Blow-Up for Schrödinger Equation with Pointwise Nonlinearity

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ABSTRACT. We compare classical results on blow-up for the standard nonlinear Schrödinger equation with some new achievements on the Schrödinger equation in dimension three with a pointwise nonlinearity. We show that the pointlike interaction model reproduce the basic features of the standard NLSE in a simpler and more tractable context. Finally, we show that the analogy does not hold for the two dimensional case, whose structure radically differs from the three dimensional one. In particular, we prove that, in two dimensions, an arbitrarily small nonlinearity power can actually produce the blow-up phenomenon.

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1. Introduction

During the last two decades, blow-up problems have been widely investigated in different contexts (semilinear parabolic and hyperbolic systems, integral equations, nonlinear Schrödinger equation). Correspondingly, the occurrence of the blow-up phenomenon in a mathematical model has been used to describe the emergence of a combustion phenomenon ([**OR**]), the focusing of a laser beam (see [**SS**] and references therein), as well as other emerging processes.

Generally speaking, a solution of a time evolution problem is said to be a blow-up solution if its norm in some suitable space diverges in a finite time. Therefore, in such a study, one is naturally led to consider questions like these: under which conditions does the blow-up occur, which are the norms that actually diverge, what are the permitted rates of explosion and how does the asymptotic profile of a blow-up solution look like, near the explosion time.

Whereas for the heat equation such problems have been essentialy solved ([**FM**], [**GK**]), for the Schrödinger equation only the first two questions have so far been answered in a satisfactory way, even if some noteworthy progresses in the study of the asymptotic profile has been recently accomplished ([**N**]).

Here our intention is to show that some helps in studying these open problems could come from the analysis of the so called nonlinear delta interactions (see [**AT1**], [**AT2**], and the treatise [**AGH-KH**] for an exhaustive study of the linear point interactions). Indeed, these systems turn out to be simpler than the standard

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nonlinear Schrödinger equation (NLSE) since they reduce to a nonlinear integral equation for a complex function of the time variable; nevertheless, in spite of their simplicity, they present a blow-up phenomenon which reproduces some features of its analogue for the NLSE.

As an example, the one dimensional NLSE with critical power nonlinearity is

$$i\partial_t \xi(t,x) = -\partial_x^2 \xi(t,x) - |\xi(t,x)|^4 \xi(t,x) \tag{1.1}$$

whereas the one dimensional Schrödinger equation with critical nonlinear delta interaction is given by ([**AT2**])

$$i\partial_t \eta(t,x) = -\partial_x^2 \eta(t,x) - |\eta(t,0)|^2 \delta_0 \eta(t,x) \tag{1.2}$$

Since for a class of blow-up solutions $\xi(t)$ of equation (1.1) it has been shown that $|\xi(t)|^2 \to \delta_0$ in the measure space ([**NT**]), it appears that, for this family of solutions, equation (1.2) approximates equation (1.1).

Unfortunately, this striking analogy does not hold in higher dimensions. However, it is possible to find further connections between standard NLSE and nonlinear pointwise interactions.

¿From now up to section 2 we will mainly deal with the three dimensional problem. Theorems 2.2 and 2.4 are the subject of a forthcoming paper ([**ADFT2**]), where detailed proofs will be presented. On the other hand, in section 3 we present new results on the two dimensional case.

Now, let us recall some basic facts on the standard NLSE.

The standard NLSE with power nonlinearity reads ([**GV**], [**K**], [**C1**])

$$i\partial_t u(t,\mathbf{x}) = -\Delta u(t,\mathbf{x}) + \kappa |u(t,\mathbf{x})|^\mu u(t,\mathbf{x}) \qquad u(0,\cdot) = u \in H^1(\mathbb{R}^3) \tag{1.3}$$

where κ is real and μ is positive.

The Cauchy problem (1.3) is well-posed if $0 \leq \mu < 4$, and the solution is global in time for any u_0 if either $0 \leq \mu < 4/3$ or $\kappa > 0$ ([**GV**], [**K**]). Mass and energy are preserved quantities, namely

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} = \|u_0\|^2_{L^2(\mathbb{R}^3)} \tag{1.4}$$

$$\mathcal{E}(u(t)) = \|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)} + \frac{\kappa}{\mu+2} \|u(t)\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R}^3)} = \mathcal{E}(u_0) \tag{1.5}$$

The problem of a nonlinear delta interaction placed at the origin of the three dimensional space is conveniently given in its integral form ([**ADFT1**]):

$$\psi(t,\mathbf{x}) = [U(t)\psi_0](\mathbf{x}) + i\int_0^t ds\, U(t-s;\mathbf{x}) q(s) \tag{1.6}$$

$$q(t) + 4\sqrt{\pi i}\gamma \int_0^t ds \frac{|q(s)|^{2\sigma} q(s)}{\sqrt{t-s}} = 4\sqrt{\pi i} \int_0^t ds \frac{[U(s)\psi_0](0)}{\sqrt{t-s}} \tag{1.7}$$

The problem (1.6), (1.7) is well posed if ψ_0 belongs to the space

$$V = \{\psi \in L^2(\mathbb{R}^3), \psi = \phi^\lambda + qG^\lambda, \phi^\lambda \in H^1_{loc}(\mathbb{R}^3), \nabla\phi^\lambda \in L^2(\mathbb{R}^3), \lambda \geq 0, q \in \mathbb{C}\} \tag{1.8}$$

where $G^\lambda = \frac{e^{-\sqrt{\lambda}|\cdot|}}{4\pi|\cdot|}$ is the Green's function of the laplacian. It has also been proved in [**ADFT1**] that the solution $\psi(t)$ satisfies the mild version of the Schrödinger equation with a nonlinear boundary condition. Then,

$$(\phi, i\partial_t \psi(t)) = (\phi, -\Delta\psi(t)) \tag{1.9}$$

for any $\phi \in C_0^\infty(\mathbb{R}^3 \backslash \{0\})$, and

$$\lim_{\mathbf{x}\to 0}\left(\psi(t,\mathbf{x}) - \frac{q(t)}{4\pi|\mathbf{x}|}\right) = \gamma q(t)|q(t)|^{2\sigma} \tag{1.10}$$

Moreover, making use of some estimates in the space V ([**ADFT1**]) together with conservation laws, it has been proven that the solution is global if either $\sigma < 1$ or $\gamma > 0$.
Mass is again a preserved quantity, while the conserved energy has the following expression:

$$E(\psi(t)) = \left\|\nabla\phi^0(t)\right\|^2_{L^2(\mathbb{R}^3)} + \frac{\gamma}{\sigma+1}|q(t)|^{2\sigma+2} = E(\psi_0) \tag{1.11}$$

2. Blow-up Solutions in Dimension Three

It is well known (see e.g. [**C2**]) that a solution of (1.3) is said to blow up if its H^1-norm diverges at a finite time. Moreover, a blow-up alternative holds: either a solution exists at any time, or it blows up.
Here is a list of some well-known results on NLSE ([**C1**], [**C2**], [**M**]):

1. There exists a critical nonlinearity power, namely, the lowest value of μ for which the blow up occurs at least for some solutions. In fact, this power is equal to $\mu_c = 4/3$ ($\mu_c(d) = 4/d$ for a generic space dimension d).
2. For any $\mu \geq \mu_c$, a sufficient condition for an initial datum to produce a blow up solution is to possess negative energy and finite moment of inertia (defined, for a square integrable function u, as the squared L^2-norm of $|\mathbf{x}|u$).
3. In the critical case $\mu = \mu_c$, the system is endowed with some additional symmetry laws ([**NT**]): if $u(t)$ is the solution associated to the initial datum u_0, then
   (a) The function $u_\varepsilon(t,\mathbf{x}) = \varepsilon^{3/2}u(\varepsilon^2 t, \varepsilon\mathbf{x})$ solves problem (1.3) with initial datum $u_{\varepsilon 0}(\mathbf{x}) = \varepsilon^{3/2}u_0(\varepsilon\mathbf{x})$ (dilation symmetry).
   (b) The function

$$u_T(t,\mathbf{x}) = \left(\frac{T}{t+T}\right)^{3/2} e^{i\frac{|\mathbf{x}|^2}{4(t+T)}}\, u\left(\frac{Tt}{t+T}, \frac{T\mathbf{x}}{t+T}\right) \tag{2.1}$$

   solves problem (1.3) with initial datum $u_{T0}(\mathbf{x}) = e^{i\frac{|\mathbf{x}|^2}{4T}}u_0(\mathbf{x})$ (gauge symmetry).
   (c) Composing the trasformation at points (a) and (b) one has

$$u^T_{pc}(t,\mathbf{x}) = \left(\frac{1}{T-t}\right)^{3/2} e^{-i\frac{|\mathbf{x}|^2}{4(T-t)}}\, u\left(\frac{1}{T-t}, \frac{\mathbf{x}}{T-t}\right) \tag{2.2}$$

   solves problem (1.3) with initial datum $u^T_{pc0}(\mathbf{x}) = T^{-3/2}e^{-i\frac{|\mathbf{x}|^2}{4(T)}}u(\frac{1}{T}, \frac{\mathbf{x}}{T})$ (pseudo-conformal symmetry).
4. In the critical case $\mu = \mu_c$, there exists a critical mass $m_c > 0$ defined as the minimal amount of L^2-norm that a solution must possess in order to blow up ([**C2**], [**M**]).
5. Still in the critical power case, there exist solutions which explose in one direction of time only, and also solutions which blow up both forward and backward in time.

Now let us recall the definition of blow-up for the problem (1.6), (1.7) ([**ADFT1**]).

DEFINITION 2.1. A solution $\psi(t)$ of the problem (1.6), (1.7) is called a blow-up solution if the associated function q, which solves equation (1.7), diverges in a finite time.

We remark that conservation of mass and energy makes this condition equivalent to the divergence of the quantity $\left\|\nabla\phi^\lambda(t)\right\|^2_{L^2(\mathbb{R}^3)}$. The result we want to prove is the following:

THEOREM 2.2. *The properties 1., 2., 3., 4. and 5. listed above for the NLSE (1.3), are shared by the model with pointlike nonlinearity (1.6, 1.7).*

We give a sketch of the proof. Details can be found in ([**ADFT2**]).

1. The critical power nonlinearity for the nonlinear delta interaction is given by $\sigma_c = 1$ as a consequence of the stated global existence result (see sec.1) and from point 2.
2. We can prove the following result.

   LEMMA 2.3. *Let* $\psi_0 = \phi_0^\lambda + q_0 G^\lambda$, $\phi_0^\lambda \in H^2(\mathbb{R}^3)$, $|\mathbf{x}|\psi_0 \in L^2(\mathbb{R}^3)$. *Then the moment of inertia* $I(t)$ *of the solution* $\psi(t)$ *is twice differentiable and*

$$\ddot{I}(t) = 8E(\psi_0) + 4\gamma|q(t)|^{2\sigma+2}\left(1 - \frac{2}{\sigma+1}\right) \tag{2.3}$$

   The proof is given in [**ADFT2**].

   Using the above lemma and a modified version of the uncertainty principle we can now state the result on the existence of blow-up solutions.

   THEOREM 2.4. *Let* $\gamma < 0$, $\sigma \geq 1$ *and* $\psi_0 = \phi_0^\lambda + q_0 G^\lambda$, *with* $\phi_0^\lambda \in H^2(\mathbb{R}^3)$, $|\mathbf{x}|\psi_0 \in L^2(\mathbb{R}^3)$, $E(\psi_0) < 0$.
   Then the corresponding solution $\psi(t)$ *blows up in both directions of time.*

   PROOF. Using the identity

$$\|g\|^2_{L^2(\mathbb{R}^3)} = -\frac{2}{3} Re \int_{\mathbb{R}^3} d\mathbf{x}\, \overline{g(\mathbf{x})}\, \mathbf{x} \cdot \nabla g(\mathbf{x}) \tag{2.4}$$

   for $g \in L^2(\mathbb{R}^3)$, $\overline{g}\,\mathbf{x}\cdot\nabla g \in L^1(\mathbb{R}^3)$, the conservation of the L^2-norm and Schwartz's inequality we have

$$\|\psi_0\|^2_{L^2(\mathbb{R}^3)} \;\leq\; \frac{2}{3}\,\||\mathbf{x}|\psi(t)\|_{L^2(\mathbb{R}^3)}\left\|\nabla\phi^\lambda(t)\right\|_{L^2(\mathbb{R}^3)} + \frac{2\sqrt{5\pi}}{3\lambda^{1/4}}|q(t)|\,\|\psi_0\|_{L^2(\mathbb{R}^3)} \tag{2.5}$$

   ¿From lemma 2.3 we know that either there exists $t_c < \infty$ such that $\lim_{t\to t_c} I(t) = 0$, or the solution ceases to exist at a time T such that $\sup_{t\in[0,T]}|I(t)| > 0$.
   In the former case, assume that there is no blow-up in $[0, t_c]$, i.e. $\|\nabla\phi_\lambda(t)\|_{L^2(\mathbb{R}^3)}$ and $|q(t)|$ remains bounded in $[0, t_c]$. Then if we choose λ satisfying the inequality

$$\frac{2\sqrt{5\pi}}{3\lambda^{1/4}} \sup_{t\in[0,t_c]} |q(t)| < \|\psi_0\|_{L^2(\mathbb{R}^3)} \tag{2.6}$$

   and take the limit for $t \to t_c$ in (2.5), we get a contradiction. Therefore the solution blows up in $[0, t_c]$.
   In the latter case, the blow-up alternative guarantees the occurrence of the blow-up.
   The same argument holds for the backward time evolution.

   $\square$

3. The symmetry laws can be directly proved on the system (1.6), (1.7).
4. As shown in [**ADFT1**], the following estimate holds:

$$E(\psi_0) \geq \|\nabla\phi_0(t)\|^2_{L^2(\mathbb{R}^3)} \left(1 - 32\pi^2|\gamma|\, \|\psi_0\|^2_{L^2(\mathbb{R}^3)}\right) \tag{2.7}$$

   The solution cannot blow up if the factor under parentheses is positive. So, $\|\psi_0\|^2_{L^2(\mathbb{R}^3)} \geq (32\pi^2|\gamma|)^{-1}$.
5. The result comes from a trivial application of the gauge symmetry to a solution that blows up both backward and forward in time. The existence of such a solution is guaranteed by point 2.

3. The Two Dimensional Case

In space dimension two the solutions of the Schrödinger equation with pointwise nonlinearity exhibit a qualitative behaviour essentially different from the solutions of the standard NLSE.

Limiting ourselves to the study of the blow up, we only would like to underline that in the problem of a pointwise nonlinearity in dimension two, it is possible to find blow up solutions for any positive value of the nonlinearity power. In other words, there is no critical power, whereas the standard NLSE exhibites a critical behaviour in correspondence to the power $\mu_c(2) = 2$.

The problem can be formulated following the same line of the three dimensional one. For a nonlinear delta interaction with strength γ and power nonlinearity σ, one gets the system

$$\psi(t,\mathbf{x}) \quad = \quad [U_2(t)\psi_0](\mathbf{x}) + i\int_0^t ds\, U_2(t-s;\mathbf{x})q(s) \tag{3.1}$$

$$q(t) \quad + \quad \frac{\pi}{2}\int_0^t ds\,(i + 8\gamma|q(s)|^{2\sigma})q(s)\nu(t-s) \;=\; 4\pi\int_0^t ds[U_2(s)\psi_0](0)\nu(t-s) \tag{3.2}$$

where $U_2(t,\mathbf{x}) = (4\pi i t)^{-1}\exp\left(i\frac{|\mathbf{x}|^2}{4t}\right)$ is the free Schrödinger propagator and $\nu(t) = \int_0^{+\infty} du\, t^{u-1}\Gamma^{-1}(u)$. The usual conservation laws still hold and the energy space is

$$V_2 = \left\{\psi \in L^2(\mathbb{R}^2), \psi = \phi^\lambda + qG_2^\lambda, \phi^\lambda \in H^1(\mathbb{R}^2), \lambda > 0, q \in \mathbb{C}\right\} \tag{3.3}$$

where $G_2^\lambda = (2\pi)^{-1}K_0(\sqrt{\lambda}|\cdot|)$ is the Green's function of the laplacian in dimension two, whereas K_0 is the Macdonald function of order zero. The conservation law of energy reads

$$E_2(\psi(t)) = \left\|\phi^1(t)\right\|^2_{H^1(\mathbb{R}^2)} + \frac{\gamma}{\sigma+1}|q(t)|^{2\sigma+2} - \|\psi_0\|^2_{L^2(\mathbb{R}^2)} = E_2(\psi_0) \tag{3.4}$$

The analogue of lemma 2.3 is given by

LEMMA 3.1. *Let $\psi_0 = \phi_0^\lambda + q_0 G_2^\lambda$, $\phi_0^\lambda \in H^2(\mathbb{R}^2)$, $|\mathbf{x}|\psi_0 \in L^2(\mathbb{R}^2)$. Then the moment of inertia $I_2(t)$ of the solution $\psi(t)$ is twice differentiable and*

$$\ddot{I}_2(t) = 8E_2(\psi_0) + 8\gamma|q(t)|^{2\sigma+2}\frac{\sigma}{\sigma+1} + \frac{2}{\pi}|q(t)|^2 \tag{3.5}$$

We are now ready to show the main result.

THEOREM 3.2. *Given $\gamma < 0$, $\sigma > 0$, let $\psi_0 \in V_2$, with $|\mathbf{x}|\psi_0 \in L^2(\mathbb{R}^2)$ and $E_2(\psi_0) < -\frac{\sigma}{4\pi(\sigma+1)}(4\pi|\gamma|\sigma)^{-1/\sigma}$.*
Then, the solution $\psi(t)$ associated to ψ_0 blows up in both directions of time.

PROOF. First we remark that such a choice for ψ_0 is always possible when $\gamma < 0$, due to the scaling properties of the energy (3.4). Moreover, we observe that the absolute maximum of the function $f(x) = 8\gamma\frac{\sigma}{\sigma+1}|x|^{2\sigma+2} + \frac{2}{\pi}|x|^2$ is given by $f_{max} = \frac{2\sigma}{\pi(\sigma+1)}(4\pi|\gamma|\sigma)^{-1/\sigma}$. Then, from lemma 3.1 we have

$$I_2(t) = 8E_2(\psi_0) + f(|q(t)|) \leq 8E_2(\psi_0) + f_{max} < 0 \tag{3.6}$$

Now, following the line of theorem 2.4 we conclude the proof. $\square$

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