EIGENVALUE ASYMPTOTICS FOR EVEN-DIMENSIONAL PERTURBED DIRAC AND SCHRÖDINGER OPERATORS WITH CONSTANT MAGNETIC FIELDS

M. Melgaard* and G. Rozenblum
Department of Mathematics
Chalmers University of Technology
and University of Gothenburg
Eklandagatan 86, S-412 96 Gothenburg, Sweden
E-mail: melgaard@math.chalmers.se
and grigori@math.chalmers.se

March 14, 2002

Abstract

The even-dimensional Dirac and Schrödinger operators with a constant magnetic field have purely essential spectrum consisting of isolated eigenvalues, so-called Landau levels. For a sign-definite electric potential V which tends to zero at infinity, *not too fast*, it is known for the Schrödinger operator that the number of eigenvalues near each Landau level is infinite and their leading (quasi-classical) asymptotics depends on the rate of decay for V. We show, both for Schrödinger and Dirac operators, that, for *any* sign-definite, bounded V which tends to zero at infinity, there still are an infinite number of eigenvalues near each Landau level. For compactly supported V we establish the *non-classical* formula, not depending on V, describing how the eigenvalues converge to the Landau levels asymptotically.

*The first author is a Marie Curie Post-Doc Fellow, supported by the European Union under grant no. HPMF-CT-2000-00973.

1 INTRODUCTION

In nonrelativistic quantum mechanics a spinless particle confined to the xy-plane, and subject to a constant magnetic field $\boldsymbol{B} = (0, 0, b)$ aligned along the z-axis as well as a real-valued electrostatic potential $\pm V(x, y)$, is described by the Schrödinger operator

$$\begin{aligned} H_0 \pm V &= (\boldsymbol{P} - \boldsymbol{A})^2 \pm V \\ &= \left(-i\frac{\partial}{\partial x} - \frac{b}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{b}{2}x\right)^2 \pm V(x, y) \end{aligned} \tag{1.1}$$

acting in the Hilbert space $L^2(\mathbb{R}^2)$. Here $\boldsymbol{P} = -i\nabla$ is the momentum operator and $\boldsymbol{A} = 1/2(\boldsymbol{B} \wedge \boldsymbol{r})$, $\boldsymbol{r} = (x, y)$, is the (magnetic) vector potential. For simplicity we have not included any physical parameters (i.e., the particle mass, the particle charge, the speed of light, or Planck's constant) in (1.1). Such "simple" systems has in recent years attracted renewed attention in connection with nanotechnology (quantum dots, boxes etc).

The unperturbed operator H_0 has rather peculiar spectral properties: Its spectrum consists of eigenvalues of infinite multiplicity, the so-called Landau levels $\Lambda_q = b(2q - 1)$, $q \in \mathbb{N}$; (we shall use the abbreviation LL for Landau level). When the operator is perturbed by a potential V which tends to zero at infinity, new eigenvalues may appear, with only possible limit points at LLs, which follows from Weyl's essential spectrum theorem. Therefore the natural question arises, whether there are a finite or infinite number of eigenvalues, and in the latter case, to find their asymptotical distribution.

We may formulate the same problem if, instead of two, the underlying Euclidean space has an even dimension $2d$. Again, the unperturbed operator H_0 has purely essential spectrum consisting of LLs $\{\Lambda_{\boldsymbol{q}}\}$, $\boldsymbol{q} \in \mathbb{N}^d$, which can be expressed as $\Lambda_{\boldsymbol{q}} = \sum_{j=1}^{d} b_j(2q_j - 1)$, for fixed numbers $b_j > 0$, $j = 1, 2, \ldots, d$, determining the magnetic field. The number $\varkappa$ of different sets $\boldsymbol{q} = \{q_1, \ldots, q_d\}$ which determine one and the same level is called the multiplicity of $\Lambda_{\boldsymbol{q}}$.

The analogous problem in three dimensions, where the spectrum of H_0 fills a semi-axis, has been studied intensively. Avron *et al* [1], Solnyshkin [12] and Sobolev [11] has shown that even for a compactly supported $V \geq 0$, there are infinitely many eigenvalues below the lowest point of the essential spectrum of $H_0 - V$. For fast, but not too fast decaying V, Sobolev obtained the leading asymptotics of power-like type.

Extending the work by Avron *et al*, Solnyshkin, Sobolev, and also Tamura [13], Raikov [7] obtained the most detailed results on the leading eigenvalue asymptotics near the boundary points of the essential spectrum in dimensions ≥ 2; e.g. in even dimensions, if V decays at infinity not too fast, the eigenvalues of $H_0 \pm V$ near each LL have power-like asymptotics. However, neither Raikov's work nor later work by Boyarchenko-Levendorski [3], Hempel-Levendorski [5] and Ivrii [6] cover very fast decaying or even compactly supported V.

The complication arising when considering very fast decaying or compactly supported potentials is related to the following circumstance. During the last two decades, the progress in studying spectral asymptotics problems was achieved by refining microlocal analysis methods, using pseudodifferential and Fourier integral operators (see, e.g. [6, 10]). Such methods were used in the papers cited above, as well. However, for very fast decaying potentials the pseudodifferential methods do not work. The reason for this can be explained by noting which properties of symbol classes are critical for the pseudodifferential analysis. In fact, usually all classes of symbols arising in various problems have the common feature: They improve their decay under differentiation (with respect to some, or all, variables). This property is observed, say, by symbols with a power-like behaviour. However, if the symbol decays rapidly, say, as $\exp(-|\xi|^2)$, differentiation worsens the behaviour of the symbol, instead of improving it. It is exactly this sort of behaviour of the symbol that one encounters when reducing the spectral problem for the magnetic Schrödinger operator with fast decaying potentials to ones for pseudodifferential operators as it was done in [11, 13, 7, 3, 6]. This is the reason why the methods of the latter works do not apply to such potentials; even the question of finiteness of the number of eigenvalues was not resolved. It is this kind of potentials we are going to consider in the present paper.

Specifically, we consider potentials satisfying the following assumption.

Assumption 1.1. Let V be a nonnegative, bounded, measurable function on $\mathbb{R}^{2d}$ having compact support. In addition, assume that $V > 0$ holds on a nonempty, open set.

Fix the LL $\Lambda = \Lambda_{\boldsymbol{q}}$, $\boldsymbol{q} \in \mathbb{N}^d$ and let $\Lambda^{\pm}$ denote the nearest LL lying to the left or right of Λ, respectively. Choose any $\mu \in (\Lambda^-, \Lambda)$ (respectively, $\mu \in (\Lambda, \Lambda^+)$) and let $N_\Lambda^-(\lambda) = N(\mu_1, \Lambda - \lambda \,|\, H_0 - V)$ (respectively, $N_\Lambda^+(\lambda) = N(\Lambda + \lambda, \mu_2 \,|\, H_0 + V)$), for $\lambda > 0$ small enough, denote the number (counting multiplicity) of eigenvalues of $H_0 \pm V$ in the gap $(\mu_1, \Lambda - \lambda)$ (respectively, $(\Lambda + \lambda, \mu_2)$).

Under Assumption 1.1 we establish the following theorem.

Theorem 1.2. *Let V satisfy Assumption 1.1 on $\mathbb{R}^{2d}$, $d \in \mathbb{N}$. Then the number of eigenvalues for $H_0 \pm V$ near any LL Λ is infinite. Moreover, if $\mu_1 \in (\Lambda^-, \Lambda)$ and $\mu_2 \in (\Lambda, \Lambda^+)$ then*

$$N(\mu_1, \Lambda - \lambda \,|\, H_0 - V) \sim N(\Lambda + \lambda, \mu_2 \,|\, H_0 + V) \sim \varkappa\, \Xi_d(\lambda) \ \text{as}\ \lambda \downarrow 0,$$

where $\varkappa$ denotes the multiplicity of the LL Λ and

$$\Xi_d(\lambda) = \frac{1}{d!}\left(\frac{|\ln\lambda|}{\ln|\ln\lambda|}\right)^d.$$

Hence, under Assumption 1.1 there still are an infinite number of eigenvalues in each gap, they converge to the LLs very fast and their leading asymptotics do not depend on V. The asymptotics is not described by the quasi-classical formulas known for the case where V admits power-like decay. In this sense it is non-classical. Such behaviour is rather exceptional for differential operators in mathematical physics.

As a spin-off we manage to strengthen the first statement in Theorem 1.2 to *any* nonnegative bounded V which tends to zero at infinity (see Proposition 9.1).

Another operator of mathematical physics with spectral structure similar to the one for the magnetic Schrödinger operator is the Dirac operator with a constant magnetic field in even dimension $2d$.

The spectrum of the unperturbed Dirac operator $\mathcal{D}_d$ is purely essential and it consists of infinitely many isolated eigenvalues, called LLs, $\mu_{\boldsymbol{q}}^{\pm}$, $\boldsymbol{q} \in \mathbb{N}^d$. Here $\mu_{\boldsymbol{q}}^{+} = -\mu_{\boldsymbol{q}}^{-}$ except for the "lowest" eigenvalue $\mu_{\boldsymbol{0}}$, which is *either* -1 *or* 1. In order to study the discrete spectrum of the operator $\mathcal{D}_d$ perturbed by an electric potential VI, where I_{2^d} is the $2^d \times 2^d$ identity matrix and the potential V is a real-valued function on $\mathbb{R}^{2d}$ satisfying Assumption 1.1, we need a detailed description of the eigenspaces corresponding to the LLs of $\mathcal{D}_d$. We were unable to find this description in the general situation in the literature. Even the analysis in [14] performed for $d = 1$ is not sufficient for our purposes since the eigenspaces are described rather implicitly there. Therefore we devote Section 4 to the spectral analysis of $\mathcal{D}_d$. Having this description at our disposal, we establish the following result.

Theorem 1.3. *Let V satisfy Assumption 1.1 on $\mathbb{R}^{2d}$. Then the number of eigenvalues for the perturbed Dirac operator $\mathcal{D}_d \pm V$ near any LL $\mu = \mu_{\boldsymbol{q}}^{\pm}$ is infinite. Moreover, if μ^- and μ^+ denote the neighboring LLs lying below and above μ, respectively, and if $s_1 \in (\mu^-, \mu)$ and $s_2 \in (\mu, \mu^+)$ then*

$$N(s_1, \mu - \lambda \,|\, \mathcal{D}_d - V) \sim N(\mu + \lambda, s_2 \,|\, \mathcal{D}_d + V) \sim \varpi\, \Xi_d(\lambda) \ \text{as}\ \lambda \downarrow 0,$$

where $\varpi = \varpi(\mu)$ is a certain integer (given explicitly in (4.17)).

Thus, just as for the Schrödinger operator, there are an infinite number of eigenvalues in each gap and the asymptotic behaviour of the eigenvalues near the LLs is expressed by the same formula (up to a multiplicative constant) as in Theorem 1.2.

Both for the Schrödinger and Dirac operators, the proofs of asymptotical eigenvalue formulas are based on the analysis of Toeplitz-like operators having the form $T = PVP$, where P is the projection onto the Landau eigenspace of the corresponding operator. The operator T was in the center of study in most of the preceding papers devoted to the subject. It is here the aforementioned microlocal methods fail for our case. To handle the problem, we apply the old-fashioned but still powerful variational method. Nevertheless, the treatment of Toeplitz operators is fairly technical, and our presentation is divided into three sections, Sections 6-7 dealing with the Schrödinger case and Section 8 with the Dirac case. The variational approach is also used in Section 9, where the perturbational reasoning carries over the results for Toeplitz operators to the perturbed Schrödinger and Dirac operators. In Appendix we establish a number theoretical result (Lemma 6.3) which is crucial in dimensions > 2.

When this paper was near its completion, Prof. G. Raikov informed the authors about the manuscript [8], where the eigenvalue asymptotics for the Schrödinger operator with compactly supported or very fast decaying potentials is considered in two and three dimensions. The results in two dimensions cover the ones herein. Higher dimensions, however, create considerable additional problems which are resolved in the present paper and, even for two dimensions, our reasoning is, probably, more transparent. Moreover, the analogous problem for the Dirac operator is not considered in [8] (or elsewhere).

Acknowledgement. We are grateful to Dr. G.R.H. Greaves for providing us with Lemma 9.3, which enable us to give a more streamlined proof of Lemma 6.3 compared to an initial, more *ad hoc* proof. Moreover, the authors thank Prof. G. Raikov for useful discussions and for sending them the manuscript [8].

2 VARIATIONAL METHOD

Let $\mathcal{H}$ be a separable Hilbert space with its norm and scalar product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. For a self-adjoint operator T the distribution function $N(l, r \,|\, T)$ counts the total number of points of the spectrum $\sigma(T)$ of T (taking into account their multiplicity) in the interval (l, r). If a point of the essential spectrum belongs to (l, r) then $N(l, r \,|\, T) = \infty$. We define $N_-(s, T) = N(-\infty, s \,|\, T)$ and $N_+(s, T) = N(s, \infty \,|\, T)$. If T is compact, we set $n_\pm(s, T) = N_\pm(s, T)$.

We shall use the min-max description of eigenvalues (following directly from the Spectral Theorem).

Lemma 2.1. *Let T be a self-adjoint operator. Then, for $l < r$,*

$$N(l,r|T) = \max\dim\left\{\mathcal{L}\subset\mathcal{D}(T) : \|(T-s)u\|^2 < t^2\|u\|^2, u\in\mathcal{L}\backslash\{0\}\right\} \quad (2.1)$$
$$= \min\operatorname{codim}\left\{\mathcal{L}\subset\mathcal{D}(T) : \|(T-s)u\|^2 \geq t^2\|u\|^2, u\in\mathcal{L}\right\}, \quad (2.2)$$

where $s = (l+r)/2$ and $t = (r-l)/2$.

For a nonnegative, compact operator T it is convenient to use another form of (2.1)-(2.2):

$$n_+(s,T) = \max\dim\left\{\mathcal{L}\subset\mathcal{H} : \langle Tu,u\rangle > s\|u\|^2,\ u\in\mathcal{L}\backslash\{0\}\right\} \quad (2.3)$$
$$= \min\operatorname{codim}\left\{\mathcal{L}\subset\mathcal{H} : \langle Tu,u\rangle \leq s\|u\|^2, u\in\mathcal{L}\right\}, s>0. \quad (2.4)$$

We shall also use the following perturbational result found in, e.g., [2].

Lemma 2.2. *Let T be self-adjoint and let S be bounded on $\mathcal{H}$ such that $\sigma(S)\subset[s_1,s_2]$. Then for any finite interval (l,r) one has that*

$$N(l+s_1, r+s_2\,|\,T+S) \geq N(l,r\,|\,T).$$

In addition to the standard order symbols we use, for two real-valued functions $f(\lambda)$ and $g(\lambda)$, the notation $f(\lambda)\lesssim g(\lambda)$ which means that $f(\lambda)\leq g(\lambda)(1+o(1))$ as $\lambda\to 0$.

3 THE UNPERTURBED SCHRÖDINGER OPERATOR

We summarize some well-known facts about the unperturbed Schrödinger operator with a constant magnetic field (see, e.g., [4]).

3.1 Two-dimensional Case

In $\mathbb{R}^2$ we consider a charged, spinless particle in a homogeneous magnetic field. We assume that the magnetic field $\boldsymbol{B}$ has constant strength $b>0$ and is aligned in the z direction, viz. $\boldsymbol{B} = (0,0,b)$. In appropiate units the Hamiltonian of the particle is $H_0 = (\boldsymbol{P}+\boldsymbol{A})^2$, where $\boldsymbol{P} = -i\nabla$ is the momentum operator and $\boldsymbol{A}$ is some vector potential associated with the field, viz. $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and defined up to a gauge transformation. We choose the gauge in which $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B} \wedge \boldsymbol{r})$, $\boldsymbol{r} = (x, y)$. The resulting Hamiltonian

$$H_0 = \left(-i\frac{\partial}{\partial x} - \frac{b}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{b}{2}x\right)^2 \tag{3.1}$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ [9]. It is well-known that H_0 has a purely essential spectrum consisting of isolated eigenvalues, viz.

$$\sigma(H_0) = \{\, \Lambda_q = b(2q-1) \,|\, q = 1, 2, 3, \dots \}. \tag{3.2}$$

In literature on physics the eigenvalues Λ_q are referred to as LLs. For a fixed LL $\Lambda_q = b(2q-1)$ one can choose an orthonormal basis in the Landau eigenspace consisting of functions

$$\begin{aligned} f_m(\rho, \theta) = f_{m,q}(\rho, \theta; b) &= (2\pi)^{-1/2} b^{(1+|m|)/2} 2^{-|m|/2} \left(\frac{q!}{(q+|m|)!}\right)^{1/2} \\ &\times e^{-im\theta} \rho^{|m|} e^{-\frac{b}{4}\rho^2} L_q^{(|m|)}\left(\frac{b}{2}\rho^2\right), \quad m = -q+1, -q+2, \dots, \end{aligned} \tag{3.3}$$

in polar coordinates (ρ, θ). Here $L_q^{(|m|)}(\xi)$ are the generalized Laguerre polynomials

$$L_q^{|m|}(\xi) = \sum_{l=0}^{q} d_{l,|m|}(-\xi)^l \,; \;\; d_{l,|m|} = \begin{pmatrix} q+|m| \\ q-l \end{pmatrix} \frac{1}{l!}. \tag{3.4}$$

3.2 Even-dimensional Case

Introduce the vector potential $\boldsymbol{a} = (a_1, a_2, \dots, a_{2d}) \in C^1(\mathbb{R}^{2d}; \mathbb{R}^{2d})$ and the 1-form $\mathcal{A} = \sum_{j=1}^{2d} a_j \, dx_j$. Define the magnetic field 2-form

$$\mathcal{B} = d\mathcal{A} = \sum_{i=1}^{2d} \sum_{j=1}^{2d} b_{ij} \, dx_i \wedge dx_j, \quad b_{ij} = \frac{1}{2}(\partial_{x_i} a_j - \partial_{x_j} a_i).$$

We assume that $\mathcal{B} \not\equiv 0$. Moreover, we suppose that all the entries of the antisymmetric matrix $\boldsymbol{B} = \{b_{ij}\}_{i,j}^{2d}$ are constant. The unperturbed Hamiltonian $H_0 = \sum_{j=1}^{2d}(-i\partial_{x_j} - a_j)^2$ defined on $C_0^\infty(\mathbb{R}^{2d})$ is essentially self-adjoint in $L^2(\mathbb{R}^{2d})$. Its unique self-adjoint extension is also denoted by H_0. The eigenvalues of $\boldsymbol{B}$ form a subset of the imaginary axis, symmetric with respect to the origin. Let $b_j \in \mathbb{R}$, $j = 1, 2, \dots, d$ be such numbers that the nonzero eigenvalues of $\boldsymbol{B}$ coincide (counting multiplicity) with the imaginary numbers $-ib_j$ and ib_j, $j = 1, \dots, n$. Thus we have $2n = \operatorname{rank} \boldsymbol{B}$, $0 < 2n \leq 2d$.

We limit ourselves to the *non-degenerate case* $n = d$. In proper coordinates $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2d}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the 2-form $\mathcal{B}$ block-diagonalizes: $\mathcal{B} = \sum_{j=1}^d b_j \, dy_j \wedge dx_j$. By changing variables $x_j \mapsto -x_j$ one may reduce the problem to the case when all b_j are positive. In these coordinates H_0 is unitary equivalent to (again we use the notation H_0) the operator

$$H_0 = \sum_{j=1}^d \tilde{H}_{0,j} = \sum_{j=1}^d (I \otimes \cdots \otimes H_{0,j} \otimes \cdots I) \tag{3.5}$$

in $L^2(\mathbb{R}^{2d}) = \bigotimes_{j=1}^d L^2(\mathbb{R}_j^2)$, where

$$H_{0,j} = \left(-i\frac{\partial}{\partial x_j} - \frac{b_j}{2} y_j\right)^2 + \left(-i\frac{\partial}{\partial y_j} + \frac{b_j}{2} x_j\right)^2 \tag{3.6}$$

and

$$\sigma(H_{0,j}) = \sigma_{ess}(H_{0,j}) = \{ b_j(2q_j - 1) \,|\, q_j = 1, 2, 3, \dots \} \tag{3.7}$$

with a basis of eigenfunctions of the form (3.3), with (ρ, θ) replaced by the polar coordinates (ρ_j, θ_j), for the corresponding eigenfunctions $f^{(j)}_{m_j,q_j}(\rho_j, \theta_j) = f_{m_j,q_j}(\rho_j, \theta_j; b_j)$.

This gives us that the operator H_0 is essentially self-adjoint on $\mathcal{D}(H_0) = \bigotimes_{j=1}^d \mathcal{D}(H_{0,j}) = \bigotimes_{j=1}^d C_0^\infty(\mathbb{R}_j^2) = C_0^\infty(\mathbb{R}^{2d})$ and, in view of (3.5) and (3.7), its spectrum equals

$$\sigma(H_0) = \sigma_{ess}(H_0) = \{ \Lambda_{\boldsymbol{q}} \,|\, \boldsymbol{q} = (q_1, \dots, q_d) \in \mathbb{N}^d \}, \tag{3.8}$$

where the isolated eigenvalues $\{\Lambda_{\boldsymbol{q}}\}$, $\boldsymbol{q} \in \mathbb{N}^d$, have the form

$$\Lambda = \Lambda_{\boldsymbol{q}} = \sum_{j=1}^d b_j(2q_j - 1), \;\; q_j \in \mathbb{N}. \tag{3.9}$$

As above, the numbers $\Lambda_{\boldsymbol{q}}$ are called LLs. The number $\varkappa$ of different sets $\boldsymbol{q} = \{q_1, q_2, \dots, q_d\}$ which determine one and the same level Λ according to (3.9) is called the multiplicity of Λ.

We discuss multiple LLs in more details. Let, e.g. $d = 2$, and let the matrix $\boldsymbol{B}$ have eigenvalues b_1, b_2, $-b_1$, $-b_2$, $b_j > 0$. Then the Schrödinger operator has the form

$$H_0 = H_1 \otimes 1 + 1 \otimes H_2 = \tilde{H}_1 + \tilde{H}_2,$$

where H_j is given in (3.6). The LLs of H_0 are sums of LLs for H_1 and H_2, i.e. for a fixed level Λ we may write

$$\Lambda = \Lambda_1 + \Lambda_2 = b_1(2q_1 - 1) + b_2(2q_2 - 1). \tag{3.10}$$

It may turn out that a given Λ can be represented in the form (3.10) in several different ways, for example

$$\Lambda = b_1(2q_1^{'} - 1) + b_2(2q_2^{'} - 1) = b_1(2q_1^{''} - 1) + b_2(2q_2^{''} - 1). \tag{3.11}$$

The number of such representations is the multiplicity $\varkappa$ of the LL Λ. According to (3.11), the Landau eigenspace of H_0 for Λ is a direct sum of eigenspaces corresponding to different representations. For example, in the case (3.11),

$$\mathcal{H}_\Lambda = \mathcal{H}_\Lambda^{'} \oplus \mathcal{H}_\Lambda^{''} = \left(\mathcal{H}^1_{q_1^{'}} \otimes \mathcal{H}^2_{q_2^{'}}\right) \oplus \left(\mathcal{H}^1_{q_1^{''}} \otimes \mathcal{H}^2_{q_2^{''}}\right) \tag{3.12}$$

where, e.g. $\mathcal{H}^j_{q_j^{'}}$, $j = 1, 2$, is the Landau eigenspace of H_j corresponding to the LL $b_j(2q_j^{'} - 1)$. Generally, for the multiplicity $\varkappa$ in dimension $2d$, we have

$$\Lambda = \sum_{j=1}^{d} \Lambda_j^{(\alpha)} = \sum_{j=1}^{d} b_j(2q_j^{(\alpha)} - 1), \quad \alpha = 1, \dots, \varkappa, \tag{3.13}$$

with corresponding eigenspace and projection given by

$$\begin{aligned} \mathcal{H}_\Lambda &= \bigoplus_{\alpha=1}^{\varkappa} \bigotimes_{j=1}^{d} \mathcal{H}^j_{q_j^{(\alpha)}} = \bigoplus \mathcal{H}_\Lambda^{(\alpha)}, \\ P &= \sum P^{(\alpha)} \text{ with } P^{(\alpha)} = \otimes_{j=1}^d P_{j,q_j^{(\alpha)}}, \end{aligned} \tag{3.14}$$

where $P_{j,q_j^{(\alpha)}}$ is the spectral projection onto $\mathcal{H}^j_{q_j^{(\alpha)}}$. For all LLs to have multiplicity 1, it is sufficient that $b_1, \dots, b_d$ are linearly independent over odd integers. Thus, in the general situation, in particular for $b_1 = \cdots = b_d$, one must expect that multiple LLs arise.

4 UNPERTURBED DIRAC OPERATOR

In this section we study the even-dimensional (unperturbed) Dirac operator $\mathcal{D}_d$ with a constant magnetic field. In analogy with the Schrödinger operator H_0 (see (3.5)) the spectrum of $\mathcal{D}_d$ turns out to be purely essential and it consists of infinitely many isolated eigenvalues $\mu_{\boldsymbol{q}}^{\pm}$, $\boldsymbol{q} \in \mathbb{N}^d$, called Landau levels (LL). Here $\mu_{\boldsymbol{q}}^{+} = -\mu_{\boldsymbol{q}}^{-}$ except for the "lowest" eigenvalue $\mu_{\boldsymbol{0}}$, which is *either* -1 *or* 1. We find, moreover, an explicit representation of the corresponding Landau eigenspaces.

4.1 Dirac Matrices and Operators

In dimension $2d$, $d \in \mathbb{N}$, there are $2d+1$ Dirac matrices, denoted $\sigma_1^{(d)}$, $\sigma_2^{(d)}$, $\dots$, $\sigma_{2d}^{(d)}$ and $\sigma_0^{(d)}$, each of size 2^d, and they satisfy the relations

$$(\sigma_j^{(d)})^* = \sigma_j^{(d)}, \quad \sigma_j^{(d)}\sigma_k^{(d)} + \sigma_k^{(d)}\sigma j^{(d)} = 2\delta_{jk} I_{2^d}, \quad j = 0, 1, 2, \dots, 2d. \tag{4.1}$$

The matrices may be defined inductively: For $d=1$ they are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \text{ and } \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\sigma_1^{(d)}, \sigma_2^{(d)}, \dots, \sigma_{2d}^{(d)}$ and $\sigma_0^{(d)}$ are given for d, the matrices for $d+1$ are

$$\begin{aligned} \sigma_1^{(d+1)} &= \begin{pmatrix} \mathbf{0} & \sigma_1^{(d)} \\ \sigma_1^{(d)} & \mathbf{0} \end{pmatrix}, \quad \dots, \quad \sigma_{2d-2}^{(d+1)} = \begin{pmatrix} \mathbf{0} & \sigma_{2d-2}^{(d)} \\ \sigma_{2d-2}^{(d)} & \mathbf{0} \end{pmatrix}, \\ \sigma_{2d-1}^{(d+1)} &= \begin{pmatrix} \mathbf{0} & \sigma_0^{(d)} \\ \sigma_0^{(d)} & \mathbf{0} \end{pmatrix}, \quad \sigma_{2d}^{(d+1)} = \begin{pmatrix} \mathbf{0} & iI \\ -iI & \mathbf{0} \end{pmatrix} \end{aligned}$$

and

$$\sigma_0^{(d+1)} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}; \quad I = I_{2^{d-1}}.$$

Vector components corresponding to upper blocks in these block-matrices will be referred to as the upper half, the rest as the lower half.

For $b_j \in \mathbb{R}$, $j = 1, 2, \dots, d$, set $\beta_j = |b_j|$. Define

$$P_{2j-1} = -i\frac{\partial}{\partial x_j} + \frac{b_j}{2}y_j, \quad P_{2j} = -i\frac{\partial}{\partial y_j} - \frac{b_j}{2}x_j.$$

Note that, since we handle first order operators now, we cannot use transformations $x_j \mapsto -x_j$ to make all b_j positive. Thus, here we have to consider b_j with arbitrary signs.

The $2d$-dimensional unperturbed Dirac operator with a constant magnetic field, acting in $L^2(\mathbb{R}^2)^d$, is defined by

$$\mathcal{D}_d = \mathcal{A}_d + \sigma_0^{(d)}, \text{ where } \mathcal{A}_d = \sum_{j=1}^{2d} \sigma_j^{(d)} P_j.$$

For $j = 1, \dots, d$ put

$$\Pi_j = P_{2j-1} + iP_{2j} = -i\frac{\partial}{\partial x_j} + \frac{b_j}{2}y_j + \frac{\partial}{\partial y_j} - i\frac{b_j}{2}x_j \tag{4.2}$$

and

$$\mathcal{T}_j = \begin{pmatrix} \Pi_j^* & \mathbf{0} \\ \mathbf{0} & -\Pi_j \end{pmatrix} = P_{2j} + \sigma_0^{(j)} P_{2j-1}. \tag{4.3}$$

It is convenient to describe $\mathcal{D}_d$ inductively as follows. For $d = 1$ we write

$$\mathcal{D}_1 = \mathcal{A}_1 + \sigma_0^{(1)}, \text{ where } \mathcal{A}_1 = \begin{pmatrix} 0 & \Pi_1^* \\ \Pi_1 & 0 \end{pmatrix} \tag{4.4}$$

and, generally for an arbitrary d,

$$\mathcal{D}_d = \mathcal{A}_d + \sigma_0^{(d)}, \text{ where } \mathcal{A}_d = \begin{pmatrix} \mathbf{0} & \mathcal{A}_{d-1} + \mathcal{T}_d^* \\ \mathcal{A}_{d-1} + \mathcal{T}_d & \mathbf{0} \end{pmatrix}. \tag{4.5}$$

The afore-defined operators satisfy the following relations:

$$\begin{aligned} [\Pi_j, \Pi_k] &= [\Pi_j, \Pi_k^*] = 0, \quad j \neq k, & (4.6)\\ \Pi_j \Pi_j^* &= \Pi_j^* \Pi_j + 2b_j = H_{0,j} + b_j,\ j = 1, 2, \dots, d, & (4.7)\\ [\,\mathcal{A}_j, \Pi_j\,] &= 0, \quad j = 1, 2, \dots, d, & (4.8)\\ \mathcal{A}_{j-1}\mathcal{T}_j + \mathcal{T}_j^* \mathcal{A}_{j-1} &= 0, \quad j = 1, 2, \dots, d, & (4.9) \end{aligned}$$

with $H_{0,j}$ being the two-dimensional Schrödinger operator (3.6) associated with b_j. Moreover,

$$\mathcal{T}_j \mathcal{T}_j^* = \mathcal{T}_j^* \mathcal{T}_j - 2b_j \sigma_0^{(j)} = H_{0,j} I_{2^j} - b_j \sigma_0^{(j)} \tag{4.10}$$

and

$$\sigma(\mathcal{T}_j \mathcal{T}_j^*) \ = \ \{\Lambda_{q_j} + b_j \,|\, q_j \in \mathbb{N}\} = \begin{cases} \{\beta_j 2q_j \,|\, q_j \in \mathbb{N}\} & \text{if } b_j > 0, \\ \{\beta_j (2q_j - 2) \,|\, q_j \in \mathbb{N}\} & \text{if } b_j < 0, \end{cases}$$

4.2 The Operator $\mathcal{A}_d^2$

From the relations (4.6)-(4.10) we deduce that

$$\begin{aligned} \mathcal{A}_d^2 \ &= \ \left(\sum_{j=1}^{d} H_{0,j}\right) \underbrace{I \otimes \cdots \otimes I}_{\#=d} - b_1 \underbrace{I \otimes \cdots \otimes I}_{\#=d-1} \otimes \sigma_0 \\ &+ b_2 \underbrace{I \otimes \cdots \otimes I}_{\#=d-2} \otimes \sigma_0 \otimes \sigma_0 + b_3 \underbrace{I \otimes \cdots \otimes I}_{\#=d-3} \otimes \sigma_0 \otimes I \otimes \sigma_0 \\ &+ \cdots + b_{d-1} I \otimes \sigma_0 \otimes \underbrace{I \otimes \cdots \otimes I}_{\#=d-3} \otimes \sigma_0 + b_d \sigma_0 \otimes \underbrace{I \otimes \cdots \otimes I}_{\#=d-2} \otimes \sigma_0. \end{aligned} \tag{4.11}$$

Thus one may represent $\mathcal{A}_d^2$ as

$$\mathcal{A}_d^2 = H_0 I_{2^d} - \Delta_d(\boldsymbol{b}). \tag{4.12}$$

Here $H_0 = \sum_{j=1}^d H_{0,j}$ is the $2d$-dimensional Schrödinger operator (3.5), $\boldsymbol{b} = (b_1, \dots, b_d)$ and $\Delta_d(\boldsymbol{b})$ is the diagonal $2^d \times 2^d$ matrix having on the diagonal the sums $b_{\boldsymbol{\varepsilon}} = \sum_{j=1}^d \varepsilon_j b_j$, where $\boldsymbol{\varepsilon}$ belongs to the set

$$E^d = \{(\varepsilon_1, \dots, \varepsilon_d) \,|\, \text{ all possible combinations of } \varepsilon_j = \pm 1\,\}.$$

The way in which $b_{\boldsymbol{\varepsilon}}$ enter on the diagonal of $\Delta_d(\boldsymbol{b})$ determines an order for elements $\boldsymbol{\varepsilon} \in E^d$. We denote by E_+^d and E_-^d the sets of those $\boldsymbol{\varepsilon} \in E^d$ for which $b_{\boldsymbol{\varepsilon}}$ enters in the upper half or lower half of $\Delta_d(\boldsymbol{b})$, respectively. The particular order in which $b_{\boldsymbol{\varepsilon}}$ enter on the diagonal is of no importance, however, one property of $b_{\boldsymbol{\varepsilon}}$, which follows easily inductively from the inductive definition of Dirac matrices and (4.11) is the following. Fix some subset $\mathbb{J} \subset \{1, 2, \dots, d\}$ with ν elements, $1 \le \nu \le d$. Then of $2^{d-\nu}$ combinations of $\boldsymbol{\varepsilon}$ with ε_j fixed for $j \in \mathbb{J}$, exactly one half, i.e. $2^{d-\nu-1}$, belong to E_+^d, and $2^{d-\nu-1}$ belong to E_-^d.

Since $\mathcal{A}_d^2$ is a direct sum of operators of the form $H_0 - b_{\boldsymbol{\varepsilon}}$, the spectrum of $\mathcal{A}_d^2$ is determined by the spectrum of H_0. To define the eigenspaces of $\mathcal{A}_d^2$, we introduce the following notation. Let $\mathcal{H}_{\boldsymbol{\varepsilon}} \subset L^2(\mathbb{R}^{2d})_{\boldsymbol{\varepsilon}}$, $\boldsymbol{\varepsilon} \in E^d$, be closed subspaces. Let $\iota_{\boldsymbol{\varepsilon}}$ be the natural embedding of $L^2(\mathbb{R}^{2d})_{\boldsymbol{\varepsilon}}$ into $L^2(\mathbb{R}^{2d}, \mathbb{C}^{2^d})$ (to an element ψ in $L^2(\mathbb{R}^{2d})_{\boldsymbol{\varepsilon}}$ we assign the 2^d-dimensional vector in $L^2(\mathbb{R}^{2d}, \mathbb{C}^{2^d})$ for which all entries are zero expect the "$\boldsymbol{\varepsilon}$th", which equals ψ). Then we set $\boxplus_{\boldsymbol{\varepsilon}} \mathcal{H}_{\boldsymbol{\varepsilon}} = \oplus_{\boldsymbol{\varepsilon}} \iota_{\boldsymbol{\varepsilon}} \mathcal{H}_{\boldsymbol{\varepsilon}}$. This will be called the exterior direct sum. The same notation will be used for elements in $\mathcal{H}_{\boldsymbol{\varepsilon}}$.

Proposition 4.1. *Let $\Lambda := \Lambda_{\boldsymbol{q}} = \sum_{j=1}^d \beta_j(2q_j - 1)$ be a LL, with multiplicity one, of the $2d$-dimensional Schrödinger operator H_0 and let $\mathcal{H}_\Lambda$ be the corresponding eigenspace. For any $\boldsymbol{\varepsilon} \in E^d$ define*

$$\mathfrak{l}_{\boldsymbol{\varepsilon}} = \sum_{j=1}^d \beta_j[2q_j - 2 - (1 - \varepsilon_j)].$$

Then the largest of these $\mathfrak{l}_{\boldsymbol{\varepsilon}}$, $\mathfrak{l} = 2\sum_{j=1}^d \beta_j(q_j - 1)$ is an eigenvalue of $\mathcal{A}_d^2$. To each $\mathfrak{l}_{\boldsymbol{\varepsilon}}$ one associates a subspace $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}$ in $L^2(\mathbb{R}^{2d})_{\boldsymbol{\varepsilon}}$ as follows: Define

$$\mathcal{E}(\boldsymbol{q}) = \{\, \boldsymbol{\varepsilon} \in E^d \,|\, 2q_j - 1 - (1 - \varepsilon_j) > 0,\ \forall \varepsilon_j \,\}.$$

If $\boldsymbol{\varepsilon} \in \mathcal{E}(\boldsymbol{q})$ then one puts $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}$ equal to the eigenspace associated with the LL $\Lambda_{\boldsymbol{\varepsilon}} := \sum_{j=1}^d \beta_j[2q_j - 1 - (1 - \varepsilon_j)]$ of H_0. Otherwise, if $\boldsymbol{\varepsilon} \notin \mathcal{E}(\boldsymbol{q})$ then $\Lambda_{\boldsymbol{\varepsilon}}$ is not a LL for H_0 and one puts $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}} = \{0\}$. Then $\mathcal{Y}_{\mathfrak{l}} = \boxplus_{\boldsymbol{\varepsilon} \in \mathcal{E}(\boldsymbol{q})} \mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}$ is the *eigenspace associated with the eigenvalue* $\mathfrak{l}$ *of* $\mathcal{A}_d^2$. *Moreover, if* $\nu(\boldsymbol{q})$ *is the number of those* q_j *which are larger than one, then* $\operatorname{Card}\mathcal{E}(\boldsymbol{q}) = 2^{\nu(\boldsymbol{q})}$ *and, if* $\nu(\boldsymbol{q}) > 0$, *exactly half of the elements in* $\mathcal{E}(\boldsymbol{q})$, *i.e.* $2^{\nu(\boldsymbol{q})-1}$, *belongs to* E_+^d, *and the other half belongs to* E_-^d.

Proof. Assume that $\mathfrak{l}$ is an eigenvalue of $\mathcal{A}_d^2$, i.e.

$$\mathcal{A}_d^2\Psi = \mathfrak{l}\Psi \tag{4.13}$$

is solved for $\Psi = (\psi_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}} \in \mathcal{Y}_{\mathfrak{l}}$, where the latter is some eigenspace to be determined. Now, (4.13) can be written as the system

$$H_0\psi_{\boldsymbol{\varepsilon}} = (\mathfrak{l} + b_{\boldsymbol{\varepsilon}})\psi_{\boldsymbol{\varepsilon}}, \tag{4.14}$$

where $b_{\boldsymbol{\varepsilon}} = \sum_{j=1}^d \varepsilon_j b_j$. Thus, at least one of these $\mathfrak{l} + b_{\boldsymbol{\varepsilon}}$ must be a LL, which implies that the largest of these $\mathfrak{l} + b_{\boldsymbol{\varepsilon}}$, namely $\Lambda = \mathfrak{l} + \sum \beta_j$ is a LL of H_0, i.e. $\Lambda = \sum \beta_j(2q_j - 1)$ for some $q_j \in \mathbb{N}$. Indeed, if the largest one is not a LL of H_0 then the smaller ones are not LLs of H_0 either. Consequently, $\mathfrak{l} = 2\sum \beta_j(q_j - 1)$. To find the associated eigenspace, we consider all other possible LLs for H_0, corresponding to this $\mathfrak{l}$, for all $\boldsymbol{\varepsilon}$. According to (4.14) they may be represented as

$$\Lambda_{\boldsymbol{\varepsilon}} = \Lambda - \left(\sum_{j=1}^d \beta_j - \sum_{j=1}^d \varepsilon_j b_j\right) = \sum_{j=1}^d \beta_j[2q_j - 1 - (1 - \varepsilon_j)]. \tag{4.15}$$

It is clear that these smaller $\Lambda_{\boldsymbol{\varepsilon}}$ need not all be LLs of H_0. Indeed, if a certain q_j equals 1 (i.e. attains its smallest possible value) and the corresponding ε_j equals -1 then the corresponding β_j is not present in the expression for $\Lambda_{\boldsymbol{\varepsilon}}$, viz. $\Lambda_{\boldsymbol{\varepsilon}}$ is not a LL of H_0. As a consequence, we define $\mathfrak{l}_{\boldsymbol{\varepsilon}} = \sum \beta_j[2q_j - 2 - (1 - \varepsilon_j)]$ (of which the largest is the eigenvalue $\mathfrak{l}$ of $\mathcal{A}_d^2$). Then we assign a subspace $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}$ to each of these $\mathfrak{l}_{\boldsymbol{\varepsilon}}$ as described in the statement of the Proposition, and it follows immediately that $\mathcal{Y}_{\mathfrak{l}} = \boxplus_{\boldsymbol{\varepsilon}\in\mathcal{E}(\boldsymbol{q})}\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}$ is the eigenspace associated with the eigenvalue $\mathfrak{l}$ of $\mathcal{A}_d^2$. This family of $\Lambda_{\boldsymbol{\varepsilon}}$ will be referred to as the "ladder" corresponding to the eigenvalue $\mathfrak{l}$.

On the other hand, any LL of H_0 can act as the largest one for some eigenvalue of $\mathcal{A}_d^2$. In particular, if we have $\Lambda = \sum_j \beta_j$ (the lowest LL of H_0) and, consequently, $\mathfrak{l} = 0$, then Λ is the only LL of H_0 included in the above-mentioned ladder.

Define $\nu(\boldsymbol{q})$ as in the Proposition, i.e. there are $d - \nu(\boldsymbol{q})$ elements in $\boldsymbol{q}$ for which $q_j = 1$. Let $\mathbb{J}$ denote the set of indices for the latter q_j's. To ensure that $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}} \neq \{0\}$ for a particular $\boldsymbol{\varepsilon}$, each of the elements ε_j, $j \in \mathbb{J}$, has to be chosen as $+1$. For the remaining $\nu(\boldsymbol{q})$ entries in each $\boldsymbol{\varepsilon}$, we may choose $\varepsilon_j = \pm 1$. Hence, there are $2^{\nu(\boldsymbol{q})}$ different $\boldsymbol{\varepsilon}$ for which $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}} \neq \{0\}$, i.e. $\operatorname{Card} \mathcal{E}(\boldsymbol{q}) = 2^{\nu(\boldsymbol{q})}$. Provided $\nu(\boldsymbol{q}) > 0$, the property mentioned just above the Proposition, asserts that exactly half of the $2^{\nu(\boldsymbol{q})}$ elements in $\mathcal{E}(\boldsymbol{q})$, i.e. $2^{\nu(\boldsymbol{q})-1}$, belongs to E^d_+. The other half belongs to E^d_-. $\square$

Define $\mathcal{Y}^{(\pm)}_{\mathfrak{l}} = \boxplus_{\boldsymbol{\varepsilon} \in \mathcal{E}(\boldsymbol{q}) \cap E^d_\pm} \mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}$. According to Proposition 4.1, the number of summands in $\mathcal{Y}^{(\pm)}_{\mathfrak{l}}$ equals $2^{\nu(\boldsymbol{q})-1}$, provided $\nu(\boldsymbol{q}) > 0$.

For multiple LLs of H_0 we have to perform the above "ladder" construction for each representation of the LL Λ.

Proposition 4.2. *Let $\Lambda = \Lambda_{\boldsymbol{q}^{(\alpha)}} = \sum_{j=1}^d \beta_j (2q_j^{(\alpha)} - 1)$, $\alpha = 1, 2, \dots, \varkappa$, be a LL, with multiplicity $\varkappa$, of the 2d-dimensional Schrödinger operator H_0. Let $\mathcal{H}_\Lambda$ be the corresponding eigenspace. For any $\boldsymbol{\varepsilon} \in E^d$ define*

$$\mathfrak{l}_{\boldsymbol{\varepsilon}} = \sum_{j=1}^d \beta_j [2q_j^{(\alpha)} - 2 - (1 - \varepsilon_j)].$$

Then the largest of these $\mathfrak{l}_{\boldsymbol{\varepsilon}}$, $\mathfrak{l} = 2\sum_{j=1}^d \beta_j (q_j^{(\alpha)} - 1)$ is an eigenvalue of $\mathcal{A}_d^2$. To each $\mathfrak{l}_{\boldsymbol{\varepsilon}}$ we associate a subspace $\mathcal{H}^{(\alpha)}_{\Lambda_{\boldsymbol{\varepsilon}}}$ in $L^2(\mathbb{R}^{2d})$ as follows: Define

$$\mathcal{E}(\boldsymbol{q}^{(\alpha)}) = \{\, \boldsymbol{\varepsilon} \in E^d \mid 2q_j^{(\alpha)} - 1 - (1 - \varepsilon_j) > 0,\ \forall \varepsilon_j \,\}. \tag{4.16}$$

If $\boldsymbol{\varepsilon} \in \mathcal{E}(\boldsymbol{q}^{(\alpha)})$ then one puts $\mathcal{H}^{(\alpha)}_{\Lambda_{\boldsymbol{\varepsilon}}}$ equal to the eigenspace associated with the LL $\Lambda_{\boldsymbol{\varepsilon}} = \sum_{j=1}^d \beta_j [2q_j^{(\alpha)} - 1 - (1 - \varepsilon_j)]$ of H_0. Otherwise, if $\boldsymbol{\varepsilon} \notin \mathcal{E}(\boldsymbol{q}^{(\alpha)})$ then $\Lambda_{\boldsymbol{\varepsilon}}$ is not a LL for H_0 and one puts $\mathcal{H}^{(\alpha)}_{\Lambda_{\boldsymbol{\varepsilon}}} = \{0\}$. If $\nu(\boldsymbol{q}^{(\alpha)})$ is the number of those q_j which are larger than one, then $\operatorname{Card} \mathcal{E}(\boldsymbol{q}^{(\alpha)}) = 2^{\nu(\boldsymbol{q})}$ and, provided $\nu(\boldsymbol{q}^{(\alpha)}) > 0$, exactly half of the elements in $\mathcal{E}(\boldsymbol{q}^{(\alpha)})$, i.e. $2^{\nu(\boldsymbol{q}^{(\alpha)})-1}$, belongs to E^d_+, and the other half belongs to E^d_-.

Define $\mathcal{Y}^{(\alpha)}_{\mathfrak{l}} = \boxplus_{\boldsymbol{\varepsilon} \in \mathcal{E}(\boldsymbol{q}^{(\alpha)})} \mathcal{H}^{(\alpha)}_{\Lambda_{\boldsymbol{\varepsilon}}}$. Then $\mathcal{Y}_{\mathfrak{l}} = \oplus_\alpha \mathcal{Y}^{(\alpha)}_{\mathfrak{l}}$ is the eigenspace associated with the eigenvalue $\mathfrak{l}$ of $\mathcal{A}_d^2$.

In Proposition 4.2 the eigenvalue $\mathfrak{l}$ of $\mathcal{A}_d^2$ can be represented in $\varkappa$ different ways, so we say that it has multiplicity $\varkappa$.

It follows from Proposition 4.2 that the number of summands in $\mathcal{Y}^{(\alpha)}_{\mathfrak{l}}$ equals $\operatorname{Card} \mathcal{E}(\boldsymbol{q}^{(\alpha)}) = 2^{\nu(\boldsymbol{q})}$ and there are $\sum_{\alpha=1}^{\varkappa} 2^{\nu(\boldsymbol{q}^{(\alpha)})}$ summands in $\mathcal{Y}_{\mathfrak{l}}$. Let $\mathcal{Y}^{(\pm,\alpha)}_{\mathfrak{l}} = \boxplus_{\boldsymbol{\varepsilon} \in \mathcal{E}(\boldsymbol{q}^{(\alpha)}) \cap E^d_\pm} \mathcal{H}^{(\alpha)}_{\Lambda_{\boldsymbol{\varepsilon}}}$ and define $\mathcal{Y}^{(\pm)}_{\mathfrak{l}} = \oplus_{\alpha=1}^{\varkappa} \mathcal{Y}^{(\pm,\alpha)}_{\mathfrak{l}}$. Clearly the number of summands in $\mathcal{Y}^{(\pm)}_{\mathfrak{l}}$ equals

$$\varpi := \sum_{\alpha=1}^{\varkappa} 2^{\nu(\boldsymbol{q}^{(\alpha)})-1}. \tag{4.17}$$

4.3 The Operator $\mathcal{D}_d$

Since $\mathcal{D}_d = \mathcal{A}_d + \sigma_0$, we have $\mathcal{D}_d^2 = \mathcal{A}_d^2 + 1$. According to Proposition 4.2 the eigenvalues of $\mathcal{A}_d^2$ equal $\mathfrak{l}_{\boldsymbol{q}} = 2\sum_{j=1}^{d} \beta_j (q_j^{(\alpha)} - 1)$. Thus, the eigenvalues of $\mathcal{D}_d^2$ equal $\mathfrak{l}_{\boldsymbol{q}} + 1$ and the associated eigenspaces are the spaces $\mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}$ also given in the proposition.

Since $\mathcal{D}_d^2 = \mathcal{A}_d^2 + 1$, the eigenvalues of $\mathcal{D}_d$ are $\mu_{\boldsymbol{q}}^{\pm} = \pm\sqrt{\mathfrak{l}_{\boldsymbol{q}} + 1}$. The eigenvalues $\mu_{\boldsymbol{q}}^{\pm}$ are called LLs in analogy with the eigenvalues of H_0. We are going now to give an explicit description of the eigenspaces of $\mathcal{D}_d$ (which are some subspaces of $\mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}$).

The equation $\mathcal{D}_d \Phi = \mu_{\boldsymbol{q}}^{\pm}\Phi$, where Φ has the form

$$\Phi(\boldsymbol{\rho}, \boldsymbol{\theta}) = \begin{pmatrix} \Phi^{(+)}(\boldsymbol{\rho}, \boldsymbol{\theta}) \\ \Phi^{(-)}(\boldsymbol{\rho}, \boldsymbol{\theta}) \end{pmatrix}, \tag{4.18}$$

can be written as

$$\begin{pmatrix} 1 & \mathcal{A}_{d-1} + \mathcal{T}_d^* \\ \mathcal{A}_{d-1} + \mathcal{T}_d & -1 \end{pmatrix} \begin{pmatrix} \Phi^{(+)} \\ \Phi^{(-)} \end{pmatrix} = \mu_{\boldsymbol{q}}^{\pm} \begin{pmatrix} \Phi^{(+)} \\ \Phi^{(-)} \end{pmatrix}.$$

Here $\Phi^{(\pm)}$ belong to some subspaces $\mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}^{(\pm)}$. The latter equation can be rewritten as

$$(1 - \mu_{\boldsymbol{q}}^{\pm})\Phi^{(+)} = -(\mathcal{A}_{d-1} + \mathcal{T}_d^*)\Phi^{(-)}, \tag{4.19}$$
$$(1 + \mu_{\boldsymbol{q}}^{\pm})\Phi^{(-)} = (\mathcal{A}_{d-1} + \mathcal{T}_d)\Phi^{(+)}. \tag{4.20}$$

Suppose first that $\mu_{\boldsymbol{q}}^{\pm} \neq \pm 1$. In other words we do not consider the "lowest" LLs of $\mathcal{D}_d$. In this case (4.19) and (4.20) are clearly equivalent: If, say, (4.20) holds then (4.19) follows from (4.13).

Of the spaces $\mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}^{(\pm)}$ at least one is non-empty (and then automatically the second is). Choose $\Phi^{(+)} \in \mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}^{(+)}$ arbitrarily and set

$$\Phi^{(-)} = (1 + \mu_{\boldsymbol{q}})^{-1}(\mathcal{A}_{d-1} + \mathcal{T}_d)\Phi^{(+)}. \tag{4.21}$$

Then the subspace $\mathcal{W}_{\mu_{\boldsymbol{q}}^{\pm}} \subset \mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}$ given by

$$\mathcal{W}_{\mu_{\boldsymbol{q}}^{\pm}} = \left\{ \begin{pmatrix} \Phi^{(+)} \\ \Phi^{(-)} \end{pmatrix} \mid \Phi^{(+)} \in \mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}^{(+)}, \Phi^{(-)} = (1 + \mu_{\boldsymbol{q}}^{\pm})^{-1}(\mathcal{A}_{d-1} + \mathcal{T}_d)\Phi^{(+)} \right\}$$

is the eigenspace corresponding to the eigenvalue $\mu_{\boldsymbol{q}}^{\pm}$ of $\mathcal{D}_d$. Thus, in $\mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}$ we find two eigenspaces corresponding to the eigenvalues $\mu_{\boldsymbol{q}}^{\pm} = \pm\sqrt{\mathfrak{l}_{\boldsymbol{q}} + 1}$, respectively. It is easy to see that $\mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}} = \mathcal{W}_{\mu_{\boldsymbol{q}}^+} \oplus \mathcal{W}_{\mu_{\boldsymbol{q}}^-}$. Furthermore, for $\mu_{\boldsymbol{q}}^{\pm} \neq \pm 1$ and any $\Phi^{(+)} \in \mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}^{(+)}$,

$$\Phi_{\mu_{\boldsymbol{q}}^{\pm}}(\boldsymbol{\rho}, \boldsymbol{\theta}) = \left(\frac{\mu_{\boldsymbol{q}}^{\pm} + 1}{2\mu_{\boldsymbol{q}}^{\pm}}\right)^{1/2} \begin{pmatrix} \Phi^{(+)}(\boldsymbol{\rho}, \boldsymbol{\theta}) \\ (1 + \mu_{\boldsymbol{q}}^{\pm})^{-1}[(\mathcal{A}_{d-1} + T_d)\Phi^{(+)}](\boldsymbol{\rho}, \boldsymbol{\theta}) \end{pmatrix}$$

is a normalized eigenvector associated to the eigenvalue $\mu_{\boldsymbol{q}}^{\pm}$ of $\mathcal{D}_d$. Moreover, the space $\mathcal{Y}^{(+)} \equiv \mathcal{Y}_{\mathfrak{l}_{\boldsymbol{q}}}^{(+)}$ splits into a direct sum of $\varkappa$ subspaces $\mathcal{Y}^{(+,\alpha)}$ corresponding to different representations in (3.9). In view of Proposition 4.1, in its turn, each $\mathcal{Y}^{(+,\alpha)}$ is the (exterior) direct sum of the subspaces $\mathcal{H}_{\Lambda\boldsymbol{\varepsilon}}^{(\alpha)}$ in $L^2(\mathbb{R}^{2d})$, which are nondegenerate Landau eigenspaces of the Schrödinger operator H_0, corresponding to the particular $\boldsymbol{q}^{(\alpha)}$ and $\boldsymbol{\varepsilon}$. Of these subspaces exactly $2^{\nu(\boldsymbol{q}^{(\alpha)})-1}$ are nonempty. According to this description, the functions in $\mathcal{Y}^{(+)}$ have the form

$$\Phi^{(+)}(\boldsymbol{\rho}, \boldsymbol{\theta}) = \sum_{\alpha=1}^{\varkappa} \Phi^{(+,\alpha)}(\boldsymbol{\rho}, \boldsymbol{\theta}), \quad \Phi^{(+,\alpha)} \in \mathcal{Y}^{(+,\alpha)}, \tag{4.22}$$

with

$$\Phi^{(+,\alpha)}(\boldsymbol{\rho}, \boldsymbol{\theta}) = \boxplus_{\boldsymbol{\varepsilon} \in E_+^d} \Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}(\boldsymbol{\rho}, \boldsymbol{\theta}), \quad \Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)} \in \mathcal{H}_{\Lambda\boldsymbol{\varepsilon}}^{(\alpha)}, \tag{4.23}$$

where

$$\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}(\boldsymbol{\rho}, \boldsymbol{\theta}) = \sum_{\boldsymbol{m} \in \mathbb{N}_{\boldsymbol{q}^{(\alpha)}}^d} c_{\boldsymbol{m},\boldsymbol{\varepsilon}} \prod_{j=1}^{d} f_{m_j, q_j^{(\alpha)}}^{(j)}(\rho_j, \theta_j), \tag{4.24}$$

$\mathbb{N}_{\boldsymbol{q}^{(\alpha)}}^d$ is defined in (6.17) and $f_{m_j, q_j^{(\alpha)}}^{(j)} = f_{m_j, q_j^{(\alpha)}}(\rho_j, \theta_j; b_j)$ are given in (3.3).

4.4 The "Lowest" LL of $\mathcal{D}_d$

The symmetry of $\mu_{\boldsymbol{q}}^{\pm}$ breaks down for the lowest LL $\mu_{\boldsymbol{0}}^{\pm} = \pm\sqrt{\mathfrak{l}_{\boldsymbol{0}} + 1} = \pm 1$ corresponding to $\boldsymbol{q} = (1, 1, \ldots, 1)$. Here only one of the numbers $\mathfrak{l}_{\boldsymbol{\varepsilon}}$ is an eigenvalue of $\mathcal{A}_d^2$, according to Proposition 4.1, actually, the one corresponding to $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon_0} = (\varepsilon_j) = (\operatorname{sign} b_j)$, so there is only one nonzero term in the direct sum $\mathcal{Y}_{\mathfrak{l}} = \boxplus_{\boldsymbol{\varepsilon}} \mathcal{H}_{\Lambda\boldsymbol{\varepsilon}} =: \mathcal{H}_{\Lambda\boldsymbol{\varepsilon_0}} \boxplus \left(\boxplus_{\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon_0}} 0\right)$. This term may enter in $\mathcal{W}_{\mu_{\boldsymbol{q}}^+}$ or $\mathcal{W}_{\mu_{\boldsymbol{q}}^-}$, depending on the particular combination of the signs of b_j. Suppose that the former case takes place. Then in (4.19), $\Phi^{(-)}$ equals zero and therefore $(1 - \mu)\Phi^{(+)} = (\mathcal{A}_d + \mathcal{T}_d)\Phi^{(-)} = 0$. Since $\Phi^{(+)} \neq 0$, this means that $\mu = 1$;

therefore the whole $\mathcal{Y}_\mathfrak{l}$ is the eigenspace of $\mathcal{D}_d$ corresponding to the eigenvalue 1, and -1 is not an eigenvalue of $\mathcal{D}_d$. In the opposite case if $\boldsymbol{\varepsilon_0}$ lies in the lower half, $\Phi^{(+)} = 0$ and therefore $(1+\mu)\Phi^{(-)} = (\mathcal{A}_d + \mathcal{T}_d)\Phi^{(+)} = 0$ according to (4.20). Thus, $\mu = -1$ is an eigenvalue, $\mu = 1$ is not an eigenvalue, and $\mathcal{Y}_\mathfrak{l}$ is the eigenspace of $\mathcal{D}_d$ corresponding to $\mu = -1$.

5 INTEGRALS CONTAINING LAGUERRE POLYNOMIALS

For further reference we establish (asymptotic) estimates for certain integrals containing Laguerre polynomials $L_q^{(m)}$ (see (3.4)).

Lemma 5.1. *Assume $V(\rho) = a\chi_{\{\rho<A\}}(\rho)$ for some $a, A > 0$. Let*

$$g_m(\rho) = \rho^m e^{-\frac{b}{4}\rho^2} L_q^{(m)}\left(\frac{b}{2}\rho^2\right), \quad m \in \mathbb{N},$$

and let X_m be the first order differential operator

$$X_m = -i\frac{\partial}{\partial\rho} - \frac{im}{\rho} - \frac{ib}{2}\rho$$

Set

$$R_m = \int_0^\infty V(\rho)|g_m(\rho)|^2\,\rho\,d\rho, \quad S_m = \int_0^\infty V(\rho)|(X_m g_m)(\rho)|^2\,\rho\,d\rho.$$

Then, for some $c > 0$,

$$|\ln R_m| \le c\,m, \quad \ln S_m \le c\,m. \tag{5.1}$$

Proof. The statement for R_m involves estimates from above and from below, the one for S_m only from above. The Laguerre polynomial $L_q^{(m)}(\xi)$ satisfies

$$|L_q^{(m)}(\xi)| = \left|\sum_{l=0}^q d_{l,m}(-\xi)^l\right| \le \sum_{l=0}^q (m+q)^{q-l}\frac{\xi^l}{l!} \le (m+q)^q e^{\xi/(m+q)}. \tag{5.2}$$

Consequently,

$$\begin{aligned} R_m &= a\int_0^A \rho^{2m+1} e^{-\frac{b}{2}\rho^2}\left(L_q^{(m)}\left(\frac{b}{2}\rho^2\right)\right)^2 d\rho \\ &\le a(m+q)^{2q} e^{bA/(m+q)}\frac{A^{2(m+1)}}{2(m+1)}, \end{aligned}$$

which, after taking the logarithm on both sides, gives the required estimate from above for $\ln R_m$. The estimate for S_m goes in the same way, since the differential operator X_m produces only extra factors of order m in the coefficients of the polynomial, so they become negligible after taking the logarithm.

To get the estimate for R_m from below, note, that for $|\xi| < A$, q fixed, the Laguerre polynomial obeys

$$L_q^m(\xi) > d_{0,m}/2 \tag{5.3}$$

provided m is large enough. Indeed, choose m large enough to ensure that $d_{l,m}/d_{l+1,m} = (m+l+1)(l+1)/(q-l) > 4A$. Then the latter clearly implies that

$$L_q^m(\xi) = \sum_{r=0}^{q/2} \left\{ d_{2r,m}\xi^{2r} - d_{2r+1,m}\xi^{2r+1} \right\} \geq d_{0,m} \left(1 - \frac{d_{1,m}}{d_{0,m}}\xi \right) > \frac{d_{0,m}}{2}.$$

From (5.3) it follows that for some $\epsilon > 0$, and for m large enough,

$$R_m \geq a \left(\frac{d_{0,m}}{2} \right)^2 \int_\epsilon^A \rho^{2m+1} e^{-\frac{b}{2}\rho^2}\, d\rho,$$

which, after taking the logarithm, gives the estimate from below. □

Remark 5.2. It follows from Lemma 5.1 that

$$\limsup_{m\to\infty} \frac{\ln S_m}{\ln R_m} \leq 1. \tag{5.4}$$

This estimate plays a key role when we establish the upper estimate in the proof of Proposition 8.1.

We introduce the short-hand notation

$$\boldsymbol{\rho}\, \boldsymbol{d\rho} = \left(\prod_{j=1}^{d} \rho_j d\rho_j \right).$$

The following technical lemma plays a crucial role in Section 7.

Lemma 5.3. *Denote by $f_q^m(\rho)$ the polynomial*

$$f_q^m(\rho) = \rho^m L_q^{(m)}(\rho^2) q!, \tag{5.5}$$

where $L_q^{(m)}$ are the generalized Laguerre polynomials in (3.4). Let $q_j^{(\alpha)}$, $\alpha = 1, \ldots, \varkappa$, $j = 1, \ldots, d$ be a collection of positive integers such that all vectors

$\boldsymbol{q}^{(\alpha)} = (q_1^{(\alpha)}, \dots, q_d^{(\alpha)})$ *are different, and let M be one more positive integer. Then there exists a function $\gamma(\boldsymbol{m})$, $\boldsymbol{m} = (m_1, \dots, m_d)$, of polynomial growth such that, for any set of coefficients $c^{(\alpha)}$, $\alpha = 1, \dots, \varkappa$,*

$$\int_{[0,M^{-1}]^d} \left| \sum_{\alpha=1}^{\varkappa} c^{(\alpha)} \prod_{j=1}^{d} f^{m_j}_{q_j^{(\alpha)}}(\rho_j) \right|^2 \boldsymbol{\rho}\, \boldsymbol{d\rho}$$

$$\geq \gamma(\boldsymbol{m})^{-1} \int_{[0,M^{-1}]^d} \sum_{\alpha=1}^{\varkappa} |c^{(\alpha)}|^2 \left(\prod_{j=1}^{d} f^{m_j}_{q^{(\alpha)}}(\rho_j) \right)^2 \boldsymbol{\rho}\, \boldsymbol{d\rho}. \tag{5.6}$$

Remark 5.4. Estimate (5.6) measures how far are the functions

$$F^{(\alpha)}(\rho_1, \dots, \rho_d) = \prod_{j=1}^{d} f^{m_j}_{q_j^{(\alpha)}}(\rho_j), \ \ \alpha = 1, \dots, \varkappa. \tag{5.7}$$

from being linearly dependent.

Proof. We reduce the inequality in (5.6) to an estimate for the eigenvalues of some matrix. For fixed $\alpha, \beta, q_j^{(\alpha)}, q_j^{(\beta)}$ we study the dependence of integrals on $\boldsymbol{m}$. For this, consider the expression

$$G(\boldsymbol{m})^{(\alpha,\beta)} = \int_{[0,M^{-1}]^d} F^{(\alpha)}(\boldsymbol{\rho}) F^{(\beta)}(\boldsymbol{\rho})\, \boldsymbol{\rho}\, \boldsymbol{d\rho}, \tag{5.8}$$

where $F^{(\alpha)}$ are defined in (5.7).

From (3.4) and (5.5) it can be observed that for $m_j \geq 0$, $f_q^{m_j}(\rho)$ is ρ^{m_j} times a polynomial in ρ variable, of degree $2q$. The coefficients of this polynomial are integer, moreover they are polynomials in m_j of degree no higher than q, again with integer coefficients:

$$\begin{pmatrix} q + m_j \\ q - l \end{pmatrix} \frac{1}{l!} q! = \begin{pmatrix} q \\ l \end{pmatrix} (m_j + q)(m_j + q - 1) \cdots (q + m_j - l + 1).$$

Multiplication of several functions $f^{m_j}_{q_j^{(\alpha)}}$ preserves this structure. Thus, $F^{(\alpha)}(\boldsymbol{\rho})$ $F^{(\beta)}(\boldsymbol{\rho})$, as a function of ρ_j, equals $\rho_j^{2m_j}$ times a polynomial of degree $2(q_j^{(\alpha)} + q_j^{(\beta)})$ with integer coefficients, being polynomials of degree not higher than $(q_j^{(\alpha)} + q_j^{(\beta)})$ in m_j. When we integrate $F^{(\alpha)}(\boldsymbol{\rho}) F^{(\beta)}(\boldsymbol{\rho}) \prod_{j=1}^{d} \rho_j$ over $[0, M^{-1}]^d$, then for each j we get an expression of the form

$$\int_0^{M^{-1}} f^{m_j}_{q_j^{(\alpha)}} f^{m_j}_{q_j^{(\beta)}} \rho_j \, d\rho_j = \sum_{l=0}^{2(q_j^{(\alpha)} + q_j^{(\beta)})} \frac{M^{-2m_j - 2 - l}}{-2m_j - 2 - l} Z_j^{(\alpha,\beta)}(m_j),$$

where $Z_j^{(\alpha,\beta)}$ is a polynomial of degree not higher than $q_j^{(\alpha)}+q_j^{(\beta)}$ with integer coefficients. Therefore

$$\int_{[0,M^{-1}]^d} F^{(\alpha)}(\boldsymbol{\rho})F^{(\beta)}(\boldsymbol{\rho})\left(\prod_{j=1}^d \rho_j\, d\rho_j\right) = M^{p(\boldsymbol{m})}\prod_{j=1}^d \frac{P_j^{(\alpha,\beta)}(m_j)}{Q_j^{(\alpha,\beta)}(m_j)},$$

where $p(\boldsymbol{m}) = -2\sum_{j=1}^d m_j - 2\sum_{j=1}^d (q_j^{(\alpha)}+q_j^{(\beta)}) - 2d$, $P_j^{(\alpha,\beta)}(m_j)$ is a polynomial in m_j, with integer coefficients, having degree at most $3(q_j^{(\alpha)}+q_j^{(\beta)})-1$, and $Q_j^{(\alpha,\beta)}(m_j) = \prod_{l=0}^{2(q_j^{(\alpha)}+q_j^{(\beta)})}(-2m_j-2-l)$, whence $Q_j^{(\alpha,\beta)}(m_j)$ has no positive integer zeros. It is convenient to define $P^{(\alpha,\beta)}(\boldsymbol{m}) = \prod_{j=1}^d P_j^{(\alpha,\beta)}(m_j)$ and $Q^{(\alpha,\beta)}(\boldsymbol{m}) = \prod_{j=1}^d Q_j^{(\alpha,\beta)}(m_j)$ such that $P^{(\alpha,\beta)}(\boldsymbol{m})/Q^{(\alpha,\beta)}(\boldsymbol{m})$ is a rational function. Note also that $G^{(\alpha,\alpha)}(\boldsymbol{m})$ is always positive.

Now introduce the quantities

$$\tau^{(\alpha)} = c^{(\alpha)}(G^{(\alpha,\alpha)}(\boldsymbol{m}))^{1/2},$$

$$\boldsymbol{M}_{\alpha,\beta} = G^{(\alpha,\beta)}(\boldsymbol{m})(G^{(\alpha,\alpha)}(\boldsymbol{m})G^{(\beta,\beta)}(\boldsymbol{m}))^{-1/2}.$$

(Note that factors, exponential in $\boldsymbol{m}$, cancel in $\boldsymbol{M}_{\alpha,\beta}$). Then the expression on the left-hand side of (5.8) can be written as $\langle \boldsymbol{M}\boldsymbol{\tau},\boldsymbol{\tau}\rangle$, where $\boldsymbol{\tau}$ is the vector composed of $\tau^{(\alpha)}$ and $\boldsymbol{M}$ is the matrix composed of $\boldsymbol{M}_{\alpha,\beta}$. The integral on the right-hand side of (5.6) then becomes $\langle\boldsymbol{\tau},\boldsymbol{\tau}\rangle$. Therefore, to establish (5.6), it is sufficient to estimate the lowest eigenvalue of the matrix $\boldsymbol{M}$.

First of all, note that due to the Schwarz inequality applied to $G^{(\alpha,\beta)}$, all elements of the matrix $\boldsymbol{M}$ are not greater than 1 by absolute value. Therefore, all eigenvalues $\lambda_\alpha(\boldsymbol{M})$ are not greater than $\varkappa$.

Let us evaluate the determinant of the matrix $\boldsymbol{M}$. Although the entries in $\boldsymbol{M}$ contain square roots of rational functions, in the process of calculation we have to consider products of elements taken one in each row and column. Thus, each $(G^{(\alpha)}(\boldsymbol{m}))^{-1/2}$ enters exactly twice in every product. Therefore $\det\boldsymbol{M}$ is a rational function of $\boldsymbol{m}$, viz. $\det\boldsymbol{M} = P(\boldsymbol{m})/Q(\boldsymbol{m})$, where, again P and Q are polynomials in $\boldsymbol{m}$, with degree depending only on $q_j^{(\alpha)}$, and with integer coefficients. The matrix $\boldsymbol{M}$ is nondegenerate; otherwise, we would have $\langle\boldsymbol{M}\boldsymbol{\tau},\boldsymbol{\tau}\rangle = 0$ for some $\boldsymbol{\tau}$, so this would mean that the functions $F^{(\alpha)}$ are linearly dependent which they are not. But since P is a polynomial with integer coefficients, the smallest positive value for $|P|$ is 1. Therefore, $\det\boldsymbol{M} \geq 1/Q(\boldsymbol{m})$. Finally, since $\det\boldsymbol{M} = \prod\lambda_\alpha(\boldsymbol{M})$, and all $\lambda_\alpha(\boldsymbol{M})$ are not greater than $\varkappa$, we get the estimate from below:

$$\lambda_{min}(\boldsymbol{M}) \geq \det\boldsymbol{M}\,\varkappa^{-(\varkappa-1)} \geq \varkappa^{-(\varkappa-1)}Q(\boldsymbol{m})^{-1}.$$

Since $Q(\boldsymbol{m})$ is a polynomial, this gives us the required estimate (5.6) with $\gamma(\boldsymbol{m}) = \varkappa^{\varkappa-1}Q(\boldsymbol{m})$. $\square$

6 TOEPLITZ OPERATORS RELATED TO THE SCHRÖDINGER OPERATOR, I

The study of eigenvalues of the Schrödinger and Dirac operators, $H_0 \pm V$ and $\mathcal{D}_d \pm V$, $V \geq 0$, near a fixed LL will be based on results on the eigenvalue asymptotics of the Toeplitz operator of the form $T = PVP$, where $P = P_\Lambda$ is the projection onto the eigenspace of the unperturbed operator corresponding to the chosen LL. In Section 8 the results for the Dirac operator will be deduced from the ones for the Schrödinger operator.

According to (2.3)-(2.4), the distribution function $n_+(\lambda, T)$, $\lambda > 0$, associated with T satisfies the relations:

$$n_+(\lambda, T) = \max\dim \left\{\mathcal{L} \subset \mathcal{H}_\Lambda : \langle Vu, u\rangle > \lambda\|u\|^2, u\in\mathcal{L}\backslash\{0\}\right\} \tag{6.1}$$
$$= \min\operatorname{codim} \left\{\mathcal{M} \subset \mathcal{H}_\Lambda : \langle Vu, u\rangle \leq \lambda\|u\|^2, u\in\mathcal{M}\right\}. \tag{6.2}$$

The formulas (6.1)-(6.2) will be systematically used in both directions.

6.1 Two-dimensional Case

In polar coordinates (ρ, θ) we set first

$$V(\rho) = a\chi_{\{\rho<M\}}(\rho) \tag{6.3}$$

for some $a, M > 0$, and consider the Toeplitz-type operator $T_q = P_qVP_q$ in the Hilbert space $\mathcal{H}_{\Lambda_q} = \operatorname{Ran} P_q$, where P_q is the projection onto the eigenspace associated with the qth LL Λ_q of H_0. Henceforth we suppress the index q in T_q, P_q, $\mathcal{H}_{\Lambda_q}$ etc. Clearly, T is a nonnegative, compact operator on $\mathcal{H}$. The following holds for its distribution function $n_+(\lambda, T)$, $\lambda > 0$.

Lemma 6.1. *Let V satisfy (6.3). Then $n_+(\lambda, T) \sim \Xi_1(\lambda)$ as $\lambda \downarrow 0$.*

Proof. We establish a lower and an upper estimate, separately.
Lower estimate. In view of (6.1) it suffices to find a subspace $\mathcal{L}(\lambda)$ of proper dimension for which

$$\langle Vu, u\rangle > \lambda\|u\|^2, \ \forall u \in \mathcal{L}(\lambda)\backslash\{0\}. \tag{6.4}$$

We take as our candidate for $\mathcal{L}(\lambda)$,

$$\mathcal{L}(\lambda) = \operatorname{span}\{f_m(\rho, \theta) \mid m = -q+1, -q+2, \dots, N\}. \tag{6.5}$$

where f_m denote the eigenfunctions in (3.3) and N will be chosen later depending on λ. Any $u \in \mathcal{L}(\lambda)$, $\|u\| = 1$, can be written as $u = \sum_{m=-q+1}^{N} c_m f_m$ with $\sum |c_m|^2 = 1$. Thus, (6.4) takes the form

$$2\pi \int_0^\infty \sum_{m=-q+1}^{N} |c_m|^2 |f_m(\rho,\theta)|^2 V(\rho)\, \rho\, d\rho > \lambda, \tag{6.6}$$

since V is radial and the functions f_m are orthogonal on circles $\rho = c$. The left-hand side of (6.9) is larger than

$$2\pi \min_{m \in \{-q+1,-q+2,\dots,N\}} \int_0^\infty |f_m(\rho,\theta)|^2 V(\rho)\, \rho\, d\rho. \tag{6.7}$$

Thus, for λ small, we have to quarantee that

$$-\ln\left(\int_0^\infty |f_m(\rho,\theta)|^2 V(\rho)\, \rho\, d\rho\right) < |\ln\lambda|$$

or, using the notation in Lemma 5.1,

$$-\ln\left(cR_m[(m+q)!]^{-1}\right) < |\ln\lambda|. \tag{6.8}$$

From Lemma 5.1 and Stirling's formula we see that the left-hand side of (6.8) can be estimated from above by

$$-m\ln m - C_1 m + C_2 \ln m \tag{6.9}$$

for positive constants C_1 and C_2. Therefore, for λ small, since the second and third terms in (6.9) are negligible compared to the leading one, (6.8) holds as soon as

$$(1+\epsilon')m\ln m < |\ln\lambda|.$$

Thus, for an arbitrary $\epsilon' > 0$, we can take $N = (1-\epsilon')|\ln\lambda|/(\ln|\ln\lambda|) = (1-\epsilon')\Xi_1(\lambda)$ in (6.5) and, consequently, as $\lambda \downarrow 0$,

$$n_+(\lambda, T) \geq \dim \mathcal{L}_q(\lambda) \sim \Xi_1(\lambda)(1+o(1)).$$

Upper estimate. Here we use (6.2). As our candidate for the subspace $\mathcal{M}(\lambda)$, we take

$$\mathcal{M}(\lambda) = \operatorname{span}\{f_m(\rho,\theta) \mid m = N_0+1, N_0+2, \dots\}. \tag{6.10}$$

where, as usual, f_m denote the eigenfunctions (3.3) and $N_0 = N_0(\lambda)$ will be chosen below. For $u \in \mathcal{M}(\lambda)$, $\|u\| = 1$, $u = \sum_{m=N_0+1}^{\infty} c_m f_m$, the inequality (6.4) reduces to

$$2\pi \int_0^\infty \sum_{m=N_0+1}^{\infty} |c_m|^2 |f_m(\rho,\theta)|^2 V(\rho)\, \rho\, d\rho \leq \lambda. \tag{6.11}$$

Using Lemma 5.1, the left-hand side of (6.11) is majorized by

$$2\pi \max_{m>N_0} \int_0^\infty |f_m(\rho,\theta)|^2 V(\rho)\, \rho\, d\rho = \max_{m>N_0} \left(cR_m[(m+q)!]^{-1} \right) \tag{6.12}$$

Thus, to quarantee (6.11), it is sufficient to have

$$\ln(m+q)! - \ln R_m - C > |\ln \lambda| \tag{6.13}$$

Due to Lemma 5.1, again, like in the estimate from below, the second and third terms in (6.13) are negligible compared with the leading one. After applying Stirling's formula, we come to

$$m \ln m - C_1 m - C_2 \ln m > |\ln \lambda|,$$

which can be obtained by taking $N_0 = (1+\epsilon'')|\ln\lambda|/(\ln|\ln\lambda|)$ in (6.10). Consequently, as $\lambda \downarrow 0$, we obtain the upper estimate

$$n_+(\lambda, T) \leq \operatorname{codim} \mathcal{M}_q(\lambda) \sim \Xi_1(\lambda)\,(1+o(1))\,, \quad \lambda \downarrow 0. \tag{6.14}$$

$\square$

Lemma 6.1 enables us to establish the following proposition.

Proposition 6.2. *Let V satisfy Assumption 1.1. Then*

$$n_+(\lambda, T) \sim \Xi_1(\lambda) \;\; \textit{as } \lambda \downarrow 0. \tag{6.15}$$

Proof. From Lemma 6.1 we know that the statement holds for the particular V in (6.3). To verify the general statement, we first make the usual shift of variables so that V becomes positive in a neighborhood of origin. Next, we can clearly find two functions $V_\pm \geq 0$ having the form in (6.3), with compact support, such that $V_- \leq V \leq V_+$. Then the variational principle implies that $n_+(\lambda, T(V_-)) \leq n_+(\lambda, T(V)) \leq n_+(\lambda, T(V_+))$ and since $n_+(\lambda, T(V_\pm))$ obeys the asymptotics (6.15) in view of Lemma 6.1, the same is true for $n_+(\lambda, T(V))$. $\square$

6.2 Even-dimensional Case for a Simple LL

A point in $\mathbb{R}^{2d}$, $d \in \mathbb{N}$, is represented by $(\boldsymbol{\rho}, \boldsymbol{\theta}) = (\rho_1, \dots, \rho_d, \theta_1, \dots, \theta_d)$. Initially we choose the potential

$$V(\boldsymbol{\rho}, \boldsymbol{\theta}) = V(\boldsymbol{\rho}) = a \prod_{j=1}^{d} V_j(\rho_j) = a \prod_{j=1}^{d} \chi_{\{\rho_j < M\}}(\rho_j) \tag{6.16}$$

for some $a, M > 0$, and consider the Toeplitz-type operator $T_{\boldsymbol{q}} = P_{\boldsymbol{q}} V P_{\boldsymbol{q}}$ in the Hilbert space $\mathcal{H}_{\Lambda_{\boldsymbol{q}}} = \operatorname{Ran} P_{\boldsymbol{q}}$, where $P_{\boldsymbol{q}}$ is the projection onto the eigenspace associated with the LL $\Lambda_{\boldsymbol{q}}$ of H_0.

Throughout this section we assume that the multiplicity of $\Lambda_{\boldsymbol{q}}$ is *one*.

In the sequel we suppress the index $\boldsymbol{q}$ in $T_{\boldsymbol{q}}$, $P_{\boldsymbol{q}}$, $\mathcal{H}_{\Lambda_{\boldsymbol{q}}}$ etc., and consider the distribution function $n_+(\lambda, T)$, $\lambda > 0$. For this purpose we introduce the set

$$\mathbb{N}_{\boldsymbol{q}}^d = \{(m_1, \dots, m_d) \,|\, m_j > -q_j + 1\,\}. \tag{6.17}$$

When V is chosen as in (6.16) and P_j denotes the operator in $L^2(\mathbb{R}^{2d})$, which acts as P in the variables (ρ_j, θ_j), then it is clear that the eigenvalues of T are products of the eigenvalues $\nu_{m_j}^{(j)} := \nu_{m_j}(T_j)$ of $T_j = P_j V_j P_j$, and the eigenfunctions of T are (tensor) products of the eigenfunctions of T_j. Hence,

$$n_+(\lambda, T) = \# \left\{ \boldsymbol{m} \in \mathbb{N}_{\boldsymbol{q}}^d \,|\, \prod_{j=1}^{d} \nu_{m_j}^{(j)} > \lambda \right\}. \tag{6.18}$$

We are going to estimate (6.18) from below and above, respectively. A key ingredient is the following number theoretical result.

Lemma 6.3. *Let $\boldsymbol{m} \in \mathbb{N}^d$ and $\lambda > 0$. Then*

$$\# \left\{ \boldsymbol{m} \in \mathbb{N}^d \,|\, \prod_{i=1}^{d} m_j^{-m_j} > \lambda \right\} \sim \Xi_d(\lambda) \quad \text{as } \lambda \downarrow 0.$$

The proof of Lemma 6.3 is given in the Appendix. Having the latter lemma at our disposal we are ready to establish the following result.

Proposition 6.4. *Let V be chosen as in (6.16). Then*

$$n_+(\lambda, T) \sim \Xi_d(\lambda) \text{ as } \lambda \downarrow 0. \tag{6.19}$$

Proof. Lemma 6.1 gives us the result for $d = 1$. We will use induction in d. Thus, we assume that the statement holds for all $d^{'} < d$ and verify it for d. From the induction hypothesis follows that one may painlessly replace $\mathbb{N}^d_{\boldsymbol{q}}$ by $\mathbb{N}^d$ in (6.18) with change of order only $O(\Xi_{d-1}(\lambda))$. For $d = 1$ Lemma 6.1 implies that, given $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that

$$\Big| |\ln \nu_m(T^{(1)})| - m \ln m \Big| < \epsilon\, m \ln m, \ \ \forall m > N_0. \tag{6.20}$$

where $\nu_m(T^{(1)})$ are eigenvalues of the operator $T = T^{(1)}$ in this lemma. In the sequel subsets of $\mathbb{I} = \{1, 2, \dots , d\}$ are denoted by $\mathbb{J}$.
Lower estimate. Using (6.20) we have for the right-hand side of (6.18):

$$\begin{aligned}
\#\left\{\boldsymbol{m}\in\mathbb{N}^d \mid \prod_{j=1}^{d}\nu^{(j)}_{m_j} > \lambda\right\} &\geq \#\left\{\boldsymbol{m}\in\mathbb{N}^d \mid m_j > N_0, j\in\mathbb{J}, \sum_{j=1}^{d} m_j \ln m_j > \frac{\ln\lambda}{1-\epsilon}\right\} \\
&\geq \#\left\{\boldsymbol{m}\in\mathbb{N}^d \mid \sum_{j=1}^{d} m_j \ln m_j > \frac{\ln\lambda}{1-\epsilon}\right\} \\
&- \sum_{j^{'}} \#\left\{\boldsymbol{m}\in\mathbb{N}^d \mid m_{j^{'}} \leq N_0, \sum_{j=1}^{d} m_j \ln m_j > \frac{\ln\lambda}{1-\epsilon}\right\} \\
&=: F_1(\lambda) - \sum_{j^{'}} F_{2,j^{'}}(\lambda).
\end{aligned}$$

Consider $F_{2,j^{'}}(\lambda)$ for, say, $j^{'} = 1$. We have, due to $m_1 \ln m_1 \leq N_0 \ln N_0$ and the induction hypothesis that, as $\lambda \downarrow 0$,

$$\begin{aligned}
F_{2,1}(\lambda) &\leq N_0 \#\left\{\boldsymbol{m}\in\mathbb{N}^{d-1} \mid \sum_{j=2}^{d} m_j \ln m_j > \frac{\ln\lambda}{1-\epsilon} - N_0 \ln N_0\right\} \\
&\leq N_0 \#\left\{\boldsymbol{m}\in\mathbb{N}^{d-1} \mid \sum_{j=2}^{d} m_j \ln m_j > \frac{\ln\lambda}{1-\epsilon}\right\} \lesssim N_0\, \Xi_{d-1}(\lambda) = o\left(\Xi_d(\lambda)\right).
\end{aligned}$$

The asymptotics for $F_1(\lambda)$ is described by Lemma 6.3, viz. $F_1(\lambda) \sim \Xi_d(\lambda)$ as $\lambda \downarrow 0$, and this shows the lower estimate.
Upper estimate. We have

$$n_+(\lambda, T) = \#\left\{\boldsymbol{m}\in\mathbb{N}^d \mid \prod_{j=1}^{d}\nu^{(j)}_{m_j} > \lambda\right\} = \sum_{\mathbb{J}\subset\mathbb{I}} F_{\mathbb{J}}(\lambda),$$

where

$$F_{\mathbb{J}}(\lambda) = \#\left\{\boldsymbol{m}\in\mathbb{N}^d \mid \prod_{j=1}^{d}\nu^{(j)}_{m_j} > \lambda, m_j \leq N_0 \text{ for } j\in\mathbb{J}, m_j > N_0 \text{ for } j\notin\mathbb{J}\right\}.$$

For every $\mathbb{J} \neq \emptyset$ (i.e. $1 \leq |\mathbb{J}| = d^{'} \leq d$) we estimate $F_{\mathbb{J}}(\lambda)$ from above:

$$F_{\mathbb{J}}(\lambda) = \# \left\{ \boldsymbol{m} \in \mathbb{N}^d \mid \sum_{j=1}^{d} \ln \nu_{m_j}^{(j)} > \ln \lambda, m_j \leq N_0, j \in \mathbb{J}, m_j > N_0, j \notin \mathbb{J} \right\}$$
$$\leq N_0^{d^{'}} \# \left\{ \boldsymbol{m} \in \mathbb{N}^{d-d^{'}} \mid \sum_{j \notin \mathbb{J}} \ln \nu_{m_j}^{(j)} > \ln \lambda \right\}.$$

For the last quantity the $d - d^{'}$ dimensional version of our induction hypothesis applies:

$$\# \left\{ \boldsymbol{m} \in \mathbb{N}^{d-d^{'}} \mid \sum_{j \notin \mathbb{J}} \ln \nu_{m_j}^{(j)} > \ln \lambda, \right\} \sim \Xi_{d-d^{'}}(\lambda) = o\left(\Xi_d(\lambda)\right) \text{ as } \lambda \downarrow 0.$$

Finally, we consider $\mathbb{J} = \emptyset$. Due to (6.20) we have that

$$F_{\mathbb{J}}(\lambda) = \# \left\{ \boldsymbol{m} \in \mathbb{N}^d \mid m_j > N_0, \prod_{j=1}^{d} \nu_{m_j}^{(j)} > \lambda \right\}$$
$$= \# \left\{ \boldsymbol{m} \in \mathbb{N}^d \mid m_j > N_0, \sum_{j=1}^{d} \ln \nu_{m_j}^{(j)} > \ln \lambda \right\}$$
$$\leq \left\{ \boldsymbol{m} \in \mathbb{N}^d \mid \sum_{j=1}^{d} m_j \ln m_j > \frac{\ln \lambda}{1+\epsilon} \right\} \sim \Xi_d(\lambda), \text{ as } \lambda \downarrow 0, \qquad (6.21)$$

where the latter asymptotics follows from Lemma 6.3. □

Proposition 6.5. *Let V satisfy Assumption 1.1. Let $T = P_{\boldsymbol{q}} V P_{\boldsymbol{q}}$ be the Toeplitz operator associated with the LL $\Lambda_{\boldsymbol{q}}$ with multiplicity one. Then*

$$n_{+}(\lambda, T) \sim \Xi_d(\lambda) \text{ as } \lambda \downarrow 0. \qquad (6.22)$$

Proof. The proof is similar to the proof of Proposition 6.2. □

To treat the Dirac operator in Section 8 we will need a slightly stronger version of the upper estimate in Proposition 6.4.

Fix some index j, say $j = i$. Suppose that τ_m is a sequence such that

$$-\ln \tau_m \gtrsim -\ln \nu_m^{(i)} \text{ as } m \to \infty. \qquad (6.23)$$

Proposition 6.6. *Under the assumptions of Proposition 6.4,*

$$\#\left\{ \boldsymbol{m} \in \mathbb{N}^d \mid \tau_{m_i} \prod_{j \neq i}^{d} \nu_{m_j}^{(j)} > \lambda \right\} \lesssim \Xi_d(\lambda). \tag{6.24}$$

Proof. The proof follows the lines of the reasoning leading to the upper estimate in the proof of Proposition 6.4, since the only thing we needed there for $\nu_{m_j}^{(j)}$ was the estimate $-\ln \nu_{m_j}^{(j)} \geq (1+\epsilon) m_j \ln m_j$, but from (6.23), the same holds for τ_{m_i}. □

7 TOEPLITZ OPERATORS RELATED TO THE SCHRÖDINGER OPERATOR, II

In the case of dimension $2d$, $d > 1$, multiple LLs can arise for the Schrödinger operator H_0. We use notations introduced in Section 3.2.

7.1 Even-dimensional Case for Multiple LLs

Let V satisfy Assumption 1.1 and let Λ be a LL of H_0 with multiplicity $\varkappa$. The main result of this section is the following Proposition.

Proposition 7.1. *Let V satisfy Assumption 1.1. Let Λ be a LL of H_0 with multiplicity $\varkappa$ and let $P = P_\Lambda$ be the projection onto the eigenspace associated with Λ. Then the eigenvalues of the Toeplitz operator $T = PVP$ obey the asymptotics*

$$n_+(\lambda, T) \sim \varkappa\, \Xi_d(\lambda) \ \text{as}\ \lambda \downarrow 0.$$

Proof. For the representation (3.14), we set $T^{(\alpha)} = P^{(\alpha)} T P^{(\alpha)}$, where $P^{(\alpha)}$ is a projection onto $\mathcal{H}_\Lambda^{(\alpha)}$.

Estimate from above. For given $\lambda > 0$, Proposition 6.5 asserts that we can find subspaces $\mathcal{L}^{(\alpha)} \subset \mathcal{H}_\Lambda^{(\alpha)}$, as in (3.14), such that

$$\int_{\mathbb{R}^{2d}} V|u|^2\, dx_1 \cdots dx_d \equiv \int_{\mathbb{R}^{2d}} V|u|^2\, \boldsymbol{dx} \leq \lambda \int_{\mathbb{R}^{2d}} |u|^2\, \boldsymbol{dx}$$

on $\mathcal{L}^{(\alpha)}$ and $\operatorname{codim}\mathcal{L}^{(\alpha)} = n_+(\lambda, T^{(\alpha)}) \sim \Xi_1(\lambda)$, $\lambda \downarrow 0$. Then, on $\mathcal{L} = \bigoplus_{\alpha=1}^{\varkappa} \mathcal{L}^{(\alpha)}$ one has

$$\begin{aligned}\int_{\mathbb{R}^{2d}} V|u|^2\,\boldsymbol{dx} &= \int_{\mathbb{R}^{2d}} V\left|\sum_{\alpha=1}^{\varkappa} u^{(\alpha)}\right|^2 \boldsymbol{dx},\ \ u^{(\alpha)} \in \mathcal{L}^{(\alpha)} \\ &\le \varkappa \int_{\mathbb{R}^{2d}} V \sum_{\alpha=1}^{\varkappa} |u^{(\alpha)}|^2\,\boldsymbol{dx} = \varkappa\,\lambda\|u\|^2. \end{aligned} \tag{7.1}$$

The inequality (7.1) shows that we have found the subspace $\mathcal{L}$ in $\mathcal{H}_\Lambda$ of codimension $\sum_{\alpha=1}^{\varkappa} n_+(\lambda, T^{(\alpha)})$, so that $\langle T_V u, u\rangle \le \varkappa\lambda\|u\|^2$ on $\mathcal{L}$. Therefore, $n_+(\varkappa\lambda, T) \le \sum_\alpha n_+(\lambda, T^{(\alpha)})$ and the estimate from above follows from the asymptotics for $n_+(\lambda, T^{(\alpha)})$ established in Proposition 6.5.
Estimate from below. The proof is based on Lemma 5.3. Equipped with this lemma we establish the lower estimate first for V having a special form and then, using monotonicity as in Section 6.1, we pass to a general V. Fix an integer M and set

$$V(\boldsymbol{\rho}) = a\prod_{j=1}^{d} V_j(\rho_j), \tag{7.2}$$

where V_j is the characteristic function of an interval $[0, \sqrt{2/b_j}M^{-1}]$. According to Proposition 6.4, for such V we can find subspaces $\mathcal{L}^{(\alpha)} \subset \mathcal{H}^{(\alpha)}$ such that

$$\int_{\mathbb{R}^{2d}} V|u|^2\,\boldsymbol{dx} > \lambda \int_{\mathbb{R}^{2d}} |u|^2\,\boldsymbol{dx},\ \ u \in \mathcal{L}^{(\alpha)}, \tag{7.3}$$

and $\dim\mathcal{L}^{(\alpha)} = n_+(\lambda, T^{(\alpha)}) \sim \Xi_d(\lambda)$ as $\lambda \downarrow 0$. Moreover $\mathcal{L}^{(\alpha)}$ consists of functions which are linear combinations of eigenfunctions in $\mathcal{H}^{(\alpha)}$, $u^{(\alpha)} = \sum_{\boldsymbol{m}} c^{(\alpha)}_{\boldsymbol{m}} \phi^{(\alpha)}_{\boldsymbol{m}}$, where $\boldsymbol{m} = (m_1, \dots, m_d)$, and

$$\phi^{(\alpha)}_{\boldsymbol{m}}(\boldsymbol{\rho}, \boldsymbol{\theta}) = \prod_{j=1}^{d} h^{(\alpha)}_{m_j}(\rho_j, \theta_j),$$

where $h^{(\alpha)}_{m_j}(\rho_j, \theta_j)$ are the eigenfunctions from (3.3), the constant factors incorporated into the afore-mentioned constants $c^{(\alpha)}_{\boldsymbol{m}}$, viz.

$$h^{(\alpha)}_{m_j}(\rho_j, \theta_j) = e^{-im_j\theta_j} L^{(m_j)}_{q_j^{(\alpha)}}\left(\frac{b_j}{2}\rho_j^2\right) \rho_j^{m_j} \exp\left(-\frac{b_j}{4}\rho_j^2\right). \tag{7.4}$$

We take a linear combination $u = \sum u^{(\alpha)}$ of the functions $u^{(\alpha)}$ and estimate $\int_{\mathbb{R}^{2d}} V|u|^2\, \boldsymbol{dx}$ from below. There are quite a lot of terms in this product, having the form

$$\int_{\mathbb{R}^{2d}} V u^{(\alpha)} u^{(\beta)}\, \boldsymbol{dx} = \sum_{\boldsymbol{m},\boldsymbol{m}'} c^{(\alpha)}_{\boldsymbol{m}} \overline{c^{(\beta)}_{\boldsymbol{m}'}} \int_{\mathbb{R}^{2d}} V \phi^{(\alpha)}_{\boldsymbol{m}} \overline{\phi^{(\beta)}_{\boldsymbol{m}'}}\, \boldsymbol{dx}. \tag{7.5}$$

However, most of these terms vanish. In fact, if the integer multi-indices $\boldsymbol{m}$ and $\boldsymbol{m}'$ differ just at some position j, the integral (7.5) equals zero because, in polar coordinates, the integral in the θ_j variable vanishes. Hence, only such terms in (7.5) survive for which $\boldsymbol{m} = \boldsymbol{m}'$. Now, after integration in θ variables (and incorporating the resulting 2π into constants $c^{(\alpha)}_{\boldsymbol{m}}$) and the change of variables $\rho_j \mapsto \rho_j\sqrt{b_j/2}$ (again incorporating the resulting constants into $c^{(\alpha)}_{\boldsymbol{m}}$), we obtain that

$$\int_{\mathbb{R}^{2d}} V|u|^2\, \boldsymbol{dx} = \sum_{\boldsymbol{m}} \int_{[0,M^{-1}]^d} V \left| \sum_{\alpha=1}^{\varkappa} c^{(\alpha)}_{\boldsymbol{m}} \prod_{j=1}^{d} L^{(m_j)}_{q_j^{(\alpha)}}(\rho_j)\rho_j^{m_j} \right|^2 e^{-\frac{1}{2}\sum \rho_j^2} \boldsymbol{\rho}\, \boldsymbol{d\rho}.$$

and, after replacing V by the function $W = c_1 \prod_{j=1}^d \exp(\frac{1}{2}\rho_j^2)\chi_{\{\rho_j < M^{-1}\}}(\rho_j)$, with constant c_1 chosen in such a way that $c_1 \exp(\frac{1}{2}M^2) = a$, the latter yields

$$\int_{\mathbb{R}^{2d}} W|u|^2\, \boldsymbol{dx} = \sum_{\boldsymbol{m}} \int_{[0,M^{-1}]^d} \left| \sum_{\alpha} c^{(\alpha)}_{\boldsymbol{m}} \prod_{j=1}^{d} L^{(m_j)}_{q_j^{(\alpha)}}(\rho_j^2)\rho_j^{m_j} \right|^2 \boldsymbol{\rho}\, \boldsymbol{d\rho} \tag{7.6}$$

and therefore

$$\frac{\int_{\mathbb{R}^{2d}} W|u|^2\, \boldsymbol{dx}}{\int_{\mathbb{R}^{2d}} V|u|^2\, \boldsymbol{dx}} \in \left[\exp(-(1/2)M^2), \exp((1/2)M^{-2})\right] \tag{7.7}$$

for any u. Using (7.6), Lemma 5.3, and (7.7), in this order, we find that

$$\int W|u|^2\, \boldsymbol{dx} \geq c_1^{-1} e^{-\frac{1}{2}M^2} \sum_{\alpha=1}^{\varkappa} \sum_{\boldsymbol{m}} \gamma(\boldsymbol{m})^{-1}$$
$$\int V \left| c^{(\alpha)}_{\boldsymbol{m}} \prod_{j=1}^{d} L^{(m_j)}_{q_j^{(\alpha)}}(\rho_j^2)\rho_j^{m_j} e^{-\frac{1}{4}\rho_j^2} \right|^2 \boldsymbol{\rho}\, \boldsymbol{d\rho}. \tag{7.8}$$

In (7.8) $\boldsymbol{m}$ runs over the set of multi-indices $\boldsymbol{m} = (m_1, \dots, m_d)$ such that the corresponding functions $\phi^{(\alpha)}_{\boldsymbol{m}}$ belong to our spaces $\mathcal{L}^{(\alpha)}$. From the estimate from above in the case of a simple LL, it follows that

$$m_j = O\left(\left(\frac{|\ln \lambda|}{\ln|\ln \lambda|}\right)^{d-1}\right),$$

and therefore $\gamma(\boldsymbol{m}) = O(|\ln\lambda|^s)$ for some $s > 0$. The latter, in conjunction with (7.8) and

$$\int V \left| \sum_{\boldsymbol{m}} c_{\boldsymbol{m}}^{(\alpha)} \phi_{\boldsymbol{m}}^{(\alpha)} \right|^2 \boldsymbol{dx} = \int V \sum_{\boldsymbol{m}} |c_{\boldsymbol{m}}^{(\alpha)}|^2 |\phi_{\boldsymbol{m}}^{(\alpha)}|^2 \, \boldsymbol{dx},$$

yields that

$$\int W|u|^2 \, \boldsymbol{dx} \geq c_1^{-1} e^{-\frac{1}{2}M^2} |\ln\lambda|^{-s} \sum_{\alpha=1}^{\varkappa} \int V|u^{(\alpha)}|^2 \, \boldsymbol{dx},$$

or, using (again) (7.7) and afterwards (7.3), we have on $\mathcal{L} = \oplus\mathcal{L}^{(\alpha)}$ that, for some $c > 0$,

$$\begin{aligned} \int V|u|^2 \, \boldsymbol{dx} &\geq c|\ln\lambda|^{-s} \sum_{\alpha=1}^{\varkappa} \int V|u^{(\alpha)}|^2 \, \boldsymbol{dx} \\ &\geq c\lambda|\ln\lambda|^{-s} \sum_{\alpha=1}^{\varkappa} \int |u^{(\alpha)}|^2 \, \boldsymbol{dx} = c\lambda|\ln\lambda|^{-s}\|u\|^2. \end{aligned} \tag{7.9}$$

This means that we have found a subspace of dimension $\sum_{\alpha=1}^{\varkappa} n_+(\lambda, T^{(\alpha)})$, where $\langle Tu, u\rangle \geq c\lambda|\ln\lambda|^{-s}\|u\|^2$, which implies that $n_+(c\lambda|\ln\lambda|^{-s}, T) \geq \sum_{\alpha=1}^{\varkappa} n_+(\lambda, T^{(\alpha)})$. Finally, let $\mu = c\lambda|\ln\lambda|^{-s}$ such that $\lambda \sim c'\mu|\ln\mu|^s$, and apply the asymptotics for $n_+(\lambda, T^{(\alpha)})$ given in Proposition 6.4. This yields the lower estimate for the V in (7.2). Passage to a general V uses monotonicity. $\square$

8 TOEPLITZ OPERATORS RELATED TO THE DIRAC OPERATOR

For the fixed LL $\mu_{\boldsymbol{q}}^{\pm} = \pm\sqrt{\mathfrak{l}_{\boldsymbol{q}} + 1}$ of the Dirac operator $\mathcal{D}_d$ and the potential $\mathbb{V} \equiv VI_{2^d}$ we associate the Toeplitz-type operator $T = T_{\boldsymbol{q}}^{\pm} = P\mathbb{V}P$, where $P = P_{\boldsymbol{q}}^{\pm}$ is the spectral projection onto the corresponding eigenspace of $\mathcal{D}_d$. The spectral asymptotics of T determines the asymptotic behaviour of eigenvalues of $\mathcal{D}_d \pm \mathbb{V}$. Due to Proposition 4.2, there is a one-to-one correspondence between $\mathfrak{l}$ and a LL Λ of the $2d$-dimensional Schrödinger operator H_0 with multiplicity $\varkappa$, so that $\Lambda = \Lambda_{\boldsymbol{q}^{(\alpha)}}$, $\alpha = 1, \dots, \varkappa$. Keeping in mind this correspondence, we *say* that $\mu_{\boldsymbol{q}}^{\pm}$ has multiplicity $\varkappa$.

In the sequel we adopt the latter terminology and, moreover, we often suppress the indices in $T_{\boldsymbol{q}}^{\pm}$ etc.

Proposition 8.1. *Let V satisfy Assumption 1.1 and let $\mu \neq \pm 1$ be a LL of $\mathcal{D}_d$ with multiplicity $\varkappa$. Then the eigenvalues of the Toeplitz operator $T = PVP$ obey the asymptotics*

$$n_+(\lambda, T) \sim \varpi\, \Xi_d(\lambda) \ \textit{as}\ \lambda \downarrow 0, \tag{8.1}$$

where $\varpi = \varpi(\mu)$ is given in (4.17). The same formula with $\varpi = \varpi(\pm 1) = 1$ holds for $\mu = 1$ or $\mu = -1$, depending on which of them is the eigenvalue of $\mathcal{D}_d$.

Proof. We consider $\mu \neq \pm 1$ first. Recall the description of the eigenspaces of $\mathcal{D}_d$ given in Section 4.3, in particular (4.18)-(4.24). Due to (4.18), we have for $\Phi \in \mathcal{W}$,

$$\langle V\Phi, \Phi\rangle = \langle V\Phi^{(+)}, \Phi^{(+)}\rangle + \langle V\Phi^{(-)}, \Phi^{(-)}\rangle. \tag{8.2}$$

We establish a lower and an upper estimate, separately. Again, as before, we first suppose that V has the form (6.16).
Lower estimate. In view of (6.1) we are going to find a subspace $\mathcal{L} = \mathcal{L}(\lambda) \subset \mathcal{W}$ such that

$$\langle V\Phi, \Phi\rangle > \lambda \|\Phi\|^2, \qquad \forall \Phi \in \mathcal{L}(\lambda)\backslash\{0\} \tag{8.3}$$

and

$$\dim \mathcal{L}(\lambda) \gtrsim \varpi\, \Xi_d(\lambda). \tag{8.4}$$

We start by finding a subspace $\mathcal{L}^{(+)}(\lambda)$ in $\mathcal{Y}^{(+)}$ of proper dimension, satisfying

$$\langle V\Phi^{(+)}, \Phi^{(+)}\rangle > \lambda \|\Phi^{(+)}\|^2, \qquad \forall \Phi^{(+)} \in \mathcal{L}^{(+)}(\lambda)\backslash\{0\}. \tag{8.5}$$

The existence of the latter follows from our considerations for the Schrödinger operator. For each $\boldsymbol{\varepsilon} \in \cup_{\alpha=1}^{\varkappa} \mathcal{E}(\boldsymbol{q}^{(\alpha)})$, where $\mathcal{E}(\boldsymbol{q}^{(\alpha)})$ is defined in (4.16), Proposition 7.1 quarantees that we can find a subspace $\mathcal{L}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)$ such that

$$\langle V\Phi_{\boldsymbol{\varepsilon}}^{(+)}, \Phi_{\boldsymbol{\varepsilon}}^{(+)}\rangle > \lambda \|\Phi_{\boldsymbol{\varepsilon}}^{(+)}\|^2, \qquad \forall \Phi_{\boldsymbol{\varepsilon}}^{(+)} \in \mathcal{L}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)\backslash\{0\} \tag{8.6}$$

and

$$\dim \mathcal{L}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda) \gtrsim \varkappa_{\boldsymbol{\varepsilon}}\, \Xi_d(\lambda), \tag{8.7}$$

where $\varkappa_{\boldsymbol{\varepsilon}}$ is the multiplicity of the LL $\Lambda_{\boldsymbol{\varepsilon}}$, defined in (4.15), for the Schrödinger operator H_0 (in particular, $\varkappa_{\boldsymbol{\varepsilon}} = 0$ if $\Lambda_{\boldsymbol{\varepsilon}}$ is *not* a LL for H_0). Taking the exterior direct sum of these $\mathcal{L}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)$ over $\boldsymbol{\varepsilon} \in (\cup_{\alpha=1}^{\varkappa} \mathcal{E}(\boldsymbol{q}^{(\alpha)})) \cap E_+^d$, we get the required subspace $\mathcal{L}^{(+)}(\lambda)$, with dimension $\gtrsim \sum_{\varepsilon} \varkappa_{\varepsilon}\, \Xi_d(\lambda)$. It remains to notice that $\sum_{\varepsilon} \varkappa_{\varepsilon} = \sum_{\alpha} 2^{\nu(\boldsymbol{q}^{(\alpha)})-1} =: \varpi$.

Having $\mathcal{L}^{(+)}(\lambda)$ at our disposal, we set

$$\mathcal{L}(\lambda)=\left\{\, (\Phi^{(+)}, (1+\mu)^{-1}(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)}) \mid \Phi^{(+)} \in \mathcal{L}^{(+)}(\lambda) \,\right\}\subset \mathcal{W}.$$

The dimension of $\mathcal{L}(\lambda)$, obviously, equals the dimension of $\mathcal{L}^{(+)}(\lambda)$. At the same time, on $\mathcal{L}(\lambda)$,

$$\langle V\Phi, \Phi\rangle \geq \langle V\Phi^{(+)}, \Phi^{(+)}\rangle. \tag{8.8}$$

On the other hand, (4.9)-(4.10) and (4.21) imply that

$$\begin{aligned} \|\Phi\|^2 &= \|\Phi^{(+)}\|^2 + \|(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)}\|^2 \\ &= \|\Phi^{(+)}\|^2 + \langle \Phi^{(+)}, (\mathcal{A}_{d-1}+\mathcal{T}_d)^*(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)}\rangle \\ &\leq \|\Phi^{(+)}\|^2 + \Lambda\|\Phi^{(+)}\|^2 \leq C\|\Phi^{(+)}\|^2. \end{aligned} \tag{8.9}$$

Therefore, (8.8)-(8.9) yield that

$$\langle V\Phi, \Phi\rangle \geq C\lambda\|\Phi\|^2, \quad \forall \Phi \in \mathcal{L}(\lambda)\backslash\{0\},$$

which gives the required lower estimate for the asymptotical eigenvalue distribution of T.

Upper estimate. We begin by constructing some special subspaces in the Schrödinger Landau eigenspaces. Recall the structure of the operator $\mathcal{A}_{d-1}+\mathcal{T}_d$, discussed in Section 4.1. It is a $2^{d-1}\times 2^{d-1}$ operator-valued matrix, each entry $X_{\varepsilon',\varepsilon}$, $\varepsilon' \in E^d_+$, $\varepsilon \in E^d_-$, being either a first order differential operator in *one* pair of variables (ρ_j, θ_j) or the zero operator.

We fix some $\varepsilon', \varepsilon$, as well as some α, and consider the subspace $\mathcal{H}^{(\alpha)}_{\Lambda_{\varepsilon}} \subset L^2(\mathbb{R}^{2d})$. If this subspace is nonempty, i.e. $\varepsilon \in \mathcal{E}(\boldsymbol{q}^{(\alpha)})$, the eigenfunctions $\Phi^{(+,\alpha)}_{\varepsilon}(\boldsymbol{\rho}, \boldsymbol{\theta})$ have the form (4.24). We use the short-hand notation $X = X_{\varepsilon',\varepsilon}$, and consider the expression

$$\int V|\Phi^{(+,\alpha)}_{\varepsilon}|^2\, \boldsymbol{dx} + \int V|X\Phi^{(+,\alpha)}_{\varepsilon}|^2\, \boldsymbol{dx}. \tag{8.10}$$

We write out the the second term:

$$\int V|X\Phi^{(+,\alpha)}_{\varepsilon}|^2\, \boldsymbol{dx} = \int V|\sum_{\boldsymbol{m}} Xc_{\boldsymbol{m}} f_{\boldsymbol{m}}|^2\, \boldsymbol{dx}$$

(we suppress all nonessential sub- and superscripts). Now it follows from (4.2)-(4.5) that the differential operator X acting in variables $(\rho_{j_0}, \theta_{j_0})$ on the function $f_{\boldsymbol{m}} = \prod_{j=1}^{d} f_{m_j}^{(j)}(\rho_j, \theta_j) = \prod_j \exp(i m_j \theta_j) \tilde{f}_{m_j}^{(j)}(\rho_j)$ replaces all circular factors $\exp(i m_{j_0} \theta_{j_0})$ by $\exp(i(m_{j_0}+1)\theta_{j_0})$ or all of them by $\exp(i(m_{j_0}-1)\theta_{j_0})$. Each pair of functions $f_{\boldsymbol{m}}$, $f_{\boldsymbol{m}'}$, $\boldsymbol{m} \neq \boldsymbol{m}'$, are orthogonal with respect to $\boldsymbol{\theta}$-integration. The above property of X then implies that $X f_{\boldsymbol{m}}$ and $X f_{\boldsymbol{m}'}$ are $\boldsymbol{\theta}$-orthogonal as well. This enables us to transform the integrals in (8.10) (for V on the form (6.16)),

$$
\begin{aligned}
&\int V|\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}|^2 \,\boldsymbol{dx} + \int V|X\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}|^2 \,\boldsymbol{dx} \\
&= \sum_{\boldsymbol{m}} |c_{\boldsymbol{m}}|^2 \int V|f_{\boldsymbol{m}}|^2 \,\boldsymbol{dx} + \sum_{\boldsymbol{m}} |c_{\boldsymbol{m}}|^2 \int V|X f_{\boldsymbol{m}}|^2 \,\boldsymbol{dx} \\
&= \sum_{\boldsymbol{m}} |c_{\boldsymbol{m}}|^2 \int V(\rho_{j_0}) \left(|f_{m_{j_0}}^{(j_0)}(\rho_{j_0}, \theta_{j_0})|^2 + |X f_{m_{j_0}}^{(j_0)}(\rho_{j_0}, \theta_{j_0})|^2\right) \rho_{j_0}\, d\rho_{j_0} d\theta_{j_0} \\
&\qquad \times \prod_{j \neq j_0} \int V(\rho_j)|f_{m_j}^{(j)}(\rho_j, \theta_j)|^2 \rho_j \, d\rho_j d\theta_j. \qquad (8.11)
\end{aligned}
$$

We denote the first integral on the right-hand side of (8.11) by $\tau_{m_{j_0}}$, and the terms in the product by $\nu_{m_j}^{(j)}$. From Lemma 5.1 (see also Remark 5.2), $-\ln \tau_{m_{j_0}} \gtrsim -\ln \nu_{m_{j_0}}^{(j_0)}$. Now, from Proposition 6.6, it follows that one can find such subset $\mathbb{M}(\lambda)$ of integer vectors $\boldsymbol{m} \in \mathbb{N}_{\boldsymbol{q}}^d$ so that

$$
\tau_{m_{j_0}} \prod_{j \neq j_0} \nu_{m_j}^{(j)} < \lambda, \quad \boldsymbol{m} \notin \mathbb{M}(\lambda) \qquad (8.12)
$$

and $\operatorname{Card} \mathbb{M}(\lambda) \lesssim \Xi_d(\lambda)$. We consider the subspace $\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+,\alpha)}(\lambda)$ in $\mathcal{H}_{\Lambda_{\boldsymbol{\varepsilon}}}^{(\alpha)}$ spanned by $f_{\boldsymbol{m}}$, $\boldsymbol{m} \notin \mathbb{M}(\lambda)$. This subspace has codimension not larger than $\Xi_d(\lambda)(1 + o(1))$, and, in view of (8.12), for any $\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)} \in \mathcal{M}_{\boldsymbol{\varepsilon}}^{(+,\alpha)}(\lambda)$,

$$
\int V|\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}|^2 \,\boldsymbol{dx} + \int V|X\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}|^2 \,\boldsymbol{dx} \leq \lambda \|\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}\|^2. \qquad (8.13)
$$

Next, still for $\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}$ fixed, consider all α involved in the representation of the LL $\Lambda_{\boldsymbol{\varepsilon}}$, and construct, according to the above reasoning, a subspace $\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+,\alpha)}(\lambda)$. Then $\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda) = \oplus_{\alpha=1}^{\varkappa} \mathcal{M}_{\boldsymbol{\varepsilon}}^{(+,\alpha)}(\lambda)$ is a subspace in $\mathcal{Y}_{\boldsymbol{\varepsilon}}^{(+)}$, having there codimension not larger than the sum of codimensions of summands in $\mathcal{Y}_{\boldsymbol{\varepsilon}}^{(+,\alpha)}$, i.e. $\varkappa_{\boldsymbol{\varepsilon}}\, \Xi_d(\lambda)(1 + o(1))$. For any element $\Phi_{\boldsymbol{\varepsilon}}^{(+)} = \sum_{\alpha=1}^{\varkappa} \Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}$ on $\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)$, we have that

$$
\begin{aligned}
&\int V|\Phi_{\boldsymbol{\varepsilon}}^{(+)}|^2 \,\boldsymbol{dx} + \int V|X\Phi_{\boldsymbol{\varepsilon}}^{(+)}|^2 \,\boldsymbol{dx} \\
&= \int V|\sum_{\alpha} \Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}|^2 \,\boldsymbol{dx} + \int V|\sum_{\alpha} X\Phi_{\boldsymbol{\varepsilon}}^{(+,\alpha)}|^2 \,\boldsymbol{dx}. \qquad (8.14)
\end{aligned}
$$

From the latter equality, in combination with the simple inequality $|\sum_{\alpha=1}^{\varkappa} a^{(\alpha)}|^2 \le \varkappa \sum |a^{(\alpha)}|^2$ and (8.13), we get that

$$\int V|\Phi_{\boldsymbol{\varepsilon}}^{(+)}|^2\,\boldsymbol{dx} + \int V|X\Phi_{\boldsymbol{\varepsilon}}^{(+)}|^2\,\boldsymbol{dx} \le \varkappa\,\lambda\|\Phi_{\boldsymbol{\varepsilon}}^{(+)}\|^2. \tag{8.15}$$

Now construct the subspace $\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)$ for each $\boldsymbol{\varepsilon} \in (\cup_{\alpha=1}^{\varkappa}\mathcal{E}(\boldsymbol{q}^{(\alpha)}))\cap E_+^d$ and set $\mathcal{M}^{(+)}(\lambda) = \boxplus_{\boldsymbol{\varepsilon}\in(\cup_{\alpha=1}^{\varkappa}\mathcal{E}(\boldsymbol{q}^{(\alpha)}))\cap E_+^d}\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)$ with codimension in $\mathcal{Y}^{(+)} = \boxplus\mathcal{Y}_{\boldsymbol{\varepsilon}}^{(+)}$ not larger than the sum of codimensions of $\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)$ in $\mathcal{Y}_{\boldsymbol{\varepsilon}}^{(+)}$, i.e.

$$\begin{aligned}\operatorname{codim}\mathcal{M}^{(+)}(\lambda) &= \sum_{\boldsymbol{\varepsilon}}\operatorname{codim}\mathcal{M}_{\boldsymbol{\varepsilon}}^{(+)}(\lambda)\\ &\lesssim \sum_{\boldsymbol{\varepsilon}}\varkappa_{\boldsymbol{\varepsilon}}\,\Xi_d(\lambda)(1+o(1)) = \varpi\,\Xi_d(\lambda)(1+o(1)).\end{aligned}$$

At the same time, due to (8.15) and the estimate

$$|\sum_{\boldsymbol{\varepsilon}} X_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}}\Phi_{\boldsymbol{\varepsilon}}^{(+)}|^2 \le 2^d\sum_{\boldsymbol{\varepsilon}}|X_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}}\Phi_{\boldsymbol{\varepsilon}}^{(+)}|^2,$$

any $\Phi^{(+)} = \boxplus_{\boldsymbol{\varepsilon}\in(\cup_{\alpha=1}^{\varkappa}\mathcal{E}(\boldsymbol{q}^{(\alpha)}))\cap E_+^d}\Phi_{\boldsymbol{\varepsilon}}^{(+)} \in \mathcal{M}^{(+)}(\lambda)$ satisfies

$$\begin{aligned}&\langle V\Phi^{(+)},\Phi^{(+)}\rangle + \langle V(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)},(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)}\rangle\\ &\quad\le 2^d\sum_{\boldsymbol{\varepsilon}'}\left(\sum_{\boldsymbol{\varepsilon}}\langle V\Phi_{\boldsymbol{\varepsilon}}^{(+)},\Phi_{\boldsymbol{\varepsilon}}^{(+)}\rangle + \sum_{\boldsymbol{\varepsilon}}\langle VX_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\Phi_{\boldsymbol{\varepsilon}}^{(+)},X_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\Phi_{\boldsymbol{\varepsilon}}^{(+)}\rangle\right)\\ &\quad\le 2^d\sum_{\boldsymbol{\varepsilon}'}\left(\sum_{\boldsymbol{\varepsilon}}\varkappa\,\lambda\|\Phi_{\boldsymbol{\varepsilon}}^{(+)}\|^2\right) = 2^d\sum_{\boldsymbol{\varepsilon}'}\varkappa\lambda\|\Phi^{(+)}\|^2\\ &\quad\le 2^d2^d\varkappa\,\lambda\|\Phi^{(+)}\|^2 = C\lambda\|\Phi^{(+)}\|^2.\end{aligned} \tag{8.16}$$

Finally, set

$$\mathcal{M}(\lambda) = \left\{\Phi^{(+)},(1+\mu)^{-1}(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)}\,|\,\Phi^{(+)}\in\mathcal{M}^{(+)}(\lambda)\right\}.$$

Due to (4.18), (4.21), the codimension of $\mathcal{M}^{(+)}(\lambda)$ in $\mathcal{Y}^{(+)}$ equals the codimension of $\mathcal{M}(\lambda)$ in $\mathcal{W}$. On $\mathcal{M}(\lambda)$, moreover, it follows from (8.16) that

$$\begin{aligned}\langle V\Phi,\Phi\rangle &= \langle V\Phi^{(+)},\Phi^{(+)}\rangle + (1+\mu)^{-2}\langle V(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)},(\mathcal{A}_{d-1}+\mathcal{T}_d)\Phi^{(+)}\rangle\\ &\le C\lambda\|\Phi^{(+)}\|^2 \le C\lambda\|\Phi\|^2.\end{aligned}$$

This means that we have constructed the subspace required in (6.2).

For the LL $\mu = 1$ or $\mu = -1$, a much more simple reasoning goes through. Let, say $\mu = -1$ be the eigenvalue of $\mathcal{D}_d$. Then, according to Section 4.4, $\mathcal{A}_{d-1} + \mathcal{T}_d = 0$ on $\mathcal{Y}_0^{(+)}$ and the eigenspace of $\mathcal{D}_d$ coincides with the one for the Schrödinger operator. Consequently, the Toeplitz operators coincide as well, and therefore the results of Section 6.2 apply (since the lowest LL is, obviously, simple). $\square$

9 THE PERTURBED OPERATORS

The proof of asymptotical formulas for eigenvalues goes in the same way for Schrödinger and Dirac operators. Therefore, we let $\mathbb{T}_0$ stand for the unperturbed Schrödinger operator H_0 or the unperturbed Dirac operator $\mathcal{D}_d$, and we let $\mathfrak{l}$ be a LL of $\mathbb{T}_0$ i.e. $\mathfrak{l}$ represents either $\Lambda_{\boldsymbol{q}}$ or $\mu^{\pm}_{\boldsymbol{q}}$.

We will prove the asymptotics for $\mathbb{T}_0 - V$, $V \geq 0$; the proof for $\mathbb{T}_0 + V$ is exactly the same, up to a few changes of signs.

In Propositions 9.1 and 9.2 we give a lower and an upper estimate, respectively, of the asymptotic behaviour of the eigenvalues of $\mathbb{T}_0 - V$ accumulating near each LL $\mathfrak{l}$ of the operator $\mathbb{T}_0$. Together the propositions establish the main result for the perturbed operators, namely Theorems 1.2-1.3.

Proposition 9.1. *Let V satisfy Assumption 1.1 and let $\mathfrak{l}$ be a LL of $\mathbb{T}_0$ with multiplicity $\varkappa$. Let $\mathfrak{l}^{\pm}$ be the neighboring LLs of $\mathbb{T}_0$ lying below and above $\mathfrak{l}$, respectively (or any number smaller than $\mathfrak{l}$ if the latter is the lowest LL of H_0). Then, for $\lambda > 0$, $\mathfrak{s}_1 \in (\mathfrak{l}^-, \mathfrak{l})$ and $\mathfrak{s}_2 \in (\mathfrak{l}, \mathfrak{l}^+)$, the number of eigenvalues for $\mathbb{T}_0 \pm V$ in $(\mathfrak{s}_1, \mathfrak{l} - \lambda)$ or $(\mathfrak{l} + \lambda, \mathfrak{s}_2)$, respectively, satisfies the following inequality as $\lambda \downarrow 0$:*

$$N(\mathfrak{s}_1, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \sim N(\mathfrak{l} + \lambda, \mathfrak{s}_2 \,|\, \mathbb{T}_0 + V) \gtrsim \varrho \, \Xi_d(\lambda), \tag{9.1}$$

where the constant ϱ either equals the multiplicity $\varkappa$ of $\mathfrak{l}$ if $\mathbb{T}_0$ stands for H_0, or the constant ϖ in (4.17) if $\mathbb{T}_0$ stands for $\mathcal{D}_d$. Moreover, (9.1) holds for any nonnegative, bounded V on $\mathbb{R}^{2d}$ which tends to zero at infinity.

Proof. We use a simple perturbation argument based on Lemma 2.2. In [8] an alternative approach is used, based on Weyl inequalities and an analysis of the Birman-Schwinger operator.

Let $\mathfrak{s} = \mathfrak{s}_1$ and let $\mathfrak{g} = \mathfrak{l} - \mathfrak{s}$ be the length of the interval $(\mathfrak{s}, \mathfrak{l} - \lambda)$.
Step 1: Small V. In this step we show the assertion for a function V satisfying $0 \leq V < a$ for some $a > 0$, small enough to ensure that $(1/2) - (a/\mathfrak{g}) > 0$. Introduce the parameters

$$\mu = (\mathfrak{l} - \lambda + \mathfrak{s})/2, \quad \tau = (\mathfrak{l} - \lambda - \mathfrak{s})/2.$$

In view of Lemma 2.1 it suffices to find some subspace $\mathcal{L} \equiv \mathcal{L}_{\boldsymbol{q}}$ for which the condition on the right-hand side of (2.1) is fulfilled for all elements in $\mathcal{L}$. As the candidate for the subspace $\mathcal{L}$ we take the span of some finite set of the Landau eigenfunctions of $\mathbb{T}_0$ corresponding to the LL $\mathfrak{l}$,

$$\mathcal{L} = \operatorname{span}\{\Phi_n(\boldsymbol{\rho}, \boldsymbol{\theta}) \mid n = 1, 2, \dots, N\},$$

where $N = N(\lambda)$ will be chosen later. Any $u \in \mathcal{L}$, $\|u\| = 1$, can be written as $u = \sum_{n=1}^{N} c_n \Phi_n$. Since each Φ_n satisfies $\mathbb{T}_0 \Phi_n = \mathfrak{l} \Phi_n$,

$$\begin{aligned} \|(\mathbb{T}_0 - V - \mu)u\|^2 &= \langle (\mathbb{T}_0 - V - \mu)u, (\mathbb{T}_0 - V - \mu)u \rangle \\ &= (\mathfrak{l} - \mu)^2 \|u\|^2 + \|Vu\|^2 - 2(\mathfrak{l} - \mu)\langle u, Vu \rangle. \end{aligned}$$

Since $\mathfrak{l} - \mu = \tau + \lambda$ and $(\mathfrak{l} - \mu)^2 - \tau^2 = \mathfrak{g}\lambda$, the inequality (2.1) becomes $(2/\mathfrak{g})(\tau + \lambda)\langle u, Vu \rangle - (1/\mathfrak{g})\|Vu\|^2 > \lambda$. The latter is fulfilled if

$$\int \left(\frac{V}{2} - \frac{V^2}{\mathfrak{g}} \right) |u|^2 \left(\prod_{j=1}^{d} \rho_j \, d\rho_j d\theta_j \right) > \lambda, \ \ \|u\| = 1, \tag{9.2}$$

is satisfied. Due to $(1/2) - (a/\mathfrak{g}) > 0$, we have $(V/2) - (V^2/\mathfrak{g}) \geq 0$ and we arrive at the problem of finding a subspace of $u = \sum_{n=1}^{N} c_n \Phi_n$ such that

$$\left\langle W \left(\sum_{n=1}^{N} c_n \Phi_n \right), \left(\sum_{n=1}^{N} c_n \Phi_n \right) \right\rangle > \lambda, \quad W = \frac{V}{2} - \frac{V^2}{\mathfrak{g}} \geq 0,$$

holds for all $N > N_0 = N_0(\lambda)$. This problem is already solved in Section 6-7 or Section 8, respectively, depending on whether $\mathbb{T}_0$ plays the role of H_0 or $\mathcal{D}_d$, with the required estimate for the number of elements in $\mathcal{L}$. In this way we obtain the estimate (9.1) for small V.

Step 2: General V. We assume that V is a nonnegative, bounded function on $\mathbb{R}^{2d}$, which tends to zero at infinity. In particular, V is $\mathbb{T}_0$-compact and, therefore, $\sigma_{ess}(\mathbb{T}_0 - V) = \sigma_{ess}(\mathbb{T}_0)$.

Choose $0 \leq V_0 < \mathfrak{g}/2$ having compact support such that $V - V_0 \geq 0$. The assumptions on V allow us to choose J large enough to ensure that $V_j := (V - V_0)/J$ satisfy $\|V_j\| < \mathfrak{g}/2$, $j = 1, \dots, J$. In addition, $V = \sum_{j=0}^{J} V_j$ and $V_j \geq 0$. We apply Lemma 2.2 as follows. Put

$$\mathbb{T}_1 = \mathbb{T}_0 - V_0 - \sum_{j=2}^{J} V_j, \quad S_1 = V_0 + \sum_{j=2}^{J} V_j - V = -V_1$$

such that $S_1 \leq 0$ and $\gamma_1 := \|S_1\| < \mathfrak{g}/2$. Moreover, let

$$s_1 = -\gamma_1, \quad s_2 = 0, \quad l_1 = \mathfrak{s} + \gamma_1, \quad r_1 = \mathfrak{l} - \lambda.$$

Then Lemma 2.2 yields that

$$N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \geq N(l_1, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V_0 - \sum_{j=2}^{J} V_j)$$

Since

$$N(l_1, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V_0 - \sum_{j=2}^{J} V_j) = N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V_0 - \sum_{j=2}^{J} V_j)$$
$$- N(\mathfrak{s}, l_1 \,|\, \mathbb{T}_0 - V_0 - \sum_{j=2}^{J} V_j)$$

where the last term on the right-hand side is finite (from above the only possible limits points of the eigenvalues for $\mathbb{T}_0 - V$ are the LLs, and here the interval $(\mathfrak{s}, l_1)$ is away from the latter), it follows that

$$N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \geq N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V_0 - \sum_{j=2}^{J} V_j) - C_1$$

for some constant $C_1 > 0$. Then we repeat this argument $J - 1$ times (introducing positive constants $C_2, C_3, \ldots, C_{J-1}$), using Lemma 2.2 on each step in a way similar to the above, and obtain that

$$N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \geq N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V_0 - V_J) - \sum_{j=1}^{J-1} C_j$$

for some constants C_j. Taking the next step in this procedure (applying Lemma 2.2 with $\mathbb{T}_J = \mathbb{T}_0 - V_0$ and $S_J = -V_J$) we obtain that

$$N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \geq N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V_0) - \sum_{j=1}^{J} C_j \tag{9.3}$$

for some constant $C_J > 0$. For the operator $\mathbb{T}_0 - V_0$ the required estimate from below is obtained in Step 1, which proves the proposition. $\square$

Next we establish the upper estimate.

Proposition 9.2. *Let V satisfy Assumption 1.1 and let $\mathfrak{l}$, $\mathfrak{l}^{\pm}$ be the same as in Proposition 9.1. Then, for $\lambda > 0$, $\mathfrak{s}_1 \in (\mathfrak{l}^-, \mathfrak{l})$ and $\mathfrak{s}_2 \in (\mathfrak{l}, \mathfrak{l}^+)$, the number* *of eigenvalues for $\mathbb{T}_0 \pm V$ in $(\mathfrak{s}_1, \mathfrak{l} - \lambda)$ or $(\mathfrak{l} + \lambda, \mathfrak{s}_2)$, respectively, satisfies the following inequality as $\lambda \downarrow 0$:*

$$N(\mathfrak{s}_1, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \sim N(\mathfrak{l} + \lambda, \mathfrak{s}_2 \,|\, \mathbb{T}_0 + V) \lesssim \varrho\, \Xi_d(\lambda), \tag{9.4}$$

where the constant ϱ either equals the multiplicity $\varkappa$ of $\mathfrak{l}$ if $\mathbb{T}_0$ stands for H_0, or the constant ϖ in (4.17) if $\mathbb{T}_0$ stands for $\mathcal{D}_d$.

Proof. Let $\mathfrak{s} = \mathfrak{s}_1$ and set

$$\mu = (\mathfrak{s} + \mathfrak{l} - \lambda)/2, \quad \tau = (\mathfrak{l} - \lambda - \mathfrak{s})/2.$$

We are going to construct such a subspace $\mathcal{M}_\mathfrak{l}$ that

$$\|(\mathbb{T}_0 - V - \mu)u\|^2 \geq \tau^2 \|u\|^2 \tag{9.5}$$

holds for every $u \in \mathcal{M}_\mathfrak{l}$. According to Lemma 2.1, this would mean

$$N(\mathfrak{s}, \mathfrak{l} - \lambda \,|\, \mathbb{T}_0 - V) \leq \operatorname{codim} \mathcal{M}_\mathfrak{l}.$$

Let $\mathcal{H}_\mathfrak{l}$ be the eigenspace associated with the eigenvalue $\mathfrak{l}$ of $\mathbb{T}_0$. Introduce its complement $\mathcal{H}_\mathfrak{l}^{'} = \mathcal{D}(\mathbb{T}_0) \ominus \mathcal{H}_\mathfrak{l}$. Any $u \in \mathcal{D}(\mathbb{T}_0)$ has the form $u = u_1 + u_2$, $u_1 \in \mathcal{H}_\mathfrak{l}$, $u_2 \in \mathcal{H}_\mathfrak{l}^{'}$. Then (9.5) gives

$$\|(\mathbb{T}_0 - V - \mu)(u_1 + u_2)\|^2 = -\tau^2 \|u_1\|^2 - \tau^2 \|u_2\|^2 \geq 0. \tag{9.6}$$

Since $(\mathbb{T}_0 - \mu)u_1 = (\mathfrak{l} - \mu)u_1$ and $\langle u_1, u_2 \rangle = 0$, the left-hand side of (9.6) can be written as follows:

$$[(\mathfrak{l} - \mu)^2 - \tau^2]\|u_1\|^2 + \|(\mathbb{T}_0 - \mu)u_2\|^2 - \tau^2\|u_2\|^2 - 2\mathrm{Re}\,\langle(\mathfrak{l} - \mu)u_1, V(u_1 + u_2)\rangle \\ -2\mathrm{Re}\,\langle(\mathbb{T}_0 - \mu)u_2, Vu_2\rangle - 2\mathrm{Re}\,\langle(\mathbb{T}_0 - \mu)u_2, Vu_1\rangle + \|V(u_1 + u_2)\|^2. \tag{9.7}$$

Take some $\epsilon > 0$ (to be chosen later). From Section 6 or Section 8, depending on whether $\mathbb{T}_0$ stands for H_0 or $\mathcal{D}_d$, we know that there exists a subspace $\mathcal{M}_{1,\mathfrak{l}} \subset \mathcal{H}_\mathfrak{l}$ such that

$$|\langle u_1, Vu_1 \rangle| \leq \lambda\epsilon \|u_1\|^2 \tag{9.8}$$

holds for every $u_1 \in \mathcal{M}_{1,\mathfrak{l}}$ and, moreover,

$$\operatorname{codim} \mathcal{M}_{1,\mathfrak{l}} \sim \varrho\, \Xi_d(\lambda) \text{ as } \lambda \downarrow 0. \tag{9.9}$$

From hereon we suppose that $u_1 \in \mathcal{M}_{1,\mathfrak{l}}$. Next we estimate each of the terms on the right-hand side of (9.7) separately.
Term 1. Clearly,

$$[(\mathfrak{l} - \mu)^2 - \tau^2]\|u_1\|^2 = |\mathfrak{l} - \mathfrak{s}|\lambda \|u_1\|^2. \tag{9.10}$$

Term 2. Since $u_2 \in \mathcal{D}(\mathbb{T}_0) \ominus \mathcal{H}_{\mathfrak{l}}$ we have that

$$\|(\mathbb{T}_0 - \mu)u_2\|^2 \geq [\operatorname{dist}(\mu, \mathfrak{s}(\mathbb{T}_0)\backslash\{\mathfrak{l}\})]^2\|u_2\|^2 \geq [(\mathfrak{s} - \mathfrak{l}^-)^2 + \tau^2]\|u_2\|^2 \quad (9.11)$$

and, therefore,

$$\|(\mathbb{T}_0 - \mu)u_2\|^2 - \tau^2\|u_2\|^2 \geq c_1\|(\mathbb{T}_0 - \mu)u_2\|^2 \quad (9.12)$$
$$\text{where} \quad c_1 = \frac{(\mathfrak{s} - \mathfrak{l}^-)^2}{2(\mathfrak{s} - \mathfrak{l}^+)^2}.$$

Term 3. For any $u_1 \in \mathcal{M}_{1,\mathfrak{l}}$ we have that

$$|2\mathrm{Re}\,\langle(\mathfrak{l} - \mu)u_1, Vu_1\rangle| \leq 2(\mathfrak{l} - \mu)|\langle u_1, Vu_1\rangle| \leq 2(\mathfrak{l} - \mu)\lambda\epsilon\|u_1\|^2, \quad (9.13)$$

due to the Schwarz inequality and (9.8).
Term 4. For any $u_1 \in \mathcal{M}_{1,\mathfrak{l}}$ and $K > 0$ we have, due to (9.8) and (9.11), that

$$\begin{aligned}
|2\mathrm{Re}\,\langle(\mathfrak{l} - \mu)u_1, Vu_2\rangle| &\leq 2|\mathfrak{l} - \mu||\langle Vu_1, u_2\rangle| \leq 2|\mathfrak{l} - \mu|\|Vu_1\|\,\|u_1\| \\
&\leq 2|\mathfrak{l} - \mu|(\lambda\epsilon\|u_1\|^2)^{1/2}\left((\mathfrak{s} - \mathfrak{l}^-)^{-2}\|(\mathbb{T}_0 - \mu)u_2\|^2\right)^{1/2} \\
&= \left(64K\frac{|\mathfrak{l} - \mu|^2}{(\mathfrak{s} - \mathfrak{l}^-)^2}\lambda\epsilon\|u_1\|^2\right)^{1/2}\left(\frac{1}{16K}\|(\mathbb{T}_0 - \mu)u_2\|^2\right)^{1/2} \\
&\leq 128K\frac{|\mathfrak{l} - \mu|^2}{(\mathfrak{s} - \mathfrak{l}^-)^2}\lambda\epsilon\|u_1\|^2 + \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2 \\
&= c_2(K)\lambda\epsilon\|u_1\|^2 + \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2. \qquad (9.14)
\end{aligned}$$

Term 5. The Schwarz inequality yields

$$\begin{aligned}
|2\mathrm{Re}\,\langle(\mathbb{T}_0 - \mu)u_2, Vu_2\rangle| &\leq 2\|(\mathbb{T}_0 - \mu)u_2\|\,\|Vu_2\| \\
&\leq \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2 + 8K\|Vu_2\|^2.
\end{aligned}$$

In the Hilbert space $\mathcal{H}_{\mathfrak{l}}^-$ with norm $|u_2|^2 = \|(\mathbb{T}_0 - \mu)u_2\|^2$, we consider the operator, denoted T, generated by the quadratic form $\|Vu_2\|^2$. This operator is, obviously, compact:

$$\begin{aligned}
n_+(t, T) &= \max\dim\{\mathcal{L} \subset \mathcal{H}_{\mathfrak{l}}^- : \|Vu_2\|^2 > t|u_2|^2\} \\
&\leq \max\dim\left\{\mathcal{M} \subset \mathcal{D}(\mathbb{T}_0) : \|Vu\|^2 > t\|(\mathbb{T}_0 - \mu)u\|^2\right\} < \infty,
\end{aligned}$$

Taking $t = 1/64K^2$ then we may find a subspace $\mathcal{M}_{2,\mathfrak{l}} \subset \mathcal{H}_{\mathfrak{l}}^-$ with finite codimension, say

$$N := \operatorname{codim}\mathcal{M}_{2,\mathfrak{l}} < \infty \quad (9.15)$$

such that $\|Vu_2\|^2 \le \frac{1}{64K^2}\|(\mathbb{T}_0 - \mu)u_2\|^2$ for every $u_2 \in \mathcal{M}_{2,\mathfrak{l}}$.

Therefore, on $\mathcal{M}_{2,\mathfrak{l}} \subset \mathcal{H}_{\mathfrak{l}}^-$ we have

$$\begin{aligned}|2\mathrm{Re}\,\langle(\mathbb{T}_0 - \mu)u_2, Vu_2\rangle| &\le \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2 + 8K\|Vu_2\|^2 \\ &\le \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2 + \frac{8K}{64K^2}\|(\mathbb{T}_0 - \mu)u_2\|^2 = \frac{1}{4K}\|(\mathbb{T}_0 - \mu)u_2\|^2 \quad (9.16)\end{aligned}$$

for every $u_2 \in \mathcal{M}_{2,\mathfrak{l}}$ and, moreover, $\operatorname{codim}\mathcal{M}_{2,\mathfrak{l}} = N < \infty$.

Term 6. For every $u_1 \in \mathcal{M}_{1,\mathfrak{l}}$ and $K > 0$ we have that

$$\begin{aligned}&|2\mathrm{Re}\,\langle(\mathbb{T}_0 - \mu)u_2, Vu_1\rangle| \le 2\|(\mathbb{T}_0 - \mu)u_2\|\,\|Vu_1\| \\ &\le \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2 + 8K\|Vu_1\|^2 \le \frac{1}{8K}\|(\mathbb{T}_0 - \mu)u_2\|^2 + 8K\lambda\epsilon\|u_1\|^2, \quad (9.17)\end{aligned}$$

where, again, we used (9.8).

We substitute the estimates (9.10)-(9.13), (9.16) and (9.17) into (9.7):

$$\begin{aligned}\|(\mathbb{T}_0 - V - \mu)(u_1 + u_2)\|^2 - \tau^2\|u_1\|^2 - \tau^2\|u_2\|^2 &\ge (c_1 - \frac{1}{2K})\|(\mathbb{T}_0 - \mu)u_2\|^2 \\ &+\lambda\left((\mathfrak{l} - \mathfrak{s}) - 2\epsilon(\mathfrak{l} - \mu) - c_2(K)\epsilon - 8K\epsilon\right)\|u_1\|^2.\end{aligned}$$

Take K, $c_1 - 1/2K > 0$ and, afterwards choose ϵ such that

$$\lambda\left((\mathfrak{l} - \mathfrak{s}) - 2\epsilon(\mathfrak{l} - \mu) - c_2(K)\epsilon - 8K\epsilon\right) > 0.$$

This gives us (9.6) for $u \in \mathcal{M}_{1,\mathfrak{l}} \oplus \mathcal{M}_{2,\mathfrak{l}}$. In conclusion, if we take $\mathcal{M}_{\mathfrak{l}} = \mathcal{M}_{1,\mathfrak{l}} \oplus \mathcal{M}_{2,\mathfrak{l}}$ then (9.6) holds for every $u \in \mathcal{M}_{\mathfrak{l}}$ and, in view of (9.9) and (9.15), we clearly have that

$$\begin{aligned}\operatorname{codim}\mathcal{M}_{\mathfrak{l}} &\le \operatorname{codim}\mathcal{M}_{1,\mathfrak{l}} + \operatorname{codim}\mathcal{M}_{2,\mathfrak{l}} \\ &\sim \varrho\,\Xi_d(\lambda) + N \lesssim \varrho\,\Xi_d(\lambda) \text{ as } \lambda \downarrow 0.\end{aligned}$$

□

APPENDIX: LATTICE POINTS INSIDE A CLOSED HYPERSURFACE

We establish Lemma 6.3. Thus, we are interested in the number $N(\mu)$ of solutions $\boldsymbol{n} \in \mathbb{N}^d$ to

$$\sum_{i=1}^{d} n_i \ln n_i < \mu = \ln \lambda^{-1} \quad (9.18)$$

as $\mu \to \infty$ (i.e. as $\lambda \downarrow 0$). In other words, we seek the number of lattice points (with positive integer coordinates) in the region Γ between the x_i-axes, $i = 1, \ldots, d-1$ and the hypersurface

$$C: \quad \sum_{i=1}^{d} x_i \ln x_i = \mu, \quad x_i \geq 1 \text{ for } 1 \leq i \leq d. \tag{9.19}$$

The hypersurface C is "flat" in the following sense.

Lemma 9.3. *There exist positive numbers $b(\mu)$ and $B(\mu)$such that*

$$b(\mu) \leq \sum_{i=1}^{d} x_i \leq B(\mu) \quad \textit{for all points } \boldsymbol{x} \in C, \tag{9.20}$$

where

$$b(\mu) \sim \frac{\mu}{\ln \mu}, \quad B(\mu) \sim \frac{\mu}{\ln \mu} \quad \textit{as } \mu \to \infty. \tag{9.21}$$

Proof. Without loss of generality suppose $x_1 \geq x_2 \geq \cdots \geq x_d$.

Note that on C we have $\mu \geq x_1$, so that $\ln x_i \leq \ln x_1 \leq \ln \mu$. Then we observe that $\mu = \sum_{i=1}^{d} x_i \ln x_i \leq \sum_{i=1}^{d} x_i \ln \mu$. This proves the first inequality in the lemma, with $b(\mu) = \mu/\ln \mu$.

To establish the second inequality in the lemma, consider the case $d = 2$ first. We will show that on C, which is now a curve given by (9.19), the maximum value of x_1+x_2 is attained at the point where $x_1 = x_2$. This follows by implicit differentiation of (9.19), giving $(1+\ln x_1)dx_1+(1+\ln x_2)dx_2 = 0$, so that $dx_2/dx_1 = -(1+\ln x_1)/(1+\ln x_2)$. As we move away from the point on C with $x_1 = x_2$ into the region $x_1 > x_2$, the value of $|dx_2/dx_1|$ increases. Hence $x_1 + x_2$ attains its maximum when $x_1 = x_2$. The dual version of this fact is that if $x_1 + x_2 = B$ is specified then $x_1 \ln x_1 + x_2 \ln x_2$ is minimized by taking $x_1 = x_2$.

To see that an analogous result holds in the d-dimensional case, consider the dual problem of minimizing $\mu = \sum_{i=1}^{d} x_i \ln x_i$ subject to the constraint $\sum_{i=1}^{d} x_i = B$. If any two of the numbers x_i are unequal, $x_1 > x_2$, say, then we can increase x_2 and decrease x_1 by the same amount δ, leaving the constraint fulfilled but reducing $x_1 \ln x_1 + x_2 \ln x_2$, and therefore reducing $\sum_{i=1}^{d} x_i \ln x_i$. Thus the minimum of this sum occurs when all the numbers x_i are the same. Dually, the maximum of $\sum_{i=1}^{d} x_i$ subject to $\sum_{i=1}^{d} x_i \ln x_i = \mu$ occurs where $x_i \ln x_i = \mu/d$ for each i. Then $x_i \sim \mu/d \ln \mu$ as $\mu \to \infty$, so that $\sum_{i=1}^{d} x_i \sim \mu/\ln \mu$. This completes the proof of the lemma. $\square$

Equipped with Lemma 9.3 we are able to establish Lemma 6.3.

Proof of Lemma 6.3. First, we use Lemma 9.3 to estimate the hypervolume V of the region Γ. We find that

$$\frac{b(\mu)^d}{d!} \le V \le \frac{B(\mu)^d}{d!}, \qquad \text{so that} \qquad V \sim \frac{1}{d!}\left(\frac{\mu}{\ln \mu}\right)^d. \tag{9.22}$$

Indeed, the second inequality in (9.20), e.g., is an expression for the hypervolume of a polyhedron for which all the angles at a vertex O are right angles. The value of this hypervolume equals $\frac{B(\mu)^d}{d!}$. Likewise, we determine the hypervolume of the polyhedron associated with the first inequality in (9.20). In this way we obtain (9.22).

To estimate the number $N(\mu)$ of lattice points satisfying (9.18), we attach to each positive integer point $\boldsymbol{n}$ satisfying (9.18) the hypercube given by

$$n_i - 1 < x_i < n_i \qquad \text{when } 1 \le i \le d. \tag{9.23}$$

Next we observe that these hypercubes all lie in the region Γ, so that $N(\mu) \le V$, which establishes the upper bound.

To obtain a lower bound, we proceed as follows. Lemma 9.3 implies that the $N(\mu)$ hypercubes fill the region

$$\sum_{i=1}^{d} x_i < b(\mu) - d. \tag{9.24}$$

Indeed, if a hypercube (9.23) is not counted by $N(\mu)$ then the point $\boldsymbol{n}$ lies outside Γ, so that, in view of Lemma 9.3, $\sum_{i=1}^{d} n_i \ge b(\mu)$. Consequently all the points $\boldsymbol{x} \in \Gamma$ not in one of the $N(\mu)$ hypercubes satisfy $\sum_{i=1}^{d} x_i \ge b(\mu) - d$, so that all $\boldsymbol{x}$ in the hypercubes satisfy (9.24) and $x_i > 0$ when $1 \le i \le \varkappa$.

The region (9.24) has volume $(b(\mu) - d)^d/d!$ (again we compute the hypervolume of a polyhedron for which all the angles at a vertex O are right angles), so that $N(\mu) \ge (b(\mu) - d)^d/d!$. Using Lemma 9.3 one more time completes the verification of the lower bound which, in conjunction with the upper bound, proves the assertion. $\square$

References

[1] Avron, J. E.; Herbst, I.; Simon, B. Schrödinger operators with magnetic fields. I. General interactions. Duke Math. J. **1978**, *45*, 847-883.

[2] Birman, M. Sh.;Solomjak, M. Z. *Spectral theory of selfadjoint operators in Hilbert space*. Mathematics and its Applications (Soviet Series); D. Reidel Publishing Co.: Dordrecht, 1987.

[3] Boyarchenko, S. I.; Levendorski, S. Z. Precise spectral asymptotics for perturbed magnetic Schrödinger operator. J. Math. Pures Appl. (9) **1997**, *76*, no. 3, 211–236.

[4] Helffer, B. *Semi-Classical Analysis for the Schrödinger operator and Applications*. Lecture Notes in Mathematics **1336**; Springer-Verlag: Berlin, 1988.

[5] Hempel, R.; Levendorski, S. Z. On eigenvalues in gaps for perturbed magnetic Schrödinger operators. J. Math. Phys. **1998**, *39*, no. 1, 63–78.

[6] Ivrii, V. *Microlocal analysis and precise spectral asymptotics.* Springer Monographs in Mathematics; Springer-Verlag: Berlin, 1998.

[7] Raikov, G. D. Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips. Comm. Partial Differential Equations **1990**, *15* no. 3, 407–434.

[8] Raikov, G. D.; Warzel, S. Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials. Preprint, ArXiv:Math-ph/0201006.

[9] Reed, M.; Simon, B. *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness*; Academic Press Inc.: London, 1975.

[10] Safarov, Yu.; Vassiliev, D. *The asymptotic distribution of eigenvalues of partial differential operators.* Translated from the Russian manuscript by the authors. Translations of Mathematical Monographs 155; American Mathematical Society: Providence, Rhode Island, 1997.

[11] Sobolev, A.V. Asymptotic behaviour of the energy levels of a quantum particle in a homogeneous magnetic field, perturbed by a decreasing electric field. I. Probl. Mat. Anal. **1984**, *9*, 67-84 [in Russian]; English translation in: J. Sov. Math. **1986**, *35*, 2201–2212.

[12] Solnyshkin, S. N. Asymptotic behavior of the energy of bound states of the Schrödinger operator in the presence of electric and homogeneous magnetic fields, in *Spectral theory. Wave processes*, p 266-278, 302, Probl. Mat. Fiz. **1982**, *10*, Leningrad. Univ., Leningrad [in Russian]; English translation in: Sel. Math. Sov. **1986**, *5*, 297–306.

[13] Tamura, H.; Asymptotic distribution of eigenvalues for Schrödinger operators with homogeneous magnetic fields. Osaka J. Math. **1988**, *25*, 633-647.

[14] Thaller, B. *The Dirac equation.* Texts and Monographs in Physics; Springer-Verlag: Berlin, 1992.