Rakib–Sivashinsky and Michelson–Sivashinsky Equations for Upward Propagating Flames: A Comparison Analysis

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Abstract

We establish a comparison between Rakib–Sivashinsky and Michelson–Sivashinsky quasi-linear parabolic differential equations governing the weak thermal limit of upward flame front propagating in a channel. For the former equation, we give a complete description of all steady solutions and present their local and global stability analysis. For the latter, multi–coalescent unstable steady solutions are introduced and shown to be exponentially more numerous than the previous known coalescente solutions. This fact is argued to be responsible for the disagreement of the observed dynamics in numerical experiments with the exact (linear) stability analysis and also gives the ingredients to describe the quasi–stable behavior of parabolic steadily propagating flame with centered tip.

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Rakib–Sivashinsky quasilinear parabolic differential equation governing the weak thermal limit of an upward flame interface propagating in a channel is given by

$$\phi_t = \varepsilon\phi_{xx} - \frac{1}{2}\phi_x^2 + \phi - \bar{\phi}\,, \tag{1}$$

where $y = \phi(t,x)$, $0 < x < \pi$, defines an instantaneous flame profile in dimensionless variables, $\bar{\phi} = \dfrac{1}{\pi}\displaystyle\int_0^\pi \phi(t,x)\,dx$ denotes the space average, $\varepsilon > 0$ is a physical parameter (Markstein length) and Neumann (adiabatic) boundary condition at channel walls is imposed: $\phi_x(t,0) = \phi_x(t,\pi) = 0$. According to Darrieus–Landau's hydrodynamic flame theory[RS] there are two competing sources of instabilities in this model given by the last two terms of (1). A plane flame front separating the cold and hot gas is subject to Rayleigh–Taylor instability, due to the thermal expansion, and to the buoyancy effect caused by external acceleration. These two ingredients lead the front to become convex toward the cold gas. As in [RS], the acceleration vector points toward positive y–values and the upward propagating direction is negative. For the purpose of comparison such convention will be maintained.

Paraboloid profiles with the tip located around the center of the channel are observed, both experimentally and in computational simulations, to remain "stable" for long time (see e.g. [MiS] and references therein). There are, however, experiments whose paraboloid tip eventually slides to the channel wall. A mathematical description of the former configuration as a "quasi–equilibrium transient state" was provided by Berestycki, Kamin and Sivashinsky [BKS] in which work stationary solutions of (1), their respective stability properties and the nonlinear dynamics were described for sufficiently small ε. Metastable dynamics has been studied in details in [SW]. The proofs of the other theorems stated in [BKS] remain, to the best of our knowledge, unpublished.

Sivashinsky's previous equation of weak thermal expansion[S, MS]

$$\phi_t = \varepsilon\phi_{xx} - \frac{1}{2}\phi_x^2 + I(\phi)\,, \tag{2}$$

hasn't considered buoyancy effect and Darrieus–Landau instability has been taken into account replacing $\phi - \bar{\phi}$ in (1) by a linear singular integral operator[1] given by a multiplication by $|k|$ in the Fourier representation

$$I(\phi)(t,x) = \frac{1}{\pi}\sum_{k=1}^{\infty} k \int_{-\pi}^{\pi} \cos\left[k(x-y)\right]\,\phi(t,y)\,dy\,.$$

In contradistinction, the only trace of $I(\phi)$ term in equation (1) comes from the removal of $k=0$ mode

$$\phi - \bar{\phi} = \frac{1}{\pi}\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}\cos\left[k\,(x-y)\right]\,\phi(t,y)\,dy\ .$$

Michelson–Sivashinsky equation (2) has been studied by many authors after Thual–Frish–Hénon's application[TFH] of pole decomposition. For a more recent survey of this method, see Vaymblat and Matalon[VM1, VM2] and references therein.

Despite of the fact that equations (1) and (2) differ in many respects, it is our purpose to expose the similarities and distinctions of their solutions. We shall see, by the introduction of so called bi–coalescent steady states, that equation (2) may also admit quasi–equilibrium transient "parabolic" profile with centered tip. Our comparison relies on two recent analysis. Firstly, Vaymblat and Matalon [VM1] have determined all steady solutions whose poles coalesce into a line parallel to the imaginary axis and solved explicitly the eigenvalue problem of (2) linearized about those states. They conclude, in a second paper[VM2], the existence of only one linearly stable steady coalescent pole solution.

For equation (1) with a different parametrization, we have given [GM] a complete description of all equilibrium solutions and provided their local and global stability in an appropriated Sobolev space. Since our results are major extensions of those stated in [BKS] and the mathematical presentation in [GM] may cause certain difficulties in translating to the present application, we shall here restate them with a brief explanations of their proofs. A detailed presentation including the metastable states analysis will appear elsewhere [MG].

We shall first restrict ourselves to the Rakib–Sivashinsky equation. The details presented after each statement (detached by brackets) are essentials to establish the subsequent comparison but may be skipped in a first reading. Multi–coalescent steady states of Michelson–Sivashinsky equation will be considered next. A conclusion will be presented at the end.

To discuss our results let us consider the equation

$$u_t = \varepsilon u_{xx} - u\,u_x + u\ , \tag{3}$$

for the derivative $u=\phi_x$, with Dirichlet boundary conditions $u(t,0)=u(t,\pi)=0$. There is a one–to–one correspondence between

$$\theta(t,x) := \int_0^x u(t,y)\,dy\,, \tag{4}$$

with u a solution of (3), and a solution ϕ of (1) given as follows. Notice that θ satisfy (1) with $\bar{\phi}$ replaced by $\varepsilon\phi_{xx}(t,0)$. By definition, $\theta(t,0)=0$ for all $t\geq 0$ and their steady solutions do not propagate (they are equilibrium). We have $\theta(t,x)=\phi(t,x)-\phi(t,0)\,,\ \theta_x=\phi_x$ and $\theta_{xx}=\phi_{xx}$. For the opposite relation, we have

$$\phi+\int_0^t \bar{\phi}\;(s)\,e^{t-s}\,ds=\theta+\varepsilon\int_0^t \theta_{xx}(s,0)\,e^{t-s}\,ds\,,$$

because both sides satisfy equation (1) without $\bar{\phi}$ term, which yields

$$\phi(t,x)=\theta(t,x)-\int_0^t\left(\bar{\theta}\;(\tau)\;-\varepsilon\,\theta_{xx}(\tau,0)\right)\;ds\,. \tag{5}$$

This and (4) will be used to discuss the steadily propagating solutions of (1).

If A denotes the operator given by r.h.s. of (3) linearized about the trivial solution $u_0=0$, let $\mathcal{B}^{1/2}$ be the Banach space equipped with the graph norm $\|f\|_{1/2}=\displaystyle\int_0^\pi\left|A^{1/2}f(x)\right|^2dx$ (equivalent to the Sobolev space $H_0^1(0,\pi)$). In [GM], Theorems 3.2 and 5.10, we have proven:

The initial value problem (3) with $u(0,\cdot)=u_0\in\mathcal{B}^{1/2}$ has a unique solution for all $t>0$ and the trajectories $\{u(t,\cdot)\}_{t\geq 0}$ lie in a compact set in $\mathcal{B}^{1/2}$.

By (4) and (5), this provides the existence and uniqueness of the initial value problem (1) for all $t>0$. Compactness property will be useful for the global stability analysis.

The steadily propagating solutions of (1) can be obtained from the equilibrium solution of (3). The quantity inside the parenthesis in (5) remains positive under the dynamics of (3) and the front propagates upward (toward negative y–values). As a consequence, the flame front propagates steadily with velocity

$$V=-\;\bar{\vartheta}\;+\varepsilon\,\vartheta_{xx}(0)=-\frac{1}{2}\,\overline{\vartheta^2}, \tag{6}$$

provided $\vartheta(x)=\int_0^x v(y)\,dy$, where v is a non trivial solution of

$$\varepsilon v_{xx}-v\,v_x+v=0\,, \tag{7}$$

with $v(0)=v(\pi)=0$. The next result is Theorem 4.1 of [GM]:

For $\varepsilon > 1$, $v_0 = 0$ is the unique solution of (7). For $\varepsilon < 1$ such that $\dfrac{1}{k+1} \leq \varepsilon < \dfrac{1}{k}$ holds for some $k \in N$, there exist $2k$ nontrivial solutions $u_j^{\pm}$, $j = 1, \ldots, k$, with $j-1$ zeros in $(0,\pi)$ and such that $u_j^-(x) = -u_j^+(\pi - x)$ for j odd and $u_j^-(x) = u_j^+(\pi/j + x)$ mod π for j even. Each pair $u_j^{\pm}$ bifurcate from the trivial solution $v_0 = 0$ at $\varepsilon_j = 1/j^2$ with $\lim_{\varepsilon \uparrow \varepsilon_j} u_j^{\pm} = 0$. In the phase space $\mathbb{R}^2$, $\left(u_j^{\pm\prime}, u_j^{\pm}\right)$ describes closed orbits around $(0,0)$ whose distance from the origin increases monotonically as ε decreases.

[Equation (7) can be written as a dynamical system

$$\left\{ \begin{array}{lcl} w' & = & \varepsilon^{-1} p\,(w-1) \\ p' = & w\,, & \end{array} \right. \tag{8}$$

where $p = v$ and $w = v'$. Since (8) remains unaltered by changing $p \to -p$ and $x \to -x$, the orbits are symmetric with respect to the w–axis and Dirichlet boundary conditions $p(0) = p(\pi) = 0$ hold for any periodic orbit with period 2π such that $p = 0$ at $x = 0$. In [GM] we have shown that **(i)** $\mathbb{R}^2$ is foliated by non–overlapping orbits

$$\gamma_{w_0} = \{(w(x), p(x)) : x \in \mathbb{R} \text{ and } (w(0), p(0)) = (w_0, 0)\}$$

labeled by the coordinate w_0 of the positive w–axis; **(ii)** as ε varies, the orbit varies continuously from one to another; **(iii)** the orbits are closed if $w_0 < 1$ and open if $w_0 \geq 1$; **(iv)** the origin $(0,0)$ is enclosed by any closed orbits. The trajectories γ_{w_0}, $w_0 \geq 0$, are portrayed in Figure 1.

The period $T = T(\varepsilon, w_0)$ of a closed orbit varies continuously in ε and w_0 and is given by

$$T = \int_{\gamma_{w_0}} dx = 2 \int \frac{dp}{w}\,,$$

where the second integral is calculated in the semi–orbit. Hence, $T = 2\pi$ defines implicitly a continuous function $w_0 = w_0(\varepsilon)$, whose corresponding semi–orbit $\gamma_{w_0} = \{(w(x), p(x)) : 0 \leq x \leq \pi\}$ determines the solution $v(x) = p(x)$ of (7). The question to be addressed now is how many 2π–periodic orbits there are for each ε. If w_0 is small, γ_{w_0} can be approximated by an ellipsoid: $\varepsilon^{-1}p^2 + w^2 = w_0^2$;

$$T_{\text{ellipse}} = 4 \int_0^{\sqrt{\varepsilon}} \frac{dp}{\sqrt{1 - \varepsilon^{-1}p^2}} = 2\pi\sqrt{\varepsilon} \tag{9}$$

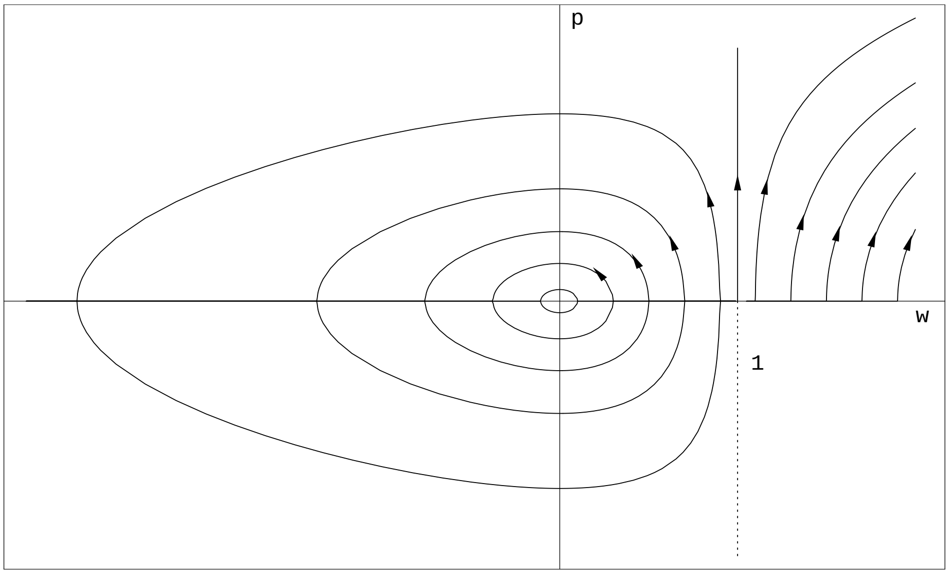

Figure 1: Trajectories of the dynamical system (8).

uniformly in w_0 and we have

$$\lim_{w_0 \to 0} T(\varepsilon, w_0) = 2\pi\sqrt{\varepsilon}\,. \tag{10}$$

In [GM], we have shown

$$\frac{\partial T}{\partial w_0} > 0 \tag{11}$$

holds form any $\varepsilon > 0$. Consequently, $T = T(\varepsilon, w_0)$ is a (strictly) monotonic increasing function of w_0 and ε, since $T(\varepsilon, w_0) = \sqrt{\varepsilon}T(1, w_0)$. As T increases when ε and w_0 increase, equation (10) implies that there is no nontrivial solutions ($v \neq 0$) for $\varepsilon \geq 1$, since the period of each nontrivial orbit exceeds 2π. Now, let

$$G_1(\varepsilon, w_0) = T(\varepsilon, w_0) - 2\pi\,,$$

be defined for $0 < \varepsilon \leq 1$ and $0 \leq w_0 < 1$. Under the condition (11), there exist a **unique** function $g_1 : [0,1] \longrightarrow \mathbb{R}_+$ with $g_1(1) = 0$ such that $G_1(\varepsilon, g(\varepsilon)) = 0$ for all $0 < \varepsilon \leq 1$. In view of (11), $g(\varepsilon)$ is a monotonically (strictly) decreasing function and $\lim_{\varepsilon\to 0} g(\varepsilon) = 1$.

Now, $v \equiv 0$ is the unique equilibrium solution of (3) for any $\varepsilon \geq 1$. For $\varepsilon < 1$ such that $1/(k+1)^2 \leq \varepsilon < 1/k^2$ holds for some integer $k \geq 1$, one can apply the implicit function theorem to equation

$$G_j(\varepsilon, w_0) := T(\varepsilon, w_0) - \frac{2\pi}{j} = 0\,,$$

with $j = 1, \dots, k$ and conclude, exactly as in the case for G_1, the existence of a unique monotone decreasing function $g_j : [0, 1/j^2] \longrightarrow \mathbb{R}_+$ with $g_j(1/j^2) = 0$ and $\lim_{\varepsilon \to 0} g_j(\varepsilon) = 1$, such that $G_j(\varepsilon, g_j(\varepsilon)) = 0$. The branch $g_j(\varepsilon)$ defines a nontrivial equilibrium solution v_j of (3) given by the p–component of the orbit $\gamma_{g_j(\varepsilon)}$ with $(g_j(\varepsilon), 0)$ at $x = 0$ and $p(x) = 0$ at each semi–period $x = m\pi/j$, $m = 1, \dots, j$. Hence, if ε is such that $1/(k+1)^2 \leq \varepsilon < 1/k^2$, there exist k equilibrium (not identically 0) solutions of (3), $v_1^+, \dots, v_k^+$, with v_j^+ having $j - 1$ zeros in the interval $(0, \pi)$. Notice that $g_j(\varepsilon) = v_j'(0) > 0$, $j = 1, \dots, k$, and $v_1(x) > 0$ for all $0 < x < \pi$. These will be useful for the stability analysis. There is an additional set of k equilibrium solution $v_1^-, \dots, v_k^-$, with $v_j^-(x) = p(\pi/j + x)$.]

When ε is small enough the orbit spends most of time at right semi–plane, $p(x) \simeq x \bmod 2\pi/j$ and $\vartheta_j^\pm(x) := \int_0^x v_j^\pm(y)\, dy$ has wrinkles of parabolic shape separated by "cusps". From (6) the steadily propagating solutions of (1) has the profile of ϑ_j propagating with the velocity V:

$$\varphi_j(x) = \vartheta_j(x) - \frac{t}{2}\, \overline{\vartheta_j^2}\,. \tag{12}$$

The plane flame front solution $\varphi_0 = \text{Const}$ is an asymptotically stable solution if $\varepsilon \geq 1$ for both equations (1) and (2). Its local stability can be read directly from the spectrum $\sigma(A)$ of the operator A, given by (1) and (2) linearized about φ_0, which consist of simple eigenvalues

$$\lambda_n = 1 - \varepsilon n^2 \qquad \text{and} \qquad \eta_n = n - \varepsilon n^2, \qquad n = 1, 2 \dots, \tag{13}$$

respectively, with corresponding eigenfunction $\sqrt{1/\pi} \cos nx$. So, $\lambda_n < 0$ and $\eta_n < 0$ if $\varepsilon > 1$. At $\varepsilon = 1$, the trivial solution φ_0 bifurcates [2] into a steadily propagating front in which, when extended periodically to the real line as an even function, there is one "parabolic" tip centered at one wall and a single "cusp" at another. The next result of [GM], Theorems 5.1 and 5.14, states that such configuration is the only one globally asymptotically stable solution of (1) for all $\varepsilon < 1$. Parabolic front with centered tip or wrinkled flame fronts may be described as quasi–equilibrium states discussed in refs. [MS, BKS, SW]. By (5), it is enough to examine the stability of the nontrivial equilibrium solutions of (3).

If $u_0 \in \mathcal{B}^{1/2}$ and $\varepsilon > 1$, then the initial value problem (3) with $u(0, \cdot) = u_0$ satisfies $\lim_{t \to \infty} \|u(t, \cdot)\|_{1/2} = 0$ and the trivial solution is globally stable. *For $\varepsilon < 1$ so that $1/(k+1)^2 \leq \varepsilon < 1/k^2$ holds, there* *exist* $\rho > 0$ *such that, if* $\|u_0 - v\|_{1/2} \leq \rho$, *then* $\lim_{t\to\infty} \|u(t,\cdot) - v\|_{1/2} = 0$ *for* $v = v_1^{\pm}$ *and, for any sequence* $\{u_n\}_{n\geq 1}$ *with* $\lim_{n\to\infty} \|u_n - v\|_{1/2} = 0$, *we have* $\sup_{t>0} \|u(t,\cdot) - v\|_{1/2} \geq \delta > 0$ *for all* n *and* v *equal to* 0 *or any equilibrium* $v_j^{\pm}$, $j = 2, \ldots, k$. *Moreover, there is an open dense set* $U \subset \mathcal{B}^{1/2}$ *containing the origin and* $\left\{v_j^{\pm}\right\}_{j=1}^{k}$ *such that, if* $u_0 \in U$ *and* $\varepsilon < 1$, *then* $\lim_{t\to\infty} \left\|u(t,\cdot) - v_1^{\pm}\right\|_{1/2} = 0$.

[Let us first examine the linear stability. If $u(t,x) = v(x) + \zeta(t,x)$ then equation (3) can be written as

$$\zeta_t = L\zeta + \zeta\zeta_x \,,$$

where[3]

$$L\xi = L\,[v]\,\xi = \varepsilon\xi'' - v\xi' + (1 - v')\xi$$

is the linearization of the right hand side of (3) about v. Acting on the space of functions ξ satisfying $\xi(0) = \xi(\pi) = 0$, L is symmetric, $(\eta, L\xi)_\rho = (L\eta, \xi)_\rho$, with respect to the inner product

$$(\eta,\xi)_p = \int_0^\pi \eta(x)\,\xi(x)\,\rho(x)\,dx$$

with weight $\rho(x) = \exp\left\{-\varepsilon^{-1}\int_0^x v(y)\,dy\right\}$. As a consequence, we can apply the comparison theorem to establish the following criterium (see [GM] for details).

Stability Criterium. *If* φ *is the solution of*

$$L\,[v]\,\varphi = 0\,,$$

on $0 < x < \pi$, *satisfying* $\varphi(0) = 0$ *and* $\varphi'(0) = 1$, *the largest eigenvalue* λ *of* $L\,[v]$ *is negative if* $\varphi > 0$ *on* $(0,\pi)$ *and positive if there exist* x' *such that* $0 < x' < \pi$ *and* $\varphi(x') = 0$.

For an equilibrium solution v of (3), let

$$\chi = c\,(2v - v'')\ ,$$

where $c > 0$ is chosen so that $\chi'(0) = 1$. It follows from equilibrium equation $\varepsilon v'' = -\,(1 - v')\,v$ that $\chi(0) = 0$ and $\chi > 0$ whenever $v > 0$ (recall $v(0) = 0$ and $(1 - v') > 0$ for all closed orbits). Moreover, if $v > 0$, an explicit calculation gives (see [GM])

$$L\,[v]\,\chi = -2\,(v')^2\,v < 0\ ,$$

and this implies, by applying the comparison theorem once more,

$$\varphi(x) \geq \chi(x)$$

for all $0 \leq x < \pi$. It thus follows from the stability criterium that v_1 is a stable equilibrium solution for all $\varepsilon < 1$ (recall $v_1(x) > 0$ for all $0 < x < \pi$, implying $\chi(x) > 0$ in the same domain). To show $v_2, \dots, v_k$ are unstable, we observe

$$L\,[v_j]\,v_j' = \left(\varepsilon v_j'' + (1 - v_j')v_j\right)' = 0\,, \tag{14}$$

by the equilibrium equation. It thus follows that v_j with $j \geq 2$ is unstable in view of the equilibrium criterium and the fact that v_j has at least one zero in $(0, \pi)$ and the same holds for φ by the Wronskian positivity $W(\varphi, v_j'; x) > 0$. It is important to note that v_j' fails to be an eigenfunction of $L\,[v_j]$ with 0 eigenvalue because it does not satisfies the boundary conditions required. Approximate eigenfunction with exponentially small (in ε) eigeinvalue can, however, be constructed using boundary layer techniques (see e. g. [SW]). Equation (14) holds also for any stationary solution v of Michelson–Sivashinsky equation.

We turn to the global stability. A Liapunov function for equation (3) can be constructed via the generalized Euler–Lagrange method due to Zelenyak, Lavrentiev and Vishnevskii [ZLV]. Let

$$U(u) = \int_0^\pi \Phi(u, u_x)\, dx$$

where $\Phi(p, w) = -p^2/(2\varepsilon) + (1 - w)\ln(1 - w) + w$ is an appropriated "Lagrangian" . The total derivative of U with respect to t is obtained by the calculus of variation

$$\begin{aligned} \dot{U}\,(u) &= -\int_0^\pi \left(\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u_x} - \frac{\partial \Phi v}{\partial u}\right) u_t\, dx \\ &= -\int_0^\pi \rho(u_x)\,(\varepsilon u_{xx} - u\, u_x + u)\, u_t\, dx \end{aligned}$$

where $\rho(w) = \varepsilon^{-1}/(1-w)$ is a positive weight. Note $\dot{U}$ is negative in view of (3) and U is a Liapunov functional. Due to the fact that the trajectories $\{u(t,\cdot)\}_{t\geq 0}$ lie in a compact set, LaSalle's invariance principle can be applied to show that all solutions of (3) in $\mathcal{B}^{1/2}$ converge to an equilibrium solution as $t \to \infty$. From equations (5), (4) and the two results stated before one concludes that $\varphi_0 = 0$

is globally asymptotically stable solution of (1) for $\varepsilon \geq 1$. In addition, if $\varepsilon < 1$, there is an open dense set $\mathcal{U} \subset \mathcal{B}^{1/2}$ of initial condition such that all solutions $\phi(t,x)$, with initial condition on $\mathcal{U}$, is asymptotically of the form $\vartheta_1 + Vt$ where $\vartheta_1 = \int_0^x v_1(y)\, dy$, the velocity $V = -\overline{\vartheta_1^2}\,/2$ and v_1 is the stable equilibrium solution of (3).]

MULTI–COALESCENT POLE SOLUTIONS

The same scenario seems to hold for equation (2), at least if one restricts to the space of coalescent pole solutions. According to [TFH, VM1], if ε is such that $\dfrac{1}{2n+1} \leq \varepsilon < \dfrac{1}{2n-1}$, $n = 1, 2, \ldots$, there exist n steadily propagating fronts, $\varphi_j(t,x) = \vartheta_j(x) + Vt$, with $\vartheta_j(x) = \displaystyle\int_0^x v_j(y)\, dy$ and

$$v_j(x) = \varepsilon \sum_{k=1}^{2j} \cot\left(\frac{x - z_k}{2}\right)$$

for $j = 1, \ldots, n$, each corresponding to the coalescent j–pole solution of (2), but only the one with largest number of poles, φ_n, is asymptotically stable. When ε crosses $1/(2n+1)$ from above, φ_n becomes unstable and a new solution φ_{n+1} bifurcates from the former solution.

φ_j was called coalescent j–pole solution because of the dynamic of poles $\{z_k(t)\}_{k=1}^{2j}$, induced by (2), tends to align them parallel to the imaginary axis. In [TFH, VM1], (2) was considered with periodic boundary conditions. With Newmann (adiabatic) boundary conditions there are two sets of steadily propagating fronts $\left\{\varphi_j^{\pm}\right\}_{j=1}^{n}$ distinguished by the location of their poles: φ_j^+ and φ_j^- has poles aligned at $\Re e\, z = 0$ and $\Re e\, z = \pi$, respectively.

Note that there is a relation between the number of steadily propagating fronts and of positive eigenvalues in the spectrum $\sigma(A) = \{\eta_j\}_{j \geq 1}$ of the linear operator $Av = \varepsilon v'' + I(v)$, since the latter increases by one unit at $\varepsilon_n = 1/n$ and the former at $\widetilde{\varepsilon}_n = 1/\left(2n-1\right)$. For equation (1), the number of steady solutions and the dimension of the unstable manifold $\mathcal{M}_n$ of the trivial solution $\varphi_0 = $ const, are equal with the instability value now located at $1/n^2$. There, in contradistinction, $\varphi_1^{\pm}$ are the only asymptotically stable solution for all $\varepsilon < 1$.

Although (12) holds for $\varphi_j^{\pm}$ with $v_j^{\pm}$ a coalescent pole solution of

$$\varepsilon v_{xx} - v\, v_x + I(v) = 0\,, \tag{15}$$

$v_j^{\pm}$ does not vanish in $(0,\pi)$. One may define a coalescent j–pole solution $v_j^{(k)\pm}$ with $k-1$ zeros in $(0,\pi)$, $k=1,2,\ldots$, by setting $v_j^{(k)\pm}(x)=\dfrac{1}{k}v_j^{\pm}(kx) \bmod \pi$. Note $v_j^{(k)\pm}$ solves (15) provided $v_j^{\pm}$ solves the same equation with ε replaced by $k\varepsilon$. As a consequence, a sets of steadily propagating fronts $\left\{\varphi_j^{(k)\pm}\right\}_{j=1}^{n}$ exists if ε is such that $\dfrac{1}{2n+1}\leq k\varepsilon<\dfrac{1}{2n-1}$ holds for some $n=1,2,\ldots$ (see Figure 3). In total, there are $2\sum_{m=1}^{n}[2n/(2m-1)]=o(n^2)$ coalescent steady solutions for $\varepsilon\gtrsim\dfrac{1}{2n+1}$, where $[z]$ means the integer part of real number z. The solutions $\varphi_j^{(k)\pm}$ with $k\geq 2$, are not stable and may represent the cellular profile observed experimentally provided an associate quasi–equilibrium solution described in [MS, BKS, SW] can be defined.

In the following, for each ε such that $\dfrac{1}{n+1}\leq\varepsilon<\dfrac{1}{n}$ holds, a new family $\mathcal{F}_n$ of steadily propagating flame front solutions, denominated multi–coalescent n–pole solutions, will be introduced. Our preliminary investigation indicates that there are at least an exponential number c^{n-1} of solutions in $\mathcal{F}_n$ and all, but 2 of them, seems to be unstable by numerical computation. As a consequence, the invariant set $\mathcal{K}_n^{MS}=\bigcup_{\varphi}\mathcal{M}_n(\varphi)$, defined as the union of the unstable manifold of all equilibrium solutions, for equation (2) differs enormously from the invariant set $\mathcal{K}_n^{RS}=\mathcal{M}_n(\varphi_0)$ for equation (1). Here, n indicates the number of bifurcations with respect to the trivial solution φ_0. In particular, $\mathcal{K}_n^{MS}$ may have dimension exponentially more numerous than the dimension of $\mathcal{K}_n^{RS}$ (for comparison, see Figures 2 and 3). We believe that this crucial distinction is responsible for the disagreement between the numerical study by Gutman–Sivashinky [GS] and the exact calculation by Vaymblat–Matalon [VM1].

The bi–coalescent n–pole solutions φ_{n_0,n_π} are indexed by (n_0,n_π) with $n_0+n_\pi=n$ indicating the number of pairs of complex conjugate poles[4] in each line $\Re e\, z=0$ and $\Re e\, z=\pi$. Note that the system of equations governing the dynamics of the poles $z_j=x_j+iy_j$, $j=1,\ldots,2n$, in the complex plane, given by

$$\dot{z_j}=-\varepsilon\sum_{l\neq j}\cot\left(\frac{z_j-z_l}{2}\right)-i\,\frac{y_j}{|y_j|}\,,$$

preserves the location of real part x_j since, in this case, $\Re e\,\cot\left(\dfrac{z_j-z_l}{2}\right)=0$. The poles of the bi–coalescent solution φ_{n_0,n_π} thus satisfies

$$\dot{x_j}=0\qquad\text{and}\qquad\dot{y_j}=F_j \tag{16}$$

where

$$F_j = \varepsilon \sum_{l \neq j} \left(\coth \left(\frac{y_j - y_l}{2} \right) \right)^{\eta_{jl}} - \frac{y_j}{|y_j|} . \tag{17}$$

with $\eta_{jl} := \cos(x_j - x_l)$ taking ± 1 values according the poles z_j and z_l are in the same or different line. Because of the real parte x_j of the pole gives the "cusp" position of a propagating flame profile, the bi–coalescent solution has its tip centered somewhere in the interior of channel.

Thual, Frisch and Hénon [TFH] have proven that, provided n is such that $\varepsilon(2n-1) < 1$, there exist one and only one coalescent steady solution and any solution of (16) with $\eta_{jl} = 1$ for all j and l, tends toward this steady state as $t \to \infty$. The proof of these properties is based on the existence of a Liapunov function with negative curvature in every direction. Here, there exists a Liapunov function for bi–coalescent solutions

$$U = \varepsilon \sum_{j,l:j \neq l} \ln \left| \frac{e^{(y_j - y_l)/2} + \eta_{jl} \, e^{-(y_j - y_l)/2}}{2} \right| - \sum_j |y_j| \ ,$$

satisfying $\dot{U} = \sum_{j=1}^{2n} F_j^2 \geq 0$ whose Hessian matrix $H = [U_{y_i y_j}]_{i,j=1}^{2n}$ cannot be proven to be negative definite in the case of strictly bi–coalescent solution (n_0, $n_\pi \neq 0, n/2$) since its Geršgorin discs may have non–vanishing intersection with the semi–plane $\Re e\, z \geq 0$. As U may have several local maxima and saddle points in this case (see Figure 2), any solution of equations (16) tends toward to a steady bi–coalescent state as $t \to \infty$ but uniqueness cannot be guaranteed.

Let the poles $z_1, \ldots, z_{2n}$ of a bi–coalescente steady solution φ_{n_0, n_π} be indexed as follows: $x_j = 0$ if $j = 1, \ldots, n_0$, $x_j = \pi$ if $j = n_0 + 1, \ldots, n$ and $y_{n+j} = -y_j$ for $j = 1, \ldots, n$. The case with $n_0 = n_\pi = n/2$ plays special role to describe the stability of coalescent solutions. Note that, if $y_j = y_{j+n/2}$ for $j = 1, \ldots, n/2$, then

$$\begin{aligned} v_{n/2,n/2}(x) &= \varepsilon \sum_{j=1}^{n/2} \sum_{\eta \in \{-1,1\}} \left\{ \cot \left(\frac{x - i\eta y_j}{2} \right) + \cot \left(\frac{x - \pi - i\eta y_j}{2} \right) \right\} \\ &= 2\varepsilon \sum_{j=1}^{n/2} \sum_{\eta \in \{-1,1\}} \cot (x - i\eta y_j) \end{aligned}$$

corresponds to a coalescent $n/2$–pole solution $v_{n/2}^2$ with one zero in $(0, \pi)$. As ε varies from $1/(2n-1)$ to $1/(2n+1)$, a point $\{z_j\}_{j=1}^{2n}$ satisfying $y_j = y_{j+n/2}$ can be shown to change from saddle point to a global maximum of U, turning a local maximum somewhere in between.

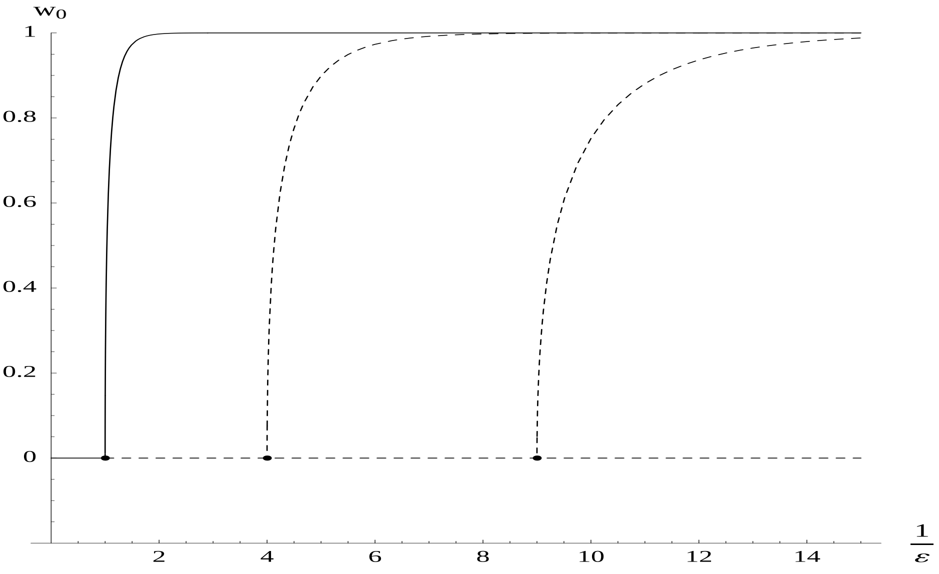

Figure 2: Bifurcation diagram for the distance $\Delta\varphi$ from cusp to tip of steadily state solutions of (1). Solid and dashed lines refer, respectively, to stable and unstable solutions.

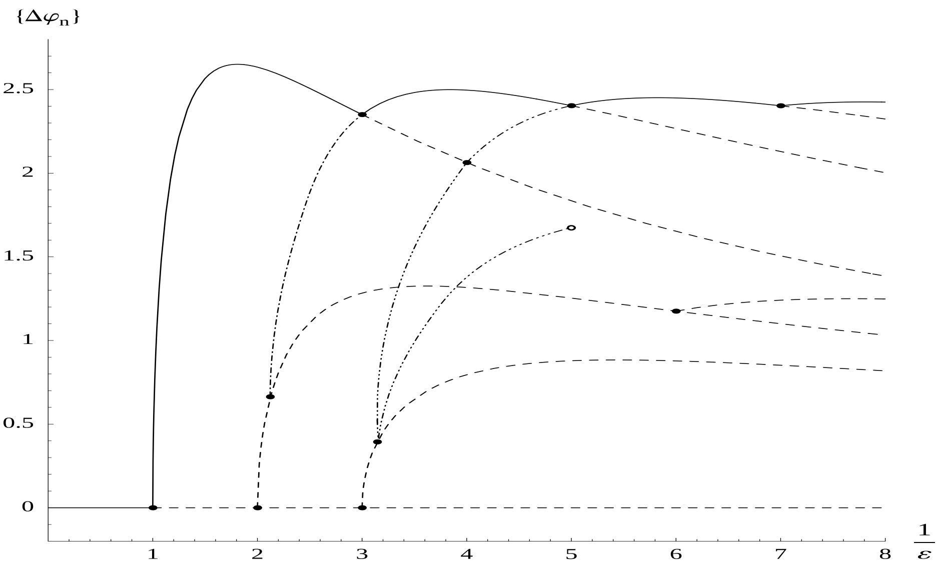

Figure 3: Bifurcation diagram for the distance $\Delta\varphi$ from cusp to tip of steadily state solutions of (2). Solid and dashed lines refer, respectively, to stable and unstable coalescente solutions. Doted-dashed lines refer to unstable multi–coalescente solutions.

Two conclusions can immediately be extracted from these observations. Because (16) tends to align the poles along either the line $\Re e\, z = 0$ or $\Re e\, z = \pi$, the coalescent n–pole solution φ_n is more stable than $\varphi^2_{n/2}$ since $\varphi^2_{n/2}$ is equivalent to a bi–coalescent n–pole solution $\varphi_{n/2,n/2}$ which is unstable under small perturbation which involves the real part x_j of the poles. Moreover, we may construct from a coalescent n–pole solution φ^k_n with $k-1$ zeros in $(0,\pi)$ a family of bi–coalescent solutions $\varphi^k_{n_0,n_{\pi/k}}$ with n_0 poles aligned in $\Re e\, z = 0$ and $n - n_0$ poles aligned in $\Re e\, z = \pi/k$ which agree with the coalescent solution at $\varepsilon \gtrsim 1/\left(2kn + k\right)$ if $n_0 = n_{\pi/k} = n/2$ and $y_j = y_{j+n/2}$. Proceeding in a similar fashion, one can introduce muti–coalescent n–pole solutions $\varphi_{n_0,n_{2\pi/k},\dots n_{2(k-1)\pi/k}}$, with $n_{2j\pi/k} \geq 1$ poles at $\Re e\, z = 2j\pi/k$ and $\sum_{j=0}^{k-1} n_{2j\pi/k} = n$ and which vanishes at $k-1$ points in $(0,\pi)$. Hence, a trajectory $\{z_j(t)\}_{t\geq 0}$ of (16) in the phase space $\mathbb{C}^n$, with $\{z_j(0)\}$ close to the poles of a multi–coalescent steady solution may go along many intermediate steady states before it reaches the final equilibrium.

CONCLUSIONS

Whether the cellular structure (flame profile with many wrinkles), observed numerically by Gutman–Sivashinky [GS] and experimentally by [G], could be produced by the instability of the coalescent pole solutions for ε small, has been debated in the literature (see e.g.[KOP, RAS]). The work of Vaymblat–Matalon [VM1, VM2] has resolved the controversies by proving that there always exist a unique (linearly) stable coalescent pole solution for $\varepsilon < 1$. In [VM1], the discrepancy between the numerical and the exact results is explained as an artefact of truncation and we shall not observe different profiles if more modes were included. According to Joulin [J], once equation (2) is incapable to describe observed wrinkled propagating flames (as an equilibrium solution) it should be replaced by another model.

A different scenario has been presented for Rakib–Sivashinsky equation. Numerical integrations of (1) have agreed with the analytic prediction since the beginning, although truncated equation has been used. Besides, parabolic profile with centered tip and cellular profiles can be successfully explained as metastable states [MiS, BKS, SW]. So, the question to be addressed is why equation (2) is more sensitive than (1) to be treated numerically and whether the cellular profiles can be described as a quasi–equilibrium solution of (2).

Based in the present analysis of equation (1) and in the existence of multi–coalescent steady states of (2), we argue in the following that many questions remain to be investigated before Michelson–Sivashinsky equation is abandoned.

Using the analysis in [GM], global existence in a Sobolev space H_0^1 (for all times $t > 0$) and uniqueness can be established for equation (2). A basic question is: *Does the solution ϕ of (2) with initial condition ϕ_0 in a dense subset of H_0^1 converge, as $t \to \infty$ and for all $\varepsilon < 1$, to one of coalescent steady state described in [VM1]?*

To answer this question a geometric analysis, as given for equation (1), must be carried out for equation (2). A family $\mathcal{F}_n$ of steady multi–coalescent poles solutions have been described. It may be difficult to determine whether $\mathcal{F}_n$ exhausts all steady solutions of (2) but it is already a remarkable difference between both equations. If all states of $\mathcal{F}_n$, except the states $\varphi_n^{\pm}$ described by [VM1], are shown to be unstable, then the unstable manifold $\mathcal{K}_n^{MS}$ containing the origin would have dimension at least exponentially large with n.

The method of [ZLV] may be useful to construct a Liapunov function. To establish the existence of a dense subspace of initial conditions from which the solutions of (2) converge to a steady solution requires, besides a Liapunov function, that the trajectories remain in a compact set. For this, it is enough that the trajectories remain bounded, which follows if the maximum principle can be shown to be adapted for equation (2).

Finally, the existence of multi–coalescent pole solutions would explain the discrepancy between the numerical integration by Sivashinky and the linear stability analysis in [VM1]. For Rakib–Sivashinsky equation, there exist metastable solutions whose time interval they remain "stable" becomes exponentially long when ε is small, creating the illusion that they have reached the equilibrium. If metastable states can be constructed from the bi–coalescent states is a question to be investigate. It would, in particular, describe the quasi–stable behavior of parabolic steadily propagating flame with centered tip. Moreover, the effect of truncation would become more sensitive than for equation (1) in view of the fact there is exponentially more numerous (meta)states available.

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[1] $I(\phi) = \text{P.V.} \int \frac{dy}{x-y}\, \phi_x(t,y)$ is the composition of minus the Hilbert transform and the derivative of ϕ with respect to x.

[2] In [VM1], the Darrieus–Landau instability term $I(\phi)$ of (2) has a $1/2$ factor in front so, the bifurcation point is at $\varepsilon = 1/\gamma = 1/2$. In [BKS], equation (1) is defined for $x \in (0,1)$ and the bifurcation occurs at $\varepsilon = \pi^{-2}$.

[3] Here, L is defined to be minus the L operator in [GM].

[4] Because we are looking for real valued solutions of (2), poles must occur in pairs of complex conjugate.