On complex functions analyticity

S.B. Karavashkin and O.N. Karavashkina
Special Laboratory for Fundamental Elaboration SELF
187 apt., 38 bldg., Prospect Gagarina, Kharkov 61140, Ukraine
Phone: +38 (0572) 276624; e-mail: selflab@go.com , selflab@mail.ru
Web site of Laboratory SELF: http://angelfire.lycos.com/la3/SELFlab
Web site of the electronic journal SELF Transactions: http://angelfire.lycos.com/la3/selftrans

This is the second paper of the cycle devoted to the new theory of non-conformal mapping. It introduces an important for further investigations concept of dynamical non-conformal mapping and by a simple example shows some new scope available now for the dynamical researchers. This paper was published first in our electronic journal SELF Transactions, **2** (2002), No 1, pp.101-110. We would like to mark for mp_arch readers that here we cannot present the dynamical non-conformal mapping animated, i.e., in its real appearance. All interested are invited to visit our journal and archive at http://angelfire.lycos.com/la3/selftrans We will be pleased if you find there a response to your thoughts and research interests. We will be grateful, receiving your opinion of our results. We are willing to answer your questions and discuss with you any problems having something in common with our developments.

Abstract

We analyse here the conventional definitions of analyticity and differentiability of functions of complex variable. We reveal the possibility to extend the conditions of analyticity and differentiability to the functions implementing the non-conformal mapping. On this basis we formulate more general definitions of analyticity and differentiability covering those conventional. We present some examples of such functions. By the example of a horizontal belt on a plane Z mapped non-conformally onto a crater-like harmonic vortex, we study the pattern of trajectory variation of a body motion in such field in case of field power function varying in time. We present the technique to solve the problems of such type with the help of dynamical functions of complex variable implementing the analytical non-conformal mapping.

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1. Introduction

In [1] we analysed the ways to find the derivative of the function of complex variable in case when the function does not satisfy the Caushy – Riemann conditions, but is continuous and one-valued in the investigated domain. We showed that the broadening concept of functions differentiability opens new unexpected scope to solve differential equations. However we made no correlation with other classes of functions of a complex variable – implementing, e.g., the quasi-conformal mapping. At the same time we should mark that some of approaches presented in [1] were used before in the formalism of quasi-conformal mapping (see, e.g., [2], [3], [4]). As we will show in this paper, a generalised class of mappings, being one-valued and continuous in the studied domain but not satisfying the Caushy – Riemann conditions, is well wider and can be limited by no specific re-

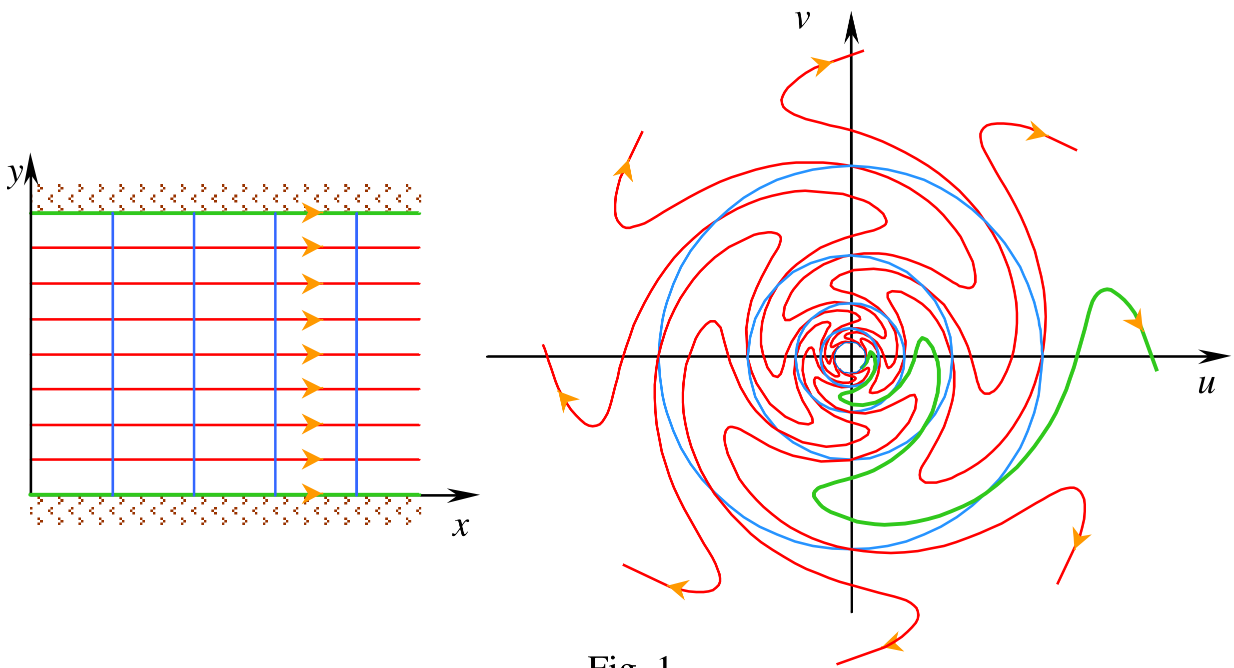

Fig. 1

$w = \exp\left[ax + j\left(b\sin cx + py\right)\right]$,

where a, b, c, p are the coefficients

lationship like Karleman condition. Because in any specific relationship containing an equality, it will be sufficient to change a little the condition, to yield a new class of functions being also one-valued and continuous in the domain of a function but satisfying neither Caushy – Riemann conditions nor the initial relationship. And up to infinity. To avoid such situation, it is desirable to establish a relationship or a system of relationships whom all classes of such functions will obey, in that number the classes of functions implementing the conformal and quasi-conformal mapping.

2. The limitation of the class of functions implementing the quasi-conformal mapping

Conventionally, the Karleman system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} + au + bv \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} + cu + dv \tag{1}$$

is the basis of quasi-conformal mapping. "The system (1) is the generalisation of Caushy – Riemann conditions (with $a = b = c = d = 0$ we yield these conditions); some problems of elastic shells, gas dynamics and other sections of continuous media mechanics are reduced to it" [2, p.316].

There is some other concept of such type of mappings, in the form of "system of the elliptic first-order linear differential equations

$$\begin{gathered} \frac{\partial v}{\partial y} = a_f \frac{\partial u}{\partial x} + b_f \frac{\partial u}{\partial y} \; ; \\ -\frac{\partial v}{\partial x} = c_f \frac{\partial u}{\partial x} + d_f \frac{\partial u}{\partial y} \; , \end{gathered} \tag{2}$$

where a_f, b_f, c_f, d_f are the known functions with respect to variables x and y, for which everywhere in the studied domain D the condition of ellipticity

$$A = a_f c_f - \left(\frac{b_f + d_f}{2} \right)^2 > 0 \tag{3}$$

is true" [2, p.320]. "Compose of the solution $u(x, y)$ and $v(x, y)$ of the system (2) a function of complex variable $f(z) = u + iv$. We will name the mapping that it implements *the quasi-conformal mapping connected with this system*" [2, p.321].

We can prove easy that the conditions defining the classes of functions that implement the conformal and quasi-conformal mapping are incomplete to define the general set of analytical functions. In such way we show in Fig. 1 the mapping implemented by the function

$$w = \exp\left[ax + j\left(b\sin cx + py \right) \right], \tag{4}$$

where a, b, c, p are some constants. As we see from the construction, this function implements the one-valued continuous mapping of a semi-finite belt of a plane Z onto outside of the circumference of the plane W (the negative semi-finite belt will map inside this circumference, and in this domain the mapping will be also one-valued and continuous). Of course, (4) does not satisfy the Caushy – Riemann conditions.

Apply the above conditions of "quasi-conformity" (2) and (3) to (4). The first particular derivatives of w with respect to x and y will have the following form:

$$\begin{aligned} &\frac{\partial u}{\partial x} = e^{ax}(a\cos\xi - bc\cos cx \sin\xi); \quad \frac{\partial u}{\partial y} = pe^{ax}\sin\xi; \\ &\frac{\partial v}{\partial x} = e^{ax}(a\sin\xi + bc\cos cx\cos\xi); \quad \frac{\partial v}{\partial y} = pe^{ax}\cos\xi; \quad \xi = b\sin cx + py. \end{aligned} \tag{5}$$

Substituting (5) into (2), we can determine the coefficients that still were unknown:

$$\begin{aligned} &a_f = \frac{bc}{a}\cos cx; \quad b_f = -\left(\frac{a}{p} + \frac{b^2c^2}{ap}\cos^2 cx \right); \\ &c_f = -\frac{p}{a}; \qquad d_f = \frac{bc}{a}\cos cx. \end{aligned} \tag{6}$$

And substituting (6) into (3), we yield

$$A = -\left(\frac{a^2}{2p^2} + \frac{bcp}{a^2}\cos cx + \frac{2a^2 + p^2}{2a^2p^2}\cos^2 cx + \frac{b^4c^4}{2a^2p^2}\cos^4 cx\right). \tag{7}$$

We see from (7) that the condition (3) is not true for (4). With definite values of parameters and coefficients included in A, its value can be negative or sign-alternative. At the same time the construction in Fig. 1 displays that the mapping is one-valued and continuous; hence, it satisfies the analyticity conditions. Along with the fact that (4) does not satisfy (3), we can easy show that (4) maps the infinitesimal circumferences of the plane Z into the infinitesimal ellipses of W, which is typical just for quasi-conformal mappings. Actually, for any δ-vicinity of the point $z_0 = x_0 + jy_0$ in the domain of the plane Z being mapped into the ε-vicinity of the point $w_0 = u_0 + jv_0$ of the plane W, we can write for (4) as follows:

$$\begin{aligned} &\Delta u = e^{ax}(a\cos\xi_0 - bc\cos cx\sin\xi_0)\Delta x + pe^{ax}\sin\xi_0\,\Delta y; \\ &\Delta v = e^{ax}(a\sin\xi_0 + bc\cos cx\cos\xi_0)\Delta x + pe^{ax}\cos\xi_0\,\Delta y; \\ &\xi_0 = b\sin cx_0 + py_0; \quad \Delta x, \Delta y \in O(\delta); \ \Delta u, \Delta v \in O(\varepsilon). \end{aligned} \tag{8}$$

For any fixed point of the domain the system (8) is equivalent to the system

$$\begin{aligned} &\Delta u = A_1\,\Delta x + B_1\,\Delta y; \\ &\Delta v = C_1\,\Delta x + D_1\Delta y, \end{aligned} \tag{9}$$

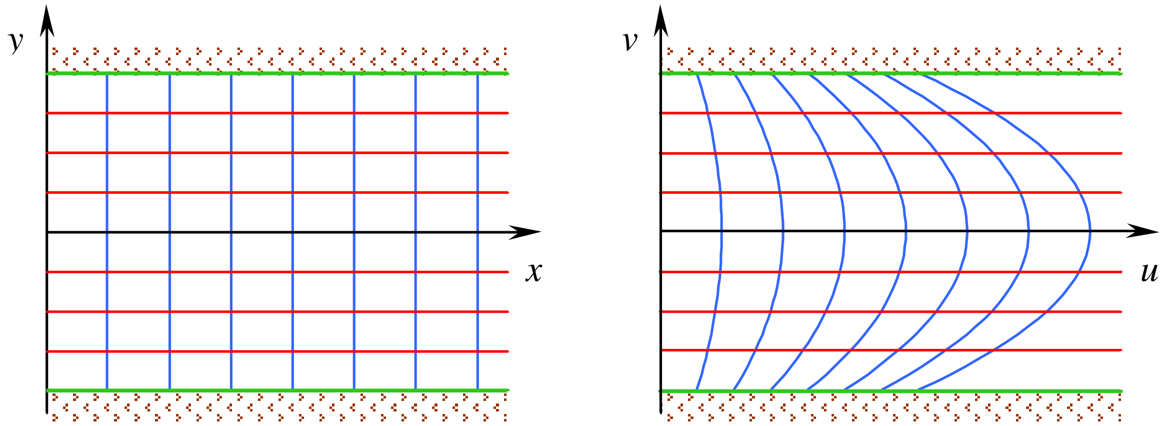

Fig. 2

$$w = x\left(1 - ay^2\right) + jy$$

where a is the coefficient

where A_1, B_1, C_1, D_1 are the constants varying only with the changing co-ordinates of the selected point on the plane Z. But (9) maps (in case of the studied function) a circumference into ellipse, because, in accord with (8), all above coefficients in the domain Z are limited functions; therefore one always can find such small $\rho(\Delta x, \Delta y) \in \delta$ that will satisfy the condition $w(\rho) \in \varepsilon$. This just proves the stated.

Thus we see that on one hand the studied function (4) satisfies neither conformity nor quasi-conformity conditions. But on the other hand, it maps the infinitesimal circumferences of Z into infinitesimal ellipses of W, which is typical just for quasi-conformal mappings. We can easy generalise this last for the case of any function of a complex variable having continuous first particular derivatives with respect to x and y in the studied point vicinity. To generalise (8) and its obvious transition to (9) will suffice for it, because in the transition to small increments of the function in the point vicinity, we will neglect all powers of an increment higher than the first. Hence, with the finite increment we will always yield the mapping onto the circumference or ellipse. This result is important, as (4) is far from being the only, and the more, it is not an exception from the common rule. To

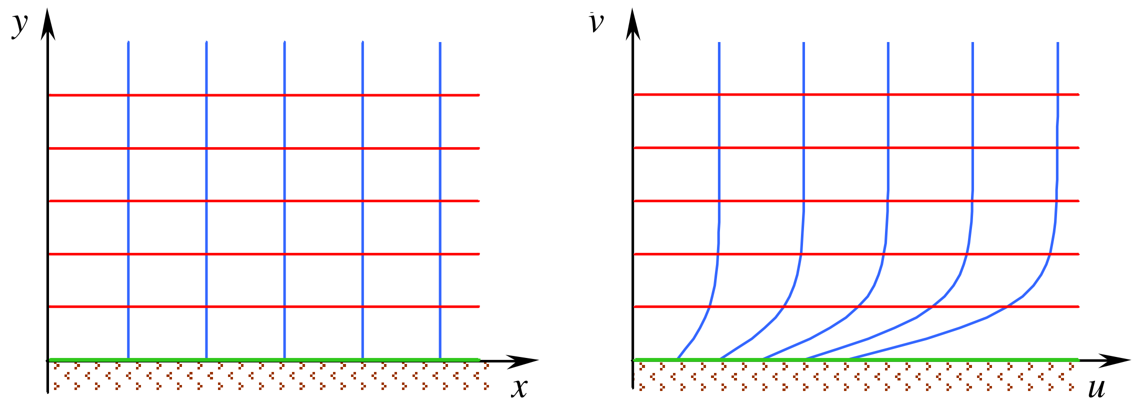

Fig. 3

$$w = x\left(1 + \operatorname{th} ay\right) + jy$$

where a is the coefficient

illustrate the said, we show in Figures 2 and 3 two alike functions implementing the mappings of other types. And while in Fig. 2 as a result of mapping the transformation of the entire mapping domain of W took place, in Fig. 3 only quite narrow its part transforms, remaining the conformity in the rest part of mapping domain with accuracy to infinitesimal values. The second corollary of the generalisation is that the main theorems of complex function presentation having been proved for quasi-conformal mapping (see, in particular, [2, pp.316-317]) are applicable in a more general case.

At the same time, this corroborates the statement of introduction that the condition of function analyticity cannot be stated as an equality. This condition can be found only on the basis of some more general principles similar to those which are used when the real-variable functions differentiability was defined. The further studying will be devoted to this.

3. Analysis of current definitions of analyticity and differentiability of a function of complex variable

Definition 1. "The one-valued function *f(z)* is named **analytical** (regular, holomorphic) at the point $z = a$, if it is *differentiable in some vicinity of the point a*" [5, p.197].

In its turn,

Definition 2. "The function $w = f(z)$ is named **differentiable** at the point $z = a$, if the limit

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{10}$$

exists at $z = a$ and does not depend on the way in which Δz tends to zero" [5, p.197, underlined by us – authors].

Note that in the definition of analyticity of a complex-variable function only the requirement, it to be differentiable, is present. The requirement, the limit of (10) to be independent of the way in which Δz tends to zero, being the basis to derive the Caushy – Riemann conditions (and in the transformed form also the Karleman conditions (1)), appears only in the definition 2. Hence, if some function satisfied the condition that defines the existence of the differential of function and requires this function to be one-valued and continuous in the studied point vicinity, but did not satisfy the condition, the limits of (10) to be equal with Δz tending to zero along different directions, then, in the view of basic definition of analyticity, this function is analytical too. It means that both functions considered above and such as

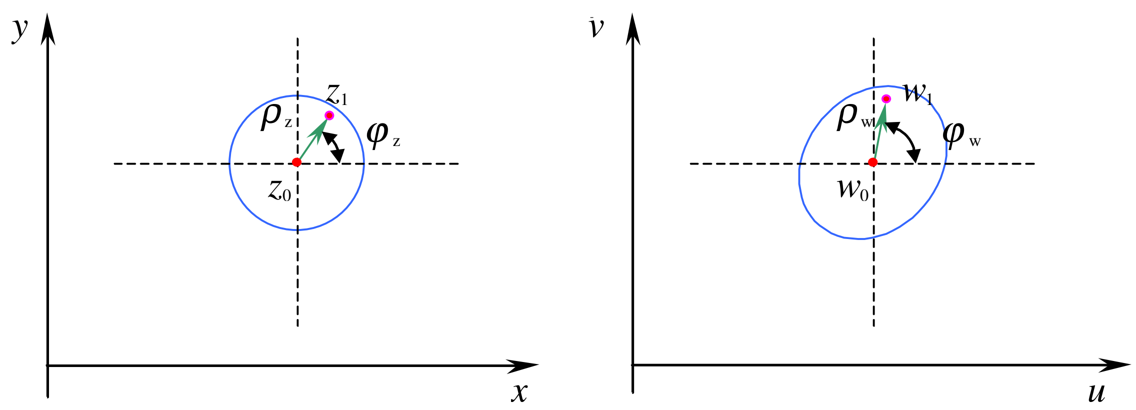

Fig. 4. Mapping of the point z_1 in the δ-vicinity of z_0 of the plane Z into the point w_1, located in the ε-vicinity of the point w_0 of the plane W.

$$\begin{aligned} &w = u(x, y) + jv(x),\ w = u(x) + jv(y), \\ &w = u(y) + jv(x, y),\ w = u(y) + jv(x) \end{aligned} \tag{11}$$

etc., which generally satisfy neither Caushy − Riemann nor Karleman conditions, but have one-valued and continuous derivatives with respect to x and y, are analytical too.

The possibility itself to generalise the definition of differentiability of complex-variable functions, with the analyticity definition retaining, means not only broadening the domain in which the definition 1 is true, but reflects the essential transformation of the concept of differential and derivative of complex function. To explain, consider some point z_1 in the δ-vicinity of the point z_0 and its mapping into the point w_1 located in the ε-vicinity of the point w_0 (see Fig. 4). Basing on this construction, we can write as follows:

$$\begin{aligned} &\Delta z = z_1 - z_0 = \Delta x + j\Delta y = \Delta\rho_z e^{j\varphi_z}; \\ &\Delta w = w_1 - w_0 = \Delta u + jvy = \Delta\rho_w e^{j\varphi_w}. \end{aligned} \tag{12}$$

Then

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta\rho_w}{\Delta\rho_z} e^{j(\varphi_w - \varphi_z)} = \frac{d\rho_w}{d\rho_z} e^{j(\varphi_w - \varphi_z)}. \tag{13}$$

The expression (13) shows that when mapping in the complex plane, the derivative *does not define the tangent of inclination angle of the tangent line at the studied point*, as it was in case of real variable, but characterises the geometrical transformation of a path with the mapping. And this naturally must reflect in the formulation of definitions.

At the same time, we can easy trace the reason, why the limits in the definition 2 must be equal. This definition was constructed by analogy with the differentiability condition of functions of one real variable, where the requirement, the limits to be equal, was formulated in the following way:

Definition 3. "The number b is the *limit value of the function* $y = f(x)$ *at the point* $x = a$ *(or the limit of a function at* $x \to a$*)*, if for any sequence $x_1, x_2, ..., x_n, ...$ of argument x converging to a, whose elements x_n distinguish from a ($x_n \ne a$), the corresponding sequence $f(x_1)$, $f(x_2)$, …, $f(x_n)$, … of the function values converged to b" [5, p.98, with our italicisation − authors]. Shilov confirms: "This definition ((2) − authors) of a derivative of a complex-variable function is by its form alike the definition of a real-function derivative in a real domain. The definition of a derivative of a real-variable function is a particular case of this presented… However with the external similarity between the derivatives in real and complex domains, there is a number of essential distinctions" [6, p.397].

For real-variable functions this severity of definition 3 is quite justified, due to the fact that in this case by the concept of any converging sequence one means some countable set $x_1, x_2, ..., x_n, ...$ mapped onto the supposed smooth curve in the ε-vicinity of a point $b = f(a)$. Due to it one can say that the set $\boldsymbol{S}$ constituted of the elements of a sequence $x_1, x_2, ..., x_n, ...$ is everywhere dense and nested for the countable set $\boldsymbol{C}$ of all sequences converging at the point a.

In case of complex functions we deal with 2D mapping of the domain Z onto 2D domain of values W. With it each arc in the δ-vicinity of the point z_0 has its only (in case of 1D mapping) image in the ε-vicinity of the point w_0, and the δ- and ε-vicinities themselves become 2D. Hence, the set $\boldsymbol{S}$ constituted of elements of the sequence $z_1, z_2, ..., z_n, ...$, though it remains nested into the set of sequences $\boldsymbol{C}$ converging to z_0, stops be everywhere dense in the set $\boldsymbol{C}$, as it is constituted only of elements belonging to one arc of a set of arcs crossing at the point z_0. This feature requires to take into account both the convergence of all sequences along one direction and all directions of contraction. However this last does not effect on the substantiation of the limit itself for each sequence, since they all converge to the common point z_0, and their images – to w_0. But the contraction velocities in different directions can be generally different.

In the view of transformation of a path about the point z_0 with its mapping into W, the contraction velocity gains the first-order importance, because just relation of the velocity of contraction to the point w_0 on the plane W to that to the point z_0 on the plane Z determines the value of a derivative along the picked out domain. With it, the different velocities in different directions do not mean yet the break, if it was (and for an analytical function must be) a smooth function with respect to angle φ_z. This is just our task to find the conditions which might select from multitude functions the classes having the above properties.

For it, consider in a plane Z some small δ-vicinity of the point z_0. In this vicinity give parametrically some arc $x(\varsigma)$, $y(\varsigma)$ passing through the point z_0. Then for the function which maps the δ-vicinity of z_0 into ε-vicinity of the point $w(z_0)$, we can write in the most general form so:

$$w(x,y)=w(x(\varsigma),y(\varsigma))=w(\varsigma). \tag{14}$$

In other words, we presented the studied function $w(x,y)$ as a complex function with respect to ς. Now if we differentiate this function with respect to ς, we yield

$$w'_\varsigma=\frac{\partial w}{\partial x}x'_\varsigma+\frac{\partial w}{\partial y}y'_\varsigma . \tag{15}$$

In its turn,

$$\frac{dw}{dz}=\frac{w'_\varsigma}{z'_\varsigma} . \tag{16}$$

We can see from (15) and (16) that, if the chosen arc (or rather a set of arcs) was *regular* in the studied δ-vicinity, then in (15) all particular derivatives x and y with respect to ς exist, and general differentiability of the function $w(x,y)$is defined by the existence of particular derivatives with respect to x and y. In this case it suffices simply to use the definition of the existence and continuity of the particular derivative, to substantiate the differentiability of the function $w(x,y)$ itself. But if the arc was irregular, one cannot conclude, is or is not this function differentiable. This proves that the regularity of an arc in the δ-vicinity of the point z_0 offers one to find, is the function $w(x,y)$ differentiable along the given arc (or a family of arcs).

Noting the proved, the function to be differentiable in δ-vicinity of z_0, it is sufficient, it to be differentiable for *any regular arcs in the δ-vicinity of the point z_0*. If in the δ-vicinity there is at least one arc along which the function $w(z_0)$ is non-differentiable, then naturally, this function on the whole cannot be thought differentiable along the selected direction (though it can be thought a partially differentiable along the selected direction or in the sector). And vice versa, if a regular arc along which the function $w(z_0)$ is non-differentiable is absent, then any families of curves being smooth in the δ-vicinity of z_0 map one-valuedly into the families of $w(z_0)$ being smooth in the ε-vicinity. Noting the proved statement of regular arcs, it speaks of the full differentiability of $w(z_0)$ at the point z_0.

On the basis of the carried out investigation and taking into account the conventional definition 2, we can formulate the definition of general differentiability so:

Definition 4. The function of complex variable $w = f(z)$ is *differentiable in general sense* at the point $z = a$, if the limit

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{17}$$

existed at $z = a$ for any *regular* arc in the δ-vicinity of the point $z = a$ passing through the point $z = a$.

We can easy make sure that the conventional condition of the function differentiability after Caushy − Riemann is a particular case of definition 4 with additional condition, the limits along all regular arcs to be equal.

Having defined the conditions of differentiability in general sense, we in fact generalised the conditions of functions analyticity in general sense, because, according to the conventional definition 1, the differentiability condition is the principal criterion of the function analyticity. So, following the division of the differentiability definition into that general and that after Caushy − Riemann, the function analyticity can be also defined now in general sense and after Caushy − Riemann. With it the class of functions analytical in general sense will be naturally covering for that analytical after Caushy − Riemann. And the analyticity definition itself will remain invariable, with accuracy to the refinements connected with the function differentiability.

Finally, our new definition of complex-variable functions analyticity can be easy extended from the δ-vicinity of the point z_0 to some connective domain of the plane Z. And it does not require a new proof, because we did not transform the concept of both δ- and ε-vicinities. In this connection, "the function is **analytical in an open domain *D***, if it was *analytical at each point* of this domain" [1, p.197].

Note however that far from every extension of the analyticity concept in general sense will be so simple. In particular, there will be problems with the analytical continuation through the border. But this is the subject of another large investigation.

4. Dynamical functions of complex variable

As one of important applications of analytical non-conformal mapping, we can refer to the mapping with the help of dynamical functions of complex variable. To illustrate, in Fig. 5 we show the dynamical analogue of the mapping shown before in Fig. 1:

$$w = \exp\left[ax + j\left(b\sin(cx + \omega t) + py\right)\right], \tag{18}$$

where t is a time parameter, and ω is the circular frequency of dynamical transformations of a complex function.

In Fig. 5 the Readers of SELF Transaction can see animated, and the Readers of Mathphys Archive can imagine that the time multiplier appearance in the right-hand part of (18) has led to the translational shift in time of the field lines of power in the plane W into the central region of a field. With it the location of their prototypes in the plane Z remains time-invariable, and therefore they still correspond to the stationary process. This feature just distinguishes the fields description by dynamical functions from the conventional description with which the function describing the power field characteristics depends directly on the studied domain co-ordinates. As the result, to describe completely the process in such fields, one needs using not one but two non-conformal mappings, both stationary and dynamical. The first of them introduces a one-valued *analytical correspondence* between the domain, where the plane Z is determined, and domain of values of the plane W. The second shows the power field time-transformation degree, so this is intended to describe the dynamical power function.

Despite the necessity to use two mappings at the same time, the property of dynamical complex functions to describe the field power function in a plane enables us to solve the dynamical problems

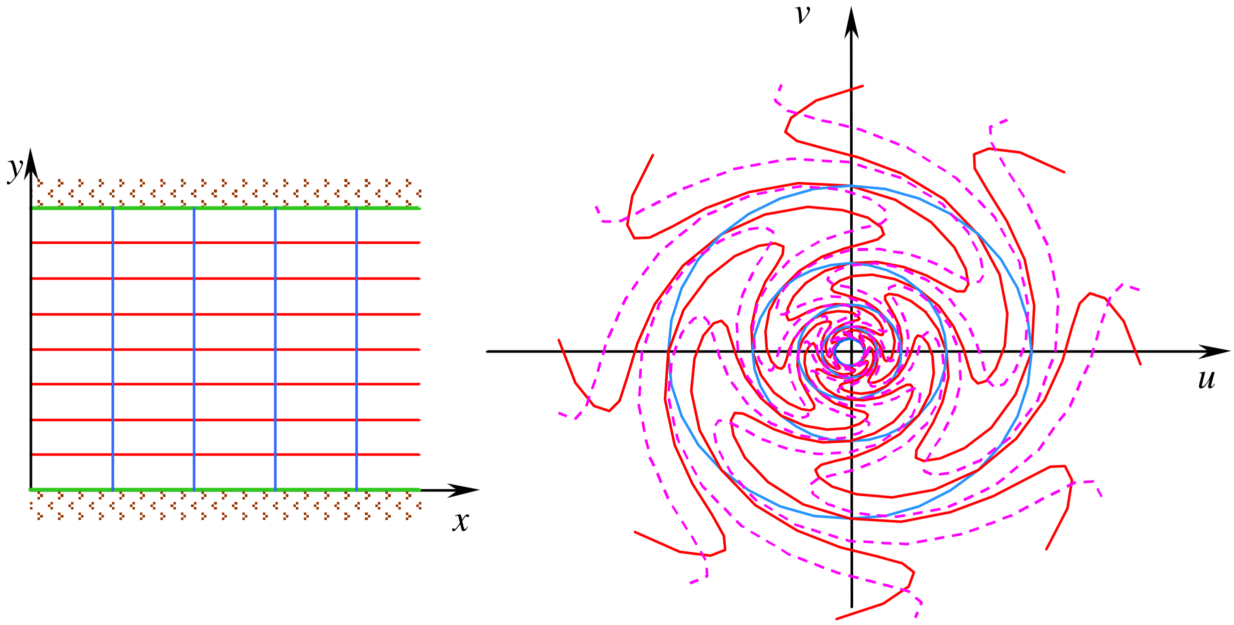

Fig. 5

$w = \exp\left[ax + j\left(b\sin(cx - \omega t) + py\right)\right]$,

where a, b, c, p, ω are the coefficients

in complex dynamical fields which one meets often in hydro- and aerodynamics, electrophysics, acoustics, geophysics and so on. With it, naturally, the necessity of using the double mapping brings its peculiarities; we will try showing them by a simple specific example, determining the motion trajectory for a body having the mass m within the field of forces described by (18) and Fig. 5.

Suppose that at the initial time moment $t = 0$ the body was located at the point $w_0(x_0, y_0)$. Associate the stationary mapping between the planes Z and W with this moment. In our case this mapping will be described by (4). Since the problem is plane, suppose that the force field amplitude decreases *along the lines of force* (which is very complicated to be taken into account by conventional methods in case of complex configuration of the lines of force) and is proportional to the first power of distance to the field centre, also along the lines of force.

Since in the considered problem the lines $x = \text{const}$ of the plane Z correspond to the lines of force of the plane W, the attenuation degree is proportional to $1/\sqrt{a^2 + c^2b^2}\,x$. The constant multiplier in the denominator was introduced to correlate the metrics of planes W and Z *along the field line of force*. It is determined conventionally:

$$\xi = \left|\frac{dw}{dz}\right| = \sqrt{\left(\frac{\partial u}{\partial x}\bigg|_{y=\text{const}}\right)^2 + \left(\frac{\partial v}{\partial x}\bigg|_{y=\text{const}}\right)^2} = \sqrt{a^2 + c^2b^2} \tag{19}$$

The force direction at each point is known to be determined by the direction of a tangent to the power line of force. Noting also that in our problem the field acts on the body towards the field centre, the tangent of force inclination ψ can be determined as follows:

$$\tan(\psi + \pi) = \frac{dv}{du} = \frac{\partial v/\partial x\big|_{y=\text{const}}}{\partial u/\partial x\big|_{y=\text{const}}}. \tag{20}$$

This last transformation is caused by, in the considered problem in the plane Z the equation $y = \text{const}$ corresponds to the lines of force. Substituting (18) into (20), we yield

$$\cos\psi = \frac{a\cos\Theta - cb\cos(cx+\omega t)\sin\Theta}{\sqrt{a^2+c^2b^2}};$$
$$\sin\psi = \frac{a\sin\Theta + cb\cos(cx+\omega t)\cos\Theta}{\sqrt{a^2+c^2b^2}}, \qquad (21)$$

where $\Theta = b\sin\left(cx+\omega t\right)+py$.

Taking into account the additional definition which we have made for the power field, we may write the differential equation of the body motion so:

$$m\frac{d^2w}{dt^2} = \frac{K_F}{\xi x}e^{j(\psi+\pi)} = -\frac{K_F}{\xi x}e^{j\psi}, \qquad (22)$$

where K_F is the coefficient determining the amplitude of field action on the studied body.

In (22) the dependence $w\left(z\right)$ corresponds to the stationary non-conformal mapping, because we completely took into account the features of the dynamics of process, when defined the power field additionally. At the same time, for a correct approach to solving (22) we have to note that despite the stationary pattern of mapping of $w\left(z\right)$, the variation of a prototype co-ordinates in the plane Z will correspond to the time shift of the body in the plane W. So in finding the solution of the differential equation (22) we have to note that the time dependence exists not only for the points of the body trajectory in the plane W, but for their prototypes in the plane Z such dependence also takes place.

Thus, to find the body trajectory within the studied power field, we have to solve the following system of equations:

$$\begin{cases} m\dfrac{d^2w}{dt^2} = -\dfrac{K_F}{\xi x}e^{j\psi}; \\ \sin\psi = \dfrac{a\sin\Theta + cb\cos(cx+\omega t)\cos\Theta}{\sqrt{a^2+c^2b^2}}; \\ \Theta = b\sin\left(cx+\omega t\right)+py; \\ w = \exp\left[ax + j\left(b\sin(cx+\omega t)+py\right)\right]. \end{cases} \qquad (23)$$

As we made without any assumptions all pre-conditions which take into account the features of body motion in the studied power field, conveniently solve (23) numerically.

To compose the calculation scheme, divide the studied time interval into a large number n of equal sections Δt. Then, using the simplest Euler method, integrate the right-hand part of (22) and yield

$$I_{1k} = \sum_{i=0}^{k-1}\frac{K_F}{\xi x_i}e^{j\psi_i}\Delta t + C_{10};\ k = 1,2,\ldots,n, \qquad (24)$$

where ψ_i, x_i, y_i are taken in the beginning of i-th time interval, in that number x_0, y_0 are the co-ordinates of the prototype of the body initial location at $t=0$, and C_{10} is the integration constant determining the initial velocity of a body.

The second integral over the time we can found the same:

$$I_{2r} = \sum_{k=1}^{r} I_{1k}\Delta t + C_{20};\ r = 1,2,\ldots,n, \qquad (25)$$

where the constant C_{20} is determined by the co-ordinates of initial location of the body in the plane W.

In fact, the recurrent relationships (24) and (25) determine the velocity and location of the body in the plane W in the end of each small time interval. To make them working, we have to establish

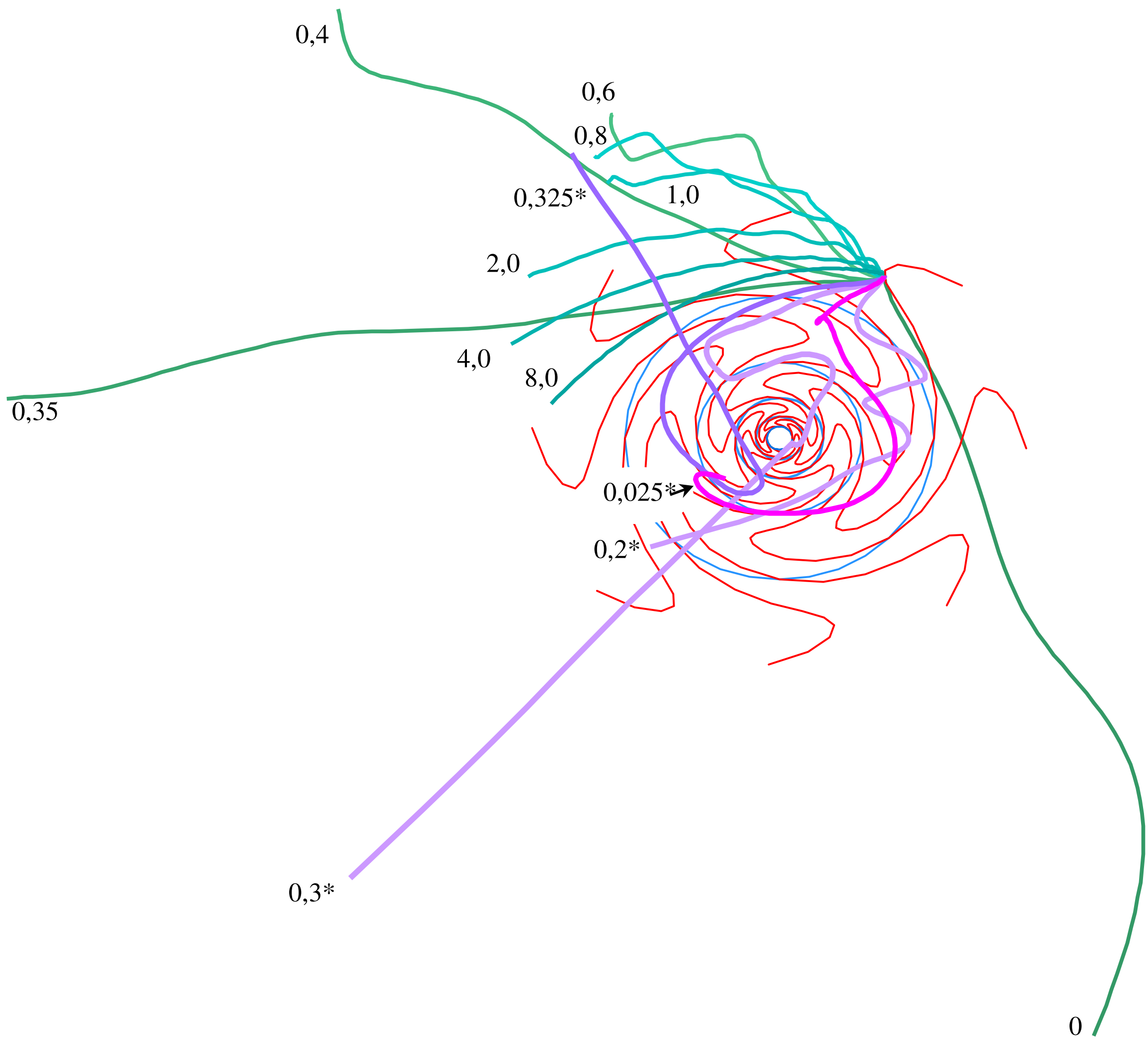

Fig. 6. The motion trajectory of the body having the mass m=1,0 kg in a dynamical central sink field (the intensity of action K_F = 20 N) presented in Fig. 5, with different frequencies ω of this field time variation

the relation between the prototype and image co-ordinates of the point location in the beginning of each small time interval. Conveniently use the stationary non-conformal mapping (the last expression of (23)). As the selected interval is small, we can write enough accurately:

$$\begin{aligned} &\mathrm{Re}(I_{2r} - I_{2(r-1)}) = \Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y; \; I_{20} = C_{20} \\ &\mathrm{Im}(I_{2r} - I_{2(r-1)}) = \Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \; . \end{aligned} \tag{26}$$

Passing sequentially the time intervals, at each of them determine the left-hand part of (26), calculating (24) and (25). We can determine the particular derivatives of the right-hand part of the system, differentiating the last expression of the system (26) with respect to x and y and substituting the values, corresponding to the beginning of each time interval. So, solving (26) with respect to Δx and Δy, we determine the corresponding shifts of the point prototype and make (24) and (25) recurrent.

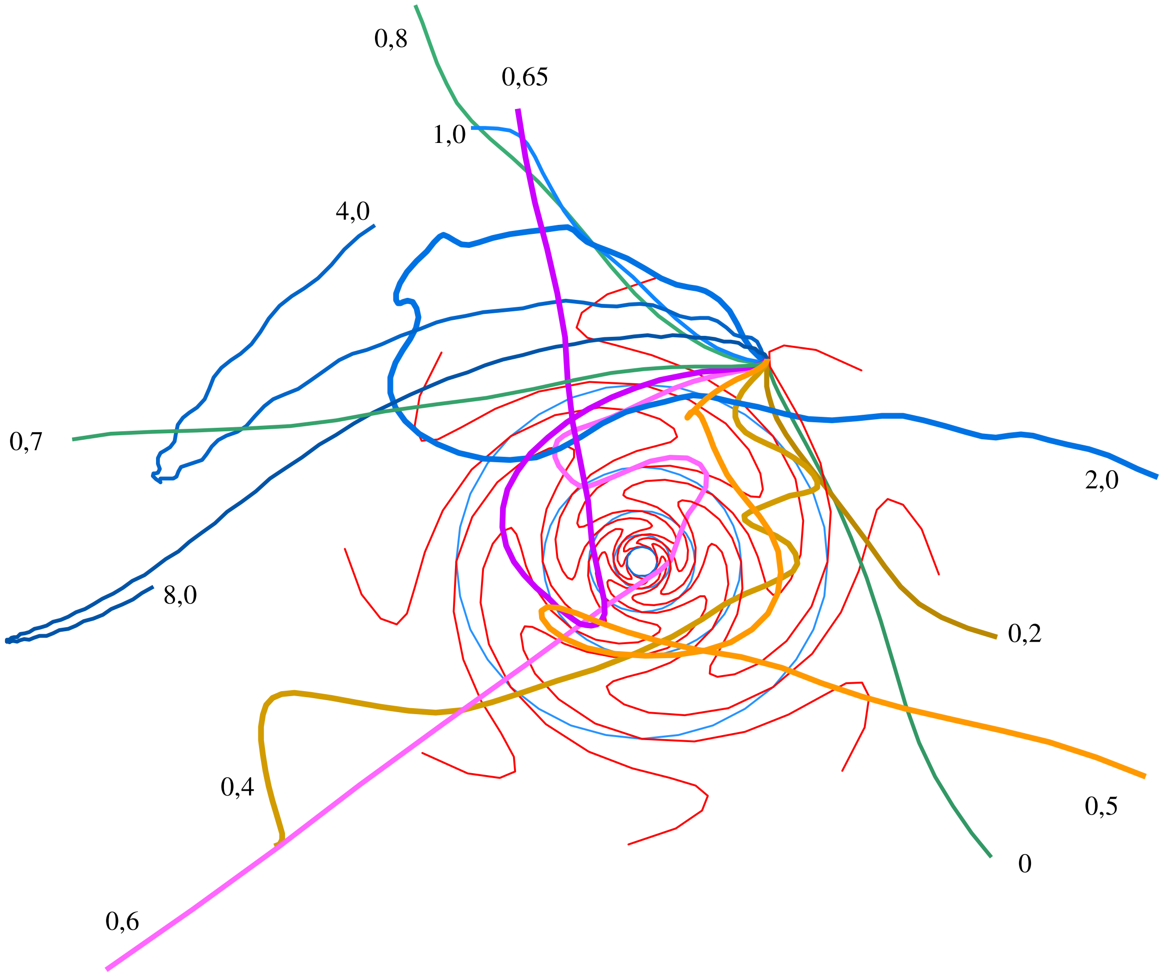

Fig. 7. The motion trajectory of the body having the mass m=0,25 kg in a dynamical central sink field (the intensity of action K_F = 20 N) presented in Fig. 5, with different frequencies ω of this field time variation

In Fig. 6 we show the body motion trajectories calculated with respect to the field transformation frequency ω for the problem stated in the beginning of this item. The initial data for the calculations were taken the following: $K_F = 20\,\mathrm{N}$; $m = 1,0\,\mathrm{kg}$; $x_0 = 14\,\mathrm{m}$; $y_0 = 0$; $a = 0,2\,\mathrm{m}^{-1}$; $b = \pi/3$; $c = 1\,\mathrm{m}^{-1}$; $p = \pi/4\ \mathrm{m}^{-1}$; $t_0 = 0$; $\Delta t = \pi/1200\,\mathrm{s}^{-1}$; $n = 9600$ (four periods of the field time variation). The length of all shown trajectories, except asterisked, corresponds to three periods of the field time variation. To make a visual relation to the field characteristics, we show the initial location of the field equipotential and force lines.

We see that in a stationary power field ($\omega = 0$) the body moves along the tangent to the line of force on which it located at the initial moment. In the motion the body displaces from the sink axis towards the weak field region; the field action falls and is able only to bend the trajectory insufficiently.

With the time transformation of the field, the trajectory displaces to the sink axis (clockwise), and its length diminishes, but the pattern complicates. The trajectories corresponding to $\omega = 0,2; 0,25;\ 0,3;\ 0,325\ s^{-1}$ have been calculated for four periods of the field variation. At these frequencies, before passing to the field periphery, the body undergoes a few repulsions whose pattern is determined first of all by the field variation frequency. At $\omega = 0,2\ s^{-1}$ the body moves within a "corridor", being four times repulsed by the lines of force. At $\omega = 0,25\ s^{-1}$, after a complicated

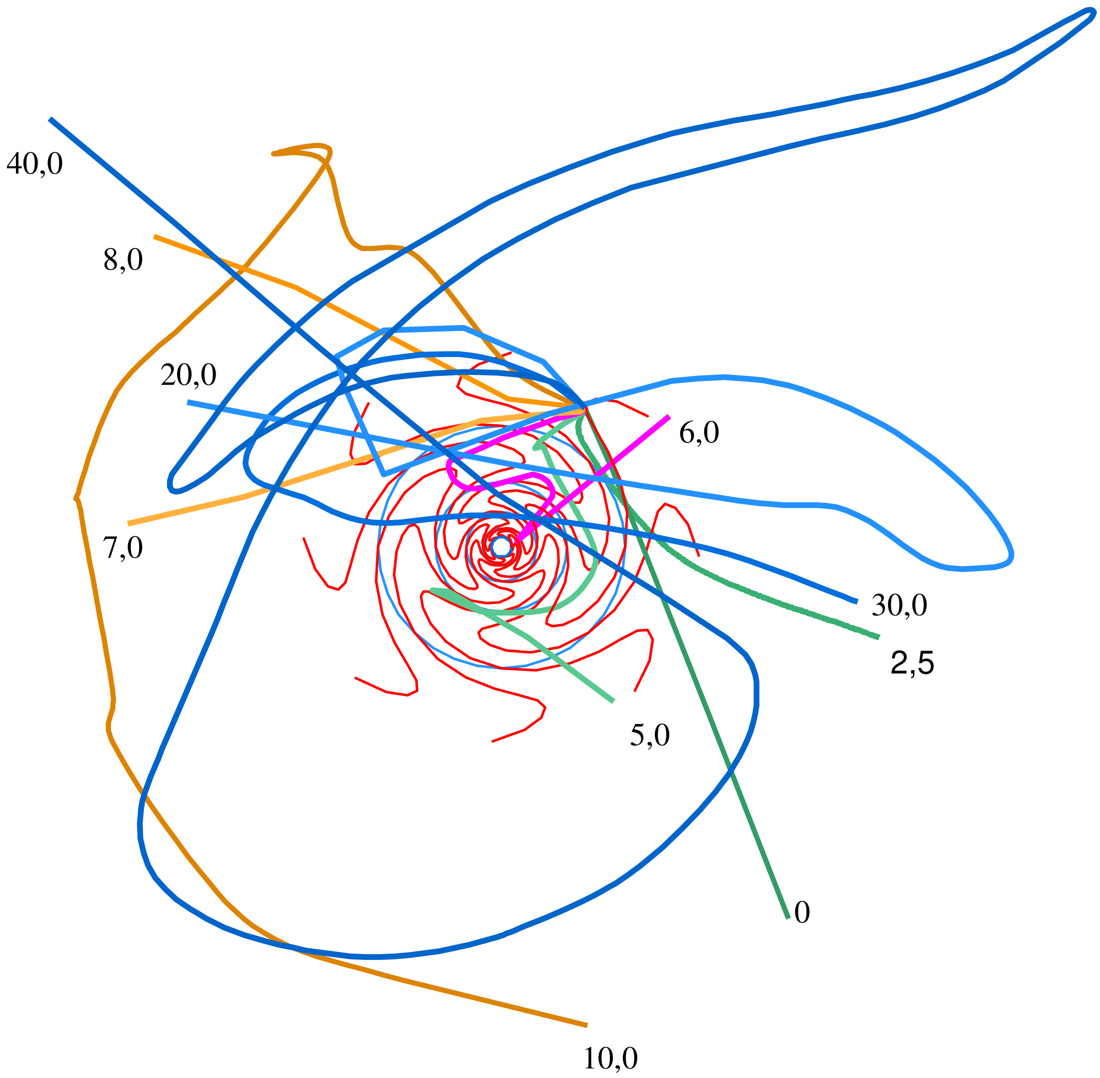

Fig. 8. The motion trajectory of the body having the mass m=0,0025 kg in a dynamical central sink field (the intensity of action K_F = 20 N) presented in Fig. 5, with different frequencies ω of this field time variation

double repulsion, the body moves along the field equipotential lines, until, repulsing again, it leaves the central domain. At $\omega = 0,3\ s^{-1}$ the body first bends around the field centre from the right, then repulses by the lines of force, goes on moving reverse along the equipotential line, repulses again towards the central domain, passes closely near the core and leaves to the periphery from the left of the field centre. And at $\omega = 0,325\ s^{-1}$ the body moves along the tangent towards the field centre, repulses abruptly towards the centre and leaves to the periphery. However we see that the body is never drawn into the field "crater", which is caused by the non-central pattern of the field action, with the central pattern of the field as a whole. We can expect alike phenomena in other fields, such as an exponential vortex sink.

In the following frequency raising, the trajectory length grows again and the trajectory itself goes on turning clockwise, but only up to $\omega \sim 0,4\ s^{-1}$. Further the trajectory turns, but the body braking grows (though it already has not a form of repulsions by the lines of force, as it was at the previous band). The trajectory diminishes in length with it. Finally, at frequencies higher than

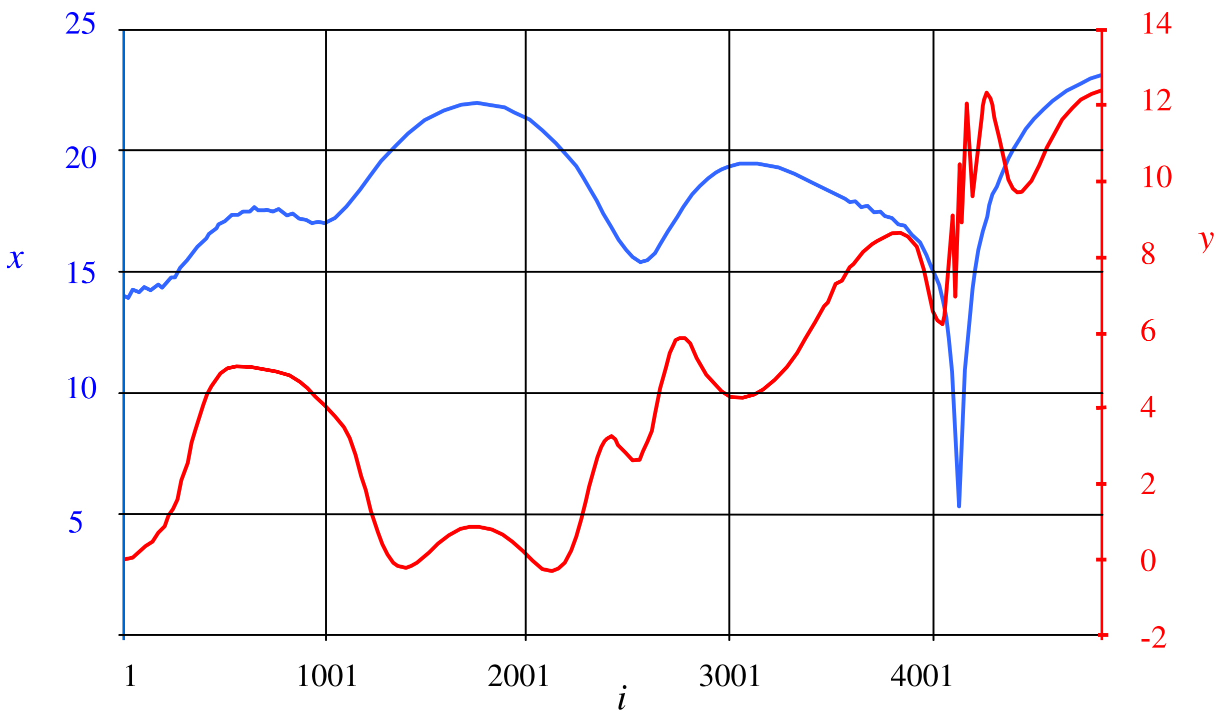

Fig.9. The time dependency of x- and y-components (with respect to the step of numerical integration i) for Z-prototype of the trajectory presented in Fig. 8 $(\omega = 40\ \mathrm{s}^{-1})$

$\omega = 0,6\ s^{-1}$, not only the trajectory length diminishes, but it turns reverse. The body first actively responses to the field variation by the small transversal oscillation motions, but as frequency grows, these oscillations smooth and the trajectory smoothly bends the field core along the tangent line.

With diminishing mass, the body motion in a crater-like harmonic field retains its described regularities, though becomes more chaotical and the lines of force repulse the body more often.

In Fig. 7 we show the body trajectories in the same field, only the body mass is four times less, $m = 0,25$ kg. The same as above, the frequency range can be schematically divided into four bands.

Within that first of a quasi-stationary field, the body moves along the tangent to the line of force at the initial time moment. If the field varied in time slowly ($\omega = 0,2\ s^{-1}$), the trajectory displaces clockwise and bends a little, due to small deviation by the lines of force.

Within the second band, the body moves first as if in a channel formed by the field lines of force ($\omega = 0,4\ s^{-1}$); then along the equipotential lines with an abrupt repulsion to the field periphery ($\omega = 0,5\ s^{-1}$); then it changes the direction of passing by the field core ($\omega = 0,6\ s^{-1}$), and finally it passes by the core from the right along the equipotential lines, abruptly repulsing to the weak-field domain ($\omega = 0,65\ s^{-1}$). Thus, with the smaller mass but other frequencies of the field variation we trace the same typical trajectories.

Within the third band ($\omega = 0,7 \div 2,0\ s^{-1}$) the chaotic motion effects on the trajectory much more. At $\omega = 2,0\ s^{-1}$ we can trace an abrupt turn and the motion towards a stronger field. However the main regularities retain here. The initial sections of the trajectory displace clockwise with the frequency growing.

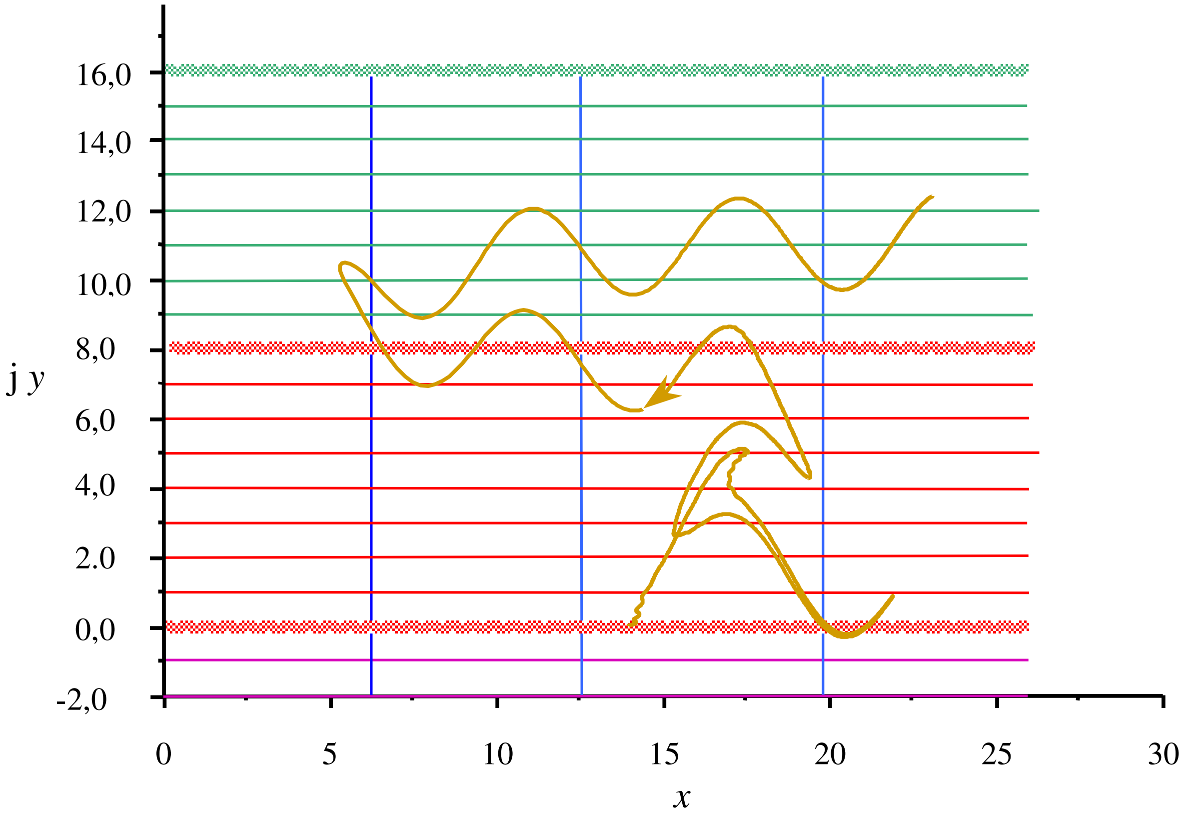

Fig. 10. The prototype of the trajectory presented in Fig. 8 $(\omega = 40\ \mathrm{s}^{-1})$ in the plane Z

And at the fourth band ($\omega > 2{,}0\ s^{-1}$) the trajectory displacement direction reverses, and the body transversal oscillations gradually smooth. The natural feature of the motion at this band and with the diminishing body mass is that the body repulsions emerge, when leaving to the weak field; we can trace it by both shown trajectories of this band.

Note that even if we take the mass very small, the above features of motion in a crater-like field will retain as a whole, though the repulsions by the lines of force will impact in this case. To illustrate, we show in Fig. 8 the trajectory of a body having a mass $m = 0{,}0025$ kg, 100 times lighter than that previous. Its motion in the fourth band ($\omega \geq 20{,}0\ s^{-1}$) is the most chaotic. However, despite the complicated pattern of trajectories being much alike the Brownian motion, their shape also has some order and regularity. There are clearly seen all four bands described above. As to chaotic pattern of trajectories shown in Fig. 8, it would be interesting to correlate their shape with the prototype – for example, for the case $\omega = 40{,}0\ s^{-1}$. We show the parametrical form of time regularity of the x- and y-co-ordinates of the prototype in Fig. 9, and the trajectory prototype in the plane Z is shown in Fig. 10. In Fig. 9 and especially in Fig. 10 we see quite regular motion of the studied body prototype. The trajectory prototype in Fig. 10 is a terraced ascending curve, some distorted at the lower level. Notable that in the end of the first level the trajectory prototype passes downwards, outside the belt of the initial one-sheet mapping. The second level is located between the initial belt and that located above, and the third level is located completely in the above belt. Thus we see that the complicated trajectory of the body motion in the plane W at $\omega = 40{,}0\ s^{-1}$ is an analytical three-sheet continuation of the trajectory prototype in the plane Z. This enables us saying of possibility of analytical continuation of many-sheet prototypes when using the dynamical non-conformal mappings.

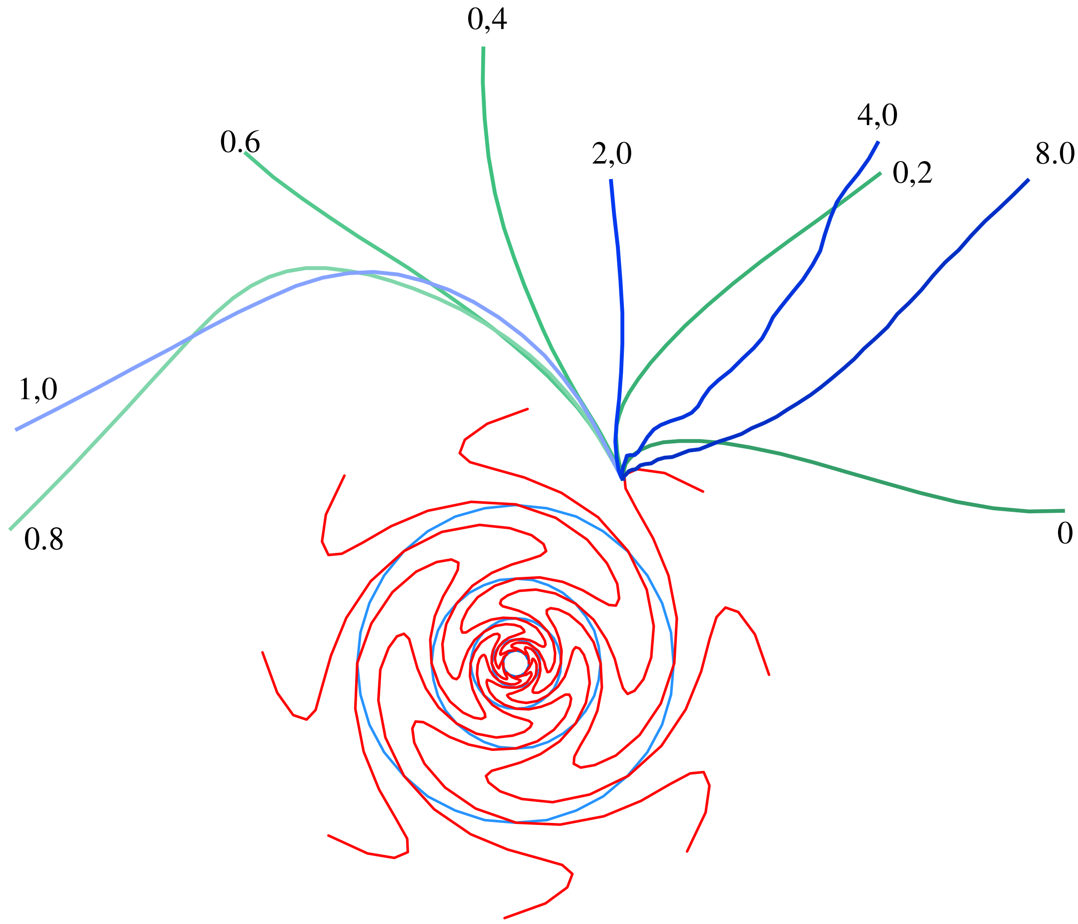

Fig. 11. The motion trajectories of the body having the mass m=0,25 kg in a dynamical central source field (the intensity of action K_F = 20 N) presented in Fig. 5, with different frequencies ω of this field time variation

Speaking of a body chaotical motion in the sink field, note, how the field lines of force effect on the direction of this motion. In the reverse changing of this motion (i.e., in the sink field transformation into the source field), the motion becomes well less chaotical, because in this case the force radial projection will be directed to the weak field. The construction of the Fig. 11 in which we show the body motion trajectory in the source field corroborate this. We see in this construction that in comparison with Fig. 7 the motion trajectories become much more smooth and the lines of force do not repulse the body.

At the same time, we see also that the trajectory still turns with the growing frequency, but now reverse. The same as it retains the transformation direction variation at $\omega = 1,0\ s^{-1}$ occurring at the same field frequency. This corroborates that the trajectory displacement pattern with the field increase is a general regularity of the body motion in such fields.

As we see, stationary and dynamical non-conformal mappings can be helpful in studying the processes in complex dynamical power fields of the most different nature. In this connection, the proved above definitions of analyticity of functions implementing the non-conformal mapping can serve a reliable tool helping to approach more accurately, and the main, more correctly to the studying the processes in which these functions are used.

5. Conclusions

In the carried out investigation we have revealed that the requirement, the limits to be equal for all sequences converging to the δ-vicinity of the point z_0, is excessive for the definition of differentiability of a function of complex variable. It suffices, the limits along all arcs regular in the δ-vicinity of the point z_0 and passing through z_0 to exist.

General definition of differentiability can be presented in the form of two definitions – in general sense and after Caushy – Riemann correspondingly. The definition after Caushy – Riemann adds to the requirement, the limits for all arcs regular in the δ-vicinity of the point z_0 to exist, the requirement, they to be equal. The condition of differentiability in general sense covers that after Caushy – Riemann. The class of functions differentiable in general sense covers that after Caushy – Riemann.

In accord with the broadening definition of the functions differentiability, the definition of their analyticity broadens too. The general definition of analyticity is presentable in the form of two definitions – in general sense and after Caushy – Riemann correspondingly. The condition of analyticity in general sense covers that after Caushy – Riemann. The class of functions analytical in general sense covers that after Caushy – Riemann.

To investigate the processes in complex dynamical fields, one needs to use two types of mappings, stationary and dynamical. This first serves to introduce the one-valued correspondence between the domain, where the function is determined, and the domain of values of non-conformal mapping, and the second serves to describe the pattern of the field power function variation in time. With it the structure of functions used for stationary and dynamical mapping must be the same.

In this studying we have revealed that when the body moves in complex central fields whose lines of force have a tangential component, with the growing in time frequency of transformation, the body trajectory displaces to the field centre (in case of a sink field) and vice versa (in case of a source field). Up to a definite frequency of the field, the trajectory displacement retains its direction even after it crosses the radial axis of the field. At the frequencies higher than that the displacement direction reverses.

As in case of source as in case of sink, if the field had a tangential component, under the field force action the body repulses to the periphery, to the weak field.

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