and} \quad \lambda_{\nu+1}(x,\xi) - \lambda > C'm(x,\xi) \tag{2.12}$$

for all $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d \setminus W$. It follows from these assumptions that for $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d \setminus W$ one can replace the contour $\Gamma_< := \bigcup_{\mu=1}^{\nu} \Gamma_\mu$ in

$$P_<(x,\xi) := \sum_{\mu=1}^{\nu} P_\mu(x,\xi) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_<} Q(x,\xi,z) \, \mathrm{d}z, \tag{2.13}$$

see (2.10), by a straight line $\Gamma_+ := \{z \in \mathbb{C};\ z = \lambda + \mathrm{i}t,\ t \in \mathbb{R}\}$ that avoids the eigenvalues of the principal symbol H_0 as well as the spectrum of $\mathcal{H}$. Correspondingly, $\Gamma_> := \bigcup_{\mu=\nu+1}^{l} \Gamma_\mu$

is deformed into Γ_- given by Γ_+ with reversed orientation. Thus, for $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d \setminus W$

$$P_{\lessgtr}(x,\xi) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\pm} Q(x,\xi,z)\ \mathrm{d}z.$$

If this relation held true for all $(x,\xi) \in \mathrm{T}^*\mathbb{R}^d$, $\tilde{\mathcal{P}}_< = \mathrm{op}^W[P_<]$ would be semiclassically equivalent to the spectral projection of $\mathcal{H}$ onto the interval $(-\infty,\lambda)$ given by

$$\mathbb{1}_{(-\infty,\lambda)}(\mathcal{H}) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_+} (\mathcal{H}-z)^{-1}\ \mathrm{d}z,$$

whereas $\tilde{\mathcal{P}}_>$ would correspond to $\mathbb{1}_{(\lambda,\infty)}(\mathcal{H})$. For $(x,\xi) \in W$, however, it might happen that $\Gamma_\pm$ crosses an eigenvalue of H_0. But the contribution to $P_{\lessgtr}(x,\xi)$ coming from the region W, where the eigenvalue functions have no sufficient distance from λ, can be shown to be semiclassically small. Therefore:

Proposition 2.2. *If the eigenvalues $\lambda_1,\dots,\lambda_l$ of the principal symbol H_0 are separated according to (H0) and the condition (2.12) is fulfilled, the almost projection operators $\tilde{\mathcal{P}}_{\lessgtr} := \mathrm{op}^W[P_{\lessgtr}]$, whose symbols are defined in (2.13), can be semiclassically identified with the spectral projections $\mathbb{1}_{(-\infty,\lambda)}(\mathcal{H})$ and $\mathbb{1}_{(\lambda,\infty)}(\mathcal{H})$ of the operator $\mathcal{H}$ to the intervals $(-\infty,\lambda)$ and (λ,∞), respectively. This means*

$$\|\tilde{\mathcal{P}}_< - \mathbb{1}_{(-\infty,\lambda)}(\mathcal{H})\| = O(\hbar^\infty) \quad \textit{and} \quad \|\tilde{\mathcal{P}}_> - \mathbb{1}_{(\lambda,\infty)}(\mathcal{H})\| = O(\hbar^\infty).$$

A corresponding statement holds for the related orthogonal projectors $\mathcal{P}_{\lessgtr}$,

$$\|\mathcal{P}_< - \mathbb{1}_{(-\infty,\lambda)}(\mathcal{H})\| = O(\hbar^\infty) \quad \textit{and} \quad \|\mathcal{P}_> - \mathbb{1}_{(\lambda,\infty)}(\mathcal{H})\| = O(\hbar^\infty).$$

Proof. We use that fact that the contour of integration in the definition of $P_{\lessgtr}$ in (2.13) can be deformed into $\Gamma_\pm$ outside the compact region $W \subset \mathrm{T}^*\mathbb{R}^d$. To cut off the region W we choose a smooth and compactly supported function $\chi \in C_0^\infty(\mathrm{T}^*\mathbb{R}^d)$ equal to one on W and use the corresponding partition of unity, $1 = \chi + (1-\chi)$, to write

$$(2\pi\mathrm{i}) P_{\lessgtr}(x,\xi) \sim \chi(x,\xi) \# \int_{\Gamma_{\lessgtr}} Q(x,\xi,z)\mathrm{d}z + (1-\chi(x,\xi)) \# \int_{\Gamma_{\lessgtr}} Q(x,\xi,z)\ \mathrm{d}z.$$

In the second contribution, whose support is contained in $\mathrm{T}^*\mathbb{R}^d \setminus W$, one can replace $\Gamma_{\lessgtr}$ by $\Gamma_\pm$. We are thus left with the first term, which represents a symbol $p(x,\xi)$ in $\mathrm{S}^0_{\mathrm{cl}}(1)$ with compact support $\operatorname{supp} p \subset \operatorname{supp}\chi$. Here we apply a translation on $\mathrm{T}^*\mathbb{R}^d$ mapping $p(x,\xi)$ to $\tilde{p}(x,\xi) := p(x-x_0,\xi-\xi_0)$, with x_0 and ξ_0 chosen such that $\xi = 0$ is no longer contained in the support of $\tilde{p}$, i.e., $\tilde{p}(x,0) = 0$ for all $x \in \mathrm{T}^*\mathbb{R}^d$. Using the fact that the Weyl operators corresponding to p and $\tilde{p}$ are unitarily equivalent, see [Fol89], we therefore obtain

$$\|\mathrm{op}^W[p]\| = \|\mathrm{op}^W[\tilde{p}]\|.$$

In order to estimate $\mathrm{op}^W[\tilde{p}]$ in operator norm, we consider its action on a function $\psi \in \mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ given by

$$\big(\mathrm{op}^W[\tilde{p}]\psi\big)(x) = \frac{1}{(2\pi\hbar)^d} \iint \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\xi} \tilde{p}\Big(\frac{x+y}{2}, \xi\Big)\psi(y)\, \mathrm{d}y\, \mathrm{d}\xi.$$

Since the symbol $\tilde{p}$ vanishes in a neighbourhood of $\xi = 0$, one can perform an integration by parts after having inserted the operator $(\mathrm{i}\hbar|\xi|^{-2}(\xi \cdot \partial_y))^N$, $N \in \mathbb{N}$. We thus obtain that $\mathrm{op}^W[\tilde{p}]\psi$ vanishes up to terms of order $\hbar^\infty$ and, since $\mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ is dense in $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, conclude

$$\| \mathrm{op}^W[\tilde{p}]\| = \| \mathrm{op}^W[p]\| = O(\hbar^\infty).$$

To finish the proof we now have to estimate

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\pm} (\mathcal{H} - z)^{-1}\, \mathrm{d}z - \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\pm} \mathrm{op}^W[(1-\chi)\#Q(x,\xi,z)]\, \mathrm{d}z$$

in operator norm. To this end consider for $z \in \Gamma_\pm$

$$\big((\mathcal{H} - z)^{-1} - \mathrm{op}^W[1-\chi]\, \mathrm{op}^W[Q(x,\xi,z)]\big)(\mathcal{H} - z) = (1 - \mathrm{op}^W[1-\chi]) + O(\hbar^\infty) = O(\hbar^\infty),$$

which holds since χ has compact support, and thus $\mathrm{op}^W[\chi]$ can be treated in the same manner as $\mathrm{op}^W[p]$ above. At this point the proof is complete, since for $z \in \Gamma_\pm$ the operator $(\mathcal{H} - z)$ is invertible and its inverse has a norm that exceeds $O(\hbar^\infty)$. □

3 Invariant algebra and Egorov theorem

In this section our aim is to identify a suitable class of operators that is left invariant by the time evolution. Recalling the reasoning from the beginning of section 2, we are interested in a subalgebra of $\mathrm{S}^\infty_{\mathrm{cl}}(1)$ whose filtration is respected by the time evolution generated by the one-parameter group $\mathcal{U}(t) = \exp\big(-\frac{\mathrm{i}}{\hbar}\mathcal{H}t\big)$, where $\mathcal{H}$ is an essentially selfadjoint pseudodifferential operator with symbol H in the class $\mathrm{S}^0_{\mathrm{cl}}(m)$. The following assumptions on the symbol H guarantee the essential selfadjointness of $\mathcal{H}$ on $\mathscr{S}(\mathbb{R}^d) \otimes \mathbb{C}^n$ (see [DS99]):

(H1) $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ is hermitian,

(H2) $H_0 + \mathrm{i}$ is elliptic in the sense that $\|(H_0(x,\xi) + \mathrm{i})^{-1}\|_{n\times n} \leq c m(x,\xi)^{-1}$.

Under the assumptions (H1) and (H2), $\mathcal{U}(t)$ therefore defines a strongly-continuous unitary one-parameter group.

We now consider the time evolution of an operator $\mathcal{B} \in B(L^2(\mathbb{R}^d) \otimes \mathbb{C}^n)$ given by

$$\mathcal{B}(t) := \mathcal{U}(t)^*\mathcal{B}\mathcal{U}(t), \tag{3.1}$$

which is, of course, a bounded operator on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. According to Proposition 1.4 the boundedness of $\mathcal{B}$ is guaranteed by choosing $B\in \mathrm{S}^q_{\mathrm{cl}}(1)$. Moreover, a conjugation of (3.1) with $\sum_{\mu=1}^l \mathcal{P}_\mu = \mathrm{id}_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^n} + O(\hbar^\infty)$ results in a bounded operator so that

$$\mathcal{B}(t) = \sum_{\nu,\mu=1}^{l} \mathcal{P}_\mu \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\mu t}\mathcal{B}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\mathcal{P}_\nu = \sum_{\nu,\mu=1}^{l} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\mu t}\mathcal{P}_\mu\mathcal{B}\mathcal{P}_\nu\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t} \mod O(\hbar^\infty) \tag{3.2}$$

in the operator norm. Here we have used the property $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}t}\mathcal{P}_\nu = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\mathcal{P}_\nu$ modulo $O(\hbar^\infty)$ that follows from the Duhamel principle. Now, the principal symbol[3] of $\mathcal{H}\mathcal{P}_\mu$ is a scalar multiple of the identity in the eigenspace $P_{\mu,0}\mathbb{C}^n$ of H_0 corresponding to λ_μ, i.e., $H_0P_{\mu,0} = \lambda_\mu P_{\mu,0}$. Thus, for $\mu=\nu$ the operator $\exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\mu t\right)\mathcal{B}\exp\left(-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t\right)$ is a pseudodifferential operator with symbol in the class $\mathrm{S}^0(1)$, see [Ivr98, BG00]. But when $\mu\neq\nu$ the corresponding expressions are semiclassical Fourier integral operators. In that case the semiclassical limit at time $t\neq 0$ is different in nature from that at time zero. For a Dirac-Hamiltonian this phenomenon is related to the so-called "Zitterbewegung" which we will discuss in more detail in [BG]. Therefore, we are here interested in operators $\mathcal{B}$ with symbols in $B\in\mathrm{S}^q_{\mathrm{cl}}(1)$ for which $\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$ is again a semiclassical pseudodifferential operator with symbol $B(t)\in\mathrm{S}^q_{\mathrm{cl}}(1)$. We hence introduce the following notion:

Definition 3.1. A symbol $B\in\mathrm{S}^q_{\mathrm{cl}}(1)$ is in the invariant subalgebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ of the algebra $\mathrm{S}^\infty_{\mathrm{cl}}(1)$, if and only if for all finite t the (bounded) operator $\mathcal{B}(t)=\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$, $\mathcal{B}=\mathrm{op}^W[B]$, is a semiclassical pseudodifferential operator with symbol $B(t)\in\mathrm{S}^q_{\mathrm{cl}}(1)$, i.e.,

$$\mathrm{S}^\infty_{\mathrm{inv}}(1) := \left\{B\in\mathrm{S}^q_{\mathrm{cl}}(1)\,;\,\mathrm{symb}^W[\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)]\in\mathrm{S}^q_{\mathrm{cl}}(1)\quad\text{for}\quad t\in[0,T],\ q\in\mathbb{Z}\right\}.$$

This means that the invariant algebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ has a filtration, induced by the filtration of $\mathrm{S}^\infty_{\mathrm{cl}}(1)$, which is invariant under conjugation of the corresponding operators with $\mathcal{U}(t)$. Due to the results of [BG00] we expect that operators which are block-diagonal with respect to the projections $\mathcal{P}_\mu$ are in the associated invariant operator algebra. This statement is made precise in Theorem 3.2 which is a variant of the Egorov theorem [Ego69] for general hyperbolic systems.

Let us first consider an operator $\mathcal{B}$ with symbol $B\in\mathrm{S}^q_{\mathrm{cl}}(1)$ that is block-diagonal with respect to the semiclassical projections, i.e.,

$$B\sim\sum_{\mu=1}^{l} P_\mu\# B\# P_\mu \quad\text{in}\quad \mathrm{S}^q_{\mathrm{cl}}(1).$$

According to the Heisenberg equation of motion (2.2) its time evolution $B(t)$ is governed by

$$\frac{\partial}{\partial t}B(t)\sim\frac{\mathrm{i}}{\hbar}[H,B(t)]_{\#}. \tag{3.3}$$

[3]We remark that before transferring operators to symbol level one can replace $\mathcal{P}_\mu$ by $\tilde{\mathcal{P}}_\mu$ and employ the classical asymptotic expansion of the symbol P_μ. This will only amount to an error of order $\hbar^\infty$.

Suppose now that $B(t)$ has a (formal) asymptotic expansion

$$B(t) \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_j$$

and use the composition formula of Lemma 1.2 together with the fact that the block-diagonal form of an operator $\mathcal{B}$ is preserved under the time evolution, see (3.2). On the symbol level the diagonal blocks $\mathcal{P}_\nu \mathcal{B}(t) \mathcal{P}_\nu$ then obey the following equation:

$$\frac{\partial}{\partial t} \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_{\nu\nu,j} \sim \\ \sum_{l,j=0}^{\infty} \sum_{|\alpha|+|\beta|\geq 0} \gamma(\alpha,\beta) \hbar^{-q+l+j+|\alpha|+|\beta|-1} \left(B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} \, H_{\nu,j}{}^{(\alpha)}_{(\beta)} - (-1)^{|\alpha|-|\beta|} H_{\nu,j}{}^{(\alpha)}_{(\beta)} B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} \right).$$

Here we introduced the notation $F^{(\alpha)}_{(\beta)} := \partial_\xi^\alpha \partial_x^\beta F$ for $F \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$, as well as

$$\gamma(\alpha,\beta) := \frac{\mathrm{i}^{|\alpha|-|\beta|-1}}{2^{|\alpha|+|\beta|}|\alpha|!|\beta|!},$$

$$H_\nu := P_\nu \# H \# P_\nu \sim H \# P_\nu \sim \sum_{j=0}^{\infty} \hbar^j H_{\nu,j},$$

$$B(t)_{\nu\nu} := P_\nu \# B(t) \# P_\nu \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_{\nu\nu,j}.$$

One hence has to solve, by taking $[H_{\nu,0}, B(t)_{\nu\nu,0}] = 0$ into account,

$$[H_{\nu,0}, B(t)_{\nu\nu,n+1}] = -\frac{\partial}{\partial t} B(t)_{\nu\nu,n} - \frac{1}{2}\Big(\{B(t)_{\nu\nu,n}, H_{\nu,0}\} - \{H_{\nu,0}, B(t)_{\nu\nu,n}\} \Big) - \mathrm{i}[B(t)_{\nu\nu,n}, H_{\nu,1}] \\ + \sum_{\substack{0\leq l\leq n-1 \\ j+|\alpha|+|\beta|=n-l+1}} \gamma(\alpha,\beta) \left(B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} H_{\nu,j}{}^{(\alpha)}_{(\beta)} - (-1)^{|\alpha|-|\beta|} H_{\nu,j}{}^{(\alpha)}_{(\beta)} B(t)_{\nu\nu,l}{}^{(\beta)}_{(\alpha)} \right). \tag{3.4}$$

Upon multiplying this commutator equation with the projection matrices $P_{\mu,0}$ from both sides one first realises that it is only solvable, if the diagonal blocks of the right-hand side, that we denote by $R_{n,\nu}(t)$, vanish. The off-diagonal blocks on both sides of the relation (3.4) then yield the general structure of the solution, which reads

$$B(t)_{\nu\nu,n+1} = \sum_{\mu=1}^{l} P_{\mu,0} B(t)_{\nu\nu,n+1} P_{\mu,0} + \sum_{\mu\neq\eta} \frac{P_{\mu,0} R_{n,\nu}(t) P_{\eta,0}}{\lambda_\mu - \lambda_\eta}, \tag{3.5}$$

see also [Cor95]. This demonstrates that one obtains the off-diagonal parts of $B(t)_{\nu\nu,n+1}$ with respect to the projection matrices $P_{\mu,0}$ from the preceding coefficients of the asymptotic expansion of $B(t)_{\nu\nu}$. The diagonal parts then have to be determined by the condition that the commutator equation (3.4) must possess a (non-trivial) solution with initial value $B(t)_{\nu\nu,n+1}|_{t=0} = B_{\nu\nu,n+1}$. Starting with $n = 0$, where the sum in (3.4) is empty, one has to solve

$$P_{\mu,0}\left(\frac{\partial}{\partial t}B(t)_{\nu\nu,0} + \frac{1}{2}\Big(\{B(t)_{\nu\nu,0}, H_{\nu,0}\} - \{H_{\nu,0}, B(t)_{\nu\nu,0}\}\Big) + \mathrm{i}[B(t)_{\nu\nu,0}, H_{\nu,1}]\right) P_{\mu,0} = 0.$$

Expressions of this type have already been considered in [Spo00], where it was shown that the above equation is equivalent to

$$\frac{\partial}{\partial t}(P_{\mu,0}B(t)_{\nu\nu,0}P_{\mu,0}) - \delta_{\nu\mu}\{\lambda_\nu, P_{\mu,0}B(t)_{\nu\nu,0}P_{\mu,0}\} - \mathrm{i}[\tilde{H}_{\nu\mu,1}, P_{\mu,0}B(t)_{\nu\nu,0}P_{\mu,0}] = 0, \tag{3.6}$$

see also appendix A. Here we have defined the hermitian $n \times n$ matrix

$$\tilde{H}_{\nu\mu,1} := \mathrm{i}(-1)^{\delta_{\nu\mu}}\frac{\lambda_\nu}{2}P_{\mu,0}\{P_{\nu,0}, P_{\nu,0}\}P_{\mu,0} - \mathrm{i}\delta_{\nu\mu}[P_{\nu,0}, \{\lambda_\nu, P_{\nu,0}\}] + P_{\mu,0}H_{\nu,1}P_{\mu,0} \tag{3.7}$$

according to (A.4) and (A.5) of appendix A. Now, equation (3.6) is trivially fulfilled for $\nu \neq \mu$, and the case $\nu = \mu$ has already been considered in [Ivr98, BN99, BG00], where it was shown that the solution reads

$$B(t)_{\nu\nu,0}(\xi, x) = d_{\nu\nu}^{-1}(x, \xi, t)B_{\nu\nu,0}\big(\Phi_\nu^t(x, \xi)\big)d_{\nu\nu}(x, \xi, t).$$

In this expression $\Phi_\nu^t : \mathrm{T}^*\mathbb{R}^d \to \mathrm{T}^*\mathbb{R}^d$ denotes the Hamiltonian flow generated by the eigenvalue λ_ν of H_0, and the transport matrix $d_{\nu\nu}$ is determined by the equation

$$\dot{d}_{\nu\nu}(x, \xi, t) + \mathrm{i}\tilde{H}_{\nu\nu,1}\big(\Phi_\nu^t(x, \xi)\big)d_{\nu\nu}(x, \xi, t) = 0, \quad d_{\nu\nu}(x, \xi, 0) = \mathrm{id}_{\mathbb{C}^n}. \tag{3.8}$$

One has thus fixed the coefficients $B(t)_{\nu\nu,0} = P_{\nu,0}B(t)_0P_{\nu,0}$, i.e., the principal symbol of $\mathcal{B}(t)$, since the off-diagonal terms $B(t)_{\nu\mu,0} = P_{\nu,0}B(t)_0P_{\mu,0}$ vanish and therefore trivially fulfill (3.4). According to (3.5) we hence have also determined the off-diagonal parts of the subprincipal term $B(t)_{\nu\nu,1}$, which vanish as well. The diagonal contributions $P_{\mu,0}B(t)_{\nu\nu,1}P_{\mu,0}$ with respect to the projection matrices obey $[P_{\eta,0}, P_{\mu,0}B(t)_{\nu\nu,1}P_{\mu,0}] = 0$ and thus can be determined from the relation (3.4). As in [Ivr98, BG00], we hence obtain a recursive Cauchy problem for the coefficients $B(t)_{\nu\nu,n}$ and are now in a position to state:

Theorem 3.2. *Let $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ be hermitian with the property*

$$\|H_j{}^{(\alpha)}_{(\beta)}(x, \xi)\|_{n\times n} \leq C_{\alpha,\beta} \quad \textit{for all} \quad (x, \xi) \in \mathrm{T}^*\mathbb{R}^d \ \textit{and}\ |\alpha| + |\beta| + j \geq 2 - \delta_{j0}, \tag{3.9}$$

and such that the conditions (H0) and (H2) are fulfilled. Furthermore, suppose that $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ *is block-diagonal with respect to the semiclassical projections defined in (2.9),*

$$B \sim \sum_{\mu=1}^{l} P_\mu \# B \# P_\mu .$$

Then B *is in the invariant algebra* $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ *introduced in Definition 3.1, i.e.,* $\mathcal{B}(t)$ *is again a semiclassical pseudodifferential operator with symbol* $B(t) \in \mathrm{S}^q_{\mathrm{cl}}(1)$*. Furthermore, the principal symbol of* $\mathcal{B}(t)$ *is given by*

$$B(t)_0(x,\xi) = \sum_{\nu=1}^{l} d^*_{\nu\nu}(x,\xi,t) B_{\nu\nu,0}\big(\Phi^t_\nu(x,\xi)\big) d_{\nu\nu}(x,\xi,t), \tag{3.10}$$

where Φ^t_ν *is the Hamiltonian flow generated by the eigenvalue* λ_ν *of* H_0*, and* $d_{\nu\nu}$ *is a unitary* $n \times n$ *matrix which is determined by the transport equation (3.8).*

Proof. As in [Ivr98, BG00] we start by rewriting (3.4) for the diagonal block of $B(t)_{\nu\nu,n}$ with respect to $P_{\mu,0}$ in the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[d^{-1}_{\nu\mu}(x,\xi,-t)(P_{\mu,0}B(t)_{\nu\nu,n}P_{\mu,0}) \circ \Phi_\nu^{-t\delta_{\nu\mu}}(x,\xi) d_{\nu\mu}(x,\xi,-t)\right] \\ &\quad = \sum_{\substack{0 \le l \le n-1 \\ j+|\alpha|+|\beta| = n-l+1}} \gamma(\alpha,\beta) P_{\mu,0} \left(B(t)_{\nu\nu,l}\,{}^{(\beta)}_{(\alpha)} H_{\nu,j}\,{}^{(\alpha)}_{(\beta)} - (-1)^{|\alpha|-|\beta|} H_{\nu,j}\,{}^{(\alpha)}_{(\beta)} B(t)_{\nu\nu,l}\,{}^{(\beta)}_{(\alpha)} \right) P_{\mu,0}, \end{aligned} \tag{3.11}$$

where $d_{\nu\mu}$ is determined by the transport equation

$$\dot{d}_{\nu\mu}(x,\xi,t) + \mathrm{i}\tilde{H}_{\nu\mu,1}(\Phi_\nu^{t\delta_{\nu\mu}}(x,\xi)) d_{\nu\mu}(x,\xi,t), \quad d_{\nu\mu}(x,\xi,0) = \mathrm{id}_{\mathbb{C}^n},$$

that generalises (3.8) also to the off-diagonal transport. And since $\tilde{H}_{\nu\mu,1}$ is hermitian, the solution $d_{\nu\mu}$ is a unitary $n \times n$ matrix, which in the case $\nu \neq \mu$ is obviously given by

$$d_{\nu\mu}(x,\xi,t) = \mathrm{e}^{-\mathrm{i}\tilde{H}_{\nu\mu,1}(x,\xi)t}.$$

In order to obtain estimates on the derivatives of the symbols $P_{\mu,0}B(t)_{\nu\nu,n}(t)P_{\mu,0}$ one has to control the behaviour of the flow Φ^t_ν generated by the eigenvalue λ_ν of H_0. To this end we first notice that $H_0 \in \mathrm{S}(m)$ implies the bound $|\lambda_\nu(x,\xi)| \le cm(x,\xi)$ on its eigenvalues. Furthermore, due to the hyperbolicity condition (H0) the projections $P_{\nu,0}$ onto the eigenspaces of H_0 are in $\mathrm{S}(1)$. We then consider the first order derivatives ($|\alpha| + |\beta| = 1$) of the relation

$$H_0(x,\xi)P_{\nu,0}(x,\xi) = \lambda_\nu(x,\xi)P_{\nu,0}(x,\xi),$$

which exist since the eigenvalues λ_ν are smooth functions on the phase space $\mathrm{T}^*\mathbb{R}^d$, see equation (2.8). One thus obtains

$$\lambda_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)P_{\nu,0}(x,\xi) = \big(H_0(x,\xi)P_{\nu,0}(x,\xi)\big)^{(\alpha)}_{(\beta)} - \lambda_\nu(x,\xi)P_{\nu,0}{}^{(\alpha)}_{(\beta)}(x,\xi).$$

Now, since $P_{\nu,0}(x,\xi)P_{\nu,0}{}^{(\alpha)}_{(\beta)}(x,\xi)P_{\nu,0}(x,\xi) = 0$, a multiplication of the above equation with $P_{\nu,0}(x,\xi)$ from both sides yields

$$\lambda_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)P_{\nu,0}(x,\xi) = P_{\nu,0}(x,\xi)H_0(x,\xi)^{(\alpha)}_{(\beta)}P_{\nu,0}(x,\xi), \tag{3.12}$$

and hence

$$\big|\lambda_\nu{}^{(\alpha)}_{(\beta)}\big| = c\big\|\lambda_\nu{}^{(\alpha)}_{(\beta)}P_{\nu,0}\big\|_{n\times n} = c\big\|P_{\nu,0}H_0{}^{(\alpha)}_{(\beta)}P_{\nu,0}\big\|_{n\times n} \leq \tilde{c}\big\|H_0{}^{(\alpha)}_{(\beta)}\big\|_{n\times n}.$$

$H_0 \in \mathrm{S}(m)$ therefore implies that the first order derivatives of λ_ν are bounded by the order function m. One can continue this argument by successively differentiating equation (3.12), and concludes that $\lambda_\nu\,\mathrm{id}_{\mathbb{C}^n} \in \mathrm{S}(m)$ for all $\nu = 1,\dots,l$. In particular, the property

$$\big\|H_0{}^{(\alpha)}_{(\beta)}(x,\xi)\big\|_{n\times n} \leq C_{\alpha,\beta} \quad \text{for} \quad |\alpha|+|\beta| \geq 1,$$

which follows from (3.9), transfers to a corresponding growth property of the eigenvalues of H_0:

$$\big|\lambda_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)\big| \leq C_{\alpha,\beta} \quad \text{for} \quad |\alpha|+|\beta| \geq 1.$$

Therefore, the Hamiltonian flows Φ^t_ν exist globally on $\mathrm{T}^*\mathbb{R}^d$ such that $|\Phi^t_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_{\alpha,\beta}$ for all α, $\beta \in \mathbb{N}_0^d$ and for all finite times $t \in [0,T]$, see [Rob87]. This property guarantees that $B \circ \Phi^t_\nu \in \mathrm{S}(1)$ for all $B \in \mathrm{S}(1)$. Concerning the unitary matrices $d_{\nu\mu}$ the following is true:

Lemma 3.3. *If the subprincipal symbol H_1 of $\mathcal{H}$ satisfies $\|H_1{}^{(\alpha)}_{(\beta)}\|_{n\times n} \leq C_{\alpha,\beta}$ for all $|\alpha|+|\beta| \geq 1$, then $\|d_{\nu\mu}{}^{(\alpha)}_{(\beta)}(x,\xi,t)\|_{n\times n} \leq C'_{\alpha,\beta}$ for all $t \in [0,T]$, $|\alpha|+|\beta| \geq 1$ and ν, $\mu = 1,\dots,l$.*

For the proof of this lemma see [Ivr98]. With these properties at hand one can integrate equation (3.11) and solve for $P_{\mu,0}B(t)_{\nu\nu,n}P_{\mu,0}$ by conjugating with $d_{\nu\mu}(x,\xi,-t)$ and shifting the arguments by $\Phi_\nu^{\delta_{\nu\mu}t}$ (which only amounts to an actual shift in the case $\nu = \mu$). For the principal symbol of $\mathcal{B}(t)$ one thus obtains

$$B(t)_{\nu\nu,0}(x,\xi) = d_{\nu\nu}(\Phi^t_\nu(x,\xi),-t)B_{\nu\nu,0}(\Phi^t_\nu(x,\xi))d^{-1}_{\nu\nu}(\Phi^t_\nu(x,\xi),-t),$$

which is the only block of $B(t)_{\nu\nu,0}$ with respect to $P_{\mu,0}$, $\mu = 1,\dots,l$, that is different from zero. Using

$$d_{\nu\mu}(\Phi_\nu^{t\delta_{\nu\mu}}(x,\xi),-t) = d^{-1}_{\nu\mu}(x,\xi,t) = d^*_{\nu\mu}(x,\xi,t), \tag{3.13}$$

see [BN99], one finally obtains (3.10). For the higher coefficients $B(t)_{\nu\nu,n}$, $n \geq 1$, one employs the Duhamel principle and uses that fact that the sum in (3.11) is taken over indices with $|\alpha| + |\beta| + j \geq 2$, and thus involves terms in $\mathrm{S}(1)$, in order to conclude that $B(t)_{\nu\nu,n} \in \mathrm{S}(1)$. This shows that one has found an asymptotic expansion in $\mathrm{S}^q_{\mathrm{cl}}(1)$ for the symbol of $\mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$ that can be summed with the Borel method to yield a complete symbol. $\square$

This theorem shows that, for finite times t, one can associate to a (semiclassically) block-diagonal symbol $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ a symbol $B(t) \in \mathrm{S}^q_{\mathrm{cl}}(1)$ whose quantisation $\mathrm{op}^W[B(t)]$ is semiclassically close to $\mathcal{B}(t) = \mathcal{U}^*(t)\mathcal{B}\mathcal{U}(t)$, i.e.,

$$\|\mathcal{B}(t) - \mathrm{op}^W[B(t)]\| = O(\hbar^\infty) \quad \text{for all} \quad t \in [0, T].$$

This is a semiclassical version of the Egorov theorem [Ego69], which was originally formulated for the case of scalar symbols. A weaker version that is also sometimes referred to as an Egorov theorem (see, e.g., [PST02]) would only assert that one can evolve the principal symbol B_0 of $\mathcal{B}$ into a symbol $B(t)_0$, as given in (3.10), such that its quantisation $\mathrm{op}^W[B(t)_0]$ is $\hbar$-close to the time-evolved operator $\mathcal{B}(t)$, i.e.,

$$\|\mathcal{B}(t) - \mathrm{op}^W[B(t)_0]\| = O(\hbar).$$

This statement is clearly covered by Theorem 3.2, since the quantisation of the difference $B(t) - B(t)_0 \in \mathrm{S}^{q-1}_{\mathrm{cl}}(1)$ yields a bounded operator with norm of order $\hbar$, see Proposition 1.4.

We will now show (generalising results of Cordes [Cor83a, Cor00, Cor01]) that the semiclassical block-diagonal operators exhaust all operators with symbols in the invariant algebra $\mathrm{S}^\infty_{\mathrm{inv}}(1)$.

Proposition 3.4. *The invariant subalgebra* $\mathrm{S}^\infty_{\mathrm{inv}}(1)$ *of* $\mathrm{S}^\infty_{\mathrm{cl}}(1)$ *consists of precisely those* $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ *which are semiclassically block-diagonal with respect to the projections* P_μ, $\mu = 1, \dots, l$, *defined in (2.11) of Proposition 2.1, i.e.,*

$$B \in \mathrm{S}^\infty_{\mathrm{inv}} \subset \mathrm{S}^\infty_{\mathrm{cl}}(1) \quad \Leftrightarrow \quad B \sim \sum_{\mu=1}^{l} P_\mu \# B \# P_\mu .$$

Proof. Consider an operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^\infty_{\mathrm{cl}}(1)$, whose equation of motion is given by (3.3). For the symbol of the time-evolved operator we now assume an asymptotic expansion

$$B(t) \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_j$$

in $\mathrm{S}^q_{\mathrm{cl}}(1)$. Furthermore, one can use (2.7) to separate (3.3) into blocks with respect to P_μ, $\mu = 1, \dots, l$. For the off-diagonal blocks ($\nu \neq \mu$) one therefore obtains

$$\frac{\partial}{\partial t} B(t)_{\nu\mu} \sim \frac{\mathrm{i}}{\hbar} [H, B(t)_{\nu\mu}]_\# , \tag{3.14}$$

where $B(t)_{\nu\mu} := P_\nu \# B(t) \# P_\mu \sim \sum_{j=0}^{\infty} \hbar^{-q+j} B(t)_{\nu\mu,j}$. In leading semiclassical order the factor $\hbar^{-1}$ on the right-hand side of equation (3.14) enforces the condition

$$[H_0, B(t)_{\nu\mu,0}] = (\lambda_\nu - \lambda_\mu) B(t)_{\nu\mu,0} = 0.$$

Since $\lambda_\mu \neq \lambda_\nu$ for $\mu \neq \nu$, this immediately yields $B(t)_{\nu\mu,0} = 0$. Furthermore,

$$\frac{\partial}{\partial t} \sum_{j=1}^{\infty} \hbar^{-q+j} B(t)_{\nu\mu,j} \sim \mathrm{i} \Big[H, \sum_{j=1}^{\infty} \hbar^{-q+j-1} B(t)_{\nu\mu,j} \Big]_{\#}.$$

Again the leading order on the right-hand side has to vanish, i.e.,

$$[H_0, B(t)_{\nu\mu,1}] = 0.$$

This means that the symbol $B(t)_{\nu\mu,1}$ must be block-diagonal with respect to the projection matrices $P_{\mu,0} \in \mathrm{S}(1)$. But

$$P_{\lambda,0} B(t)_{\nu\mu,1} P_{\lambda,0} = \mathrm{symb}_P^W \left[\hbar^{-1} \left(P_\lambda \# (B(t)_{\nu\mu} - B(t)_{\nu\mu,0}) \# P_\lambda \right) \right] = 0,$$

since $B(t)_{\nu\mu,0} = 0$ for $\nu \neq \mu$. Iterating the above procedure we see that if $B \in \mathrm{S}_{\mathrm{inv}}(1)$, then it has to be block-diagonal with respect to P_μ, $\mu = 1, \dots, l$. This proves one direction asserted in the proposition. The other direction, that the block-diagonal operators form a subset of the invariant algebra, is contained in the Egorov theorem 3.2. $\square$

At this point we want to add a comment on the transport equation (3.8) that not only occurs in connection with an Egorov theorem, but also in a WKB-type framework. In this context Littlejohn and Flynn [LF91] introduced a splitting of the analogue to $\tilde{H}_{\nu\nu,1}$ (defined in equation (3.7)) into two contributions, one of which is related to a Berry connection [Ber84]. Subsequently Emmrich and Weinstein [EW96] generalised the approach of [LF91] and gave a geometrical interpretation for the second contribution, which they related to a Poisson curvature. We now want to identify the two contributions in the present situation, i.e., in $\tilde{H}_{\nu\nu,1}$. To this end we calculate $H_{\nu,1} = P_{\nu,1} H_0 + P_{\nu,0} H_1 + \frac{\mathrm{i}}{2}\{P_{\nu,0}, H_0\}$ using

$$-P_{\nu,0} P_{\nu,1} P_{\nu,0} + (1 - P_{\nu,0}) P_{\nu,1} (1 - P_{\nu,0}) = \frac{\mathrm{i}}{2} \{P_{\nu,0}, P_{\nu,0}\},$$

which follows from the condition $P_\nu \# P_\nu \sim P_\nu$ and the composition formula in Lemma 1.2. Thus

$$P_{\nu,0} H_{\nu,1} P_{\nu,0} = P_{\nu,0} H_1 P_{\nu,0} + \mathrm{i} \frac{\lambda_\nu}{2} P_{\nu,0} \{P_{\nu,0}, P_{\nu,0}\} P_{\nu,0} + \frac{\mathrm{i}}{2} \sum_{\eta=1}^{l} \lambda_\eta P_{\nu,0} \{P_{\nu,0}, P_{\eta,0}\} P_{\nu,0}.$$

The relation $P_{\nu,0}\{P_{\nu,0}, P_{\eta,0}\} P_{\nu,0} = -P_{\nu,0}\{P_{\eta,0}, P_{\eta,0}\} P_{\nu,0}$ and the spectral representation $H_0 = \sum_\mu \lambda_\mu P_{\mu,0}$ now allow to rewrite the expression (3.7) for $\tilde{H}_{\nu\nu,1}$ as

$$\tilde{H}_{\nu\nu,1} = H_{\nu,\mathrm{Berry}} + H_{\nu,\mathrm{Poisson}} + P_{\nu,0} H_1 P_{\nu,0}$$

with

$$\begin{aligned} H_{\nu,\mathrm{Berry}} &:= -\mathrm{i}[P_{\nu,0}, \{\lambda_\nu, P_{\nu,0}\}], \\ H_{\nu,\mathrm{Poisson}} &:= \frac{\mathrm{i}}{2}\Big(\lambda_\nu P_{\nu,0}\{P_{\nu,0}, P_{\nu,0}\}P_{\nu,0} + P_{\nu,0}\{P_{\nu,0}, H_0 - \lambda_\nu P_{\nu,0}\}P_{\nu,0}\Big). \end{aligned}$$

This corresponds exactly to the splitting discussed in [EW96], see also [Spo00].

4 Dynamics in the eigenspaces

According to the Egorov theorem 3.2, the semiclassical calculus outlined above results not only in a transport of the principal symbols of observables by the Hamiltonian flows Φ^t_ν, but also in a conjugation by the (unitary $n \times n$) transport matrices $d_{\nu\nu}$. The latter define the dynamics of those degrees of freedom that on the quantum mechanical level are described by the factor $\mathbb{C}^n$ of the total Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. Our intention in this section now is to develop combined classical dynamics of both types of degrees of freedom, i.e., those described by the Hamiltonian flows and those that are represented by the conjugations. In this context the fact that the conjugations enter along integral curves of the Hamiltonian flows introduces a hierarchy among the two types of degrees of freedom.

In a first step we confirm that the dynamics represented by the transport matrices $d_{\nu\nu}$ take place in the eigenspaces of the principal symbol H_0 in $\mathbb{C}^n$. To this end we notice that since at every point $(x, \xi) \in \mathrm{T}^*\mathbb{R}^d$ the projection matrices $P_{\nu,0}(x, \xi)$ yield an orthogonal splitting of $\mathbb{C}^n$ and have constant rank k_ν, they define k_ν-dimensional subbundles $\pi_\nu : E^\nu \to \mathrm{T}^*\mathbb{R}^d$ of the trivial vector bundle $\mathrm{T}^*\mathbb{R}^d \times \mathbb{C}^n$ over phase space. The fibre $E^\nu_{(x,\xi)} = \pi_\nu^{-1}(x, \xi)$ over $(x, \xi) \in \mathrm{T}^*\mathbb{R}^d$ is given by the range of the projection, i.e., $E^\nu_{(x,\xi)} = P_{\nu,0}(x, \xi)\mathbb{C}^n$. Furthermore, the canonical hermitian structure of $\mathbb{C}^n$ induces a hermitian structure on the fibres. We now intend to interpret the conjugation by $d_{\nu\nu}$ as a dynamics in the eigenvector bundle E^ν, and for this purpose notice:

Lemma 4.1. *The restricted transport matrices $d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi)$ provide unitary maps between the fibres $E^\nu_{(x,\xi)}$ and $E^\nu_{\Phi^t_\nu(x,\xi)}$.*

Proof. In order to see that $d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi)$ maps $E^\nu_{(x,\xi)}$ into $E^\nu_{\Phi^t_\nu(x,\xi)}$ we show

$$P_{\nu,0}(\Phi^t_\nu(x, \xi))d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi) = d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi). \tag{4.1}$$

This relation is certainly true for $t = 0$ where both sides yield $P_{\nu,0}$. Moreover, the derivative with respect to t of the left-hand side reads

$$\{\lambda_\nu, P_{\nu,0}\}(\Phi^t_\nu(x, \xi))d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi) - \mathrm{i}P_{\nu,0}(\Phi^t_\nu(x, \xi))\tilde{H}_{\nu\nu,1}(\Phi^t_\nu(x, \xi))d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi),$$

which equals

$$-\mathrm{i}\tilde{H}_{\nu\nu,1}(\Phi^t_\nu(x, \xi))P_{\nu,0}(\Phi^t_\nu(x, \xi))d_{\nu\nu}(x, \xi, t)P_{\nu,0}(x, \xi),$$

since the commutator $[P_{\nu,0}, \tilde{H}_{\nu\nu,1}]$ can be calculated as (see equation (3.7))

$$-\mathrm{i}[P_{\nu,0},[P_{\nu,0},\{\lambda_\nu, P_{\nu,0}\}]] = -\mathrm{i}\left(P_{\nu,0}\{\lambda_\nu, P_{\nu,0}\} + \{\lambda_\nu, P_{\nu,0}\}P_{\nu,0}\right) = -\mathrm{i}\{\lambda_\nu, P_{\nu,0}\};$$

here we have used (A.2) and $P_{\nu,0}^2 = P_{\nu,0}$. Thus, $P_{\nu,0}(\Phi_\nu^t(x,\xi))d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$ fulfills the same differential equation with respect to t as $d_{\nu\nu}(x,\xi,t)P_{\nu,0}$, and this finally implies the validity of equation (4.1).

In order to see the unitarity, one has to show that $d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$ is an isometry whose range is the complete fibre $E^\nu_{\Phi_\nu^t(x,\xi)}$. The first point is clear since $d_{\nu\nu}$ is unitary on $\mathbb{C}^n$ and the fibres inherit their hermitian structures from $\mathbb{C}^n$. The second point follows from the observation that the transport provided by $d_{\nu\nu}$ can be reversed: Given $v(\Phi_\nu^t(x,\xi)) \in E_{\Phi_\nu^t(x,\xi)}$, the vector $P_{\nu,0}(x,\xi)d_{\nu\nu}(\Phi_\nu^t(x,\xi),-t)v(\Phi_\nu^t(x,\xi))$ lies in $E^\nu_{(x,\xi)}$ and is mapped to $v(\Phi_\nu^t(x,\xi))$ by $d_{\nu\nu}(x,\xi,t)P_{\nu,0}(x,\xi)$, see (3.13). $\square$

According to the above, the action of $d_{\nu\nu}(x,\xi,t)$ on a section in E^ν can be viewed as a parallel transport along the integral curves of the flow Φ_ν^t. If one now introduces sections of E^ν that yield orthonormal bases $\{e_1(x,\xi),\ldots,e_{k_\nu}(x,\xi)\}$ of the fibres $E^\nu_{(x,\xi)}$, the representations of $d_{\nu\nu}(x,\xi,t)$ in these bases are unitary $k_\nu \times k_\nu$ matrices $D_\nu(x,\xi,t)$. Since the principal symbol H_0 of the operator $\mathcal{H}$ is hermitian (on $\mathbb{C}^n$), a preferred choice for the sections $\{e_1,\ldots,e_{k_\nu}\}$ would consist of orthonormal eigenvectors of H_0. However, this choice is obviously not unique because it amounts to fixing an isometry $V_\nu(x,\xi) : \mathbb{C}^{k_\nu} \to E^\nu_{(x,\xi)}$, such that $V_\nu(x,\xi)V_\nu^*(x,\xi) = P_{\nu,0}(x,\xi)$ and $V_\nu^*(x,\xi)V_\nu(x,\xi) = \mathrm{id}_{\mathbb{C}^{k_\nu}}$. Here one still has a freedom to change the isometry by an arbitrary unitary automorphism of $\mathbb{C}^{k_\nu}$. Having chosen an isometry $V_\nu(x,\xi)$ for every fibre $E^\nu_{(x,\xi)}$ in a smooth way, the $n\times n$ transport matrices $d_{\nu\nu}(x,\xi,t)$ are mapped to unitary $k_\nu \times k_\nu$ matrices

$$D_\nu(x,\xi,t) := V_\nu^*\big(\Phi_\nu^t(x,\xi)\big)d_{\nu\nu}(x,\xi,t)V_\nu(x,\xi). \tag{4.2}$$

Their dynamics follows from the transport equation (3.8) as

$$\dot{D}_\nu(x,\xi,t) + \mathrm{i}\tilde{H}_\nu\big(\Phi_\nu^t(x,\xi)\big)D_\nu(x,\xi,t) = 0 \quad \text{with} \quad D_\nu(x,\xi,0) = \mathrm{id}_{\mathbb{C}^{k_\nu}}, \tag{4.3}$$

where the hermitian $k_\nu \times k_\nu$ matrix $\tilde{H}_\nu$ is derived from (3.7) for $\mu = \nu$,

$$\tilde{H}_\nu = -\mathrm{i}\frac{\lambda_\nu}{2}V_\nu^*\{P_{\nu,0},P_{\nu,0}\}V_\nu + \mathrm{i}\{\lambda_\nu, V_\nu^*\}V_\nu + V_\nu^* H_{\nu,1}V_\nu.$$

What is of more importance for later purposes than the non-uniqueness of this representation, however, is the fact that the above construction allows to introduce a skew-product flow over the Hamiltonian flow Φ_ν^t, thus reflecting the hierarchy of the two types of degrees of freedom. See [CFS82] for a definition of skew-product flows and cf. [BK99b] where these occur in the context of a semiclassical trace formula for matrix valued operators. At this stage now provisionally consider

$$\hat{Y}_\nu^t : \mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu) \to \mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu),$$

defined by $\hat{Y}_\nu^t(x,\xi,g) := (\Phi_\nu^t(x,\xi), D_\nu(x,\xi,t)g)$, which yields a flow on the product space $\mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu)$ due to the cocycle relation $D_\nu(x,\xi,t+t') = D_\nu(\Phi_\nu^t(x,\xi),t')D_\nu(x,\xi,t)$. Later we are interested in ergodic properties of such skew-product flows, and these are independent of the particular choice of the sections $\{e_1,\dots,e_{k_\nu}\}$. Here we remark that in some cases the point of view advertised above might turn out too general. It can indeed happen that the fibre part of the skew-product flow does not require the complete group $\mathrm{U}(k_\nu)$. E.g., in [BGK01] a situation was considered where $k_\nu = 2j+1$, $j \in \frac{1}{2}\mathbb{N}$, and the transport matrices D_ν were operators in a $2j+1$-dimensional unitary irreducible representation of SU(2). This fact could be identified by the observation that when (x,ξ) ranges over $\mathrm{T}^*\mathbb{R}^d$, the skew-hermitian matrices $\mathrm{i}\tilde{H}_\nu(x,\xi)$ generate a Lie subalgebra of $\mathrm{u}(2j+1)$ which is isomorphic to su(2).

In the general case one therefore should not necessarily expect that the transport matrices D_ν generate all of $\mathrm{U}(k_\nu)$, but only a certain Lie subgroup. In order to identify this group we consider the Lie subalgebra

$$\left\langle \mathrm{i}\tilde{H}_\nu(x,\xi);\ (x,\xi) \in \mathrm{T}^*\mathbb{R}^d \right\rangle \subset \mathrm{u}(k_\nu) \tag{4.4}$$

generated by the skew-hermitian matrices $\mathrm{i}\tilde{H}_\nu(x,\xi)$. Via exponentiation of this subalgebra one hence obtains a Lie subgroup $G \subset \mathrm{U}(k_\nu)$ that is compact and connected. To be more precise, the result of the exponentiation is a k_ν-dimensional unitary representation ρ of G. Its Lie algebra $\mathfrak{g}$ then is embedded in (4.4) via the derived representation $\mathrm{d}\rho$. In this setting the transport matrices D_ν are operators in the representation ρ, i.e., $D_\nu(x,\xi,t) = \rho(g_\nu(x,\xi,t))$. Hence we are now in a position to define the skew-product flows

$$\tilde{Y}_\nu^t : \mathrm{T}^*\mathbb{R}^d \times G \to \mathrm{T}^*\mathbb{R}^d \times G \tag{4.5}$$

through

$$\tilde{Y}_\nu^t(x,\xi,g) = \big(\Phi_\nu^t(x,\xi), g_\nu(x,\xi,t)g\big). \tag{4.6}$$

These flows leave the product measure $\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}g$ on $\mathrm{T}^*\mathbb{R}^d \times G$ invariant, which consists of Lebesgue measure $\mathrm{d}x\,\mathrm{d}\xi$ on $\mathrm{T}^*\mathbb{R}^d$ and the normalised Haar measure $\mathrm{d}g$ on G. Moreover, if one restricts the Hamiltonian flows Φ_ν^t to compact level surfaces of the eigenvalue functions λ_ν at non-critical values E,

$$\Omega_{\nu,E} := \lambda_\nu^{-1}(E) = \big\{(x,\xi) \in \mathrm{T}^*\mathbb{R}^d;\ \lambda_\nu(x,\xi) = E\big\},$$

the restrictions of the skew-product flows $\tilde{Y}_\nu^t$ to $\Omega_{\nu,E} \times G$ leave the measures $\mathrm{d}\ell(x,\xi)\,\mathrm{d}g$ invariant, where $\mathrm{d}\ell(x,\xi)$ denotes the normalised Liouville measure on $\Omega_{\nu,E}$.

Below we are interested in the question under which conditions imposed on suitable classical dynamics quantum ergodicity holds, see section 6. In analogy to [BG00] one approach to this problem would be to consider the restriction of the skew-product flow $\hat{Y}_\nu^t$ to $\Omega_{\nu,E} \times \mathrm{U}(k_\nu)$: its ergodicity with respect to the product measure that consists of Liouville measure on $\Omega_{\nu,E}$ and Haar measure on $\mathrm{U}(k_\nu)$ implies quantum ergodicity. Since, however, the dynamics in the eigenspaces is completely fixed by a restriction to the group G, the dynamical behaviour of the flow $\hat{Y}_\nu^t$ is determined by that of $\tilde{Y}_\nu^t$. One hence concludes that in order to proof quantum ergodicity one requires the following condition (see Remark 6.3):

(Irr_ν) The representation $\rho : G \to \mathrm{U}(k_\nu)$ is irreducible.

In the sequel we always assume this to be the case.

Our intention now is to relate the dynamics in the eigenspaces, given by the conjugation with the transport matrices $d_{\nu\nu}$, to proper classical dynamics. To this end we require a symplectic manifold with the dynamics realised in a Hamiltonian fashion. For this purpose Kirillov's orbit method [Kir76] provides the necessary tools: it relates the unitary irreducible representation $(\rho, \mathbb{C}^{k_\nu})$ to a coadjoint orbit $\mathcal{O}$ of G, which is a symplectic manifold. Moreover, the conjugation dynamics is realised in terms of the coadjoint action of G on $\mathcal{O}$. As in the case of $G = \mathrm{SU}(2)$ considered in [BGK01], this setting then also allows to introduce a Moyal-type quantisation such that hermitian matrix valued symbols can be uniquely related to real valued functions on the symplectic product phase space $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}$.

Let us now recall some properties of coadjoint orbits [Kir76]: The adjoint representation $\mathrm{Ad} : G \to \mathrm{aut}(\mathfrak{g})$, $g \mapsto (\mathrm{T}_e \mathrm{I})(g)$, of a Lie group G on its Lie algebra $\mathfrak{g} \cong \mathrm{T}_e G$ is defined as the differential $\mathrm{T}_e \mathrm{I}$ of the inner automorphism $\mathrm{I}(g) : G \to G$, $x \mapsto gxg^{-1}$, $g \in G$, at the identity $e \in G$. The coadjoint representation of G on the dual Lie algebra $\mathfrak{g}^*$ is then provided by the dual $\mathrm{Ad}_g^* := (\mathrm{Ad}_{g^{-1}})^*$ of the linear map $\mathrm{Ad}_{g^{-1}}$, i.e.,

$$(\mathrm{Ad}_g^*(\lambda), X) = (\lambda, \mathrm{Ad}_{g^{-1}} X),$$

for $X \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$; here $(\ ,\ ) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ denotes the dual pairing between the vector spaces $\mathfrak{g}$ and $\mathfrak{g}^*$. A coadjoint orbit $\mathcal{O}_\lambda$ through $\lambda \in \mathfrak{g}^*$ then is an orbit of this group action,

$$\mathcal{O}_\lambda := \{\mathrm{Ad}_g^*(\lambda);\ g \in G\} \subset \mathfrak{g}^*.$$

If G is compact, $\mathcal{O}_\lambda$ is a smooth embedded and compact submanifold of $\mathfrak{g}^*$. One of the main features of coadjoint orbits is their symplectic structure [Kir76].

Proposition 4.2. *Let G be a connected Lie group and $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit. Then $\mathcal{O}$ is a symplectic manifold and there exist unique symplectic forms $\sigma^\pm$ on $\mathcal{O}$ such that*

$$\sigma^\pm(\lambda)\big(\mathrm{ad}_X^* \lambda, \mathrm{ad}_Y^* \lambda\big) = \pm\big(\lambda, [X, Y]\big)$$

for all $\lambda \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$. Here ad^* denotes the differential of the coadjoint action and $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$. The forms $\sigma^\pm$ are referred to as the coadjoint orbit symplectic structures.

Furthermore, let $G_\lambda := \{g \in G;\ \mathrm{Ad}_g^* \lambda = \lambda\}$ denote the isotropy subgroup of $\lambda \in \mathfrak{g}^*$ under the coadjoint action. Then this is a closed subgroup of G, and so the quotient G/G_λ is a smooth manifold with smooth projection $\pi : G \to G/G_\lambda$ such that one can identify $G/G_\lambda \cong \mathcal{O}_\lambda$ via the diffeomorphism $\kappa : gG_\lambda \mapsto \mathrm{Ad}_g^* \lambda$. Moreover, since the coadjoint action on $\mathcal{O}_\lambda$ preserves its symplectic structure and is obviously transitive, $\mathcal{O}_\lambda$ is a symplectic homogeneous space. In the following we only need one symplectic structure that turns $\mathcal{O}_\lambda$ into a symplectic homogeneous space and therefore now fix $\sigma := \sigma^+$.

Our next goal is to construct a certain quantisation of the symplectic manifold $\mathcal{O}_\lambda$. On the classical side one considers suitable functions on the phase space $\mathcal{O}_\lambda$ as observables;

here we choose functions that are integrable with respect to the volume form $\mathrm{d}\eta$ that arises as the maximal exterior power of the symplectic form σ. A Hamiltonian dynamics is then generated by a smooth real valued function h on $\mathcal{O}_\lambda$ through the association of a Hamiltonian vector field $\mathcal{X}_h$ according to $\sigma(\mathcal{X}_h, \cdot) = \mathrm{d}h$. On the quantum side observables are hermitian endomorphisms of the representation space V with inner product $\langle \cdot, \cdot \rangle_V$. A Moyal quantiser now assigns to a hermitian $A \in \mathcal{L}(V)$ a function $a \in L^1(\mathcal{O}_\lambda)$ such that $\rho(g)A\rho(g^{-1})$ is mapped to $a \circ \mathrm{Ad}^*_{g^{-1}}$. This covariance property then ensures that the dynamics given by the conjugation with $D_\nu = \rho(g_\nu)$ is represented on the phase space $\mathcal{O}_\lambda$ through the coadjoint action of g_ν. Quantisations of this type were constructed by Simon [Sim80], who introduced suitable Berezin symbols representing $A \in \mathcal{L}(V)$. However, here we will closely follow [FGV90], where it is demonstrated that one can obtain a quantisation with an additional tracial property that turns out to be useful later.

In the present context G is a matrix Lie group, i.e., a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, and its Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathrm{M}_n(\mathbb{C})$ with the matrix commutator as Lie bracket. Thus there also exists a non-degenerate symmetric bilinear form B on $\mathfrak{g}$, given by $B(X,Y) = \mathrm{Re}\ \mathrm{tr}(XY)$, that allows to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. Consider now the unitary irreducible representation (ρ, V) of G and fix a highest (real) weight $\lambda \in \mathfrak{t}^*$ corresponding to this representation, where $\mathfrak{t}$ is the Lie algebra of a suitable maximal torus $T \subset G$. Since one can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the bilinear form B, one can regard the highest weight as $\lambda \in \mathfrak{g}^*$. Up to a phase, to this highest weight there corresponds a unique normalised weight vector $w_\lambda \in V$. Now define the map $J : V \to \mathfrak{g}^*$ by

$$(J(v), X) := \langle v, \mathrm{d}\rho(X)v \rangle_V,$$

such that in particular $J(w_\lambda) = \lambda$. This map is equivariant in the sense that $J(\rho(g)v) = \mathrm{Ad}^*_g J(v)$. Since the weight space of the maximal weight is one-dimensional one obtains

$$J^{-1}(\lambda) = \{ z w_\lambda;\ z \in \mathbb{C},\ |z| = 1 \},$$

and therefore

$$J^{-1}(\mathcal{O}_\lambda) = \{ \rho(g) w_\lambda;\ g \in G \}.$$

This setting now allows to associate to points $\eta \in \mathcal{O}_\lambda$ vectors $v_\eta \in V$ that are unique up to a phase. For this purpose one chooses a measurable section $\eta \mapsto g_\eta$ in $G \to G/G_\lambda$ with $\mathrm{Ad}^*_{g_\eta}(\lambda) = \eta$ and $g_\lambda = e$, which is possible due to the fact that $\mathcal{O}_\lambda \cong G/G_\lambda$ is an orbit of the coadjoint action. Then define for every $\eta \in \mathcal{O}_\lambda$ a vector $v_\eta := \rho(g_\eta) w_\lambda$, which can also be viewed as a coherent state [Per86]. The equivariance of the map J now implies that $J(v_\eta) = \eta$, such that v_η is unique up to a phase. This finally allows to define for every $A \in \mathcal{L}(V)$ the unique covariant symbol

$$Q_A(\eta) := \langle v_\eta, A v_\eta \rangle_V. \tag{4.7}$$

In fact, $Q_A : \mathcal{O}_\lambda \to \mathbb{C}$ is continuous. We denote the space of covariant symbols that are constructed according to the above scheme by $\mathcal{S}_\lambda := \{ Q_A;\ A \in \mathcal{L}(V) \}$ and recall from [FGV90]

Lemma 4.3. *The map $\mathcal{L}(V) \to \mathcal{S}_\lambda$ defined in equation (4.7) is one-to-one.*

Now consider a normalised vector $w \in V$ and the associated orthogonal projector Π_w onto $\mathbb{C}w \subset V$. Since, in the language of quantum mechanics, expectations of an observable $A \in \mathcal{L}(V)$ in the state w read $\langle w, Aw\rangle_V = \operatorname{tr}(A\Pi_w)$, one would like a Moyal quantisation to represent $\operatorname{tr}(AB)$ as

$$\int_{\mathcal{O}_\lambda} \overline{Q}_A(\eta) Q_B(\eta)\ \mathrm{d}\eta.$$

This relation, however, does not hold. Considering $\mathcal{L}(V)$ as a (finite-dimensional) Hilbert space with inner product $\operatorname{tr}(A^*B)$, we are hence looking for an isometry $\mathcal{L}(V) \to L^2(\mathcal{O}_\lambda)$. To this end one notices that $\mathcal{S}_\lambda$ being a finite dimensional subspace of $L^2(\mathcal{O}_\lambda)$, the Riesz representation theorem ensures for every $A \in \mathcal{L}(V)$ that the linear form $L_A : \mathcal{S}_\lambda \to \mathbb{C}$ given by $L_A(Q_B) := \operatorname{tr}(A^*B)$ can be represented in terms of a unique function $P_A \in \mathcal{S}_\lambda$ such that

$$L_A(Q_B) = \operatorname{tr}(A^*B) = \int_{\mathcal{O}_\lambda} \overline{P}_A(\eta) Q_B(\eta)\ \mathrm{d}\eta.$$

Since according to Lemma 4.3 the spaces $\mathcal{L}(V)$ and $\mathcal{S}_\lambda$ have the same (finite) dimension, the map $\mathcal{L}(V) \ni A \mapsto P_A \in \mathcal{S}_\lambda$ is a (linear) bijection that can as well serve as a symbol map; P_A is then called contravariant symbol. We remark that in order to satisfy the natural condition $P_{\mathrm{id}_V} = 1$, the volume form on $\mathcal{O}_\lambda$ has to be normalised such that $\operatorname{vol}(\mathcal{O}_\lambda) = \dim V$. Both the covariant and the contravariant symbol of $A \in \mathcal{L}(V)$ fulfill the covariance condition

$$\begin{aligned} Q_{\rho(g)A\rho(g^{-1})}(\eta) &= Q_A(\operatorname{Ad}^*_{g^{-1}}\eta), \\ P_{\rho(g)A\rho(g^{-1})}(\eta) &= P_A(\operatorname{Ad}^*_{g^{-1}}\eta), \end{aligned}$$

for all $g \in G$ and all $\eta \in \mathcal{O}_\lambda$. However, in order to obtain the desired isometry from $\mathcal{L}(V)$ into $L^2(\mathcal{O}_\lambda)$, one is forced to introduce a symbol map that in a certain sense lies in between Q and P.

In [FGV90] it is shown that the operator K on $\mathcal{S}_\lambda$ that maps Q_A to P_A is bijective and positive. It therefore allows for a (positive) square-root $K^{1/2}$ which can be used to define a symbol map with all desired properties:

Definition 4.4. For $A \in \mathcal{L}(V)$ the Stratonovich-Weyl symbol $\operatorname{symb}^{SW}[A] \in \mathcal{S}_\lambda$ is given by

$$\operatorname{symb}^{SW}[A] := K^{1/2} Q_A = K^{-1/2} P_A.$$

Summarising the above finally yields [FGV90]:

Proposition 4.5. *The symbol map $A \mapsto \operatorname{symb}^{SW}[A]$ has the following properties:*

(i) It is a linear one-to-one map from $\mathcal{L}(V)$ to $\mathcal{S}_\lambda$,

(ii) $\mathrm{symb}^{SW}[A^*] = \overline{\mathrm{symb}^{SW}[A]}$,

(iii) $\mathrm{symb}^{SW}[\mathrm{id}_V] = 1$,

(iv) $\mathrm{symb}^{SW}[\rho(g)A\rho(g^{-1})](\eta) = \mathrm{symb}^{SW}[A](\mathrm{Ad}^*_{g^{-1}}\,\eta)$ *for all* $\eta \in \mathcal{O}_\lambda,\ g \in G$,

(v) $\displaystyle\int_{\mathcal{O}_\lambda} \mathrm{symb}^{SW}[A](\eta)\,\mathrm{symb}^{SW}[B](\eta)\ \mathrm{d}\eta = \mathrm{tr}(AB)$.

In order to make the relation between $A \in \mathcal{L}(V)$ and its symbol $\mathrm{symb}^{SW}[A]$ explicit, one introduces a (hermitian) quantiser $\Delta_\lambda : \mathcal{O}_\lambda \to \mathcal{L}(V)$ such that

$$\mathrm{symb}^{SW}[A] = \mathrm{tr}\big(A\Delta_\lambda\big) \quad \text{and} \quad A = \int_{\mathcal{O}_\lambda} \mathrm{symb}^{SW}[A](\eta)\Delta_\lambda(\eta)\ \mathrm{d}\eta. \tag{4.8}$$

As shown in [FGV90], the quantiser can be expressed in terms of generalised spherical harmonics associated with those unitary irreducible representations that appear in the regular representation $(\tau(g)f)(\eta) = f(\mathrm{Ad}^*_{g^{-1}}\,\eta)$ of G on $\mathcal{S}_\lambda$.

With this formalism at hand one can now transfer the dynamics of a (hermitian) $B \in \mathcal{L}(V)$ given by a conjugation with $D(t) = \rho(g(t))$, $B \mapsto B(t) = D^{-1}(t)BD(t)$, to the coadjoint action of $g(t)$ on the symplectic manifold $\mathcal{O}_\lambda$ via the relation $\mathrm{symb}^{SW}[B(t)](\eta) = \mathrm{symb}^{SW}[B](\mathrm{Ad}^*_{g(t)}\,\eta)$. The symplectic structure on $\mathcal{O}_\lambda$ defined by the form σ, furthermore, allows to identify the dynamics $\eta \mapsto \mathrm{Ad}^*_{g(t)}\,\eta$ as being Hamiltonian. To see this assume that $D(t)$ is determined by

$$\dot{D}(t) + \mathrm{i}HD(t) = 0 \quad \text{with} \quad D(0) = \mathrm{id}_V, \tag{4.9}$$

where $H \in \mathcal{L}(V)$ is hermitian; compare equation (4.3). On the one hand now, a Hamiltonian flow $\eta \mapsto \eta(t)$ can be introduced on $\mathcal{O}_\lambda$ that is generated by the Stratonovich-Weyl symbol of H. The associated Hamiltonian vector field $\mathcal{X}_{\mathrm{symb}^{SW}[H]}$ is then defined through

$$\sigma(\mathcal{X}_{\mathrm{symb}^{SW}[H]},\cdot) = \mathrm{d}\,\mathrm{symb}^{SW}[H],$$

so that the time evolution $f(t)(\eta) = f(\eta(t))$ of a function $f \in C^\infty(\mathcal{O}_\lambda)$ is governed by the equation

$$\dot{f}(t) = \big\{\mathrm{symb}^{SW}[H], f(t)\big\} = \sigma\big(\mathcal{X}_{\mathrm{symb}^{SW}[H]}, \mathcal{X}_{f(t)}\big) = -\mathrm{d}f(t)\big(\mathcal{X}_{\mathrm{symb}^{SW}[H]}\big). \tag{4.10}$$

On the other hand, differentiating $\mathrm{Ad}^*_{g(t)}\,\eta$ with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}^*_{g(t)}\,\eta\Big|_{t=0} = -\,\mathrm{ad}^*_{X_H}\,\eta, \tag{4.11}$$

where $X_H \in \mathfrak{g}$ is the generator of the curve $g(t)$ in G which, according to equation (4.9), is related to H via $\mathrm{d}\rho(X_H) = -\mathrm{i}H$. A comparison of (4.10) and (4.11) then shows that the dynamics provided by the coadjoint action $\mathrm{Ad}^*_{g(t)}$ coincides with the Hamiltonian flow generated by the symbol $\mathrm{symb}^{SW}[H]$.

As an ultimate outcome of the above formalism we are now in a position to introduce a skew-product flow on the symplectic phase space $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$ that completely determines the time evolution of the ν-th diagonal block of an observable on the level of its principal symbol. Explicitly, this flow is given by

$$Y_\nu^t : \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \to \mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda \tag{4.12}$$

with

$$Y_\nu^t(x,\xi,\eta) := \big(\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta\big); \tag{4.13}$$

it leaves the product measure $\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\eta$ invariant.

Consider now a semiclassical pseudodifferential operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^\infty_{\mathrm{cl}}(1)$. Mod $O(\hbar^\infty)$ the quantum dynamics preserves the diagonal structure of its blocks $\mathcal{P}_\nu\mathcal{B}\mathcal{P}_\nu$. According to the Egorov theorem 3.2, together with the definition (4.2), the principal symbol of $\mathcal{P}_\nu\mathcal{B}(t)\mathcal{P}_\nu$ hence reads

$$V_\nu(x,\xi)D_\nu^*(x,\xi,t)\big(V_\nu^* B_0 V_\nu\big)\big(\Phi_\nu^t(x,\xi)\big)D_\nu(x,\xi,t)V_\nu^*(x,\xi). \tag{4.14}$$

We now exploit the possibility, explicitly provided by (4.8), to uniquely represent the value of $V_\nu^* B_0 V_\nu : \mathrm{T}^*\mathbb{R}^d \to \mathcal{L}(\mathbb{C}^{k_\nu})$ in terms of a Stratonovich-Weyl symbol,

$$b_{0,\nu}(x,\xi,\eta) := \mathrm{symb}^{SW}\big[(V_\nu^* B_0 V_\nu)(x,\xi)\big](\eta). \tag{4.15}$$

The dynamics of the principal symbol in this representation is now summarised in the following variant of the Egorov theorem:

Proposition 4.6. *The Stratonovich-Weyl symbol $b(t)_{0,\nu}$ associated with the principal symbol of the operator $\mathcal{P}_\nu\mathcal{B}(t)\mathcal{P}_\nu$ is the time evolution of $b_{0,\nu}$ under the skew-product flow Y_ν^t defined in equations (4.12)–(4.13), i.e.,*

$$b(t)_{0,\nu}(x,\xi,\eta) = b_{0,\nu}\big(Y_\nu^t(x,\xi,\eta)\big).$$

Proof. According to (4.14) and (4.15), $b(t)_{0,\nu}$ is given by

$$b(t)_{0,\nu}(x,\xi,\eta) = \mathrm{symb}^{SW}\big[\rho(g_\nu^{-1}(x,\xi,t))\big(V_\nu^* B_0 V_\nu\big)\big(\Phi_\nu^t(x,\xi)\big)\rho(g_\nu(x,\xi,t))\big](\eta),$$

which due to the covariance property (iv) of Proposition 4.5 reads

$$\begin{aligned} b(t)_{0,\nu}(x,\xi,\eta) &= \mathrm{symb}^{SW}\big[\big(V_\nu^* B_0 V_\nu\big)\big(\Phi_\nu^t(x,\xi)\big)\big](\mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta) \\ &= b(t)_{0,\nu}\big(\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta\big). \end{aligned}$$

□

5 Trace asymptotics and a limit formula for averaged expectation values

A fundamental ingredient in the asymptotics of eigenvectors we are aiming at is a semiclassical limit formula for the expectation values of bounded operators $\mathcal{B}$ on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. Below we will obtain a Szegö-type formula which connects semiclassically averaged expectation values with objects that can be calculated from the principal symbol B_0 of the operator $\mathcal{B}$ and therefore allow for a classical interpretation. On the so defined classical side we fix a value E for all eigenvalue functions λ_ν, $\nu = 1, \dots, l$, of the principal symbol H_0 with the following properties:

(H3$_\nu$) There exists some $\varepsilon > 0$ such that all $\lambda_\nu^{-1}([E-\varepsilon, E+\varepsilon]) \subset \mathrm{T}^*\mathbb{R}^d$ are compact.

(H4$_\nu$) The functions λ_ν shall possess no critical values in $[E-\varepsilon, E+\varepsilon]$.

(H5$_\nu$) Among the level surfaces $\Omega_{\nu,E} = \lambda_\nu^{-1}(E)$, $\nu = 1, \dots, l$, at least one is non-empty.

In addition to (H1) and (H2), which imply the essential selfadjointness of the operator $\mathcal{H}$, these conditions guarantee as in the scalar case [DS99] that for sufficiently small $\hbar$ the spectrum of $\mathcal{H}$ is discrete in any open interval contained in $[E-\varepsilon, E+\varepsilon]$. This setting now allows us to generalise the constructions made in [BG00] to Hamiltonians with non-scalar principal symbols: The expectation values of an operator $\mathcal{B}$ will be considered in normalised eigenvectors ψ_j of $\mathcal{H}$ with corresponding eigenvalues E_j in an interval $I(E,\hbar) = [E-\hbar\omega, E+\hbar\omega]$, $\omega > 0$, such that $I(E,\hbar) \subset [E-\varepsilon, E+\varepsilon]$ if $\hbar$ is small enough. On the classical side the Hamiltonian flows Φ_ν^t generated by the eigenvalue functions λ_ν will enter on the level surfaces $\Omega_{\nu,E}$. Regarding these we assume:

(H6$_\nu$) The periodic points of Φ_ν^t with non-trivial periods form a set of Liouville measure zero in $\Omega_{\nu,E}$.

The quantities appearing on the classical side of the limit formula turn out to be averages of smooth matrix valued functions $B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ over $\Omega_{\nu,E}$ with respect to Liouville measure, for which we introduce the notation

$$\ell_{\nu,E}(B) := \int_{\Omega_{\nu,E}} B(x,\xi)\ \mathrm{d}\ell(x,\xi).$$

The main result of this section is now summarised in the following Szegö-type limit formula:

Proposition 5.1. *Let $\mathcal{H}$ be a semiclassical pseudodifferential operator with symbol $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$, such that the principal symbol H_0 satisfies the assumptions (H0)–(H2) and (H3$_\nu$)–(H6$_\nu$) for all $\nu = 1, \dots, l$. Furthermore, let $\mathcal{B}$ be an operator with symbol $B \in \mathrm{S}^0_{\mathrm{cl}}(1)$ and principal symbol B_0. Then the limit formula*

$$\lim_{\hbar \to 0} \frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \langle \psi_j, \mathcal{B}\psi_j \rangle = \frac{\sum_{\nu=1}^l \mathrm{vol}\,\Omega_{\nu,E}\, \mathrm{tr}\, \ell_{\nu,E}(P_{\nu,0} B_0 P_{\nu,0})}{\sum_{\nu=1}^l k_\nu\, \mathrm{vol}\,\Omega_{\nu,E}} \tag{5.1}$$

holds.

Proof. Adapted to the spectral localisation mentioned above we choose a smooth and compactly supported function $g \in C_0^\infty(\mathbb{R})$ such that $g(\lambda) = \lambda$ on a neighbourhood of $[E-\varepsilon, E+\varepsilon]$. Furthermore, we apply the semiclassical splitting of the Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ given by the projection operators $\mathcal{P}_\nu$,

$$L^2(\mathbb{R}^d)\otimes\mathbb{C}^n = \operatorname{ran}\mathcal{P}_1 \oplus\cdots\oplus \operatorname{ran}\mathcal{P}_l \mod \hbar^\infty, \tag{5.2}$$

and the corresponding decomposition $\mathcal{H} = \sum_{\nu=1}^{l} \mathcal{H}\mathcal{P}_\nu$ (mod $O(\hbar^\infty)$) of the Hamiltonian. By employing the generalisation of the Helffer-Sjöstrand formula to matrix valued operators developed in [Dim93, Dim98], we represent $g(\mathcal{H}) = \sum_{\nu=1}^{l} g(\mathcal{H}\mathcal{P}_\nu)\mathcal{P}_\nu$ (mod $O(\hbar^\infty)$) with

$$g(\mathcal{H}\mathcal{P}_\nu)\mathcal{P}_\nu = -\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde g(z)(\mathcal{H}-z)^{-1}\mathcal{P}_\nu\,\mathrm{d}z,$$

where $\tilde g$ is an almost-analytic extension of g. Since the principal symbol $H_0P_{\nu,0}$ of $\mathcal{H}\mathcal{P}_\nu$ is scalar, $H_0P_{\nu,0} = \lambda_\nu P_{\nu,0}$, when considered to act on sections in the eigenvector bundle E^ν, one can use the methods of [DS99] to show that on $\lambda_\nu^{-1}([E-\varepsilon, E+\varepsilon])$ the asymptotic expansions of $\mathrm{symb}^W[g(\mathcal{H}\mathcal{P}_\nu)]$ and of $\mathrm{symb}^W[\mathcal{H}\mathcal{P}_\nu]$ coincide. Below we will always employ the spectral localisation to the interval $I(E,\hbar)$, and since $\mathrm{symb}^W[g(\mathcal{H}\mathcal{P}_\nu)] \in \mathrm{S}^0_{\mathrm{cl}}(1)$, one can therefore now assume that $H \in \mathrm{S}^0_{\mathrm{cl}}(1)$. Furthermore, the decomposition (5.2) allows us to employ the techniques of [DS99] in the same manner as in [BG00]. Hence, if $\chi \in C_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $I(E,\hbar)$ and $\operatorname{supp}\chi \subset [E-\varepsilon, E+\varepsilon]$, the operator

$$\mathcal{U}_\chi(t) := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}t}\chi(\mathcal{H})\sum_{\nu=1}^{l}\mathcal{P}_\nu = \sum_{\nu=1}^{l}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\chi(\mathcal{H}\mathcal{P}_\nu)\mathcal{P}_\nu \mod O(\hbar^\infty),$$

has a pure point spectrum. Moreover, each of the operators $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathcal{H}\mathcal{P}_\nu t}\chi(\mathcal{H}\mathcal{P}_\nu)$ can be approximated in trace norm up to an error of $O(\hbar^\infty)$ by a semiclassical Fourier integral operator with a kernel of the form

$$K_\nu(x,y,t) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} a_\nu(x,y,t,\xi)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\nu(x,\xi,t)-\xi y)}\,\mathrm{d}\xi. \tag{5.3}$$

Here, as in [BK99a], the phases S_ν have to fulfill the Hamilton-Jacobi equations

$$\lambda_\nu\big(x, \partial_x S_\nu(x,\xi,t)\big) + \partial_t S_\nu(x,\xi,t) = 0, \quad S_\nu(x,\xi,0) = x\xi.$$

The amplitudes $a_\nu \in \mathrm{S}^0_{\mathrm{cl}}(1)$ with asymptotic expansions $a_\nu \sim \sum_{j=0}^{\infty}\hbar^j a_{\nu,j}$ are determined as solutions of certain transport equations [BK99a] with initial conditions $a_\nu|_{t=0} = \chi(\lambda_\nu)P_{\nu,0} + O(\hbar)$. Following [BG00] further, we choose test functions $\rho \in C^\infty(\mathbb{R})$ with compactly supported Fourier transforms $\hat\rho \in C_0^\infty(\mathbb{R})$ such that

$$\operatorname{Tr}\frac{1}{2\pi}\int_{\mathbb{R}}\hat\rho(t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}Et}\mathcal{B}\mathcal{U}_\chi(t)\,\mathrm{d}t = \sum_j \chi(E_j)\langle\psi_j, \mathcal{B}\psi_j\rangle\rho\left(\frac{E_j - E}{\hbar}\right),$$

where Tr denotes the operator trace on the Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. Using the semiclassical approximation (5.3) one now has to calculate

$$\frac{1}{2\pi(2\pi\hbar)^d}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\rho}(t)\sum_{\nu=1}^{l}\mathrm{tr}\big(B_0(x,\partial_x S_\nu)a_{\nu,0}(x,x,t,\xi)\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\nu(x,\xi,t)-x\xi+Et)}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \tag{5.4}$$

in leading semiclassical order. This can be done with the method of stationary phase, where the stationary points $(x_{\nu,\mathrm{st}},\xi_{\nu,\mathrm{st}},t_{\nu,\mathrm{st}})$ of the phase $S_\nu(x,\xi,t)-x\xi+Et$ determine periodic points $(x_{\nu,\mathrm{st}},\xi_{\nu,\mathrm{st}})\in\Omega_{\nu,E}$ of the Hamiltonian flow Φ^t_ν with periods $t_{\nu,\mathrm{st}}$. Since the eigenvalue function λ_ν is supposed to be non-critical at E, the periods $t_{\nu,\mathrm{st}}$ of the flow Φ^t_ν cannot accumulate at zero, see [Rob87]. One can hence split $\hat{\rho}=\hat{\rho}_1+\hat{\rho}_2$ in such a way that $\hat{\rho}_1$ is supported only in a small neighbourhood of zero and $\hat{\rho}_2=0$ in the vicinity of zero, so that the only period in $\mathrm{supp}\,\hat{\rho}_1$ is the trivial one, $t_{\nu,\mathrm{st}}=0$. The contribution coming from $\hat{\rho}_1$ to (5.4) is therefore determined by the periodic points with $t_{\nu,\mathrm{st}}=0$. These build up the entire level surface $\Omega_{\nu,E}$ which, according to assumption (H3$_\nu$), is compact. The result then reads (see [DS99, BG00])

$$\sum_j\chi(E_j)\langle\psi_j,\mathcal{B}\psi_j\rangle\rho_1\left(\frac{E_j-E}{\hbar}\right)=\chi(E)\frac{\hat{\rho}_1(0)}{2\pi}\sum_{\nu=1}^{l}\frac{\mathrm{vol}\,\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}}\big(\mathrm{tr}\,\ell_{\nu,E}(P_{\nu,0}B_0P_{\nu,0})+O(\hbar)\big). \tag{5.5}$$

Coming to the contribution of the term with $\hat{\rho}_2$ to the expression (5.4), we recall that $\hat{\rho}_2$ has been chosen to vanish in a neighbourhood of zero. The relevant stationary points are hence related to periodic orbits of the flow Φ^t_ν with non-vanishing periods. The condition (H6$_\nu$) now allows us to employ the methods of [DS99], leading to the estimate

$$\sum_j\chi(E_j)\langle\psi_j,\mathcal{B}\psi_j\rangle\rho_2\left(\frac{E_j-E}{\hbar}\right)=o(\hbar^{1-d}). \tag{5.6}$$

The relations (5.5) and (5.6) together therefore imply that for every test function $\rho\in C^\infty(\mathbb{R})$ with Fourier transform $\hat{\rho}\in C_0^\infty(\mathbb{R})$ the estimate (5.5) holds with ρ_1 replaced by ρ. Hence, the Tauberian argument developed in [BPU95] can be applied to yield

$$\sum_{E_j\in I(E,\hbar)}\langle\psi_j,\mathcal{B}\psi_j\rangle=\frac{\omega}{\pi}\sum_{\nu=1}^{l}\frac{\mathrm{vol}\,\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}}\,\mathrm{tr}\,\ell_{\nu,E}(P_{\nu,0}B_0P_{\nu,0})+o(\hbar^{1-d}). \tag{5.7}$$

In this relation one can set the operator $\mathcal{B}$ equal to the identity and thus obtains a semiclassical expression for the number N_I of eigenvalues of $\mathcal{H}$ in $I(E,\hbar)$,

$$N_I:=\#\{E_j\in I(E,\hbar)\}=\frac{\omega}{\pi}\sum_{\nu=1}^{l}k_\nu\frac{\mathrm{vol}\,\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}}+o(\hbar^{1-d}), \tag{5.8}$$

where $k_\nu=\mathrm{tr}\,P_{\nu,0}$ denotes the dimension of the fibre $\mathrm{ran}\,P_{\nu,0}=E^\nu$ corresponding to the eigenvalue λ_ν of H_0. The proof is now finished by combining the expressions (5.7) and (5.8). $\square$

Let us add two comments:

1. Under the additional assumption (Irr_ν) the Stratonovich-Weyl calculus discussed in section 4 can be applied. It allows to express $\mathrm{tr}(P_{\nu,0}B_0P_{\nu,0}) = \mathrm{tr}(V_\nu^* B_0 V_\nu)$ in terms of the symbol $b_{0,\nu}$ introduced in (4.15). This then leads to the representation

$$\begin{aligned} \frac{1}{k_\nu} \mathrm{tr}\, \ell_{\nu,E}(P_{\nu,0}B_0P_{\nu,0}) &= \frac{1}{\mathrm{vol}\,\mathcal{O}_\lambda} \int_{\Omega_{\nu,E}} \int_{\mathcal{O}_\lambda} b_{0,\nu}(x,\xi,\eta)\ \mathrm{d}\eta\, \mathrm{d}\ell(x,\xi) \\ &=: \mathrm{M}_{E,\nu,\lambda}\big(b_{0,\nu}\big) \end{aligned} \tag{5.9}$$

as an integral over the product space $\Omega_{\nu,E} \times \mathcal{O}_\lambda$. Here the relation $k_\nu = \mathrm{vol}\,\mathcal{O}_\lambda$, introduced in section 4, enables one to give the right-hand side of (5.1) a genuinely classical interpretation.

2. The operators $\mathcal{B}$ considered in the limit formula (5.1) have not been restricted to those with symbols in the invariant subalgebra $\mathrm{S}^0_{\mathrm{inv}}(1) \subset \mathrm{S}^0_{\mathrm{cl}}(1)$. Nevertheless, only the diagonal blocks of their principal symbols B_0 with respect to the projection matrices $P_{\nu,0}$ enter on the right-hand side of (5.1). In particular, this implies that for an operator $\mathcal{B}$ with a purely off-diagonal principal symbol, i.e., $P_{\mu,0}B_0P_{\mu,0} = 0$ for all $\mu = 1, \dots, l$, the semiclassical average vanishes. Thus one can replace an operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^0_{\mathrm{cl}}(1)$ by its diagonal part $\sum_\mu \tilde{\mathcal{P}}_\mu \mathcal{B} \tilde{\mathcal{P}}_\mu$, whose symbol is in the invariant algebra $\mathrm{S}^0_{\mathrm{inv}}(1)$, without changing the value of the limit on the right-hand side of (5.1).

So far we have considered expectation values in normalised eigenvectors of $\mathcal{H}$. Our intention now is to discuss the projections $\mathcal{P}_\nu \psi_j$ of the eigenvectors of $\mathcal{H}$ to a fixed almost invariant subspace of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$. One thus expresses averaged expectation values in the projected eigenvectors in terms of classical quantities related to the single Hamiltonian flow Φ^t_ν. In order to achieve this one applies Proposition 5.1 to operators $\mathcal{P}_\nu \mathcal{B} \mathcal{P}_\nu$ and exploits the selfadjointness of $\mathcal{P}_\nu$. This results in

Corollary 5.2. *Under the assumptions stated in Proposition 5.1, for each $\nu \in \{1, \dots, l\}$ the restricted limit formula*

$$\lim_{\hbar \to 0} \frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \langle \mathcal{P}_\nu \psi_j, \mathcal{B} \mathcal{P}_\nu \psi_j \rangle = \frac{\mathrm{vol}\,\Omega_{\nu,E}\, \mathrm{tr}\, \ell_{\nu,E}(P_{\nu,0}B_0P_{\nu,0})}{\sum_{\mu=1}^l k_\mu\, \mathrm{vol}\,\Omega_{\mu,E}}. \tag{5.10}$$

holds.

Thus the semiclassical average of the projected eigenvectors $\mathcal{P}_\nu \psi_j$, with $E_j \in I(E,\hbar)$, localises on the corresponding level surface $\Omega_{\nu,E} \subset \mathrm{T}^*\mathbb{R}^d$. If one considers (5.10) for different ν, the relative weights of the corresponding projections are determined by the relative volumes of the associated level surfaces and the dimensions of the eigenspaces E^ν, which equal the volumes of the coadjoint orbits $\mathcal{O}_\lambda$.

In general, however, the projected eigenvectors $\mathcal{P}_\nu \psi_j$ are neither normalised, nor are they genuine eigenvectors of $\mathcal{H}$. We therefore now introduce the normalised vectors

$$\phi_{j,\nu} := \frac{\mathcal{P}_\nu \psi_j}{\|\mathcal{P}_\nu \psi_j\|}. \tag{5.11}$$

Since the projectors $\mathcal{P}_\nu$ only commute with $\mathcal{H}$ up to a term of $O(\hbar^\infty)$, the pairs $(E_j, \phi_{j,\nu})$ are quasimodes with discrepancies $r_{j,\nu}$, i.e.,

$$\big(\mathcal{H} - E_j\big)\phi_{j,\nu} = \frac{[\mathcal{H}, \mathcal{P}_\nu]\psi_j}{\|\mathcal{P}_\nu \psi_j\|} \quad \text{and} \quad r_{j,\nu} = \frac{\|[\mathcal{H}, \mathcal{P}_\nu]\psi_j\|}{\|\mathcal{P}_\nu \psi_j\|}.$$

This observation only ensures the existence of an eigenvalue of $\mathcal{H}$ in the interval $[E_j - r_{j,\nu}, E_j + r_{j,\nu}]$, which is a trivial statement; it does not imply that $\phi_{j,\nu}$ is close to an eigenvector of $\mathcal{H}$, see [Laz93]. It therefore is of somewhat more interest to consider the operator $\mathcal{H}\mathcal{P}_\nu$, whose spectrum inside the interval $[E-\varepsilon, E+\varepsilon] \supset I(E,\hbar)$ is as well purely discrete. Following the above reasoning, one then concludes that $(E_j, \phi_{j,\nu})$ is a quasimode with discrepancy $r_{j,\nu}$ also for this operator. Thus, if $\|\mathcal{P}_\nu \psi_j\| \geq c\hbar^N$ for some $N \geq 0$ and hence $r_{j,\nu} = O(\hbar^\infty)$, the operator $\mathcal{H}\mathcal{P}_\nu$ has an eigenvalue with distance $O(\hbar^\infty)$ away from E_j. Since there are N_I eigenvalues $E_j \in I(E,\hbar)$ one finds as many quasimodes for $\mathcal{H}\mathcal{P}_\nu$. But this operator has only

$$N_I^\nu = \frac{k_\nu \omega}{\pi} \frac{\operatorname{vol}\Omega_{\nu,E}}{(2\pi\hbar)^{d-1}} + o(\hbar^{1-d})$$

eigenvalues in $I(E,\hbar)$, compare (5.8). This observation might suggest that only approximately N_I^ν of the N_I projected eigenvectors $\mathcal{P}_\nu \psi_j$ are of considerable size, such that the discrepancies of the associated quasimodes are smaller than the distance of E_j to neighbouring eigenvalues of $\mathcal{H}$. This expectation can be strengthened by an application of the limit formula (5.10) with the choice $\mathcal{B} = \mathrm{id}$,

$$\lim_{\hbar \to 0} \frac{1}{N_I} \sum_{E_j \in I(E,\hbar)} \|\mathcal{P}_\nu \psi_j\|^2 = \frac{k_\nu \operatorname{vol}\Omega_{\nu,E}}{\sum_{\mu=1}^{l} k_\mu \operatorname{vol}\Omega_{\mu,E}}, \tag{5.12}$$

which implies that

$$N_I^\nu = \sum_{E_j \in I(E,\hbar)} \|\mathcal{P}_\nu \psi_j\|^2 + o(1), \quad \hbar \to 0. \tag{5.13}$$

One could thus expect that roughly N_I^ν of the projected eigenvectors $\mathcal{P}_\nu \psi_j$ are close to ψ_j, and the rest is such that $\|\mathcal{P}_\nu \psi_j\|$ is semiclassically small. However, (5.13) does not rule out the other extreme situation, provided by projected eigenvectors $\mathcal{P}_\nu \psi_j$, $\nu = 1, \dots, l$, equidistributing in the sense that their squared norms are asymptotic to N_I^ν / N_I as $\hbar \to 0$. In that case the discrepancies of the associated quasimodes for the operators $\mathcal{H}\mathcal{P}_\nu$ can be estimated as $r_{j,\nu} = O(\hbar^\infty)$. In order now that these quasimodes do not produce more than N_I^ν eigenvalues of $\mathcal{H}\mathcal{P}_\nu$ in $I(E,\hbar)$, a finite fraction of the eigenvalues E_j of $\mathcal{H}$ must possess spacings to their nearest neighbours of the order $\hbar^\infty$. Since in general there exist no sufficient lower bounds on eigenvalue spacings, none of the two extreme situations discussed above can be excluded so far.

What is possible, however, is to derive from (5.12) an upper bound for the fraction of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ that are close in norm to ψ_j,

$$\lim_{\hbar\to 0}\frac{1}{N_I}\#\big\{E_j\in I(E,\hbar);\ \|\mathcal{P}_\nu\psi_j-\psi_j\|=o(1)\big\}\leq\frac{k_\nu\,\mathrm{vol}\,\Omega_{\nu,E}}{\sum_{\mu=1}^{l}k_\mu\,\mathrm{vol}\,\Omega_{\mu,E}},$$

see also [Sch01]. To obtain lower bounds is notoriously more difficult. The limit formula (5.12) only allows to estimate the fraction of projected eigenvectors with norms that tend to a finite limit as $\hbar\to 0$. One conveniently measures this fraction in units of the value that is expected for equidistributed projections. Therefore, with $\delta:=\tilde{\delta}\frac{k_\nu\,\mathrm{vol}\,\Omega_{\nu,E}}{\sum_{\mu=1}^{l}k_\mu\,\mathrm{vol}\,\Omega_{\mu,E}}$, we consider

$$N_{\nu,I}^\delta:=\#\big\{E_j\in I(E,\hbar);\ \|\mathcal{P}_\nu\psi_j\|^2\geq\delta\big\}.$$

Since

$$\begin{aligned}\frac{1}{N_I}\sum_{E_j\in I(E,\hbar)}\|\mathcal{P}_\nu\psi_j\|^2&\leq\frac{1}{N_I}\sum_{\substack{E_j\in I(E,\hbar)\\ \|\mathcal{P}_\nu\psi_j\|^2\geq\delta}}1+\frac{1}{N_I}\sum_{\substack{E_j\in I(E,\hbar)\\ \|\mathcal{P}_\nu\psi_j\|^2<\delta}}\|\mathcal{P}_\nu\psi_j\|^2\\&\leq\frac{N_{\nu,I}^\delta}{N_I}+\frac{\delta}{N_I}(N_I-N_{\nu,I}^\delta),\end{aligned}$$

the relative fraction of projected eigenvectors with finite semiclassical limit can be estimated from below as

$$\lim_{\hbar\to 0}\frac{N_{\nu,I}^\delta}{N_I}\geq\frac{(1-\tilde{\delta})k_\nu\,\mathrm{vol}\,\Omega_{\nu,E}}{\sum_{\mu=1}^{l}k_\mu\,\mathrm{vol}\,\Omega_{\mu,E}}.\tag{5.14}$$

6 Quantum ergodicity

Our intention in this section is to consider quantum ergodicity for the normalised eigenvectors ψ_j, $E_j\in I(E,\hbar)$, of the quantum Hamiltonian $\mathcal{H}$. In the case of scalar pseudodifferential operators one denotes by quantum ergodicity a weak convergence of the phase space lifts of almost all eigenfunctions to Liouville measure on the level surface $\Omega_E=H_0^{-1}(E)$, and proves this to hold if the flow generated by the principal symbol H_0 of the quantum Hamiltonian is ergodic on Ω_E. In the present situation of operators with matrix valued symbols, however, each eigenvalue λ_ν of H_0 defines its own classical dynamics. One hence can only expect quantum ergodicity to be concerned with statements about the projections $\mathcal{P}_\nu\psi_j$ of the eigenvectors to the different almost invariant subspaces of $L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$

in relation to the behaviour of the associated classical systems. In the preceding section we discussed the question of identifying those projected eigenvectors whose norms are not semiclassically small. Since presently this problem cannot be resolved directly, quantum ergodicity can only be formulated by restricting to those eigenvectors whose squared norms exceed a value of δ in the semiclassical limit, without specifying them further.

Conventionally the convergence of quantum states determined by the eigenvectors ψ_j of $\mathcal{H}$ is discussed in terms of expectation values of observables in these states. Explicit lifts of the eigenfunctions to phase space are then, e.g., provided by their Wigner transforms. The choice of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ leads to consider expectation values of diagonal blocks $\mathcal{P}_\nu\mathcal{B}\mathcal{P}_\nu$ of operators $\mathcal{B}$ with symbols $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$. On the symbol level the time evolution of these blocks is covered by the Egorov theorem 3.2. Representing then the blocks of the principal symbols by Stratonovich-Weyl symbols as described in section 4, according to Proposition 4.6 we are faced with the skew-product flows Y^t_ν on the product phase spaces $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$. Since the Stratonovich-Weyl symbols $b_{0,\nu}$ defined in equation (4.15) that are associated with symbols $B \in \mathrm{S}^q_{\mathrm{cl}}(1)$ are clearly integrable with respect to the measures $\mathrm{d}\ell\,\mathrm{d}\eta$ on the (compact) manifolds $\Omega_{\nu,E} \times \mathcal{O}_\lambda$, the (assumed) ergodicity of the flow Y^t_ν implies that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(b_{0,\nu}\circ Y^t_\nu\big)(x,\xi,\eta)\ \mathrm{d}t &= \frac{1}{\mathrm{vol}\,\mathcal{O}_\lambda}\int_{\Omega_{\nu,E}}\int_{\mathcal{O}_\lambda} b_{0,\nu}(x',\xi',\eta')\ \mathrm{d}\eta'\,\mathrm{d}\ell(x',\xi') \\ &= \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\end{aligned} \tag{6.1}$$

holds for almost all initial conditions $(x,\xi,\eta) \in \Omega_{\nu,E} \times \mathcal{O}_\lambda$. In particular, one immediately realises that the supposed ergodicity of Y^t_ν implies ergodicity for the flow Φ^t_ν on $\Omega_{\nu,E}$ with respect to Liouville measure $\mathrm{d}\ell$. As a consequence the condition (H6$_\nu$) is automatically fulfilled.

For the subsequent formulation and proof of quantum ergodicity we choose to follow in principle the approach of [Zel96, ZZ96]. This means that we investigate the variance of expectation values about their mean in the semiclassical limit. In order to avoid the problem of explicitly estimating the norms of projected eigenvectors we here consider the normalised vectors $\phi_{j,\nu}$, defined in (5.11), which have been identified as quasimodes for both the operators $\mathcal{H}$ and $\mathcal{H}\mathcal{P}_\nu$. Moreover, we concentrate on vectors corresponding to projected eigenvectors with norms that do not vanish semiclassically, i.e., with $\|\mathcal{P}_\nu\psi_j\|^2 \geq \delta$ for some fixed $\delta \in (0,1)$. This approach is similar to the one introduced by Schubert [Sch01] in the context of local quantum ergodicity, where an equidistribution was shown for quasimodes associated with ergodic components of phase space. In section 5 we estimated the relative number $N^\delta_{\nu,I}/N_I$ of the associated eigenvectors among all eigenvectors of $\mathcal{H}$ in the semiclassical limit from below, see (5.14). A non-trivial bound could only be obtained for $\tilde{\delta} < 1$ corresponding to

$$\delta < \delta_\nu := \frac{k_\nu\,\mathrm{vol}\,\Omega_{\nu,E}}{\sum_{\mu=1}^{l} k_\mu\,\mathrm{vol}\,\Omega_{\mu,E}}.$$

Therefore, from now on we confine δ to the interval $\delta \in (0,\delta_\nu)$, and are thus in a position to state our main result.

Theorem 6.1. *Let $\mathcal{H}$ be a pseudodifferential operator with hermitian symbol $H \in \mathrm{S}^0_{\mathrm{cl}}(m)$ whose principal part H_0 fulfills the conditions (H1) and (H2) of section 3. The eigenvalues $\lambda_1, \dots, \lambda_l$ of H_0 are required to have constant multiplicities and shall obey the conditions (H3$_\nu$)–(H5$_\nu$) of section 5 for all $\nu \in \{1, \dots, l\}$. Moreover, they shall be separated according to the hyperbolicity condition (H0),*

$$|\lambda_\nu(x,\xi) - \lambda_\mu(x,\xi)| \geq Cm(x,\xi) \quad \textit{for} \quad \nu \neq \mu \quad \textit{and} \quad |x| + |\xi| \geq c.$$

Assume now that the symbol $H \sim \sum_{j=0}^\infty \hbar^j H_j$ satisfies the growth condition

$$\|H_j{}^{(\alpha)}_{(\beta)}(x,\xi)\|_{n\times n} \leq C_{\alpha,\beta} \quad \textit{for all} \quad (x,\xi) \in \mathrm{T}^*\mathbb{R}^d \textit{ and } |\alpha| + |\beta| + j \geq 2 - \delta_{j0}, \tag{3.9}$$

and that the condition (Irr$_\nu$) of section 4 holds. If then the flow Y^t_ν defined in (4.13) is ergodic on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ with respect to the invariant measure $\mathrm{d}\ell\,\mathrm{d}\eta$, in every sequence of normalised projected eigenvectors $\{\phi_{j,\nu}\}_{j\in\mathbb{N}}$, with $\|\mathcal{P}_\nu\psi_j\|^2 \geq \delta$, $\delta \in (0,\delta_\nu)$ fixed, one finds a subsequence $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}$ of density one, i.e.,

$$\lim_{\hbar\to 0} \frac{\#\{\alpha;\ \|\mathcal{P}_\nu\psi_{j_\alpha}\|^2 \geq \delta\}}{\#\{j;\ \|\mathcal{P}_\nu\psi_j\|^2 \geq \delta\}} = 1,$$

such that for every operator $\mathcal{B}$ with symbol $B \in \mathrm{S}^0_{\mathrm{cl}}(1)$ and principal symbol B_0

$$\lim_{\hbar\to 0} \langle \phi_{j_\alpha,\nu}, \mathcal{B}\phi_{j_\alpha,\nu}\rangle = \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}), \tag{6.2}$$

where $b_{0,\nu}$ denotes the Stratonovich-Weyl symbol associated with $P_{\nu,0}B_0P_{\nu,0}$. Furthermore, the density-one subsequence $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}$ can be chosen to be independent of the operator $\mathcal{B}$.

Proof. We start with considering expectation values of the operator $\mathcal{B}$ taken in the quasi-modes $\{\phi_{j,\nu}\}$ and denote their variance about the mean $\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})$ of the corresponding Stratonovich-Weyl symbol $b_{0,\nu}$ defined in (4.15) as

$$S^\delta_{2,\nu}(E,\hbar) := \frac{1}{N^\delta_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar) \\ \|\mathcal{P}_\nu\psi_j\|^2 \geq \delta}} \big|\langle \phi_{j,\nu}, \mathcal{B}\phi_{j,\nu}\rangle - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\big|^2.$$

Due to the definition (5.11) of the normalised vectors $\phi_{j,\nu}$, this variance can also be written as

$$\begin{aligned} S^\delta_{2,\nu}(E,\hbar) &= \frac{1}{N^\delta_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar) \\ \|\mathcal{P}_\nu\psi_j\|^2 \geq \delta}} \big|\langle \phi_{j,\nu}, \big(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\big)\phi_{j,\nu}\rangle\big|^2 \\ &= \frac{1}{N^\delta_{\nu,I}} \sum_{\substack{E_j \in I(E,\hbar) \\ \|\mathcal{P}_\nu\psi_j\|^2 \geq \delta}} \|\mathcal{P}_\nu\psi_j\|^{-2} \big|\langle \psi_j, \mathcal{P}_\nu\big(\mathcal{B} - \mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\big)\mathcal{P}_\nu\psi_j\rangle\big|^2. \end{aligned}$$

Allowing for an error of $O(\hbar^\infty)$, in this expression the expectation values can be replaced by those of the operator $\tilde{\mathcal{P}}_\nu(\mathcal{B}-\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}))\tilde{\mathcal{P}}_\nu$ whose symbol is in the invariant subalgebra $\mathrm{S}^0_{\mathrm{inv}}(1)\subset \mathrm{S}^0_{\mathrm{cl}}(1)$. Therefore, since all further requirements are also met, the Egorov theorem 3.2 applies and yields that for finite times $t\in[0,T]$ the evolution $\mathcal{U}^*(t)\tilde{\mathcal{P}}_\nu(\mathcal{B}-\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}))\tilde{\mathcal{P}}_\nu\mathcal{U}(t)$ of this operator is again a pseudodifferential operator with symbol in the class $\mathrm{S}^0_{\mathrm{cl}}(1)$. Taking into account that the ψ_js are eigenvectors of $\mathcal{H}$ with eigenvalues E_j, the above expression can be rewritten as

$$S^\delta_{2,\nu}(E,\hbar)=\frac{1}{N^\delta_{\nu,I}}\sum_{\substack{E_j\in I(E,\hbar)\\ \|\mathcal{P}_\nu\psi_j\|^2\geq\delta}}\bigl|\langle\psi_j,\mathcal{B}_{\nu,T}\psi_j\rangle\bigr|^2\|\mathcal{P}_\nu\psi_j\|^{-2},$$

where we have defined the auxiliary operator

$$\mathcal{B}_{\nu,T}:=\frac{1}{T}\int_0^T\mathcal{U}^*(t)\tilde{\mathcal{P}}_\nu\bigl(\mathcal{B}-\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})\bigr)\tilde{\mathcal{P}}_\nu\mathcal{U}(t)\ \mathrm{d}t. \tag{6.3}$$

Furthermore, by using using the Cauchy-Schwarz inequality and the lower bound on the norms $\|\mathcal{P}_\nu\psi_j\|^2\geq\delta>0$ we obtain as an upper bound

$$S^\delta_{2,\nu}(E,\hbar)\leq\frac{1}{\delta}\frac{N_I}{N^\delta_{\nu,I}}\frac{1}{N_I}\sum_{E_j\in I(E,\hbar)}\langle\psi_j,\mathcal{B}^2_{\nu,T}\psi_j\rangle.$$

According to equation (5.14) the factor $N_I/N^\delta_{\nu,I}$ can be estimated from above in the semiclassical limit. We hence now consider the semiclassical limit of the expression

$$\frac{1}{N_I}\sum_{E_j\in I(E,\hbar)}\langle\psi_j,\mathcal{B}^2_{\nu,T}\psi_j\rangle,$$

to which Proposition 5.1 can be applied. To this end one requires the principal symbol $B_{\nu,T,0}$ of the auxiliary operator $\mathcal{B}_{\nu,T}$, which follows from Theorem 3.2 as

$$B_{\nu,T,0}=\frac{1}{T}\int_0^T d^*_{\nu\nu}\bigl((P_{\nu,0}B_0P_{\nu,0})\circ\Phi^t_\nu\bigr)d_{\nu\nu}\ \mathrm{d}t-\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})P_{\nu,0}.$$

Given this, the limit formula (5.1) and the estimate (5.14) yield

$$\begin{aligned}\lim_{\hbar\to0}S^\delta_{2,\nu}(E,\hbar)&\leq\frac{1}{\delta}\,\frac{\sum_{\mu=1}^l k_\mu\,\mathrm{vol}\,\Omega_{\mu,E}}{(1-\tilde{\delta})k_\nu\,\mathrm{vol}\,\Omega_{\nu,E}}\ \frac{\mathrm{vol}\,\Omega_{\nu,E}\,\mathrm{tr}\,\ell_{\nu,E}(B^2_{\nu,T,0})}{\sum_{\mu=1}^l k_\mu\,\mathrm{vol}\,\Omega_{\mu,E}}\\&=\frac{1}{\delta}\,\frac{1}{1-\tilde{\delta}}\ \mathrm{M}_{E,\nu,\lambda}\bigl((\mathrm{symb}^{SW}[B_{\nu,T,0}])^2\bigr),\end{aligned} \tag{6.4}$$

when employing the tracial property (v) of Proposition 4.5.

According to Proposition 4.6 the Stratonovich-Weyl symbol of $B_{\nu,T,0}$ can now be easily calculated as

$$\mathrm{symb}^{SW}[B_{\nu,T,0}(x,\xi)](\eta)=\frac{1}{T}\int_0^T\big(b_{0,\nu}\circ Y_\nu^t\big)(x,\xi,\eta)\ \mathrm{d}t-\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}).$$

Since we assume the skew-product flow Y_ν^t to be ergodic with respect to $\mathrm{d}\ell\,\mathrm{d}\eta$, the relation (6.1) implies that $\mathrm{symb}^{SW}[B_{\nu,T,0}(x,\xi)](\eta)$ vanishes in the limit $T\to\infty$ for almost all points $(x,\xi,\eta)\in\Omega_{\nu,E}\times\mathcal{O}_\lambda$. Now, on the right-hand side of (6.4) the square of $\mathrm{symb}^{SW}[B_{\nu,T,0}]$ enters integrated over $\Omega_{\nu,E}\times\mathcal{O}_\lambda$, so that this expression vanishes as $T\to\infty$. We hence conclude that

$$\lim_{\hbar\to 0}S_{2,\nu}^{\delta}(E,\hbar)=0.$$

This, in turn, is equivalent to the existence of a subsequence $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}\subset\{\phi_{j,\nu}\}_{j\in\mathbb{N}}$ of density one, such that equation (6.2) holds. Finally, by a diagonal construction as in [Zel87, CdV85] one can extract a subsequence of $\{\phi_{j_\alpha,\nu}\}_{\alpha\in\mathbb{N}}$ that is still of density one in $\{\phi_{j,\nu}\}_{j\in\mathbb{N}}$, such that (6.2) holds independently of the operator $\mathcal{B}$. □

The version of quantum ergodicity asserted in Theorem 6.1 means that in the semiclassical limit the lifts of almost all quasimodes $\phi_{j,\nu}$ to the phase space $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}_\lambda$ equidistribute in the sense that suitable Wigner functions (weakly) converge to an invariant measure on $\Omega_{\nu,E}\times\mathcal{O}_\lambda$ that is proportional to $\mathrm{d}\ell\,\mathrm{d}\eta$. In order to identify the proper Wigner transform consider

$$\langle\phi_{j_\alpha,\nu},\mathcal{B}\phi_{j_\alpha,\nu}\rangle=\frac{1}{(2\pi\hbar)^d}\iint_{\mathrm{T}^*\mathbb{R}^d}\mathrm{tr}\Big(W[\phi_{j_\alpha,\nu}](x,\xi)P_\nu(x,\xi)B(x,\xi)P_\nu(x,\xi)\Big)\ \mathrm{d}x\,\mathrm{d}\xi+O(\hbar^\infty),$$

with the matrix valued Wigner transform

$$W[\psi](x,\xi):=\int_{\mathbb{R}^d}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\xi y}\overline{\psi}(x-\tfrac{y}{2})\otimes\psi(x+\tfrac{y}{2})\ \mathrm{d}y$$

defined for $\psi\in L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$. We now exploit the Stratonovich-Weyl calculus to conclude that on the level of principal symbols

$$\begin{aligned}\mathrm{tr}\Big(W[\phi_{j_\alpha,\nu}]P_{\nu,0}B_0P_{\nu,0}\Big)&=\mathrm{tr}\Big(\big(V_\nu^*W[\phi_{j_\alpha,\nu}]V_\nu\big)\big(V_\nu^*B_0V_\nu\big)\Big)\\&=\int_{\mathcal{O}_\lambda}\mathrm{symb}^{SW}[V_\nu^*W[\phi_{j_\alpha,\nu}]V_\nu](\eta)\,\mathrm{symb}^{SW}[V_\nu^*B_0V_\nu](\eta)\ \mathrm{d}\eta.\end{aligned}$$

The second factor in the integral has been defined as $b_{0,\nu}$ in (4.15). In analogy to this we therefore introduce for $\psi\in L^2(\mathbb{R}^d)\otimes\mathbb{C}^n$ the scalar Wigner transform (see also [BGK01])

$$w_\nu[\psi](x,\xi,\eta):=\mathrm{symb}^{SW}[V_\nu^*(x,\xi)W[\psi](x,\xi)V_\nu(x,\xi)](\eta),$$

that indeed provides a lift of ψ to the phase space $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}_\lambda$. The statement of Theorem 6.1 can thus be rephrased in that under the given conditions one obtains (in the sense of a weak convergence),

$$\lim_{\hbar\to 0}\frac{1}{(2\pi\hbar)^d}\,w_\nu[\phi_{j_\alpha,\nu}](x,\xi,\eta)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\eta=\frac{1}{\mathrm{vol}\,\mathcal{O}_\lambda}\,\mathrm{d}\ell(x,\xi)\,\mathrm{d}\eta$$

along the subsequence of density one. However, since in $\phi_{j,\nu}$ the normalisation of $\mathcal{P}_\nu\psi_j$ is hidden, an equivalent equidistribution for the lifts of the projected eigenvectors is only shown up to a constant. In analogy to the discussion in [Sch01] this means that in the sequence $\{\psi_j;\ E_j\in I(E,\hbar)\}$ there exists a subsequence $\{\psi_{j_\alpha}\}$ of density one such that as $\hbar\to 0$,

$$\langle\psi_{j_\alpha},\mathcal{P}_\nu\mathcal{B}\mathcal{P}_\nu\psi_{j_\alpha}\rangle=\|\mathcal{P}_\nu\psi_{j_\alpha}\|^2\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})+o(1),$$

with a corresponding statement for the scalar Wigner transforms $w_\nu[\mathcal{P}_\nu\psi_{j_\alpha}]$. Notice that the factor $\|\mathcal{P}_\nu\psi_{j_\alpha}\|^2$ is independent of the operator $\mathcal{B}$ so that the subsequence can again be chosen independently of $\mathcal{B}$. Therefore, a non-vanishing semiclassical limit only exists for those subsequences along which the norms $\|\mathcal{P}_\nu\psi_{j_\alpha}\|$ do not tend to zero as $\hbar\to 0$. These subsequences are excluded in the formulation of Theorem 6.1 since δ is fixed and positive.

The difficulties with estimating norms of the projected eigenvectors $\mathcal{P}_\nu\psi_j$ arise from the presence of several level surfaces $\Omega_{\nu,E}$ on which the lifts of eigenfunctions potentially condense in the semiclassical limit. The situation simplifies considerably, if at the energy E all of the l level surfaces except one are empty.

Corollary 6.2. *If under the conditions stated in Theorem 6.1 only the level surface $\Omega_{\nu,E}\subset\mathrm{T}^*\mathbb{R}^d$ is non-empty, there exists a subsequence $\{\psi_{j_\alpha}\}$ of density one in $\{\psi_j;\ E_j\in I(E,\hbar)\}$, independent of the operator $\mathcal{B}$, such that*

$$\lim_{\hbar\to 0}\langle\psi_{j_\alpha},\mathcal{P}_\mu\mathcal{B}\mathcal{P}_\mu\psi_{j_\alpha}\rangle=\delta_{\mu\nu}\,\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu}).$$

In this situation the norms $\|\mathcal{P}_\mu\psi_{j_\alpha}\|$ converge to one for $\mu=\nu$ and to zero otherwise as $\hbar\to 0$ along the subsequence. The lifts of the eigenvectors therefore condense on the only available level surface in $\mathrm{T}^*\mathbb{R}^d$, as one clearly would have expected.

Remark 6.3. As a condition for quantum ergodicity to hold we have assumed the skew-product flow Y_ν^t on $\Omega_{\nu,E}\times\mathcal{O}_\lambda$ to be ergodic. The reason for introducing this flow was to formulate a genuinely classical criterion in terms of a dynamics on the symplectic phase space $\mathrm{T}^*\mathbb{R}^d\times\mathcal{O}_\lambda$. The formulation will be somewhat simpler, if one refrains from insisting on a completely classical description and employs the skew-product flow $\tilde{Y}_\nu^t$ defined on $\mathrm{T}^*\mathbb{R}^d\times G$, see (4.6), instead. Then the use of the Stratonovich-Weyl calculus can be avoided. Such a formulation is based on a hybrid of the classical Hamiltonian flow Φ_ν^t on $\mathrm{T}^*\mathbb{R}^d$ and the dynamics represented by the conjugation with the unitary matrices D_ν, which appears to be quantum mechanical in nature. Both formulations, however, are equivalent in the sense that, first, the Stratonovich-Weyl calculus relates the quantum dynamics in the eigenspace to a classical dynamics on the coadjoint orbit in a one-to-one manner. Second, in appendix B we show that the skew-product Y_ν^t on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ is ergodic, if and only if the skew-product $\tilde{Y}_\nu^t$ is ergodic on $\Omega_{\nu,E} \times G$. One can therefore formulate Theorem 6.1 without recourse to the Stratonovich-Weyl calculus once the limit $\mathrm{M}_{E,\nu,\lambda}(b_{0,\nu})$ is expressed as

$$\mathrm{M}_{E,\nu,\lambda}\big(b_{0,\nu}\big) = \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(P_{\nu,0} B_0 P_{\nu,0}),$$

see (5.9). Up to equation (6.4) the proof of Theorem 6.1 proceeds in the same manner as shown. From this point on one can then basically follow the method of [BG00], and to this end represents the principal symbol $B_{\nu,T,0}$ of the auxiliary operator (6.3) in terms of the isometries V_ν,

$$V_\nu^* B_{\nu,T,0} V_\nu = \frac{1}{T} \int_0^T D_\nu^* \big((V_\nu^* B_0 V_\nu) \circ \Phi_\nu^t\big) D_\nu \; \mathrm{d}t - \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(V_\nu^* B_0 V_\nu).$$

We now suppose that the flow $\tilde{Y}_\nu^t$ is ergodic on $\Omega_{\nu,E} \times G$ and choose the function $F(x,\xi,g) := \rho(g)^* (V_\nu^* B_0 V_\nu)(x,\xi) \rho(g) \in L^1(\Omega_{\nu,E} \times G) \otimes \mathrm{M}_{k_\nu}(\mathbb{C})$ to exploit the ergodicity. This yields for almost all initial values $(x,\xi,g) \in \Omega_{\nu,E} \times G$ that

$$\begin{aligned} &\lim_{T\to\infty} \rho(g)^* V_\nu^*(x,\xi) B_{\nu,T,0}(x,\xi) V_\nu(x,\xi) \rho(g) \\ &\quad = \int_{\Omega_{\nu,E}} \int_G \rho(h)^* (V_\nu^* B_0 V_\nu)(y,\zeta) \rho(h) \; \mathrm{d}h \, \mathrm{d}\ell(y,\zeta) - \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(V_\nu^* B_0 V_\nu). \end{aligned}$$

Furthermore, since the representation $(\rho, \mathbb{C}^{k_\nu})$ is assumed to be irreducible and the integral in the above expression is invariant under conjugation with arbitrary elements of $U(k_\nu)$, Schur's lemma implies that this integral is a multiple of the identity in $\mathbb{C}^{k_\nu}$, leading to

$$\int_{\Omega_{\nu,E}} \int_G \rho(h)^* (V_\nu^* B_0 V_\nu)(y,\zeta) \rho(h) \; \mathrm{d}h \, \mathrm{d}\ell(y,\zeta) = \frac{1}{k_\nu} \operatorname{tr} \ell_{\nu,E}(V_\nu^* B_0 V_\nu).$$

Due to the way the principal symbol $B_{\nu,T,0}$ enters on the right-hand side of (6.4), the conjugation with $V_\nu(x,\xi)\rho(g)$ as well as the restriction to almost all (x,ξ,g) is inessential, so that again one concludes a vanishing of $S_{2,\nu}^\delta(E,\hbar)$ as $\hbar \to 0$.

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Appendices

A Relations for Poisson brackets of matrix valued functions

In this appendix we collect some relations for Poisson brackets of matrix valued functions on the phase space $\mathrm{T}^*\mathbb{R}^d$ that are needed in section 3. These relations are already stated in [EW96, GMMP97, Spo00] and can be verified by straightforward calculations.

Our convention for the Poisson bracket of smooth matrix valued functions $A, B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ is

$$\{A, B\} := \partial_\xi A \, \partial_x B - \partial_x A \, \partial_\xi B.$$

The first general relation then reads

$$A\{B, C\} - \{A, B\}C = \{AB, C\} - \{A, BC\}. \tag{A.1}$$

Furthermore, for the projection matrices $P = PP$ one finds

$$P\{\lambda, P\}P = 0, \tag{A.2}$$

where λ is any smooth scalar function on $\mathrm{T}^*\mathbb{R}^d$.

For $B \in C^\infty(\mathrm{T}^*\mathbb{R}^d) \otimes \mathrm{M}_n(\mathbb{C})$ commuting with P one then derives

$$P\{\lambda, B\}P = \{\lambda, PBP\} - [PBP, [P, \{\lambda, P\}]], \tag{A.3}$$

In particular, using (A.1) for projection matrices one obtains

$$P\{P, B\} - \{P, P\}B = \{P, B\} - \{P, PB\}$$

and

$$B\{P, P\} - \{B, P\}P = \{BP, P\} - \{B, P\}.$$

Using these relations together with the condition $[B, P] = 0$ one gets

$$P\big(\{B, P\} - \{P, B\}\big)P = [B, P\{P, P\}P].$$

Furthermore, for different projection matrices P_μ and P_ν with $P_\mu P_\nu = 0$ for $\nu \neq \mu$ the general relation (A.1) implies

$$P_\mu\{P_\nu, P_\nu\} = -\{P_\mu, P_\nu\}(1 - P_\nu)$$

and

$$\{P_\nu, P_\nu\}P_\mu = -(1 - P_\nu)\{P_\nu, P_\mu\}.$$

In the case $[P_\nu, B] = 0 = [P_\mu, B]$ one finds

$$P_\nu\{P_\mu, B\} - \{P_\nu, P_\mu\}B = -\{P_\nu, P_\mu B\},$$
$$B\{P_\mu, P_\nu\} - \{B, P_\mu\}P_\nu = \{BP_\mu, P_\nu\}.$$

These equations imply

$$P_\nu\big(\{B, P_\mu\} - \{P_\mu, B\}\big)P_\nu = -[B, P_\nu\{P_\mu, P_\mu\}P_\nu].$$

One can now apply the above relations to expressions of the type arising in section 3, i.e.,

$$\begin{aligned} P_\mu &\left(\frac{\partial}{\partial t}B(t) + \frac{1}{2}\Big(\{B(t), \lambda_\nu P_\nu\} - \{\lambda_\nu P_\nu, B(t)\}\Big) + \mathrm{i}[B(t), H_1]\right) P_\mu \\ &= \frac{\partial}{\partial t}P_\mu B(t) P_\mu - \delta_{\nu\mu}\{\lambda_\nu, P_\mu B(t) P_\mu\} \\ &\quad + \left[\frac{\lambda_\nu}{2}(-1)^{\delta_{\nu\mu}} P_\mu\{P_\nu, P_\nu\}P_\mu - \delta_{\nu\mu}[P_\nu, \{\lambda_\nu, P_\nu\}] - \mathrm{i}P_\mu H_1 P_\mu, P_\mu B(t) P_\mu\right]. \end{aligned}$$

Therefore, the definition

$$\tilde{H}_1 := \mathrm{i}(-1)^{\delta_{\nu\mu}}\frac{\lambda_\nu}{2}P_\mu\{P_\nu, P_\nu\}P_\mu - \mathrm{i}\delta_{\nu\mu}[P_\nu, \{\lambda_\nu, P_\nu\}] + P_\mu H_1 P_\mu \tag{A.4}$$

allows to conclude that

$$\frac{\partial}{\partial t}P_\mu B P_\mu - \delta_{\nu\mu}\{\lambda_\nu, P_\mu B P_\mu\} - \mathrm{i}[\tilde{H}_1, P_\mu B P_\mu] = 0. \tag{A.5}$$

B A relation between the ergodicity of two skew-product flows

In section 4 we considered two types of skew-product dynamics built over the Hamiltonian flows Φ^t_ν on $\mathrm{T}^*\mathbb{R}^d$. Both derive from the dynamics in the eigenvector bundles $E^\nu \to \mathrm{T}^*\mathbb{R}^d$ given by conjugating the diagonal blocks of principal symbols with the transport matrices $d_{\nu\nu}$ along integral curves of the Hamiltonian flows. After having fixed local orthonormal bases in the fibres, or isometries $V_\nu(x,\xi) : \mathbb{C}^{k_\nu} \to E^\nu(x,\xi)$, respectively, the transport matrices $d_{\nu\nu}$ have been represented by unitary $k_\nu \times k_\nu$ matrices D_ν, leading to the skew-product flows $\hat{Y}^t_\nu$ on $\mathrm{T}^*\mathbb{R}^d \times \mathrm{U}(k_\nu)$. We then noticed that the dynamics in the fibres might not exhaust the whole group $\mathrm{U}(k_\nu)$, but only some subgroup G, which is then represented in $\mathrm{U}(k_\nu)$. This led us to consider the skew-product flows $\tilde{Y}^t_\nu$ on $\mathrm{T}^*\mathbb{R}^d \times G$, given as $\tilde{Y}^t_\nu(x,\xi,g) = (\Phi^t_\nu(x,\xi), g_\nu(x,\xi,t)g)$, see (4.5) and (4.6). Assuming that the representation ρ of G in $\mathrm{U}(k_\nu)$ is irreducible, we constructed a representation of the fibre dynamics on the coadjoint orbit $\mathcal{O}_\lambda$ of G determined by ρ. We thus arrived at the skew-product flows Y^t_ν on the symplectic phase spaces $\mathrm{T}^*\mathbb{R}^d \times \mathcal{O}_\lambda$, with $Y^t_\nu(x,\xi,\eta) = (\Phi^t_\nu(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}\,\eta)$, see (4.12) and (4.13). In section 6 we required either the flows $\tilde{Y}^t_\nu$ or Y^t_ν, restricted to the level surfaces $\Omega_{\nu,E} \subset \mathrm{T}^*\mathbb{R}^d$ in the base manifold, to be ergodic relative to the respective invariant measures $\mathrm{d}\ell\,\mathrm{d}g$ or $\mathrm{d}\ell\,\mathrm{d}\eta$. We now show:

Proposition B.1. *The flow* $\tilde{Y}_\nu^t : \Omega_{\nu,E} \times G \to \Omega_{\nu,E} \times G$ *is ergodic with respect to* $\mathrm{d}\ell\,\mathrm{d}g$*, if and only if the associated flow* $Y_\nu^t : \Omega_{\nu,E} \times \mathcal{O}_\nu \to \Omega_{\nu,E} \times \mathcal{O}_\nu$ *is ergodic with respect to* $\mathrm{d}\ell\,\mathrm{d}\eta$*.*

Proof. A convenient characterisation for the ergodicity of a flow Φ^t on a probability space (Σ, dm) with invariant measure dm employs the flow-invariant subsets of Σ: The flow is ergodic with respect to dm, if and only if every measurable flow-invariant set has either measure zero or full measure. We now first consider the 'if' direction asserted in the proposition and to this end assume that Y_ν^t on $\Omega_{\nu,E} \times \mathcal{O}_\lambda$ is ergodic with respect to $\mathrm{d}\ell\,\mathrm{d}\eta$. Hence every measurable Y_ν^t-invariant set $B \subset \Omega_{\nu,E} \times \mathcal{O}_\lambda$ has either measure zero or full measure. In order to relate these sets with subsets of $\Omega_{\nu,E} \times G$ we recall the composed map $G \overset{\pi}{\to} G/G_\lambda \overset{\kappa}{\to} \mathcal{O}_\lambda$ from section 4, where π denotes the canonical projection of G onto G/G_λ and κ is the diffeomorphism that identifies G/G_λ with $\mathcal{O}_\lambda$. One then realises that the following diagram commutes:

$$
\begin{tikzcd}
(x,\xi,g) \arrow[r, mapsto, "\tilde{Y}_\nu^t"] \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\pi"'] & \big(\Phi_\nu^t(x,\xi), g_\nu(x,\xi,t)g\big) \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\pi"] \\
(x,\xi,gG_\lambda) \arrow[r, mapsto, "\overline{Y}_\nu^t"] \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa"'] & (\Phi_\nu^t(x,\xi), g_\nu(x,\xi,t)gG_\lambda) \arrow[d, mapsto, "\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa"] \\
(x,\xi,\eta) \arrow[r, mapsto, "Y_\nu^t"] & \big(\Phi_\nu^t(x,\xi), \mathrm{Ad}^*_{g_\nu(x,\xi,t)}(\eta)\big)
\end{tikzcd}\ , \tag{B.1}
$$

where $\overline{Y}_\nu^t$ is induced by Y_ν^t under $\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\pi$. According to this diagram a $\tilde{Y}_\nu^t$-invariant set $A \subset \Omega_{\nu,E} \times G$ projects to a Y_ν^t-invariant subset $(\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa\circ\pi)(A)$ of $\Omega_{\nu,E} \times \mathcal{O}_\lambda$. The assumed ergodicity of Y_ν^t then implies that the measure of $(\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa\circ\pi)(A)$ is zero or one. Now the normalised Haar measure $\mathrm{d}g$ on G projects under $\kappa\circ\pi$ to the volume measure $\mathrm{d}\eta$ on the coadjoint orbit $\mathcal{O}_\lambda$. This can be obtained from the Fubini theorem (cf. [BtD85]) which states for every $f \in L^1(\mathcal{O}_\lambda)$ that

$$
\begin{aligned}
\int_G (\pi^*\kappa^* f)(g)\ \mathrm{d}g &= \int_{G/G_\lambda} \left(\int_{G_\lambda} (\kappa^* f)\circ\pi(gh)\ \mathrm{d}h \right) \mathrm{d}(gG_\lambda) \\
&= \int_{G/G_\lambda} (\kappa^* f)(gG_\lambda)\ \mathrm{d}(gG_\lambda).
\end{aligned} \tag{B.2}
$$

Here $\mathrm{d}h$ denotes the normalised Haar measure on G_λ and $\mathrm{d}(gG_\lambda)$ is the normalised left invariant volume form on G/G_λ arising from the volume form on the coadjoint orbit under the pullback κ^*. Hence, the sets A and $(\mathrm{id}_{\mathrm{T}^*\mathbb{R}^d}\times\kappa\circ\pi)(A)$ have identical measures and thus the measure of A is either zero or one. Therefore, the assumed ergodicity of Y_ν^t implies ergodicity of $\tilde{Y}_\nu^t$.

In order to prove the opposite direction one simply reverses the above argument: Starting with Y_ν^t-invariant subsets of $\Omega_{\nu,E}\times\mathcal{O}_\lambda$, one lifts these to $\Omega_{\nu,E}\times G$. Due to the commuting diagram (B.1) these lifts are $\tilde{Y}^t_\nu$-invariant and therefore, according to the assumed ergodicity of $\tilde{Y}^t_\nu$, have measure zero or one. Again the Fubini theorem (B.2) implies equal measures of the sets and their lifts. Hence Y^t_ν is ergodic. $\square$

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