The Time-Dependent Approach to Inverse Scattering*

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Abstract

In these lectures I give an introduction to the time-dependent approach to inverse scattering, that has been developed recently. The aim of this approach is to solve various inverse scattering problems with time-dependent methods that closely follow the physical (and geometrical) intuition of the scattering phenomena. This method has been applied to many linear and nonlinear scattering problems. We first discuss the case of quantum mechanical potential scattering. We give explicit limits for the high-energy behaviour of the scattering operator that offer us formulae for the unique reconstruction of the potential.
Then, we consider the case of the Aharonov-Bohm effect (Schrödinger operators with singular magnetic potentials and exterior domains). This is a particularly interesting inverse scattering problem that shows that in quantum mechanics a magnetic field acts on a charged particle -by means of the magnetic potential- even in regions where it is identically zero.
The key issue for these two problems is that at high-energies translation of the wave packet dominates over spreading during the interaction time.

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In fact, in this limit it is sufficient for the calculation of the scattering operator to consider translation of wave packets rather than their correct free evolution. Finally, we study the nonlinear Schrödinger equation with a potential. In this case, from the scattering operator we uniquely reconstruct the potential and the nonlinearity. For this purpose, we observe that in the small amplitude limit the nonlinear effects become negligible and scattering is dominated by the linear term. Using this idea we prove that the derivative at zero of the nonlinear scattering operator is the linear one. With the aid of this fact we first uniquely reconstruct the potential from the associated linear inverse scattering problem and in a second step we uniquely reconstruct the nonlinearity.

1 Potential Scattering

First we briefly discuss time-dependent direct scattering theory in the particular case of potential scattering for the Schrödinger equation. It is important to keep in mind that physical scattering is a time-dependent phenomenon that studies the interaction of a finite-energy wave packet with a target. Initially, for large negative times, the wave packet is far from the target and since the interaction is very small its evolution is well approximated by an *incoming asymptotic state*, ϕ_-, that propagates according to the free dynamics, with the interaction set to zero. During the interaction time the wave packet is close to the target and, as the interaction is strong, the evolution of the wave packet is given by an *interacting state* that evolves according to the interacting dynamics. But eventually, the wave packet flies away from the target and its evolution is again well approximated by an *outgoing asymptotic state*, ϕ_+, that evolves according to the free dynamics. The scattering operator, S, is the operator that sends ϕ_- to ϕ_+. The aim of scattering experiments is to measure the transition probabilities, $(S\phi, \psi)$.

One objective of the time-dependent approach to inverse scattering theory is to use in an essential way the physical propagation aspects to solve the inverse scattering problem and to obtain mathematical proofs that closely follow physical intuition. It is hoped that a good physical understanding of the inversion mechanisms will be reflected in more transparent mathematical methods. In the stationary (frequency domain) method the physical solution is idealized as a time-periodic solution with infinite energy. By doing so, the propagation aspects of physical scattering are lost. This loss of physics is then reflected in mathematical methods that do not give much information about the physics of the inversion. Moreover, in the stationary method the wave packets (finite-energy solutions) are obtained using a generalized Fourier transfom that integrates over the infinite-energy, time-periodic, solutions. Here the linearity of the direct scattering problem plays an essential role, because it is only in this case that linear combinations of solutions are solutions. On the contrary, the time-dependent approach does not use the linearity of the direct scattering problem in an essential way and, as we will see below, it has a natural extension to the case where the direct scattering problem is nonlinear.

But, let us be more specific. We consider a quantum-mechanical particle in $\mathbf{R}^n$ whose dynamics is described by the Schrödinger equation,

$$i\frac{\partial}{\partial t}\Phi(t,x) = \frac{\mathbf{p}^2}{2m}\Phi(t,x) + V(x)\Phi(t,x),\ \ \Phi(0,x) = \Phi_0(x) \in L^2\left(\mathbf{R}^n\right), \tag{1.1}$$

where $t \in \mathbf{R},\ x \in \mathbf{R}^n, n = 2, \cdots$, and Φ is complex valued, $\mathbf{p} := -i\nabla$ is the momentum operator, and $m > 0$ is the mass of the particle. We take Planck's constant equal to one. The target is given by the potential, V, that is a real-valued function. For simplicity of the presentation, we consider the following class of bounded continuous short-range potentials. For the general case including singular and long-range potentials see [8] and [9].

$$\mathcal{V}_{SR} := \left\{ V \in C\left(\mathbf{R}^n\right) : \sup_{|x|\geq R} |V(x)| \in L^1\left([0,\infty)\right) \right\}. \tag{1.2}$$

Both the free Hamiltonian, $H_0 := \frac{\mathbf{p}^2}{2m}$ and the interacting Hamiltonian, $H := H_0 + V$ are self-adjoint in $L^2\left(\mathbf{R}^n\right)$ with domain the Sobolev space, $W_{2,2}$ [1]. In what follows we denote by $\|\cdot\|$ the norm in $L^2\left(\mathbf{R}^n\right)$. As is well known (see [18] for a general reference in scattering theory) if $V \in \mathcal{V}_{SR}$ -and also under much more general conditions- the wave operators,

$$W_{\pm} := \mathrm{s}\text{ - }\lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}, \tag{1.3}$$

exist and are complete. That is to say, the strong limits in (1.3) exist and their range is equal to the subspace of absolute continuity of H, $\mathcal{H}_{ac}$. Moreover, $W_{\pm}$ are unitary from $L^2\left(\mathbf{R}^n\right)$ onto $\mathcal{H}_{ac}$. These facts give a mathematical basis to the physical description of scattering given above, as we explain now. The solutions to the free Schrödinger equation (1.1) with $V = 0$ are given by $e^{-itH_0}\Phi_-,\ \Phi_- \in L^2\left(\mathbf{R}^n\right)$. They correspond to free particles that travel in space without being perturbed by a potential. Since the potential is localized near zero an incoming particle that is localized at spatial infinity for very large negative times will be -to a good approximation- a solution to the free Schrödinger equation, $e^{-itH_0}\Phi_-$, with *incoming asymptotic configuration* Φ_-. As time goes by, and the incoming solution is near the scattering center (zero) it will feel the influence of the potential and we expect that it will be close to a solution of the interacting Schrödinger equation (1.1) with the potential, $e^{-itH}\Psi$, with *interacting state*, Ψ . But (1.3) with $t \to -\infty$ tells us that there is a unique $\Psi := W_- \Phi_- \in \mathcal{H}_{ac}$ such that this is true,

$$\lim_{t \to -\infty} \left\| e^{-itH}\Psi - e^{-itH_0}\Phi_- \right\| = 0. \tag{1.4}$$

Furthermore, we expect that at later times the particle will escape the influence of the potential and that as $t \to \infty$ it will travel to spatial infinity where again it will be close to an outgoing solution of the free Schrödinger equation. In fact (1.3) with $t \to \infty$ tells us that this is true and that there is a unique *outgoing asymptotic configuration* $\Phi_+ := W_+^* \Psi$ such that,

$$\lim_{t \to \infty} \left\| e^{-itH}\Psi - e^{-itH_0}\Phi_+ \right\| = 0. \tag{1.5}$$

In a scattering experiment, given the *incoming free solution* one seeks to obtain information about the *outgoing free solution*. This is actually parametrized by the corresponding Cauchy data at $t = 0$, $\Phi_\mp$. The scattering operator is then, the operator that assigns Φ_+ to Φ_-:

$$S := W_+^* W_-. \tag{1.6}$$

We prove below that the high-energy limit of the scattering operator gives the Radon (or X-ray) transform of the potential. Inverting this transform we uniquely reconstruct the potential. The mathematical proof closely follows physical intuition. The key issue is that at high energies *translation of the wave packets* dominates over *spreading* during the interaction time. In fact, in the high-energy limit it is sufficient for the calculation of the scattering operator to consider *translation of wave packets* rather than their correct free evolution. Since on this limit *spreading* occurs only when and where the interaction is negligible, i.e. when the free and the interacting time evolutions are almost the same, the effect of *spreading* does not appear on the scattering operator.

For this reason scattering simplifies on the high-energy limit and we can uniquely reconstruct the potential. We also obtain error bounds. Let us consider states, Φ_0, with compact momentum support on the open ball, $B_{m\eta}$ of radius $m\eta$ and center zero,

$$\hat{\Phi_0} \in C_0^{\infty}\left(B_{m\eta}(0)\right), \tag{1.7}$$

where $\hat{\Phi_0}$ denotes the Fourier transform of Φ_0. The boosted state,

$$\Phi_{\mathbf{v}} := e^{im\mathbf{v}\cdot x}\Phi_0 \leftrightarrow \hat{\Phi}_{\mathbf{v}} = \hat{\Phi}_0(p - m - \mathbf{v}) \in C_0^{\infty}\left(B_{m\eta}(m\mathbf{v})\right), \tag{1.8}$$

has velocity support of radius η around $\mathbf{v}$. Above we denote by $B_{m\eta}(m\mathbf{v})$ the open ball of center $m\mathbf{v}$ and radius $m\eta$. In the theorem below we use the high-velocity limit in an arbitrary fixed direction $\hat{\mathbf{v}} := \frac{\mathbf{v}}{|\mathbf{v}|}$, $v := |\mathbf{v}| \to \infty$.

THEOREM 1.1. *Suppose that $V \in \mathcal{V}_{SR}$ and that for some $0 \leq \rho \leq 1$,*

$$(1+R)^{\rho}\sup_{|x|\geq R}\ |V(x)| \in L^1\left([0,\infty)\right). \tag{1.9}$$

Then, for all $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as defined in (1.8)

$$\begin{aligned} iv\left((S-I)\,\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) &\equiv iv\left(e^{-im\mathbf{v}\cdot x}\ (S-I)\ e^{im\mathbf{v}\cdot x}\Phi_0, \Psi_0\right) \\ &= \left(\int_{-\infty}^{\infty} d\tau V(x+\hat{\mathbf{v}}\tau)\Phi_0, \Psi_0\right) + \begin{cases} o\left(v^{-\rho}\right), & 0 \leq \rho < 1, \\ O\left(v^{-1}\right), & \rho = 1. \end{cases} \end{aligned} \tag{1.10}$$

Moreover, the scattering operator, S, determines uniquely the potential $V \in \mathcal{V}_{SR}$.

■

Theorem 1.1 is proven in [7] and [9]. We give an idea of the proof. By Duhamel's formula,

$$W_{\pm} = I + i\int_0^{\pm\infty} dt\, e^{itH} V e^{-itH_0}, \tag{1.11}$$

Then,

$$W_+ - W_- = i\int_{-\infty}^{\infty} dt\, e^{iHt}\, V\, e^{-itH_0}. \tag{1.12}$$

Since the wave operators $W_\pm$ are unitary, $W_\pm^* W_\pm = I$, and moreover, as they satisfy the intertwinig relations, $W_\pm H_0 = HW_\pm$, it follows from (1.6) and (1.12) that

$$i(S-I) = i\left(W_+^* W_- - W_-^* W_-\right) = i\,(W_+ - W_-)^*\, W_- = \int dt\, e^{iH_0 t}\, V\, e^{-iHt}\, W_-$$
$$= \int dt\, e^{iH_0 t}\, V\, W_-\, e^{-iH_0 t}. \tag{1.13}$$

Taking a boost with velocity $\mathbf{v}$ and making the substitution $\tau = vt$, we obtain that,

$$e^{im\mathbf{v}\cdot x} iv(S-I) e^{-im\mathbf{v}\cdot x} = L_{\mathbf{v}} + R_{\mathbf{v}}, \tag{1.14}$$

where,

$$L_{\mathbf{v}} := \int \left[e^{-im\mathbf{v}\cdot x}\, e^{iH_0\tau/v} e^{im\mathbf{v}\cdot x}\right]\, V(x)\, \left[e^{-im\mathbf{v}\cdot x}\, e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x}\right]\, d\tau, \tag{1.15}$$

and

$$R_{\mathbf{v}} := \int d\tau\, e^{-im\mathbf{v}\cdot x}\, e^{iH_0\tau/v}\, V\,(W_- - I)\, e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x}. \tag{1.16}$$

$L_{\mathbf{v}}$ - that is the first Born approximation- is the leading term which tends to a finite limit if the velocity v goes to infinity. We will use it to reconstruct the potential. The remainder, $R_{\mathbf{v}}$, represents multiple scattering. It is intuitively clear that each instance of scattering by a short-range potential yields a factor v^{-1}: the strength of the interaction as measured by $(S-I)$ or $(W_\pm - I)$ is proportional to the time which the particle spends in the interacting region where the potential is strong. This time is inversely proportional to the speed v and since we have rescaled multiplying by v, we expect $R_{\mathbf{v}}$ to decay as $1/v$, and that we can neglect it as $v \to \infty$.

For any Borel function f let us define the operator $f(\mathbf{p})$ by functional calculus, or equivalently, as $f(\mathbf{p}) := \mathcal{F}^{-1} f(\cdot)\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform.

Under translation in momentum or configuration space, generated by x or $\mathbf{p} := -i\nabla$, respectively, we obtain

$$e^{-im\mathbf{v}\cdot x} f(\mathbf{p}) e^{im\mathbf{v}\cdot x} = f(\mathbf{p} + m\mathbf{v}), \tag{1.17}$$

in particular,

$$e^{-im\mathbf{v}\cdot x} e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x} = e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}\tau}\, e^{-iH_0\tau/v} e^{-imv\tau/2}. \tag{1.18}$$

Moreover,

$$e^{i\mathbf{p}\cdot\hat{\mathbf{v}}\tau} f(x) e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}\tau} = f(x+\hat{\mathbf{v}}\tau). \tag{1.19}$$

By (1.18) the boosted free evolution consists of the classical translation in configuration space, $e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}\tau}$ that is independent of v, the term, $e^{-iH_0\tau/v}$ -that is responsible for the spreading of the wave packet- and an unimportant phase. In the limit when $v\to\infty$ with τ fixed, $e^{-iH_0\tau/v}$ dissapears and it follows that we can replace the boosted free evolution by the classical translation $e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}\tau}\Phi = \Phi(x-\hat{\mathbf{v}}\tau)$. The point is that the classical translation is independent of v and the term responsible for the spreading goes to zero like $1/v$ and then it is irrelevant in the high velocity limit. Then, by (1.19) the pointwise limit under the integral in (1.15) gives us,

$$\lim_{v\to\infty} L_{\mathbf{v}} = \int e^{i\mathbf{p}\cdot\hat{\mathbf{v}}\tau} V(x) e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}\tau}\, d\tau = \int V(x+\hat{\mathbf{v}}\tau)\, d\tau. \tag{1.20}$$

This is the the desired limit in Theorem 1.1. To make this argument rigorous we have to prove that the integrals above exist when applied to states as in Theorem 1.1 and we need an integrable uniform bound for all large enough v that allows us to use the dominated convergence theorem to exchange limit and integration. At the same time, we will also prove that the remainder, $R_{\mathbf{v}}$, goes to zero as $1/v$.

We first prove a propagation estimate that expresses in a convenient way the fact that the solutions to the free Schrödinger equation have rapid decay away from the classically allowed region, i.e. away from the region in space where a classical particle that travels in straight lines with constant velocity would be. In fact the result follows easily from the standard stationary phase estimate (see the Corollary to Theorem XI.14 in [18]). Let us denote by $F(x\in\mathcal{M})$ the operator of multiplication by the characteristic function of $\mathcal{M}$. We designate by $\hat{f}$ the Fourier transform of f, $\hat{f}(p) := 1/(2\pi)^{n/2}\int e^{-ip\cdot x} f(x)\, dx$.

LEMMA 1.2. *For any $f\in C_0^\infty(B_{m\eta})$, for some $\eta>0$, and any $l=1,2,\cdots$, there is a constant C_l such that the following estimate holds:*

$$\left\| F(x\in\tilde{\mathcal{M}})\, e^{-itH_0} f\left(\frac{\mathbf{p}-m\mathbf{v}}{v^\rho}\right) F(x\in\mathcal{M})\right\| \le C_l(1+rv^\rho+\eta\, v^{2\rho}|t|)^{-l}, \tag{1.21}$$

for every $\mathbf{v} \in \mathbf{R}^n$, $t \in \mathbf{R}$, $v > 0$, $\rho \in \mathbf{R}$, *and any measurable sets* $\mathcal{M}$ *and* $\tilde{\mathcal{M}}$ *in* $\mathbf{R}^n$ *such that* $r := \text{dist}\left(\tilde{\mathcal{M}}, \mathcal{M} + \mathbf{v}t\right) - \eta v^\rho |t| \geq 0$.

Proof: by (1.17)-(1.19) it is enough to prove the estimate,

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p}}{v^\rho}\right)\, F(x \in \mathcal{M}) \right\| \leq C_l\, (1 + r v^\rho + \eta\, v^{2\rho} |t|)^{-l}, \tag{1.22}$$

provided that, $r := \text{dist}\left(\tilde{\mathcal{M}}, \mathcal{M}\right) - \eta v^\rho |t| \geq 0$. We have that,

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p}}{v^\rho}\right)\, F(x \in \mathcal{M})\phi \right\|^2 = \frac{1}{(2\pi)^n} \int_{\tilde{\mathcal{M}}} dx \int_{\mathcal{M}} dy \int_{\mathcal{M}} dz\, \tilde{f}_t(x-y)\, \phi(y)\, \overline{\tilde{f}_t(x-z)\, \phi(z)}, \tag{1.23}$$

where,

$$\tilde{f}_t(x) := \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot x} e^{-ip^2 t/2m}\, f(p/v^\rho)\, dp. \tag{1.24}$$

But, as $2|\phi(y)|\,|\phi(z)| \leq |\phi(y)|^2 + |\phi(z)|^2$,

$$\left\| F(x \in \tilde{\mathcal{M}})\, e^{-itH_0}\, f\left(\frac{\mathbf{p}}{v^\rho}\right)\, F(x \in \mathcal{M})\phi \right\|^2 \leq \frac{1}{(2\pi)^n} \left[\int_{|x| \geq r + \eta v^\rho |t|} |\tilde{f}_t(x)|\, dx \right]^2 \|\phi\|^2. \tag{1.25}$$

Finally, by the Corollary to Theorem XI.14 in [18], for any $N = 1, 2, \cdots$, there is a constant C_N such that,

$$|\tilde{f}_t(x)| \leq C_N\, v^{n\rho}\, (1 + |x| v^\rho + v^{2\rho} |t|)^{-N}, \tag{1.26}$$

for $|x| \geq \eta v^\rho |t|$. The lemma follows inserting (1.26) in (1.25).

COROLLARY 1.3. *For any* $f \in C_0^\infty(B_{m\eta})$, *any* $v > 0$ *and for any* $l = 1, 2, \cdots$, *there is a constant* C_l *such that,*

$$\left\| F(|x| \geq |\tau|/4 + \eta|\tau|/v)\, e^{-iH_0\tau/v}\, f(p) F(|x| \leq |\tau|/8) \right\| \leq C_l (1 + |\tau|)^{-l}. \tag{1.27}$$

Proof : the Corollary follows from Lemma 1.2 with $\rho = 0$, $\mathbf{v} = 0$ and $\tilde{\mathcal{M}} = \{|x| > |\tau|/4 + \eta|\tau|/v\}$, $\mathcal{M} = \{|x| \leq |\tau|/8\}$. Observe that $r := \text{dist}\left(\tilde{\mathcal{M}}, \mathcal{M}\right) - \eta|\tau|/v \geq |\tau|/8 \geq 0$.

$\blacksquare$

By (1.15) and(1.18) for any Φ_0, as in (1.7),

$$L_{\mathbf{v}}\Phi_0 = \int e^{iH_0\tau/v}\, V(x+\hat{\mathbf{v}}\tau)\, e^{-iH_0\tau/v}\,\Phi_0\, d\tau. \tag{1.28}$$

We prove below the bound,

$$\left\| e^{iH_0\tau/v} V(x+\hat{\mathbf{v}}\tau) e^{-iH_0\tau/v}\Phi_0 \right\| \leq h(|\tau|), \tag{1.29}$$

where h is integrable. The left-hand side of (1.29) side is dominated by

$$C\left\| V(x+\hat{\mathbf{v}}\tau) F\left(|x| \leq \frac{|\tau|}{2}\right)\right\| + C\left\| F\left(|x| \geq \frac{|\tau|}{2}\right) e^{-iH_0\tau/v}\Phi_0\right\|. \tag{1.30}$$

The first summand is bounded by $h_1(|\tau|) := \sup_{|y|\geq|\tau|/2}|V(y)|$ which is independent of v and integrable by (1.2). If $v \geq 4\eta$ (with η the radius of the velocity support of Φ_0), the second term describes free propagation into the *classically forbidden region* and it is rapidly decaying. Let $g \in C_0^\infty(B_{m\eta})$ be such that $g(\mathbf{p})\Phi_0 = \Phi_0$. Then, for $v \geq 4\eta$,

$$\begin{aligned}\left\| F\left(|x| \geq \frac{|\tau|}{2}\right) e^{-iH_0\tau/v}\Phi_0\right\| &\leq \left\| F\left(|x| \geq \frac{|\tau|}{8}\right)\Phi_0\right\| \\ +C\left\| F\left(|x| \geq \frac{|\tau|}{4} + \eta\frac{|\tau|}{v}\right) e^{-iH_0\tau/v} g(\mathbf{p}) F\left(|x| \leq \frac{|\tau|}{8}\right)\right\| &:= h_2(\tau) + h_3(\tau).\end{aligned} \tag{1.31}$$

The two terms on the right-hand side of (1.31) have rapid decay as $|\tau| \to \infty$ uniformly on v, for $v \geq 4\eta$. In the case of the first term this is obvious because Φ_0 belongs to Schwartz space. For the second term it follows from Corollary 1.3. Defining, $h := h_1 + h_2 + h_3$ we obtain the integrable bound (1.29) for all $v \geq 4\eta$. By the dominated convergence theorem we can take pointwise limit under the integral in (1.28) and we obtain,

$$\text{s - }\lim_{v\to\infty} L_{\mathbf{v}}\Phi_0 = \int_{-\infty}^{\infty} d\tau V(x+\hat{\mathbf{v}}\tau)\Phi_0. \tag{1.32}$$

We now prove that the remainder $R_{\mathbf{v}}$ (*multiple scattering*) is one order smaller, i.e., that it goes to zero as $1/v,\ v\to\infty$. By (1.11), (1.16) and (1.18) have that,

$$|(R_{\mathbf{v}}\Phi_0,\Psi_0)| \leq \left| \int_{-\infty}^{\infty} d\tau \left(\int_{-\infty}^{0} dt\, e^{itH}\, V e^{-i(t+\tau/v)H_0} e^{im\mathbf{v}\cdot x}\Phi_0, V e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x}\Psi_0 \right) \right|$$

$$\leq \frac{1}{v} \left(\int_{-\infty}^{\infty} d\tau \left\| V e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x}\Phi_0 \right\| \right) \left(\int_{-\infty}^{\infty} d\tau \left\| V e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x}\Psi_0 \right\| \right). \tag{1.33}$$

Using (1.17)-(1.19) and (1.29) we prove that,

$$\int_{-\infty}^{\infty} d\tau \left\| V(x) e^{-iH_0\tau/v} e^{im\mathbf{v}\cdot x}\Psi_0 \right\| = \int_{-\infty}^{\infty} d\tau \left\| V(x+\hat{\mathbf{v}}\tau) e^{-iH_0\tau/v}\Psi_0 \right\| \leq C \tag{1.34}$$

uniformly in v. By (1.33) and (1.34)

$$|(R_{\mathbf{v}}\Phi_0,\Psi_0)| \leq C/v, v \geq 4\eta. \tag{1.35}$$

By (1.14), (1.32) and (1.35)

$$\lim_{v\to\infty} iv\left((S-I)\,\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}\right) = \lim_{v\to\infty} iv\left(e^{-im\mathbf{v}\cdot x}\ (S-I)\ e^{im\mathbf{v}\cdot x}\Phi_0,\Psi_0\right)$$

$$= \left(\int_{-\infty}^{\infty} d\tau V(x+\hat{\mathbf{v}}\tau)\Phi_0,\Psi_0 \right). \tag{1.36}$$

This is the correct limit as required in Theorem 1.1. We estimate the error term in a similar way. See [9] for details.

Let us denote: $x^{\perp} := x - (x\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \equiv x - x^{\|}\hat{\mathbf{v}}$. Then, the integral

$$W(x^{\perp};\hat{\mathbf{v}}) := \int d\tau V(x+\hat{\mathbf{v}}\tau), \tag{1.37}$$

exists and is continuous by (1.1). The set of all Φ_0,Ψ_0 is rich enough to determine for any $\hat{\mathbf{v}}$ the continuous function $W(\cdot,\hat{\mathbf{v}})$ from the r.h.s. of (1.10). For $n=2$, $W(x^{\perp};\hat{\mathbf{v}})$ is the Radon transform of the square integrable potential $V(x)$. It is well known that the Radon transform determines V uniquely ([11], p. 115). If $n\geq 3$ it is the X-ray transform. However, one can fix arbitrarily $(x_3,x_4,\cdots x_n)$ and apply the same to the resulting two-dimensional function. Then, varying $\hat{\mathbf{v}}$ in a plane is actually sufficient to reconstruct $V(x)$ from $W(x^{\perp};\hat{\mathbf{v}})$. This completes the proof of Theorem 1.1.

The time-dependent approach is quite flexible. It has been applied to many inverse scattering problems. In [8], [9] to N-body systems with singular and long-range potentials, in [22] to the

N-body Stark effect, and in [10] to two-cluster scattering. In [23] the case of N-body systems with time-dependent potentials was treated. The case of regular magnetic fields on $\mathbf{R}^n$ was considered in [3], [4], and [5]. The relativistic Schrödinger operator, and the Dirac and Klein-Gordon equations were studied in [13] and [14]. In [12] the Dirac equation with time-dependent electromagnetic potentials was considered. For the case of the Aharonov-Bohm effect, see Section 2. For references on the stationary theory see [21] and [9]. In all of these papers the direct scattering problem is linear. For the case when the direct scattering problem is nonlinear see Section 3.

2 The Aharonov-Bohm Effect

We now discuss the Aharonov-Bohm effect [2]. Aharonov and Bohm considered the scattering of an electron off the magnetic field of a tiny solenoid, idealized as having infinite length and zero radius (scattering off a thread of magnetic flux). We assume that the solenoid is located on the vertical axis of the coordinate system, and in consequence it is enough to consider scattering in the plane orthogonal to the solenoid, as the problem is invariant under translation along the vertical direction. The magnetic field of an unshielded solenoid at zero is given by, $B_s\,\delta(x)$ with $x = (x_1, x_2) \in \mathbf{R}^2$ (this actually corresponds to a magnetic field in $\mathbf{R}^3$ with components $(0, 0, B_s\,\delta(x))$. We also assume that there is a regular magnetic field $B_R \in C_0^1(\mathbf{R}^2)$ that is continuously differentiable and has compact support. The total magnetic field is written as,

$$B := B_s\,\delta(x) + B_R, \tag{2.1}$$

where, of course, B_s is a real constant and B_R is a real-valued function. In order to define the Schrödinger operator for an electron in the presence of B we have to introduce a magnetic potential. The magnetic potential for $B_s\,\delta(x)$ in the Coulomb gauge is given by,

$$A_0 := \frac{\alpha_{\{0\}}}{|x|^2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \tag{2.2}$$

where we denote $\alpha_{\{0\}} := \frac{B_s}{2\pi}$. Observe that $\alpha_{\{0\}}$ is the flux (across zero) of the singular magnetic field normalized by 2π. It is easily checked that $\nabla \times A_s := \frac{\partial}{\partial x_1} A_{s,2} - \frac{\partial}{\partial x_2} A_{s,1} = B_s\,\delta(x)$ and that $\nabla \cdot A_s = 0$, with the derivatives taken in distribution sense in $\mathcal{D}'$. The magnetic potential in the

Coulomb gauge for B_R is given by,

$$A_R = \frac{1}{2\pi} \int B_R(x-y) \begin{bmatrix} -\hat{y}_2 \\ \hat{y}_1 \end{bmatrix} \frac{dy}{|y|}, \tag{2.3}$$

where, $\hat{y} := \frac{y}{|y|}$. Clearly, $A_R \in C^1(\mathbf{R}^2, \mathbf{R}^2)$. The magnetic potential for B in the Coulomb gauge is given by,

$$A_c := A_s + A_R. \tag{2.4}$$

As is well known, the magnetic potential is not uniquely defined by the magnetic field; there is always the possibility of a gauge transformation. We introduce below a general class of magnetic potentials that is convenient for our purposes.

DEFINITION 2.1. *We denote by $\mathcal{A}_{\{0\}}(\alpha_{\{0\}}, B_R)$ the set of all real-valued $A \in C^1(\mathbf{R}^2 \setminus 0, \mathbf{R}^2) \cap L^1_{\mathrm{loc}}(\mathbf{R}^2, \mathbf{R}^2)$, with $\nabla \times A = 2\pi\alpha_{\{0\}}\, \delta(x) + B_R$, in $\mathcal{D}'$. Moreover, we assume that $A(x) = O(|x|^{-1}), |x| \to \infty$ and that,*

$$a(r) := \sup_{|x| \geq r} |A(x) \cdot \hat{x}| \in L^1([0, \infty)). \tag{2.5}$$

Let us now study the case where a general singular magnetic field is contained inside an infinite cylinder, with axis along the vertical direction, and transversal section K, where K is a compact subset of $\mathbf{R}^2$. The purpose of the cylinder is to shield the singular magnetic field from the incoming electrons. As we will see below, we cannot hope that the scattering operator determines uniquely the magnetic field inside K. In fact, we can only determine the (normalized) flux of the magnetic field across K modulo 2. This suggests that instead of specifying the magnetic field inside K we only fix the magnetic flux across K, normalized by 2π, that by Stokes' theorem is given by the circulation of the magnetic potential, A, along ∂K,

$$\alpha_K := \frac{1}{2\pi} \int_{\partial K} A(x) \cdot dx, \tag{2.6}$$

where we integrate in counter-clockwise sense. Of course, we also specify a regular magnetic field $B_R \in C^1_0(\mathbf{R}^2 \setminus K)$ outside of K. The magnetic flux α_K could be produced, for example, by a finite number of solenoids inside K, and also by a regular magnetic field contained inside K, or by a combination of both. The considerations above suggest the definition of the following class of magnetic potentials. In what follows we denote, $\Omega := \mathbf{R}^2 \setminus K$, where for the unshielded solenoid, $K = \{0\}$.

DEFINITION 2.2. *Let K be a compact set such that $0 \in K$ and that its boundary, ∂K, is a simple, closed, C^1 curve. Then, for any $\alpha_K \in \mathbf{R}$ and any real-valued $B_R \in C_0^1\left(\overline{\Omega}\right)$ we denote by $\mathcal{A}_K\left(\alpha_K, B_R\right)$ the set of all real-valued $A \in C^1\left(\overline{\Omega}, \mathbf{R}^2\right)$ with $\nabla \times A = B_R$ and such that,*

$$\alpha_K = \frac{1}{2\pi} \int_{\partial K} A(x) \cdot dx, \tag{2.7}$$

where we integrate in counter-clockwise sense. Moreover, we assume that $A(x) = O\left(|x|^{-1}\right), |x| \to \infty$ and that,

$$a(r) := \sup_{x \in \Omega, |x| \geq r} |A(x) \cdot \hat{x}| \in L^1([0, \infty)). \tag{2.8}$$

The formal Hamiltonian is the operator,

$$h_A := \frac{(\mathbf{p} - A)^2}{2m}, \tag{2.9}$$

with domain $C_0^2(\Omega)$, $\mathbf{p} := -i\nabla$ is the momentum operator, and $m > 0$ is the mass of the electron. We take Planck's constant and the speed of light all equal to one and the charge of the electron equal to minus one. The quadratic form associated to h_A is given by,

$$q_A(\phi, \psi) := ((\mathbf{p} - A)\phi, (\mathbf{p} - A)\psi), \tag{2.10}$$

with domain $C_0^1(\Omega)$. The form q_A is non-negative and closable. The Hamiltonian, H_A, is the self-adjoint operator in $L^2(\Omega)$ associated to the closure of q_A (see [15]). H_A is the extension of h_A with Dirichlet boundary condition on $\partial\Omega$. Let J be the identification operator from $L^2\left(\mathbf{R}^2\right)$ onto $L^2(\Omega)$ given by multiplication by the characteristic function of Ω. In the case $K = \{0\}$ we take $J = I$. The unperturbed Hamiltonian is given by $H_0 := \frac{\mathbf{p}^2}{2m}$, with domain the Sobolev space $W_{2,2}\left(\mathbf{R}^2\right)$. The wave operators are defined as,

$$W_{\pm}(A) := \text{s - } \lim_{t \to \pm\infty} e^{itH_A} J e^{-itH_0}. \tag{2.11}$$

We prove in [33] that if $A \in \mathcal{A}_K(\alpha_K, B_R)$, the strong limits exist and are isometric. The scattering operator is given by,

$$S(A) := W_+^*(A)\ W_-(A)\,. \tag{2.12}$$

Note that to define H_A we only use the values of A in Ω. This means that as long as A is fixed in Ω, we can change the magnetic potential in the interior of K without changing H_A. Note however that as $A \in C^1\left(\overline{\Omega}\right)$ the flux α_K is uniquely defined by the values of A in Ω. This explains why we cannot hope to uniquely reconstruct the magnetic field inside K from the scattering operator and makes it plausible that we can reconstruct α_K.

As we said above, the only purpose of the obstacle, K, is to shield the incoming electron from the magnetic field, and in order to separate the scattering effect of the magnetic potential from that of the obstacle, we consider asymptotic configurations that have negligible interaction with K for all times in the high-velocity limit. For this purpose, given $\hat{\mathbf{v}} \in S^1$, let us denote,

$$\Omega_{\hat{\mathbf{v}}} := \{x \in \Omega : x + \hat{\mathbf{v}}\tau \in \Omega, \text{ for all } \tau \in \mathbf{R}\}\,. \tag{2.13}$$

Given $\mathbf{v} \in \mathbf{R}^2$ we take asymptotic configurations $\Phi \in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$, where $\hat{\mathbf{v}} := \mathbf{v}/v$, with $v := |\mathbf{v}|$. The free evolution boosted by $\mathbf{v}$ is given by $e^{-im\mathbf{v}\cdot x}\, e^{-itH_0}\, e^{im\mathbf{v}\cdot x} = e^{-imv^2t/2} e^{-i\mathbf{p}\cdot\mathbf{v}t} e^{-itH_0}$, and -to a good approximation- in the limit when $v \to \infty$ with $\hat{\mathbf{v}}$ fixed this can be replaced (modulo an unimportant phase) by the classical translation $e^{-it\mathbf{p}\cdot\mathbf{v}}$. Then, in the high-velocity limit it is a good approximation to assume that the free evolution of our asymptotic configuration is given by $e^{-it\mathbf{p}\cdot\mathbf{v}}\Phi_0 = \Phi_0(x - \mathbf{v}t)$, and as $\Phi_0 \in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$, it has negligible interaction with K for all times. In the following theorem we evaluate the high-velocity asymptotics of the scattering operator.

THEOREM 2.3. *Suppose that $A \in \mathcal{A}_K(\alpha_K, B_R)$ and that $\Phi_0, \Psi_0 \in C_0^\infty(\Omega_{\hat{\mathbf{v}}})$. Let $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ be the boosted asymptotic configurations,*

$$\Phi_{\mathbf{v}} := e^{im\mathbf{v}\cdot x}\Phi_0,\ \ \Psi_{\mathbf{v}} := e^{im\mathbf{v}\cdot x}\Psi_0. \tag{2.14}$$

Then,

$$(S(A)\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) = \left(e^{i\int_{-\infty}^{\infty} \hat{\mathbf{v}}\cdot A(x+\hat{\mathbf{v}}\tau)\, d\tau}\Phi_0, \Psi_0\right) + O\left(\frac{1}{|v|}\right), \quad |v| \to \infty. \tag{2.15}$$

We prove in [33] that (2.15) determines uniquely the Radon transforms,

$$\int_{-\infty}^{\infty} B_R(x+\hat{\mathbf{v}}\tau)\, d\tau, \quad x \in \Omega_{\hat{v}}. \tag{2.16}$$

By the support theorem for the Radon transform [16], B_R is uniquely determined in Ω, provided that K is convex. The fact that the magnetic flux, α_K, is determined modulo 2 follows from an explicit calculation. This gives us the following theorem.

THEOREM 2.4. *Suppose that $A^{(j)} \in \mathcal{A}_K\left(\alpha_K^{(j)}, B_R^{(j)}\right), j = 1, 2$ and that K is convex. Then, if $S\left(A^{(1)}\right) = S\left(A^{(2)}\right)$, we have that, $\alpha_K^{(1)} = \alpha_K^{(2)}$ modulo 2 and that $B_R^{(1)}(x) = B_R^{(2)}(x), x \in \Omega$.*

For a complete study of this problem including proofs and an analysis of gauge transformations see [33]. Observe that since (2.16) is obtained from (2.15) we actually only need to know the high-velocity limit of the scattering operator. Moreover, our proof gives a method for the reconstruction of B_R in Ω. Note that in spite of the fact that the scattering of the electron takes place outside of K, we determine the magnetic flux in K- modulo 2- from the scattering operator. This is the Aharonov-Bohm effect [2] that shows that in quantum mechanics the magnetic field acts on a charged particle -by means of the magnetic potential- even in regions where it is identically zero. Nicoleau has proven in [17] the following result using stationary methods. Suppose that the magnetic field, B, is infinitely differentiable in $\mathbf{R}^2$ and has compact support, and that K is compact, convex, $0 \in K$ and ∂K is smooth. Then, if $S\left(A^{(1)}\right) = S\left(A^{(2)}\right)$, with $A^{(j)}$, $j = 1, 2$, the Coulomb gauge potentials, then the two magnetic fluxes across $\mathbf{R}^2$ are equal modulo 2 and $B_R^{(1)}(x) = B_R^{(2)}(x), x \in \Omega$. The case of an unshielded solenoid with $K = \{0\}$ is not covered by the result of [17]. Note that this is actually the problem considered by Aharonov and Bohm [2]. In the case where the interior of K is non-empty, [17] considers the situation where the magnetic flux across K is produced by a magnetic field in $C_0^\infty(\mathbf{R}^2)$. Our result is considerably more general in the sense that we study the case where the magnetic flux across K is produced by any magnetic field inside K. The only restriction is that the magnetic flux across K has to be finite. We could as well consider the case where there is also a scalar potential -as is done in [17]- however, we do not pursue that direction here.

3 The Non-Linear Schrödinger Equation

We discuss now the non-linear Schrödinger equation with a potential on the line. The goal is to give a method to uniquely reconstruct the potential and the nonlinearity. This problem was solved in [24], [26], [27], [29] and [30]. In these papers the multidimensional case was also considered. For the nonlinear Klein-Gordon equation see [28], [31] and [32].

Let us consider the following non-linear Schrödinger equation with a potential

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{\partial^2}{\partial x^2}u(t,x) + V_0(x)u(t,x) + F(x,u),\ \ u(0,x) = \phi(x), \tag{3.1}$$

where $t, x \in \mathbf{R}$, the potential, V_0, is a real-valued function, and $F(x,u)$ is a complex-valued function.

Let H_0 denote the unique self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain the Sobolev space $W_{2,2}$ [1]. For any $\gamma \in \mathbf{R}$, L^1_γ denotes the Banach space of all complex-valued measurable functions, ϕ, defined on $\mathbf{R}$ and such that

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)|\,(1+|x|)^\gamma\,dx < \infty. \tag{3.2}$$

If $V_0 \in L^1_1$ the differential expression $\tau := -\frac{d^2}{dx^2} + V_0(x)$ is essentially self-adjoint. We denote by H the unique self-adjoint realization of τ. The H has a finite number of negative eigenvalues, it has no positive or zero eigenvalues, it has no singular-continuous spectrum and the absolutely-continuous spectrum is $[0,\infty)$ (for these results see [34]). The wave operators are defined as in the multidimensional case,

$$W_\pm := \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}. \tag{3.3}$$

The limits in (3.3) are taken in the strong topology in L^2. As is well known (see [19]), the $W_\pm$ exist and are complete. The scattering operator for the linear Schrödinger equation ((3.1) with $F = 0$) is defined as follows:

$$S_L := W_+^* W_-. \tag{3.4}$$

The key issue for the scattering theory for equation (3.1) is the following time-dependent $L^p - L^{\acute{p}}$ estimate that we proved in [26],

$$\left\| e^{-itH} P_c \right\|_{\mathcal{B}\left(L^p, L^{\acute{p}}\right)} \leq \frac{C}{t^{\left(\frac{1}{p}-\frac{1}{2}\right)}},\ t > 0, \tag{3.5}$$

for some constant $C,\ 1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{\acute{p}} = 1$, and where P_c denotes the projector onto the space of continuity of H. The $L^p - L^{\acute{p}}$ estimate (3.5) expresses the dispersive nature of the solutions to the linear Schrödinger equation with initial data on the continuous subspace of H. It gives a quantitative meaning to the *spreading of the wave packets.* In typical applications the nonlinearity, F, is proportional to a high-enough power of u. This type of nonlinearity makes the solutions to (3.1) even larger where they are already large. On the other hand, the *spreading* of the associated linear equation prevents the solution from becoming to large, provided that the initial data was small enough. It is the balance from these two phenomena that is at the heart of small-amplitude scattering theory. Eventually, the *spreading* prevents the solution from blowing up in a finite time, and for large times the evolution is dominated by the linear part in the sense that the solution is asymptotic to a solution of the linearized equation. This is the physical content of Theorem 3.1 below. By the same argument, on the small amplitude limit the nonlinear effects become negligible and scattering is dominated by the linear term. This fact is expressed in a quantitave way by Theorem 3.2 that allows us to reconstruct the linear scattering operator from the derivative at zero of the nonlinear scattering operator. It is interesting to remark that the *spreading of the wave packets* -that is irrelevant on the high-energy limit in the linear case- is actually essential on the low-energy (small amplitude) limit in the non-linear case.

Before we state our results we introduce some standard definitions and notations. We say that $F(x,u)$ is a C^k function of u in the real sense if for each fixed $x \in \mathbf{R}$, $\Re F$ and $\Im F$ are C^k functions of the real and imaginary parts of u. We will assume that F is C^2 in the real sense and that $\left(\frac{\partial}{\partial x} F\right)(x,u)$ is C^1 in the real sense. If $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r + is,\ r, s \in \mathbf{R}$ we denote,

$$F^{(2)}(x,u) := \sum_{j=1}^{2} \left[\left| \frac{\partial^2}{\partial r^2} F_j(x,u) \right| + \left| \frac{\partial^2}{\partial r \partial s} F_j(x,u) \right| + \left| \frac{\partial^2}{\partial s^2} F_j(x,u) \right| \right], \tag{3.6}$$

$$\left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u):=\sum_{j=1}^{2}\left[\left|\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right|+\left|\frac{\partial}{\partial s}\left(\frac{\partial}{\partial x}F_j\right)(x,u)\right|\right]. \tag{3.7}$$

For any pair u, v of solutions to the stationary Schrödinger equation:

$$-\frac{d^2}{dx^2}u+V_0u=k^2u,\ k\in\mathbf{C}^+, \tag{3.8}$$

let $[u, v]$ denote the Wronskian of u, v:

$$[u,v]:=\left(\frac{d}{dx}u\right)v-u\frac{d}{dx}v. \tag{3.9}$$

Let $f_1(x,k)\approx e^{ikx}, x\to\infty, f_2(x,k)\approx e^{-ikx}, x\to-\infty$, be the Jost solutions to (3.8) (see for example [6]). A potential V_0 is said to be *generic* if $[f_1(x,0), f_2(x,0)]\neq 0$ and V_0 is said to be *exceptional* if $[f_1(x,0), f_2(x,0)] = 0$. If V_0 is *exceptional* there is a bounded solution to (3.8) with $k^2 = 0$ (a half-bound state or a zero-energy resonance). The trivial potential, $V_0 = 0$, is *exceptional.* Let us denote,

$$M:=\left\{u\in C(\mathbf{R},W_{1,p+1}):\sup_{t\in\mathbf{R}}(1+|t|)^d\|u\|_{W_{1,p+1}}<\infty\right\},$$

$$\text{with norm}\|u\|_M:=\sup_{t\in\mathbf{R}}(1+|t|)^d\|u\|_{W_{1,p+1}}, \tag{3.10}$$

where $p\geq 1$, and $d:=\frac{1}{2}\frac{p-1}{p+1}$. For functions $u(t,x)$ defined in $\mathbf{R}^2$ we simply write $u(t)$, instead $u(t,\cdot)$. The $W_{k,p}$ are the Sobolev spaces [1]. The small-amplitude scattering operator is given in the following theorem.

THEOREM 3.1. *Suppose that $V_0\in L^1_\gamma$, where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$, *that H has no negative eigenvalues, and that*

$$N(V_0):=\sup_{x\in\mathbf{R}}\int_x^{x+1}|V_0(y)|^2\,dy<\infty. \tag{3.11}$$

Furthermore, assume that F is C^2 in the real sense, that $F(x,0) = 0$, and that for each fixed $x\in\mathbf{R}$ all the first order derivatives, in the real sense, of F vanish at zero. Moreover, suppose that $\frac{\partial}{\partial x}F$ is C^1 in the real sense. We assume that the following estimates hold:

$$F^{(2)}(x,u)=O\left(|u|^{p-2}\right),\ \left(\frac{\partial}{\partial x}F\right)^{(1)}(x,u)=O\left(|u|^{p-1}\right),\ u\to 0, \tag{3.12}$$

uniformly for $x \in \mathbf{R}$*, for some* $\rho < p < \infty$*, and where* ρ *is the positive root of* $\frac{1}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}$*. Then, there is a* $\delta > 0$ *such that for all* $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ *with* $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \leq \delta$ *there is a unique solution,* u*, to (3.1) such that* $u \in C(\mathbf{R}, W_{1,2}) \cap M$ *and,*

$$\lim_{t \to -\infty} \|u(t) - e^{-itH}\phi_-\|_{W_{1,2}} = 0. \tag{3.13}$$

Moreover, there is a unique $\phi_+ \in W_{1,2}$ *such that*

$$\lim_{t \to \infty} \|u(t) - e^{-itH}\phi_+\|_{W_{1,2}} = 0. \tag{3.14}$$

Furthermore, $e^{-itH}\phi_\pm \in M$ *and*

$$\left\|u - e^{-itH}\phi_\pm\right\|_M \leq C \left\|e^{-itH}\phi_\pm\right\|_M^p, \tag{3.15}$$

$$\|\phi_+ - \phi_-\|_{W_{1,2}} \leq C \left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}}\right]^p. \tag{3.16}$$

The scattering operator, $S_{V_0} : \phi_- \hookrightarrow \phi_+$ *is injective on* $W_{1,1+\frac{1}{p}} \cap W_{2,2}$*.*

Observe that, $\rho \approx 3.56$. Remark that we do not to restrict F in such a way that energy is conserved.

To reconstruct the potential, V_0, we introduce below the scattering operator associated with asymptotic states that are solutions to the linear Schrödinger equation with potential zero:

$$S := W_+^* S_{V_0} W_-. \tag{3.17}$$

THEOREM 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then for every* $\phi \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$

$$\frac{d}{d\epsilon} S(\epsilon\phi)\bigg|_{\epsilon=0} = S_L\phi, \tag{3.18}$$

where the derivative on the left-hand side of (3.18) exists in the sense of strong convergence in $W_{1,2} \cap W_{1,p+1}$*.*

COROLLARY 3.3. *Under the conditions of Theorem 3.1 the scattering operator,* S*, determines uniquely the potential* V_0*.*

In the case where $F(x,u)=\sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, with fixed integer j_0, we can also reconstruct the $V_j,\ \ j=1,2,\cdots$.

LEMMA 3.4. *Suppose that the conditions of Theorem 3.1 are satisfied, and moreover, that $F(x,u)=\sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, where j_0 is an integer such that, $j_0 \geq (p-3)/2$, for $|u| \leq \eta$, for some $\eta>0$, and where $V_j \in W_{1,\infty}$ with $\|V_j\|_{W_{1,\infty}} \leq M^j, j=1,2,\cdots$, for some positive constant M. Then, for any $\phi \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ there is an $\epsilon_0>0$ such that for all $0<\epsilon<\epsilon_0$:*

$$i\left((S_{V_0}-I)(\epsilon\phi),\,\phi\right)_{L^2}=\sum_{j=1}^{\infty}\epsilon^{2(j_0+j)+1}\left[\int\int dt\,dx\,V_j(x)\left|e^{-itH}\phi\right|^{2(j_0+j+1)}+Q_j\right], \tag{3.19}$$

where $Q_1=0$ and $Q_j, j>1$, depends only on ϕ and on V_k with $k<j$. Moreover, for any $\acute{x} \in \mathbf{R}$, and any $\lambda>0$, we denote, $\phi_\lambda(x):=\phi(\lambda(x-\acute{x}))$. Then, if $\phi \neq 0$:

$$V_j(\acute{x})=\frac{\lim_{\lambda\to\infty}\lambda^3\int\int\,dt\,dx\,V_j(x)\left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)}}{\int\int\,dt\,dx\,\left|e^{-itH_0}\phi\right|^{2(j_0+j+1)}}. \tag{3.20}$$

COROLLARY 3.5. *Under the conditions of Lemma 3.4 the scattering operator, S, determines uniquely the potentials $V_j, j=0,1,\cdots$.*

The method to reconstruct the potentials $V_j, j=0,1,\cdots$, is as follows. First we obtain S_L from S using (3.18). By any standard method for inverse scattering for the linear Schrödinger equation on the line we reconstruct V_0 (recall that H has no eigenvalues). We then reconstruct S_{V_0} from S using (3.17). Finally (3.19) and (3.20) give us, recursively, $V_j, j=1,2,\cdots$.

Theorems 3.1, 3.2, Lemma 3.4 and Corollaries 3.3, 3.5 are proven in [29] where also a discussion of the literature is given. We give below an idea of the proofs.

In Theorem 1.1 of [25] we proved that $W_{\pm}$ and $W_{\pm}^*$ are bounded operators on $W_{k,p}, k=0,1,\ 1<p<\infty$. By Theorem 3 in page 135 of [20],

$$\|\mathcal{F}^{-1}(1+q^2)^{k/2}(\mathcal{F}f)(q)\|_{L^p}, \tag{3.21}$$

is a norm that is equivalent to the norm of $W_{k,p}, 1<p<\infty$. In (3.21) $\mathcal{F}$ denotes the Fourier transform. Then, by the continuity of the $W_{\pm}$ and $W_{\pm}^*$ on $W_{k,p}$ (see Corollary 1.2 of [25])

$$\|(I+H)^{k/2}f\|_{L^p}, \tag{3.22}$$

defines a norm that is equivalent to the norm of $W_{k,p}, k = 0, 1, 1 < p < \infty$.

Condition (3.11) and Theorem 2.7.1 in page 35 of [19] imply that, $D(H) = D(H_0) = W_{2,2}$, and that the following norm is equivalent to the norm of $W_{2,2}$:

$$\|(H + I)\phi\|_{L^2}. \tag{3.23}$$

The weight $(I + H)^{k/2}, k = 1, 2$ has the advantage that it commutes with e^{-itH} and moreover, in the case $p = 2$ the equivalent norm is invariant under the time evolution given by e^{-itH}. We will use these equivalences without further comments. In particular, it follows from (3.22) that estimate (3.5) holds in the norm on $\mathcal{B}\left(W_{1,p}, W_{1,\acute{p}}\right)$, $1 < p \leq 2$.

By Sobolev's imbedding theorem [1], $W_{1,2}$ is continuously imbedded in L^∞. It follows that F is locally Lipschitz continuous on $W_{1,2}$. Then, by standard arguments, $u \in C(\mathbf{R}, W_{1,2}) \cap M$ is a solution to (3.1) with $\lim_{t \to -\infty} \|u(t) - e^{-itH}\phi\|_{W_{1,2}} = 0$, for some $\phi \in W_{1,2}$, if and only if u is a solution to the following integral equation:

$$u = e^{-itH}\phi + \frac{1}{i}\int_{-\infty}^{t} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau. \tag{3.24}$$

As we prove below the integral in the right-hand side of (3.24) converges absolutely in $W_{1,2}$ and in M. For $u \in M$ we denote

$$\mathcal{Q}u(t) := \frac{1}{i}\int_{-\infty}^{t} e^{-i(t-\tau)H} F(x, u(\tau))\, d\tau. \tag{3.25}$$

It follows from (3.5), and since$W_{1,p+1}$ is continuously imbedded in L^∞, that

$$\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\|_{W_{1,p+1}} \leq C\,(1 + |t|)^{-d}(\|u\|_M + \|v\|_M)^{p-1}\|u - v\|_M, \tag{3.26}$$

where we used that $p\,d > 1$. The constants C in (3.26) can be taken uniform in closed balls in M. By (3.26) with $v(t) = 0$:

$$\|\mathcal{Q}u(t)\|^2_{W_{1,2}} \leq C\Re\int_{-\infty}^{t} d\tau\,\left(\sqrt{I + H}F(x, u(\tau)), \sqrt{I + H}\mathcal{Q}u(\tau)\right)_{L^2} \leq C\int_{-\infty}^{t} d\tau\,\|F(x, u)(\tau)\|_{W_{1,1+1/p}} \times$$

$$(1 + |\tau|)^{-d}\,\|u\|^p_M \leq C\int_{-\infty}^{t} d\tau\,\|u\|^p_{W_{1,p+1}}\,(1 + |\tau|)^{-d}\|u\|^p_M \leq C\int_{-\infty}^{t} d\tau\,(1 + |\tau|)^{-d(p+1)}\,\|u\|^{2p}_M$$

$$\leq C(1 + \max[0, -t])^{-(d+dp-1)}\,\|u\|^{2p}_M. \tag{3.27}$$

By (3.26) with $v(t) = 0$, the integral in the right-hand side of (3.24) converges in M and by (3.27) the converge holds also in $W_{1,2}$.

By (3.23) and Sobolev's imbedding theorem,

$$\|e^{-itH}\phi_-\|_{W_{1,p+1}} \leq C\|e^{-itH}\phi_-\|_{W_{2,2}} \leq C\left\|(H+I)e^{-itH}\phi_-\right\|_{L^2} = C\left\|(H+I)\phi_-\right\|_{L^2} \leq C\left\|\phi_-\right\|_{W_{2,2}}. \tag{3.28}$$

Then, (3.5) and (3.28) imply that,

$$\left\|e^{-itH}\phi_-\right\|_M \leq C\left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}}\right]. \tag{3.29}$$

For $R > 0$ let us denote: $M_R := \{u \in M : \|u\|_M \leq R\}$. Let us take R so small that $C(2R)^{p-1} \leq 1/2$, with C as in (3.26), and $\delta > 0$ so small that $C\delta \leq R/4$, with C as in (3.29). It follows from (3.26) and (3.29) that the map $u \hookrightarrow e^{-itH}\phi_- + \mathcal{Q}u$ is a contraction from M_R into M_R for all $\phi_- \in W_{2,2} \cap W_{1,1+\frac{1}{p}}$ with $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W_{1,1+\frac{1}{p}}} \leq \delta$. The contraction mapping theorem implies that there is an unique solution to (3.24) in M_R. This is the solution $u(t)$ of Theorem 3.1. Moreover,

$$\|u\|_M \leq \left\|e^{-itH}\phi_-\right\|_M + \frac{1}{2}\|u\|_M. \tag{3.30}$$

Then,

$$\|u\|_M \leq C\left\|e^{-itH}\phi_-\right\|_M. \tag{3.31}$$

We define:

$$\phi_+ = \phi_- + \frac{1}{i}\int_{-\infty}^{\infty} e^{i\tau H}F(x, u(\tau))\,d\tau. \tag{3.32}$$

For further details on the proof of Theorem 3.1 see [29].

Proof of Theorem 3.2 : Since, $S(0) = 0$, and $W_\pm$ are bounded on $W_{2,2} \cap W_{1,1+\frac{1}{p}}$ [25], it is enough to prove that

$$s - \lim_{\epsilon \to 0}\frac{1}{\epsilon}\left(S_{V_0}(\epsilon\phi) - \epsilon\phi\right) = 0. \tag{3.33}$$

By (3.29) and (3.31) with ϕ_- replaced by $\epsilon\phi$:

$$\|u\|_M \leq C|\epsilon|\left[\|\phi\|_{W_{2,2}} + \|\phi\|_{W_{1,1+\frac{1}{p}}}\right]. \tag{3.34}$$

Using and (3.5) and (3.32) we obtain that,

$$\|S_{V_0}(\epsilon\phi)-\epsilon\phi\|^2_{W_{1,2}} \leq C\int_{-\infty}^{\infty} d\tau \,\left(\sqrt{I+H}F(x,u(\tau)), \sqrt{I+H}\int_{-\infty}^{\infty} d\rho\, e^{-i(\tau-\rho)H}F(x,u(\rho))\right)_{L^2}$$

$$\leq C\int_{-\infty}^{\infty} d\tau\, \|F(x,u)(\tau)\|_{W_{1,1+1/p}}\times$$

$$(1+|\tau|)^{-d}\,\|u\|^p_M \leq C\int_{-\infty}^{\infty} d\tau\,\|u\|^p_{W_{1,p+1}}\,(1+|\tau|)^{-d}\|u\|^p_M \leq C\int_{-\infty}^{\infty} d\tau\,(1+|\tau|)^{-d(p+1)}\,\|u\|^{2p}_M$$

$$\leq C\|u\|^{2p}_M. \tag{3.35}$$

Equation (3.33) follows from (3.34) and (3.35).

Proof of Corollary 3.3: By Theorem 3.2 S determines uniquely S_L. From S_L we get the reflection coefficients for linear Schrödinger scattering on the line. As H has no bound states we uniquely reconstruct V_0 from one of the reflection coefficients by using any method for inverse scattering on the line.

Proof of Lemma 3.4 : By the contraction mapping theorem,

$$u(t)=e^{-itH}\epsilon\phi+\sum_{n=1}^{\infty}\mathcal{Q}^n e^{-itH}\epsilon\phi. \tag{3.36}$$

Equation (3.19) follows from (3.32) and (3.36). By Sobolev's imbedding theorem [1], $W_{2,2}\subset L^q$, $2\leq q\leq\infty$. Then, estimating as in (3.28) we prove that, $\left\|e^{-itH}\phi\right\|_{L^q}\leq C_q\left\|e^{-itH}\phi\right\|_{W_{2,2}}\leq C_q\,\|\phi\|_{W_{2,2}}, 2\leq q\leq\infty$, and as $2(j_0+j+1)\geq p+1$ it follows from (3.5) that:

$$\int\int dt\,dx\,\left|e^{-itH}\phi\right|^{2(j_0+j+1)}\leq\left\|e^{-itH}\phi\right\|^{2(j_0+j+1)-p-1}_{L^\infty}\int dt\,dx\,\left|e^{-itH}\phi\right|^{p+1}<\infty,\;\; j=1,2,\cdots. \tag{3.37}$$

For $\lambda>0$ and $\acute{x}\in\mathbf{R}$ we denote by H_λ the following self-adjoint operator in L^2:

$$H_\lambda := H_0+V_\lambda(x),\text{ where } V_\lambda(x)=\frac{1}{\lambda^2}V_0\left(\frac{x}{\lambda}+\acute{x}\right). \tag{3.38}$$

Since H has no eigenvalues, we have that H_λ has no eigenvalues, i.e., $H_\lambda>0$. It follows from (3.11) and from Theorem 2.7.1 on page 35 of [19] that

$$C_1\,\|\phi\|_{W_{2,2}}\leq\|(H_\lambda+I)\phi\|_{L^2}\leq C_2\,\|\phi\|_{W_{2,2}}, \tag{3.39}$$

for some constants C_1, C_2. Moreover, since $N(V_\lambda) \leq \frac{1}{\lambda^3} N(V_0),\ \ \lambda \geq 1$, the proof of Theorem 2.7.1 on page 35 of [19] implies that we can take fixed C_1 and C_2 for all $\lambda \geq 1$. To prove (3.20) we denote: $\tilde{t} := \lambda^2 t$ and $\tilde{x} := \lambda(x - \acute{x})$. Then, we observe that,

$$\left(e^{-i\tilde{t}H_\lambda}\phi\right)(\tilde{x}) = \left(e^{-itH}\phi_\lambda\right)(x). \tag{3.40}$$

This can be seen as follows,

$$i\frac{\partial}{\partial t}\left(e^{-i\tilde{t}H_\lambda}\phi\right) = H\left(e^{-i\tilde{t}H_\lambda}\phi\right),\ \text{ and }\ \left(e^{-i\tilde{t}H_\lambda}\phi\right)|_{t=0} = \phi_\lambda. \tag{3.41}$$

Since the solution to the linear Schrödinger equation is unique, (3.40) is proved. It follows from (3.40) that,

$$I_j := \lambda^3 \int\int dt\, dx\, V_j(x)\left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)} = \int\int d\tilde{t}\, d\tilde{x}\, V_j(\frac{\tilde{x}}{\lambda} + \acute{x})\left|e^{-i\tilde{t}H_\lambda}\phi\right|^{2(j_0+j+1)}(\tilde{x}). \tag{3.42}$$

By (3.39)

$$s - \lim_{\lambda\to\infty} e^{-i\tilde{t}H_\lambda}\phi = e^{-i\tilde{t}H_0}\phi, \tag{3.43}$$

where the limit exists in the strong topology on $W_{2,2}$. By Sobolev's imbedding theorem, the limit in (3.43) also exists in the strong topology on L^q, $2 \leq q \leq \infty$, and moreover,

$$\left\|e^{-i\tilde{t}H_\lambda}\phi\right\|_{L^q} \leq C_q \left\|\phi\right\|_{W_{2,2}},\ \ 2 \leq q \leq \infty,\ \ \lambda \geq 1. \tag{3.44}$$

Furthermore, by (3.5) and(3.40)

$$\left\|e^{-i\tilde{t}H_\lambda}\phi\right\|_{L^{p+1}}^{p+1} = \lambda\left\|e^{-itH}\phi_\lambda\right\|_{L^{p+1}}^{p+1} \leq C\frac{1}{t^{d(p+1)}}\lambda\left\|\phi_\lambda\right\|_{L^{1+1/p}}^{p+1} = C\frac{1}{\tilde{t}^{d(p+1)}}\left\|\phi\right\|_{L^{1+1/p}}^{p+1}, \tag{3.45}$$

with $d := \frac{1}{2}\frac{p-1}{p+1}$. Equation (3.20) follows from (3.42), (3.43), (3.44), (3.45) and the dominated convergence theorem, observing that $2(j_0 + j + 1) \geq p + 1$, that $d(p+1) > 1$ and that V_j is continuous.

Proof of Corollary 3.5: By Corollary 3.3, S determines uniquely V_0. Then the wave operators, $W_\pm$, are uniquely determined, and by (3.17), S determines uniquely S_{V_0}. Finally by (3.19) and (3.20) S_{V_0} determines uniquely $V_j, j = 1, 2, \cdots$.

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