On the continuous analog of Rakhmanov's theorem for orthogonal polynomials.

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Abstract

We obtain the continuous analogs of Rakhmanov's theorem for polynomials orthogonal on the unit circle. Sturm-Liouville operators and Krein systems are considered. Under some conditions on the potential q of Sturm-Liouville operator, we prove that if absolutely continuous component of the spectrum fills the whole half-line, then q decays at the infinity.

Consider Sturm-Liouville operator on the half-line

$$H_\alpha f = -f'' + qf, \tag{1}$$

with boundary condition at zero $f(0) = \alpha f'(0)$. Assume that the potential q is uniformly locally integrable

$$\sup_{x\geq 0} \int_x^{x+1} |q(s)|ds < \infty. \tag{2}$$

Denote the spectral measure of the operator by $\rho_\alpha(\lambda)$. It is well known, that if potential $q(x)$ decays at the infinity sufficiently fast, then the absolutely continuous component of the spectral measure fills the whole positive half-line, i.e.

$$\rho'_\alpha(\lambda) > 0, \text{for a.e.}\, \lambda > 0. \tag{3}$$

For example, (3) was proved in each of the following cases

1. $q(x) \in L^2(R^+)$, [2].

2. $q(x)$ is uniformly locally square summable, i.e. $\sup\limits_{x\geq 0} \int\limits_x^{x+1} |q(s)|^2 ds < \infty$, and $e^{-x} \int\limits_0^x e^s q(s) ds \in L^2(R^+)$, [3].

We are interested in the following question. Does the property (3) imply the decay of potential at the infinity? Notice here that if (3) holds for some value α, then it holds for all real α including $\alpha = \infty$ [6]. The answer to this question is given by the following Theorem.

Theorem 2. *If $q(x)$ is bounded function, $q_-(x)\equiv\min\{0,q(x)\}\in L^1(R^+)$, and $\rho_\alpha'(\lambda)>0$ a.e. on R^+, then $\lim_{y\to\infty}\int\limits_{y-\delta}^{y+\delta} q(x)dx\to 0$ for each fixed $\delta>0$.*

Remark. The decay of potential in general does not imply (3). There are some examples of potentials with decay $q(x)<C(x+1)^{-\beta},\beta\le 1/2$, such that the spectrum is purely singular continuous on the positive half-line [7].

The analogous problem for the case of polynomials orthogonal on the unit circle was solved by Rakhmanov [13],[12]. Consider measure $\mu(\theta)$ on the unit circle and the corresponding orthonormal polynomials

$$\varphi_n(z)=\alpha_n z^n+\ldots+\beta_n, \tag{4}$$

where $\alpha_n>0$. Consider also monic orthogonal polynomials $\Phi_n(z)=\frac{\varphi_n(z)}{\alpha_n}$. Then, the following equations hold

$$\Phi_{n+1}(z)=z\Phi_n(z)-\overline{a_n}\Phi_n^*(z),\Phi_0(z)=1,\Phi_{n+1}^*(z)=\Phi_n^*(z)-a_n z\Phi_n(z),\Phi_0^*(z)=1, \tag{5}$$

where $\Phi_n^*(z)$ is the reverse polynomial $\Phi_n^*(z)=z^n\overline{\Phi_n(1/\bar z)}$. Coefficients a_n are called reflection (or Geronimus) coefficients.

Definition. Measure μ is called Erdös' measure if $\mu'(\theta)>0$ for a.e. $\theta\in[-\pi,\pi]$.

Definition. Measure μ is called Nevai's measure if $\lim\limits_{n\to\infty}a_n=0$.

Theorem [13]. *Each Erdös' measure is Nevai's measure.*

To obtain the continuous version of this statement we will use the continuous analogs of polynomials orthogonal on the unit circle. These polynomials were introduced by M.Krein in [8][1].

Assume that we are given Hermitian function $H(t)$ which is continuous on the whole line. If

$$\int\limits_0^x|\varphi(s)|^2ds+\int\limits_0^x\int\limits_0^x H(t-s)\varphi(t)\overline{\varphi(s)}dtds\ge 0 \tag{6}$$

for any function $\varphi(t)\in C[0,x]$ and any $x>0$, then the function

$$g(t)=-|t|/2-\int\limits_0^t(t-s)H(s)ds \tag{7}$$

belongs to the class G_∞ [9]. Therefore, we have the integral representation

$$\int\limits_0^t(t-s)H(s)ds=\int\limits_{-\infty}^{\infty}\left(1+\frac{i\lambda t}{1+\lambda^2}-e^{i\lambda t}\right)\frac{d\sigma(\lambda)}{\lambda^2}+\left(i\gamma t-|t|/2\right), \tag{8}$$

where γ is real constant. Nondecreasing function $\sigma(\lambda)$ is such that $\sigma(0)=0$ and

$$\int\limits_{-\infty}^{\infty}\frac{d\sigma(\lambda)}{1+\lambda^2}<\infty. \tag{9}$$

[1]For more details on the theory of Krein systems, see [15, 16, 17, 1, 14, 4].

Function $H(t)$ is called the accelerant for the function $g(t)$. Assume that for any $x > 0$, we have equality in (6) for $\varphi = 0$ only. In that case there exists the resolvent $\Gamma_x(t,s)$ for the kernel $H(t-s)$ for any positive $x > 0$. Define the continuous analogs of orthogonal polynomials by the formulas

$$P(x,\lambda) = e^{i\lambda x}\left(1 - \int_0^x \Gamma_x(s,0)e^{-i\lambda s}ds\right), P_*(x,\lambda) = 1 - \int_0^x \Gamma_x(0,s)e^{i\lambda s}ds. \tag{10}$$

Functions $P(x,\lambda), P_*(x,\lambda)$ satisfy the following system (Krein system)

$$\begin{cases} \frac{dP(x,\lambda)}{dx} &= i\lambda P(x,\lambda) - \overline{A(x)}P_*(x,\lambda), \quad P(0,\lambda) = 1, \\ \frac{dP_*(x,\lambda)}{dx} &= -A(x)P(x,\lambda), \qquad\qquad P_*(0,\lambda) = 1, \end{cases} \tag{11}$$

where $A(x) = \Gamma_x(0,x)$.

The system of functions $\{P(x,\lambda)\}$ can be considered as the orthonormal system with respect to nondecreasing function $\sigma(\lambda)$.

Proposition 1 [8]. *For each finite $f(x) \in L^2(R^+)$, we have the following equality*

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |F_P(\lambda)|^2 \, d\sigma(\lambda), \text{where } F_P(\lambda) = \int_0^{\infty} f(x)P(x,\lambda)dx.$$

Consequently, we have the isometric mapping U_P from $L^2(R^+)$ into $L^2(\sigma, R)$. Let $e^{-i\lambda x}P(2x,\lambda) = \varphi(x,\lambda) + i\psi(x,\lambda)$. Then φ, ψ are generalized eigenfunctions of the following Dirac system

$$\begin{cases} \frac{d\varphi}{dx} &= -\lambda\psi - a(x)\varphi + b(x)\psi, \quad \varphi(0,\lambda) = 1, \\ \frac{d\psi}{dx} &= \lambda\varphi + b(x)\varphi + a(x)\psi, \quad \psi(0,\lambda) = 0, \end{cases} \tag{12}$$

where $a(x) = 2\Re A(2x)$, $b(x) = 2\Im A(2x)$. The spectral measure of the Dirac system (12) is $2\sigma(\lambda)$. Notice here that if $H(t) \equiv 0$, then $A(x) \equiv 0$, $P(x,\lambda) = e^{i\lambda x}$, $P_*(x,\lambda) = 1$, $\sigma(\lambda) = \sigma_\circ(\lambda) = (2\pi)^{-1}\lambda$.

The continuous analog of Rakhmanov's theorem for Krein system with real-valued coefficient is as follows.

Theorem 1. *If the coefficient $A(x)$ in the Krein system is real-valued, $A(x)$, $A'(x)$ are bounded on R, and $\sigma'(\lambda)$ is finite and positive almost everywhere on R, then $A(x) \to 0$ as $x \to +\infty$.*

We can assume without loss of generality that $A(0) = 0$. Indeed, we can satisfy this condition by changing coefficient $A(x)$ on the finite interval, say, $[0,1]$. This change can be considered as trace-class perturbation (in the resolvent sense) for the associated Dirac system (12). Consequently, the measure σ for the modified system will still have the positive derivative almost everywhere on the whole line. We need some preliminary statements to prove Theorem 1.

Lemma 1. *Under the conditions of Theorem 1, the following limit exists*

$$\lim_{N\to\infty}\left[\int_{-\infty}^{\infty} \frac{N^2}{N^2+\lambda^2}|P(x,\lambda)|^2 d\sigma(\lambda) - \frac{N}{2}\right] = 0 \tag{13}$$

uniformly in $x \in R$.[2]

[2] Actually, this result in a sense is similar to the Levitan's results on the asymptotical behavior of spectral function for Sturm-Liouville operators [10].

Proof. It is known [8] that $|P(2x,\lambda)|^2 = |\varphi(x,\lambda)|^2 + |\psi(x,\lambda)|^2$, where

$$\begin{aligned} (a) &: \ \varphi'' - q_1\varphi + \lambda^2\varphi = 0,\ \varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = 0; \\ (b) &: \ \psi'' - q\psi + \lambda^2\psi = 0,\ \psi(0,\lambda) = 0,\ \psi'(0,\lambda) = \lambda. \end{aligned}$$

Here $q = a^2 + a'$, $q_1 = a^2 - a'$, $a(x) = 2A(2x)$. Denote the spectral measure for the Sturm-Liouville operator H_d with potential q and Dirichlet boundary condition at zero by ρ_d, and for operator H_n^1 with potential q_1 and Neumann boundary condition by ρ_n^1. The both operators are non-negative and

$$\begin{aligned} \rho_n^1(\lambda) &= 4\sigma(\sqrt{\lambda}), \\ \rho_d(\lambda) &= 4\int\limits_0^{\sqrt{\lambda}} \xi^2 d\sigma(\xi). \end{aligned}$$

Therefore, the following identity holds

$$\int\limits_{-\infty}^{\infty} \frac{N^2}{N^2+\lambda^2} |P\,(2x,\lambda)\,|^2 d\sigma(\lambda) = \frac{N^2}{2}\left[G_d(x,x,-N^2) + G_n^1(x,x,-N^2)\right],$$

where $G_d(x,y,z)$, $G_n^1(x,y,z)$ denotes the Green functions for the operators H_d, H_n^1 respectively. The second resolvent identity yields

$$G_d(x,y,-N^2) = G_d^\circ(x,y,-N^2) - \int\limits_{-\infty}^{\infty} G_d^\circ(x,s,-N^2)q(s)G_d(s,y,-N^2)ds, \tag{14}$$

$$G_n^1(x,y,-N^2) = G_n^\circ(x,y,-N^2) - \int\limits_{-\infty}^{\infty} G_n^\circ(x,s,-N^2)q_1(s)G_n^1(s,y,-N^2)ds, \tag{15}$$

where

$$\begin{aligned} G_d^\circ(x,y,-N^2) &= \frac{e^{-N|x-y|} - e^{-N|x+y|}}{2N}, \\ G_n^\circ(x,y,-N^2) &= \frac{e^{-N|x-y|} + e^{-N|x+y|}}{2N}. \end{aligned}$$

Notice that by assumption potentials $q(x)$, $q_1(x)$ are bounded. If N is large enough, we can iterate (15), (14) to obtain the trivial estimates $|G_d(x,y,-N^2)| < CN^{-1}$, $|G_n^1(x,y,-N^2)| < CN^{-1}$. Sum (15) and (14) for $y = x$ to obtain (13).□

Lemma 2. *Consider the function*

$$\widehat{f}_a(\lambda) = \int\limits_{-\infty}^{\infty} f(x)e^{i\lambda x}dx,$$

where $f(x) \in L^2(R)$ with support on $[0,a]$. Then the following identity holds

$$2\pi \int_{-\infty}^{\infty} \left|\widehat{f}_a(\lambda)\right|^2 d\sigma = \int_{-\infty}^{\infty} \left|\widehat{f}_a(\lambda)\right|^2 \frac{d\lambda}{|P(x,\lambda)|^2}, \tag{16}$$

provided that $x > a$.[3]

Proof. From [14](formulas (2.3), (3.7)), the following relation follows

$$\int_{-\infty}^{\infty} \left|\widehat{f}_a(\lambda)\right|^2 \frac{d\lambda}{|P(x,\lambda)|^2} = \int_{-\infty}^{\infty} \left|\widehat{f}_a(\lambda)\right|^2 \left[1 + 2\Re \int_0^{\infty} \overline{H_x(t)} e^{i\lambda t} dt\right] d\lambda, \tag{17}$$

where $H_x(t)$ is some function that satisfies the following conditions

$$\begin{aligned} 1 \ &: \ \int_0^{\infty} |H_x(t)|\, dt < \infty, \\ 2 \ &: \ H_x(t) = H(t), \text{if}\, t < x. \end{aligned}$$

Here $H(t)$ denotes accelerant of the Krein system. The right hand side of (17) is equal to

$$\int_{-\infty}^{\infty} \left|\widehat{f}_a(\lambda)\right|^2 d\lambda + 2\Re \int_0^{\infty} \overline{H_x(t)} dt \int_{-\infty}^{\infty} e^{i\lambda t} \left[\int_{-\infty}^{\infty} e^{-i\lambda s} \left(\int_0^a f(y)\overline{f(y+s)} dy\right) ds\right] d\lambda =$$

$$= \int_{-\infty}^{\infty} \left|\widehat{f}_a(\lambda)\right|^2 d\lambda + 4\pi\Re \int_0^{\infty} \overline{H_x(t)} dt \left[\int_0^a f(y)\overline{f(y+t)} dy\right].$$

The function $\int_0^a f(y)\overline{f(y+t)}dy$ is finite with support on $[-a,a]$. Therefore, if $x > a$, then

$$\int_0^{\infty} \overline{H_x(t)} dt \left[\int_0^a f(y)\overline{f(y+t)} dy\right] = \int_0^{\infty} \overline{H(t)} dt \left[\int_0^a f(y)\overline{f(y+t)} dy\right].$$

So we have (16) because

$$\int_0^t (t-s) H(s) ds + |t|/2 = \int_{-\infty}^{\infty} \frac{1-\cos(\lambda t)}{\lambda^2} d\sigma(\lambda).$$

□

[3]This statement is the continuous analog of the well-known fact for polynomials orthogonal on the unit circle [5].

Remark. In the same way, one can easily show that if $f(x)$ is finite $C^1(R)$ function with support on $[-a,a]$, and $\widehat{f}_a(\lambda)=\int\limits_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$, then

$$\int\limits_{-\infty}^{\infty} \widehat{f}_a(\lambda)\frac{d\lambda}{|P(x,\lambda)|^2}=2\pi\int\limits_{-\infty}^{\infty}\widehat{f}_a(\lambda)d\sigma,$$

provided that $x>a$.

The following Corollary is the consequence of Remark given above and standard approximation procedure.

Corollary. *If $f(\lambda)$ is finite continuous function on R, then*

$$\int\limits_{-\infty}^{\infty} f(\lambda)\frac{d\lambda}{|P(x,\lambda)|^2}\to 2\pi\int\limits_{-\infty}^{\infty} f(\lambda)d\sigma,$$

as $x\to\infty$.

Proof. Let C denote positive constant which value can change from one formula to another. We need to use the common approximation procedure and the fact that

$$\int\limits_{-\infty}^{\infty}\frac{d\lambda}{(\lambda^2+1)\left|P(x,\lambda)\right|^2}=2\Re\int\limits_{-\infty}^{\infty}\frac{\Omega_*(x,\lambda)d\lambda}{(\lambda^2+1)P_*(x,\lambda)}=2\pi\Re\frac{\Omega_*(x,i)}{P_*(x,i)}<C,$$

uniformly in $x\in R^+$ (see [14],[16]). Here (Ω,Ω_*) - solution of the second kind for the Krein system [14].□

Lemma 3 [12]. *Let ν be a finite positive Borel measure on R that is singular with respect to the Lebesgue measure. Then there is a sequence $\{h_n\}$ of continuous functions on the real line such that $0<h_n(x)\le 1$ for all x, $\lim_{n\to\infty}h_n(x)=1$ a.e., and $\lim_{n\to\infty}\int\limits_{-\infty}^{\infty} h_n(x)d\nu(x)=0$.*

Lemma 4. *Under the conditions of the Theorem 1, the following inequality holds*

$$N\pi-\delta_1(N)<\liminf_{x\to\infty}\int\limits_{-\infty}^{\infty}\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|\frac{N^2}{N^2+\lambda^2}d\lambda,$$

where $\delta_1(N)\to 0$ as $N\to\infty$.

Proof. The proof uses one idea from [12]. Fix any $N>0$. Let f be any nonnegative finite continuous function. Then the trivial inequality holds

$$\int\limits_{-\infty}^{\infty}\left(f\sigma'\right)^{\frac14}\frac{N^2}{N^2+\lambda^2}d\lambda \;\le\; \left[\int\limits_{-\infty}^{\infty}\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|\frac{N^2}{N^2+\lambda^2}d\lambda\right]^{\frac12}\left[N^2\int\limits_{-\infty}^{\infty}\frac{\left|P(2x,\lambda)\right|^2}{N^2+\lambda^2}d\sigma\right]^{\frac14}\times$$
$$\left[\int\limits_{-\infty}^{\infty}\frac{f(\lambda)N^2}{(N^2+\lambda^2)|P(x,\lambda)|^2}d\lambda\right]^{\frac14}.$$

Take $\liminf_{x\to\infty}$ of the both sides and use Lemma 1 for the second factor and Corollary for the third one.

$$\int_{-\infty}^{\infty}\left(f\sigma'\right)^{\frac{1}{4}}\frac{N^2}{N^2+\lambda^2}d\lambda \le \left[\liminf_{x\to\infty}\int_{-\infty}^{\infty}\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|\frac{N^2}{N^2+\lambda^2}d\lambda\right]^{\frac{1}{2}}\times$$
$$\left[2\pi\int_{-\infty}^{\infty}\frac{f(\lambda)N^2}{(N^2+\lambda^2)}d\sigma\right]^{\frac{1}{4}}(N/2+\delta(N))^{\frac{1}{4}}.$$

Fix $M>0$ such that $\pm M$ are not the point masses for the measure σ. Consider restriction of the measure σ to the segment $[-M,M]$. By Lemma 3, we can choose continuous functions $\{h_l\}$ such that $0<h_l\le 1$, $\lim_{l\to\infty}h_l=1$ a.e., and $\lim_{n\to\infty}\int\limits_{-M}^{M}h_n(\lambda)d\sigma_s=0$. For any $\varepsilon>0$, choose continuous g_k with support on $[-M,M]$ such that $0<g_k(\lambda)\le\varepsilon^{-1}$, $\lim_{k\to\infty}g_k(\lambda)=\left(\sigma'+\varepsilon\right)^{-1}$ almost everywhere on $[-M,M]$. Fix ε, then take $f=h_lg_k$. Let $l\to\infty$, then $k\to\infty$, $\varepsilon\to 0$, and $M\to\infty$. We have

$$N\pi-\delta_1(N)<\liminf_{x\to\infty}\int_{-\infty}^{\infty}\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|\frac{N^2}{N^2+\lambda^2}d\lambda,$$

where $\delta_1(N)\to 0$ as $N\to\infty$.□

Lemma 5. *Under the conditions of the Theorem 1, the following relation holds*

$$\limsup_{x\to\infty}\int_{-\infty}^{\infty}\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|^2\frac{N^2}{N^2+\lambda^2}d\lambda=2\pi\limsup_{x\to\infty}\int_{-\infty}^{\infty}|P(x,\lambda)|^2\frac{N^2}{N^2+\lambda^2}d\sigma=N\pi+\delta(N),\qquad(18)$$

where $\delta(N)\to 0$ *as* $N\to\infty$.

Proof. The second equality is consequence of Lemma 1. So we need to prove the first one. Fix any $N>0$. We have the following trivial identity

$$\frac{1}{\lambda^2+N^2}=\left|\int_0^{\infty}e^{-Nx}e^{i\lambda x}dx\right|^2=|k_1^a(\lambda)+k_2^a(\lambda)|^2,$$

where

$$k_1^a(\lambda)=\int_0^a e^{-Nx}e^{i\lambda x}dx=\frac{-1+e^{(i\lambda-N)a}}{i\lambda-N},\ |k_1^a(\lambda)|<\frac{2}{\sqrt{\lambda^2+N^2}},$$
$$k_2^a(\lambda)=\int_a^{\infty}e^{-Nx}e^{i\lambda x}dx=-\frac{e^{(i\lambda-N)a}}{i\lambda-N},\ |k_2^a(\lambda)|<\frac{e^{-aN}}{\sqrt{\lambda^2+N^2}}.$$

Notice that

$$\int_{-\infty}^{\infty} |k_1^a(\lambda)|^2 \frac{|P(x,\lambda)|^2}{|P(2x,\lambda)|^2} d\lambda = 2\pi \int_{-\infty}^{\infty} |k_1^a(\lambda)|^2 |P(x,\lambda)|^2 d\sigma(\lambda),$$

if $x > a$. It follows from the representation (10) for $P_*(x,\lambda)$, identity $|P(x,\lambda)| = |P_*(x,\lambda)|$, and Lemma 2. So we need only to prove that the integral

$$\int_{-\infty}^{\infty} \left| \frac{P(x,\lambda)}{P(2x,\lambda)} \right|^2 \frac{1}{1+\lambda^2} d\lambda$$

is bounded uniformly in $x \in R^+$, which will allow us to use approximation procedure. One can easily find functions $R_{1(2)}$ such that

$$\frac{1}{\lambda^2+1} = \left| \int_0^{\infty} e^{-x} e^{i\lambda x} dx \right|^2 = |R_1(\lambda) + R_2(\lambda)|^2 \le 2|R_1(\lambda)|^2 + 2|R_2(\lambda)|^2,$$

where

$$R_1(\lambda) = \int_0^2 r_1(x) e^{i\lambda x} dx, \ R_2(\lambda) = \int_1^{\infty} r_2(x) e^{i\lambda x} dx.$$

Here $r_1 \in C^1(R^+)$, $r_1 = 0$ for $x \ge 2$, so $|R_1(\lambda)| < C(\lambda^2+1)^{-\frac{1}{2}}$. The function r_2 is from the Schwarz class with the support on $[1,\infty)$. So $R_2(\lambda)$ is from Schwarz class as well. Integral

$$\int_{-\infty}^{\infty} \left| \frac{P(x,\lambda)}{P(2x,\lambda)} \right|^2 |R_1|^2 \, d\lambda$$

is bounded in x by the Lemma 2 and by the estimate

$$\int_{-\infty}^{\infty} \frac{|P(x,\lambda)|^2}{\lambda^2+1} d\sigma(\lambda) < C, \tag{19}$$

which is uniform in x. To estimate the other integral, we consider the auxiliary function $s(\lambda) = \int_0^1 e^{-x} e^{i\lambda x} dx = \frac{1-e^{i\lambda-1}}{1-i\lambda}$. We have $|s(\lambda)|^2 = \frac{1+e^{-2}-2e^{-1}\cos\lambda}{\lambda^2+1} > 0$. Integrating by parts, we have

$$\int_{-\infty}^{\infty} \left| \frac{P(x,\lambda)}{P(2x,\lambda)} \right|^2 |R_2|^2 d\lambda = 2 \int_0^{\infty} \left[\int_0^{\lambda} \frac{|P(x,t)|^2}{|P(2x,t)|^2} |s(t)|^2 dt \right] \left(\frac{|R_2(\lambda)|^2}{|s(\lambda)|^2} \right)' d\lambda \le$$

$$2 \int_0^{\infty} \left| \frac{P(x,\lambda)}{P(2x,\lambda)} \right|^2 |s(\lambda)|^2 \, d\lambda \int_0^{\infty} \left| \left[\left| \frac{R_2(\lambda)}{s(\lambda)} \right|^2 \right]' \right| d\lambda.$$

If x is large enough, we can use Lemma 2 to show that the first factor is bounded in x. The second integral converges because R_2 is from the Schwarz class. Therefore, (18) holds.□

Lemma 6. *Under the conditions of the Theorem 1, we have*

$$\int_{-\infty}^{\infty} \frac{|P(x,\lambda)|^2}{1+\lambda^2} d\sigma \to \int_{-\infty}^{\infty} \frac{d\sigma_\circ}{1+\lambda^2} \tag{20}$$

as $x \to \infty$.

Proof. Notice that

$$\frac{1}{\lambda^2+1} = \left| \int_0^\infty e^{-x} e^{i\lambda x} dx \right|^2 = |g_1^a(\lambda) + g_2^a(\lambda)|^2,$$

where

$$\begin{aligned} g_1^a(\lambda) &= \int_0^a e^{-x} e^{i\lambda x} dx = \frac{-1+e^{(i\lambda-1)a}}{i\lambda-1}, \ |g_1^a(\lambda)| \le \frac{2}{\sqrt{\lambda^2+1}}, \\ g_2^a(\lambda) &= \int_a^\infty e^{-x} e^{i\lambda x} dx = -\frac{e^{(i\lambda-1)a}}{i\lambda-1}, \ |g_2^a(\lambda)| \le \frac{e^{-a}}{\sqrt{\lambda^2+1}}. \end{aligned}$$

Because of the estimate (9) on the measure, inequality (19), and Lemma 1, it suffices to prove that

$$\int_{-\infty}^{\infty} |P(x,\lambda)|^2 \left|g_1^a(\lambda)\right|^2 d\sigma \to \int_{-\infty}^{\infty} \left|g_1^a(\lambda)\right|^2 d\sigma_\circ$$

for any $a > 0$.

For x large enough, we have the identity

$$2\pi \int_{-\infty}^{\infty} |P(x,\lambda)|^2 \left|g_1^a(\lambda)\right|^2 d\sigma = \int_{-\infty}^{\infty} \frac{|P(x,\lambda)|^2}{|P(2x,\lambda)|^2} \left|g_1^a(\lambda)\right|^2 d\lambda.$$

Indeed, $|P(x,\lambda)| = |P_*(x,\lambda)|$. For $P_*(x,\lambda)$, we have representation

$$P_*(x,\lambda) = 1 - \int_0^x \Gamma_x(0,s) e^{i\lambda s} ds,$$

which shows that $P_*(x,\lambda) g_1^a(\lambda)$ is the function to which Lemma 2 is applicable.

Take N any positive constant. The following estimate is trivial

$$\left| \int_{-\infty}^{\infty} \frac{|P(x,\lambda)|^2}{|P(2x,\lambda)|^2} \left|g_1^a(\lambda)\right|^2 d\lambda - \int_{-\infty}^{\infty} \left|g_1^a(\lambda)\right|^2 d\lambda \right| =$$

$$\int_{-\infty}^{\infty}\left[\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|-1\right]\frac{N}{\sqrt{N^2+\lambda^2}}\times\left[\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|+1\right]\frac{\sqrt{N^2+\lambda^2}}{N}|g_1^a(\lambda)|^2d\lambda\le$$

$$\left[\int_{-\infty}^{\infty}\left[\frac{|P(x,\lambda)|}{|P(2x,\lambda)|}-1\right]^2\frac{N^2}{N^2+\lambda^2}d\lambda\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}\left[\frac{|P(x,\lambda)|}{|P(2x,\lambda)|}+1\right]^2\frac{16(N^2+\lambda^2)}{N^2(1+\lambda^2)^2}d\lambda\right]^{\frac{1}{2}}.$$

Here we used Cauchy inequality and the estimate $g_1^a(\lambda)\le 2(\lambda^2+1)^{-1/2}$. The second factor squared is less than

$$\int_{-\infty}^{\infty}\left[2\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|^2+2\right]\frac{16}{(1+\lambda^2)}d\lambda<C$$

for all $x>0$ by Lemma 5. For the first factor, we have

$$0\le\limsup_{x\to\infty}\int_{-\infty}^{\infty}\left[\frac{|P(x,\lambda)|}{|P(2x,\lambda)|}-1\right]^2\frac{N^2}{N^2+\lambda^2}d\lambda\le\limsup_{x\to\infty}\int_{-\infty}^{\infty}\left[\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|^2-1\right]\frac{N^2}{N^2+\lambda^2}d\lambda-$$

$$-2\liminf_{x\to\infty}\left[\int_{-\infty}^{\infty}\left(\left|\frac{P(x,\lambda)}{P(2x,\lambda)}\right|-1\right)\frac{N^2}{N^2+\lambda^2}d\lambda\right]\le\delta_2(N),$$

where $\delta_2(N)\to 0$ as $N\to\infty$. Here we used Lemma 5 for the first term and Lemma 4 for the second one. Because N can be chosen as an arbitrary positive number, we have (20).□

Remark. In the same way one can prove that for any $k\ne 0$,

$$\int_{-\infty}^{\infty}\frac{|P(x,\lambda)|^2}{k^2+\lambda^2}d\sigma(\lambda)\to\int_{-\infty}^{\infty}\frac{\sigma_\circ(\lambda)}{k^2+\lambda^2}$$

as $x\to\infty$.

Proof of the Theorem 1. Using the calculations from Lemma 1, one can see that the statement of the Remark after Lemma 6 can be written in the following way. For any positive k, we have

$$\left[G_d(x,x,-k^2)+G_n^1(x,x,-k^2)\right]-\left[G_d^\circ(x,x,-k^2)+G_n^\circ(x,x,-k^2)\right]\to 0$$

as $x\to\infty$. Let us assume that $A(x)$ does not tend to zero at infinity. Because A' is bounded and $a(x)=2A(2x)$, there exists the sequence $x_n\to\infty$ such that $\min_{x\in[x_n-\delta_1,x_n+\delta_1]}|a(x)|>\delta_2$, where $\delta_{1(2)}-$ some positive constants.

From the second resolvent formula for operators H_d, H_n^1 , we have

$$G_d(x,x,-k^2)=G_d^\circ(x,x,-k^2)-\int_{-\infty}^{\infty}\left[G_d^\circ(x,s,-k^2)\right]^2q(s)ds+R_d(x,k),\qquad(21)$$

$$G_n^1(x,x,-k^2) = G_n^\circ(x,x,-k^2) - \int\limits_{-\infty}^{\infty} \left[G_n^\circ(x,s,-k^2)\right]^2 q_1(s)ds + R_n^1(x,k), \tag{22}$$

where $|R_d(x,k)| < Ck^{-5}$,$|R_n^1(x,k)| < Ck^{-5}$. Adding (21) to (22), we have an estimate

$$k^{-2}\int\limits_{-\infty}^{\infty} e^{-2k|x-s|}a^2(s)ds < C\left[I(x,k)+k^{-5}\right],$$

where $I(x,k) \to 0$ as $x \to \infty$. But, due to our assumption, we have an inequality

$$\delta k^{-3} < k^{-2}\int\limits_{-\infty}^{\infty} e^{-2k|x_n-s|}a^2(s)ds.$$

Thus, we have

$$\delta k^{-3} < C\left[I(x_n,k)+k^{-5}\right].$$

Now, take k large enough such that $\delta k^{-3} > Ck^{-5}$. After that let $x_n \to \infty$. We came to the contradiction so $A(x) \to 0$ as $x \to \infty$. □

Consider Sturm-Liouville operator H

$$Hf = -f^{''} + qf$$

on the half-line with some boundary condition at zero. Denote the spectral measure by $\rho(\lambda)$.

Theorem 2. *If $q(x)$ is bounded function, $q_-(x) \equiv \min\{0, q(x)\} \in L^1(R^+)$, and $\rho^{'}(\lambda) > 0$ a.e. on R^+, then $\lim_{y\to\infty} \int\limits_{y-\delta}^{y+\delta} q(x)dx \to 0$ for each fixed $\delta > 0$.*

Proof. Consider operators $H_{n(d)}$ with potential $q = q_+ \equiv \max\{0, q(x)\}$ and Neumann (Dirichlet) boundary condition at zero. The trace-class argument shows that the spectral measure $\rho_{n(d)}$ of these operators have positive derivative a.e. on the positive half-line. Notice also that these operators are non-negative. Let us introduce the Krein system with coefficient $A(x)$ which corresponds to H_d. We need to solve the equation

$$q_+ = a^{'} + a^2, \tag{23}$$

where $a(x) = 2A(2x)$. Consider function $\phi : \phi(0) = 1,\ \phi^{'}(0) = 0,\ \phi^{''} = q_+\phi$. It is easy to prove that $\phi(x) > 0$, $\phi(x)$ is convex, and the function $a(x) = \phi^{'}\phi^{-1}$ satisfies Riccati equation (23). Because $\phi'(x) \geq 0$, $a(x) \geq 0$. For $a^{'}$, we have $a^{'} = q_+ - a^2$. Therefore, if $a(x) > \sqrt{q_+(x)}$, then $a^{'} < 0$. Consequently, $0 \leq a(x) \leq \sup_{x\geq 0}\sqrt{q_+(x)}$. Notice also that $a^{'}$ is bounded. That follows directly from the Riccati equation (23). For the measure σ of the Krein system with coefficient $A(x) = a(x/2)/2$, we have $\sigma^{'}(\lambda) > 0$ a.e. on R. Therefore, Theorem 1 is applicable. That means $a(x) \to 0$ as $x \to \infty$. $\int\limits_{y-\delta}^{y+\delta} q_+(x)dx = a(y+\delta) - a(y-\delta) + \int\limits_{y-\delta}^{y+\delta} a^2(x)dx \to 0$ as $y \to \infty$. Because $q_- \in L^1(R^+)$, we have $\int\limits_{y-\delta}^{y+\delta} q(x)dx \to 0$ as $y \to \infty$. □

Remark. Let us notice here that there are some $q(x) \geq 0, q(x) \in L^1(R^+)$ such that $q(x)$ does not tend to zero at the infinity. Nevertheless, the spectrum is purely absolutely continuous on R^+. On the other hand, if we add the condition $q' \in L^\infty(R^+)$ to the conditions of the Theorem 2, then $q(x) \to 0$ at infinity.

We conclude the paper with the following open problem.

Open problem. Assume that $q(x)$ is uniformly locally summable on the half-line, i.e.

$$\sup_{x>0} \int_x^{x+1} |q(s)| ds < \infty.$$

Prove that if for some boundary condition at zero, the essential support of the spectrum for the corresponding Schrödinger operator is R^+, and the absolutely continuous component of the measure fills the whole R^+, then q decays at the infinity in the following sense. For any $\delta > 0$,

$$\int_{x-\delta}^{x+\delta} q(s) ds \to 0$$

as $x \to \infty$.

References

[1] N.I.Akhiezer, A.M.Rybalko, On the continuous analogues of polynomials orthogonal on the unit circle, Ukranian Math. J., 20 (1968), 1, 3–24.

[2] P.Deift, R.Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, Comm. Math. Phys., 203 (1999), 341–347.

[3] S.A.Denisov, On the existence of the absolutely continuous component for the measure associated with some orthogonal systems, (to appear in Comm. Math. Phys.).

[4] S.A.Denisov, To the spectral Theory of Krein systems, (to appear in Integral Equations Operator Theory).

[5] Y.L.Geronimus, *Polinomilas orthogonal on the unit circle,* (in Russian), 1958.

[6] D.J.Gilbert, D.B.Pearson, On the subordinacy and analysis of the spectrum of the one-dimensional Schrodinger operators, J. Math. Anal., 128 (1987), 30–56.

[7] A.Kiselev, Y.Last, and B.Simon, Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators, Comm. Math. Phys., 194 (1998), 1–45.

[8] M.G.Krein, Continuous analogues of propositions on polynomials orthogonal on the unit circle, Dokl. Akad. Nauk SSSR, 105 (1955), 637–640.

[9] M.G.Krein, On the integral representation of a continuous Hermitian-indefinite functions with finite number of negative squares, (Russian), Dokl. Akad. Nauk SSSR, 125, 1 (1959), 31–34.

[10] B.M.Levitan, On the asymptotic behavior of the spectral function of a self-adjoint equation of the second order, Izv. Akad. Nauk SSSR, Ser. Mat., 16 (1952), 325–352.

[11] A.Mate, P.Nevai, V.Totik, Asymptotics for the ratio of leading coefficients of orthogonal polynomials on the unit circle, Constr. Approx., 1 (1985), 63–69.

[12] A.Mate, P.Nevai, V.Totik, Asymptotics for the ratio of leading coefficients of orthogonal polynomials on the unit circle, Constr. Approx., 1 (1985), 63–69.

[13] E.A.Rakhmanov, On the asymptotics of the ratio of orthogonal polynomials, 2, Math. USSR, Sb. 46 (1983), 105–117.

[14] A.M.Rybalko, On the theory of continual analogues of orthogonal polynomials, Teor. Funktsii, Funktsional. Anal. i Prilozhen., 3 (1966), 42–60.

[15] L.A.Sakhnovich, Spectral theory of a class of canonical systems, Funct. Anal. Appl., No.2, 34 (2000), 119–129.

[16] L.A.Sakhnovich, On a class of canonical systems on half-axis, Integral Equations Operator Theory, 31 (1998), 92–112.

[17] L.A.Sakhnovich, *Spectral theory of canonical differential systems: method of operator identities*, Oper. Theory, Adv. and Applications, Vol.107, Birkhauser Verlag, Basel, Boston, 1999.