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MECHANICS OF STRESSEDLY-DEFORMED STATE

The features of inclined force acting on 1D homogeneous elastic lumped line and corresponding modernisation of the wave equations

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This paper was published first in the journal of National Academy of Sciences of Belarus "Materials. Technologies. Tools", **6** (2001), 4, pp.13-19 (in Russian). This is the fourth of a large cycle devoted to the new method to obtain the exact analytical solutions for vibrant elastic systems, both lumped and distributed. Though the material stated in it is quite simple, as well as the investigated problems, you can make sure from the presented solutions that in the exact analytical form and in all their completeness these solutions were not obtained up to now.

It is important that the obtained solutions match well to the generalising solution of the wave equation which we present in this paper. Also important is that these results are basic for the further investigations which we carry out for much more complicated models. There we often refer to this introducing paper, so we decided to load it also to the Archive, to make its English version available for English-language readers. You can read some of our further papers, visiting for free our electronic journal "SELF Transaction" at the URL

http://angelfire.lycos.com/la3/selftrans .

We will be thankful if you express us your opinion and the more grateful for your wish to go on with us in the investigation of this huge and prospective theme.

Abstract

We analyse the exact analytical solutions for 1D elastic lumped lines under action of an external force inclined to the line axis. We show that in this case an inclined wave being described by an implicit function propagates along the line. We extend this conclusion both to free vibrations and to distributed lines. We prove that the presented solution in the form of implicit function is a generalizing for the wave equation.

When taken into consideration exactly, the dynamical processes pattern leads to the conclusion that the divergence of a vector in dynamical fields is not zero but proportional to the scalar product of the partial derivative of the given vector with respect to time into the wave propagation direction vector.

Keywords: Mathematical physics, Wave physics, Dynamics, Elastic lumped lines, Inclined force action, General solution of the wave equation, Vector flgebra, Divergence of vector in dynamical fields, ODE systems

Classification by MSC 2000: 30E25; 70E55; 70J35; 70J60; 70K40; 70F40

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1. Introduction

In the previous papers of this cycle, [1]-[3], we have considered the longitudinal vibrations in a 1D elastic lumped line. However, in such line the vibrations of more general form are possible, if we attribute the concept of one-dimensionality only to the general shape of a line but not to the degree of freedom of the line elements vibration. In this paper we will study the features of exact analytical solutions for this class of problems.

We will also present some results developing the vector algebra for dynamical fields, as well as some essential improvements of general solution of the wave equation. Though these results seem simple, their impact on the further concept of basic models will be essential enough, they to be formulated especially. This is why, targeting to formulate the most completely the basis of the exact analytical solutions for dynamical many-body systems, we do not care, are the considered problems complicated or simple, but draw our attention to the importance of such or other aspect in general context of the studied phenomenon.

2. Vibrations in a semi-finite elastic line under an inclined external force acting on the line start

In the view of stated above, supposing an elastic line elements having two degrees of vibration freedom, we can present the model as shown in Fig. 1. In case of small vibration amplitude (linear vibrations), this model can be described by two systems of equations – with respect to x- and y-projections of the external force action correspondingly:

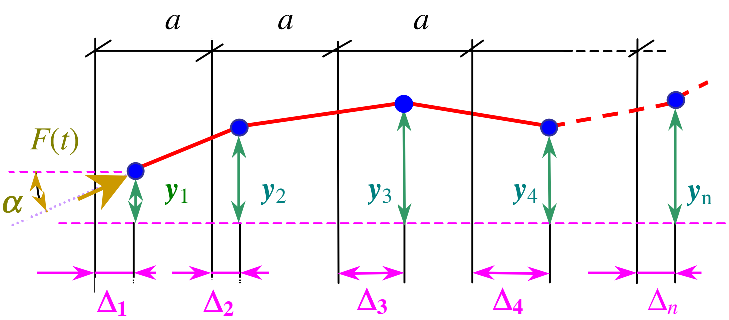

Fig.1. The model of inclined vibrations in a 1D elastic lumped line

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2} = F(t)\cos\alpha + s(\Delta_2 - \Delta_1) \\ m\dfrac{d^2\Delta_2}{dt^2} = s(\Delta_3 + \Delta_1 - 2\Delta_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ m\dfrac{d^2\Delta_n}{dt^2} = s(\Delta_{n+1} + \Delta_{n-1} - 2\Delta_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{1}$$

and

$$\begin{cases} m\dfrac{d^2 y_1}{dt^2} = F(t)\sin\alpha + s(y_2 - y_1) \\ m\dfrac{d^2 y_2}{dt^2} = s(y_3 + y_1 - 2y_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ m\dfrac{d^2 y_n}{dt^2} = s(y_{n+1} + y_{n-1} - 2y_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \end{cases} \tag{2}$$

where α is the angle of an external force inclination to the line axis.

Each of these systems is similar to those considered in [1]. Hence, we can write at once the exact analytical solutions for each of them, as follows:
for the x-component of vibration:
in the periodical regime ($\beta < 1$)

$$\Delta_n = -j\frac{F_0\cos\alpha}{\omega\sqrt{sm}}\, e^{j[\omega t-(2n-1)\tau]} , \tag{3}$$

in the aperiodical regime ($\beta > 1$)

$$\Delta_n = (-1)^n\frac{F_0\cos\alpha}{\omega\sqrt{sm}}\,\gamma_-^{(2n-1)}\, e^{j\omega t} \tag{4}$$

and in that critical ($\beta = 1$)

$$\Delta_n = (-1)^n\frac{F_0\cos\alpha}{2s}\, e^{j\omega t} . \tag{5}$$

For the y-component of vibrations we yield correspondingly:
in the periodical regime ($\beta < 1$)

$$y_n = -j\frac{F_0\sin\alpha}{\omega\sqrt{sm}}\, e^{j[\omega t-(2n-1)\tau]} , \tag{6}$$

in the aperiodical regime ($\beta > 1$)

$$y_n = (-1)^n\frac{F_0\sin\alpha}{\omega\sqrt{sm}}\,\gamma_-^{(2n-1)}\, e^{j\omega t} \tag{7}$$

and in that critical ($\beta = 1$)

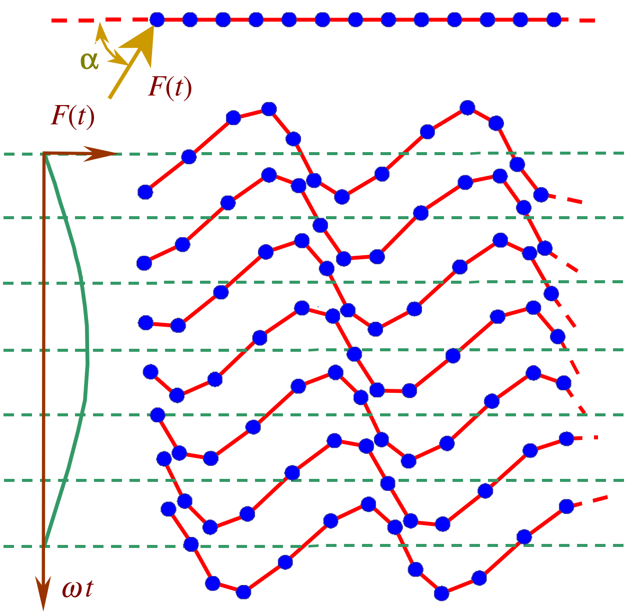

Fig.2.The diagram of vibrations in a semi-finite line being under action of a force inclined to the line axis. The line parameters: *m = 0,01 кг; s = 100 Н/м; a = 0,01 м;* $\alpha = 60^{o}$; $F_o = 2$ *Н; f = 15 Гц*

$$y_n = (-1)^n \frac{F_0 \sin\alpha}{2s} e^{j\omega t} \tag{8}$$

As a result of the superposition, there forms an inclined wave propagating in the positive direction of the axis x; this is corroborated by the vibration diagram shown in Fig. 2.

Typically, the inclined pattern of vibrations remains both with free vibrations in a lumped line and with the limiting process to a distributed line.

Basing on results presented in [1], the solution, e.g., for free vibrations has the following form: for the x - component of vibrations

$$\Delta_n = \frac{X_k \cos(2i-1)\tau}{\cos(2k-1)\tau} e^{j\omega t} \tag{9}$$

and for the y -component

$$y_n = \frac{Y_k \cos(2i-1)\tau}{\cos(2k-1)\tau} e^{j\omega t}\ , \tag{10}$$

where X_k and Y_k are the x - and y - components of vibration amplitude of the k th element whose parameters are specified, and k is the element number whose vibration is specified.

In case of limiting process to a distributed line we can present

$$\rho = \frac{m}{a}; \qquad s = \frac{T}{a}; \qquad n = \frac{x_0}{a}\ ,$$

where ρ is the density; T is the line stiffness; x_0 is the distance from the line start to the point of rest of the considered line element; m is the line elements mass. With it the solutions (3)-(8) transform into the following system:

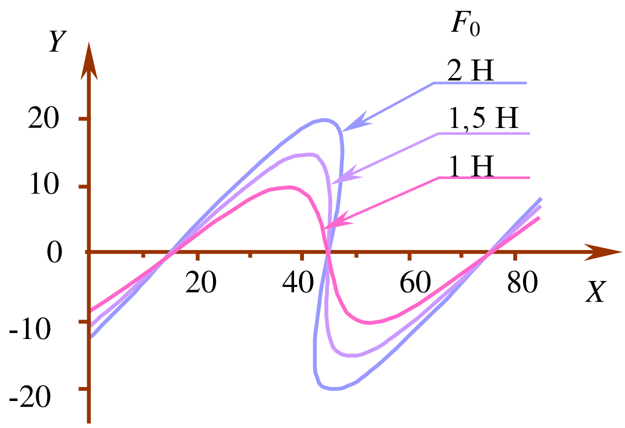

Fig.3. Vibrations in a distributed line under action of the external inclined force on the line start . The angle of force inclination α=60°

$$\begin{cases} x = -j\dfrac{F_0 \cos\alpha}{\omega\sqrt{T\rho}} e^{j\omega\left(t - x_0\sqrt{\rho/T}\right)} + x_0 \\ y = -j\dfrac{F_0 \sin\alpha}{\omega\sqrt{T\rho}} e^{j\omega\left(t - x_0\sqrt{\rho/T}\right)} \end{cases} \tag{11}$$

The obtained system describes parametrically an inclined wave propagation along the axis x, as is shown in Fig. 3.

From the carried out brief investigation we can see that far from always inclined vibrations arise as a consequence of non-linear processes in an elastic system, as it was supposed before. Inclined waves can arise quite naturally under action of a force inclined to the wave propagation direction. And this conclusion can be quite simply extended to a most wide spectrum of the vibration processes.

3. The line elements motion trajectory

Drawing our attention to a considered line elements motion trajectory, we see that this trajectory has a form of an ellipse circumscribed around the point of rest of an element. And the wave inclination forms due to the shift phase of the element-to-element motion along these elliptic trajectories. The presented vibration structure is well-known in physics, particularly in wave processes in unbounded volumes of liquid. “In a wave, the motion of liquid is non-stationary. So the separate particles trajectories are far from being time-coinciding with the lines of current. They have absolutely other form. With the small amplitudes they are circumferences in a great approximation. We find these circular trajectories both on a surface and in depth of liquid. Only in the most upper layers the diameters of circular ways are the most large” [4, pp. 300-301].

Indeed, the spatial vibration processes have their peculiarities. None the less, it is typical that the basic regularities can be run down already in a 1D model. It also follows from the obtained solutions that the elements vibration ellipsoidal pattern retains as in the critical as in aperiodical regimes. Due to it, in this last case there in the line forms a complex wave fast-damping along the line, and this is one more feature that the exact analytical solutions demonstrate.

4. The new class of functions being the solution of the wave equation [5]

We can extend the above generalisation also to the solution of the wave equation on the whole. It is known that the hyperbolic-type differential equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{k^2}{\omega^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{12}$$

has general solution [6, p. 300]

$$\Phi(x,t) = \Phi_1(kx - \omega t) + \Phi_2(kx + \omega t), \tag{13}$$

where $c = \omega/k$ is the wave propagation velocity – i.e., in the form of two explicit functions with respect to $(x - ct)$ and $(x + ct)$ correspondingly. Up to now this solution was thought the only and complete, due to the theorem of uniqueness of solution of differential equation. None the less, there exists one more class of functions being the solution of (12) but not taken into account by the solution (13). We can present the general form of this class of functions as

$$y(x,t) = \Phi_1\left(kx - \omega t + \phi_1(y)\right) + \Phi_2\left(kx + \omega t + \phi_2(y)\right), \tag{14}$$

here $\phi_1(y)$ and $\phi_2(y)$ are some double-differentiable functions. In other words, (14) belongs to the class of implicit functions whose regularities of behaviour and technique of differentiation and integration essentially differ from such for explicit functions. It is important that, while for explicit functions definite systematisation of differential equations has been created and for definite class of these equations the regularities and schemes to obtain solutions have been defined, for implicit functions all these developments are absent. Naturally, for today the correspondence of (14) to (12) can be checked only by the most simple way – by direct substitution (14) into (12).

For it, basing on the known laws of implicit functions differentiation, find first and second particular derivatives of (14) with respect to x and t. To simplify the calculation, consider a half of the right part of (14)

$$y(x,t) = \Phi_1\left(kx - \omega t + \phi_1(y)\right) = \Phi_1(A), \tag{15}$$

where

$$A = kx - \omega t + \phi_1(y).$$

The first derivatives have the form

$$\frac{\partial y}{\partial x} = \frac{k\frac{d\Phi_1}{dA}}{1 - \frac{d\phi_1}{dy}\frac{d\Phi_1}{dA}}; \qquad \frac{\partial y}{\partial t} = -\frac{\omega\frac{d\Phi_1}{dA}}{1 - \frac{d\phi_1}{dy}\frac{d\Phi_1}{dA}}. \tag{16}$$

The second derivatives after transformation and substitution of expressions (16) take the form

$$\begin{gathered}\frac{\partial^2 y}{\partial x^2} = \frac{k^2\left[\frac{d^2\Phi_1}{dA^2} + \frac{d^2\phi_1}{dy^2}\left(\frac{d\Phi_1}{dA}\right)^3\right]}{\left(1 - \frac{d\phi_1}{dy}\frac{d\Phi_1}{dA}\right)^3} \\ \frac{\partial^2 y}{\partial t^2} = \frac{\omega^2\left[\frac{d^2\Phi_1}{dA^2} + \frac{d^2\phi_1}{dy^2}\left(\frac{d\Phi_1}{dA}\right)^3\right]}{\left(1 - \frac{d\phi_1}{dy}\frac{d\Phi_1}{dA}\right)^3}.\end{gathered} \tag{17}$$

Substituting (17) into (12), we obtain the required. Similarly we can prove the second part of expression (14) to be corresponding to the equation (12).

Thus, the solution (17) determines a whole class of implicit functions satisfying the linear wave equation. And the presence of a new class of functions being the solution of differential equation

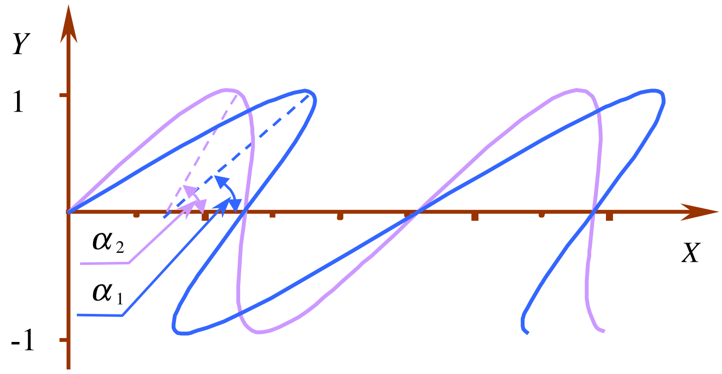

Fig.4. The inclined vibrations described by the implicit function being the solution of a wave equation at α_1=45°; α_2=60°

(12) does not violate a least the theorem of uniqueness of solution of differential equation, because under definite conditions

$$\phi_1(y) \equiv 0; \quad \phi_2(y) \equiv 0$$

the expression (14) degenerates into (13). Hereby it is proved that the solution known before is a particular case of more general class of functions.

The found class of implicit functions determines a non-linear wave; its degree of deformation depends on the type of functions $\phi_1(y)$ and $\phi_2(y)$. For example, in the following particular case of expression (14) (see Fig. 4)

$$y = c\sin(kx - \omega t + y\,\mathrm{ctg}\,\alpha) \tag{18}$$

the solution of a wave equation (12) describes a progressive wave propagating along the axis x and inclined by the angle α, which completely corresponds to the inclined vibration in an 1D line considered above.

4. The divergence theorem changing in dynamical fields

Studying the changes appearing in the wave physics solutions, we must touch briefly the changes in the vector algebra equations appearing when taking into account the dynamical processes in power fields.

The main conservation laws related to the vector flux are known to be formulated on the basis of Ostrogradsky-Gauss theorem about the flux of vector through a picked out space region. The Ostrogradsky-Gauss theorem is known in its turn to be formulated for stationary fields, i.e., for the case when the vector of flux does not depend on time directly. In dynamical fields the pattern essentially changes.

Actually, let in some bounded, connective, free of sources spatial region Ω propagate a plane-parallel wave whose power vector $\vec{F}(x,t)$ has a standard form

$$F(x,t) = F_0 \sin(\omega t - kx)\,. \tag{19}$$

Pick out of this region four surfaces a_0, a_1, a_2, a_3 perpendicular to the wave propagation direction, and form with their help three picked out volumes V_{01}, V_{02}, V_{03} bounded by the corresponding surfaces and lateral surface connecting them. Noting that the wave is 1D and $\vec{F}(x,t)$ is parallel to the lateral surface of the picked out regions, in the further consideration we will not take into account the lateral surfaces.

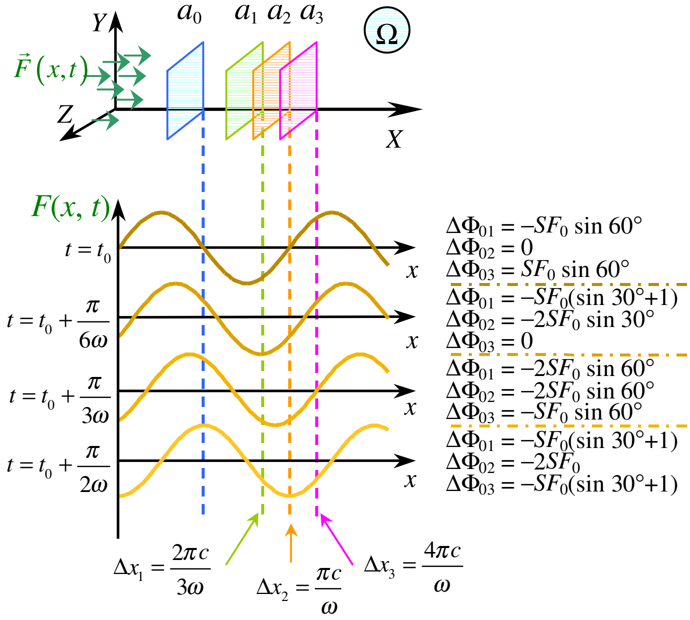

Fig.5. The time-dependent diagram for the investigation of vector flux through the picked out space

On the basis of this model, consider a conventional definition of the divergence of vector (see, e.g., [5, p.166])

$$div\vec{F}(r,t) = \vec{\nabla}\cdot\vec{F} = \lim_{\delta\to 0}\frac{\int_S \vec{ds}\cdot\vec{F}(s)}{\int_V dv}, \tag{20}$$

where V is the studied region containing the point $(\vec{r})$; S is the closed-loop surface limiting the region V; δ here denotes the most distance from the point $(\vec{r})$ to the points of the surface S.

Since the picked out regions V_{01}, V_{02}, V_{03} were finite, study first the expression

$$G_i = \frac{\int_S \vec{ds}\cdot\vec{F}(s)}{\int_V dv} = \frac{\Phi_i - \Phi_0}{\int_V dv}, \tag{21}$$

where i=1, 2, 3. For it, plot the time-spatial variation of $\vec{F}(x,t)$, noting the wave process progressive pattern, and determine $\Delta\Phi_{0i} = \Phi_i - \Phi_0$ for all picked out regions. This will be easy, noting the wave front plane pattern and that the flux of vector is 1D. Therefore the integration is reduced to a simple multiplying of the vector parameters at the fixed moment at the studied point into the square of corresponding surface of the picked out region. The obtained results are shown in Fig. 5 (bottom and right). As we see, $\Delta\Phi_{0i}$ varies in time for all picked out regions and is different for all surfaces a_0, a_1, a_2, a_3, though we calculate relatively to the surface a_0 common for all regions. Furthermore, if we plot the regularity $\Delta\Phi_{0i}(t)$ on the basis of obtained values $\Delta\Phi_{0i}$, we will see that both amplitudes and phases of the fluxes difference variation $\Delta\Phi_{0i}$ are different for all picked out regions, the same as these parameters are different for G_i. Moreover, with the diminishing size of the picked out region, G_i grows, tending to some finite limit value. This corroborates that the flux of vector through the picked out region is time-inconstant in dynamical fields. And the fact itself of $\Delta\Phi_{0i}$ variation is caused not by the spatial parameters but just by the progressive pattern of the wave process, and we can easy prove it mathematically.

Actually, for each picked out region at the moment t_0

$$\begin{aligned}\Delta\Phi_{0i} &= \Phi_i - \Phi_0 = \left(F\left(r_i, t_0\right) - F\left(r_0, t_0\right)\right)\cdot S = F_0 S\left[\sin\left(\omega t_0 - kx_i\right) - \sin\left(\omega t_0 - kx_0\right)\right] = \\ &= -2F_0 S \sin k\frac{\Delta x_i}{2}\cos\left(\omega t_0 - kx_0 - k\frac{\Delta x_i}{2}\right),\end{aligned} \tag{22}$$

where $\Delta x_i = x_i - x_0$. With it the denominator of (7) takes the form

$$\int_{V_{0i}} dv = S\Delta x_i\ . \tag{23}$$

Consequently,

$$G_i = -\frac{2F_0 \sin k\frac{\Delta x_i}{2}}{\Delta x_i}\cos\left(\omega t_0 - kx_0 - k\frac{\Delta x_i}{2}\right). \tag{24}$$

As we can see from (24), G_i variation depends as on the parameters t_0 and x_0 determining the momentary time of G_i investigation and location of the picked out region as on the wave delay phase when transiting from the surface a_0 to the surface a_i. But the wave delay has the time-dependent pattern, since, though the wave propagates in space, its delay is caused by the finiteness of its propagation velocity, which makes it just time-dependent. So we can transform Δx_i in the following standard way:

$$\frac{k\Delta x_i}{2} = \frac{kc}{2}\Delta t_i = \frac{\omega}{2}\Delta t_i\ ,$$

where Δt_i is the wave delay time when transiting along the picked out region V_{0i}. Then (10) takes the form

$$G_i = -\frac{\omega F_0}{c}\frac{\sin\frac{\omega\Delta t_i}{2}}{\frac{\omega\Delta t_i}{2}}\cos\left(\omega t_0 - kx_0 - \frac{\omega\Delta t_i}{2}\right). \tag{25}$$

It means that despite we study the wave process simultaneously on the entire surface of the picked out region at the only moment t_0, due to the wave time-delay the value G_i does not vanish but depends on the wave delay time Δt_i in the picked out region.

When diminishing the size of picked out region V_{0i}, i.e., when Δx_i and correspondingly at Δt_i tend to zero, on the basis of (20) we yield

$$div\vec{F}\left(x_0, t_0\right) = -\frac{\omega F_0}{c}\lim_{\Delta t_i\to 0}\frac{\sin\frac{\omega\Delta t_i}{2}}{\frac{\omega\Delta t_i}{2}}\cos\left(\omega t_0 - kx_0 - \frac{\omega\Delta t_i}{2}\right) = -\frac{\omega F_0}{c}\cos\left(\omega t_0 - kx_0\right). \tag{26}$$

As we see, in the considered model of a 1D wave process the divergence of vector does not vanish.

General proof of this theorem presented in [7, pp.28-33] and [8, pp.234-243] leads us to the following regularity:

$$div\vec{F}(r,t) = -\frac{1}{c}\vec{n}\cdot\frac{\partial\vec{F}(r,t)}{\partial t}, \qquad (27)$$

i.e., *when the wave flux propagates in a source-free space, the divergence of vector of flux is proportional to the scalar product of the particular derivative of this vector with respect to time into the unit vector of the studied flux direction.*

V.A. Atsukovsky [9, pp.172-173] has obtained an alike result through the differential forms.

5. Conclusions

The carried out investigation showed that in a semi-finite model of an elastic line on whose free end a force inclined to the line axis acts the inclined waves propagate, and they are described by an implicit function. This conclusion can be easy extended to the finite lines, to the free vibrations in an elastic line and to the distributed lines vibrations.

The general solution in the form of implicit function is generalising for the conventional solution of a wave equation being the superposition of travelling waves; this essentially changes the concept of the non-linear vibration pattern in the form of inclined waves propagating in elastic systems.

We also have ascertained that under an inclined force action the line elements move by an elliptic trajectories.

If taking into account the dynamics of processes, we see that in dynamical fields the divergence of vector is not zero, as conventionally, but is proportional to the scalar product of the particular derivative of this vector with respect to time into the unit vector of the studied flux direction. When transiting to the stationary processes which do not depend on time directly, the divergence naturally vanishes, and for these stationary processes the Ostrogradsky-Gauss theorem is true.

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