Relativistic Dynamic Equation in Invariant Form

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Abstract

In some dynamical systems boosts and rotations occur synchronically. Usually in analysis of such systems we separate these two types of motion. Unfortunately, the result of such analysis depends on the order in which operations are performed.

To avoid the order dependence of the above operations, we propose a new dynamic variable called the symmetric velocity. This new velocity could be calculated directly from the regular velocity and is its relativistic half. The set of all possible symmetric velocities is a three dimensional ball of radius c and the Lorentz group acts on this ball via conformal maps. The generators of these maps (elements of the Lie algebra) are second order transformations expressed by a triple product. This triple product is the one corresponding to the Bounded Symmetric Domain of type 4 in Cartan's classification, also called the spin factor. The product is connected with the Geometric (Clifford) product, explaining why use of the geometric product simplifies formulae in several areas of physics.

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1 Introduction

This paper provides another indication that bounded symmetric domains and JB*-triples may provide a model for different areas in physics. A bounded domain D in a Banach space is called a *bounded symmetric domain* if for every $z \in D$ there exists a smooth automorphism $s_z \in Aut(D)$ of period two on D, having z as the only fixed point. The smoothness of the automorphism may mean complex analytic, conformal (preserving angles) or projective (preserving linear segments). It is known that a domain D is a bounded symmetric domain if it has a symmetry about one point and is homogeneous in sense that for any two points $z, w \in D$there is an automorphism $\varphi \in Aut(D)$ such that $\varphi(z) = w$.

There is a triple product uniquely associated with any bounded symmetric domain D in a Banach space A, obtained as follows (see [11] for details): By fixing any point in D (we may assume for simplicity that this is the zero point of A) we may decompose $Aut(D)$ into rotations and translations. This implies that the Lie algebra $aut(D)$ will be a direct sum of linear terms, as generators of rotations, and generators of translations. It was shown (see [11] and [10]) that the generators of translations are of the form

$$\xi_a(z) = a - q_a(z), \quad z \in A \tag{1}$$

where $q_a(z)$ is quadratic in z and is conjugate linear in a.

The quadratic form $q_a(z)$ could be rewritten as a real trilinear form $q_a(z) = \{z, a, z\}$ and by linearization of the quadratic dependence in z (polarization) we define a triple product on A with the following properties:

(i) $\{a, b, c\}$ is linear in a, c and conjugate linear in b,

(ii) $\{a, b, c\} = \{c, b, a\}$.

The operator $D(a, b)$ defined by

$$D(a, b)c = \{a, b, c\}, \quad c \in A \tag{2}$$

satisfies:

(iii) the operator $D(a, a)$ is a hermitian with positive spectrum,

(iv)

$$D(a, a)\{x, y, z\} = \{D(a, a)x, y, z\} - \{x, D(a, a)y, z\} + \{x, y, D(a, a)z\}, \tag{3}$$

for any $a, x, y, z \in A$ and

(v) $\|a\|^3 = \|\{a, a, a\}\|$.

A Banach space A with a triple product satisfying the above properties is called a *JB*-triple*. In [10] it is shown that the category of bounded symmetric domains is equivalent to the category of JB*-triples.

Any bounded symmetric domain could be decomposed into indecomposable domains, called Cartan factors. There are six types of Cartan factors and the JB*-triples associated with them ([11] and [1]). The type 1 consists of bounded operators between two Hilbert spaces. The types 2 and 3 consist of anti-symmetric and symmetric complex matrices, respectively, representing bounded operators on a Hilbert space. In all above types the triple product is defined by

$$\{a, b, c\} = \frac{ab^*c + cb^*a}{2}, \tag{4}$$

where b^* is the adjoint operator to b. The types 5 and 6 are exceptional of dimensions 16 and 27 respectively, and we will not be concerned with them here.

The Cartan factor of type 4, sometimes called a *spin factor* will be mainly used here. This factor is defined as the space R^n or C^n with the triple product

$$\{a, b, c\} =< a|b > c+ < c|b > a- < a|\bar{c} > \bar{b}. \tag{5}$$

As it was shown in [4] any JB*-triple is isomorphic to the sum of an exceptional part and a subspace of operators on a Hilbert space with the triple product (4).

Why are the bounded symmetric domains and JB*-triples a good model for physics? Any law in physics must satisfy the symmetry or invariance principle. This principle states that the law should not change if we change the point from which we observe the phenomena. Such a principle imposes homogeneity of the state space of the system, that could be expressed as a symmetry of the domain representing the possible states of the system. This suggest that the state space is a symmetric domain. In order to be able to obtain numerical results we need an algebraic structure. Currently there is only one algebraic structure that has an origin only in geometry - the JB*-triple product.

In [3] and [2] it was shown that bounded symmetric domains occur in Transmission Line theory and in Special Relativity. Note that the state space of any quantum system must contain some geometry coming from the measuring process. In [5] and [6] it was shown that this geometry implies that the state space of the system is a bounded symmetric domain. Günaydin noticed [8] that all physically meaningful quantities in Quantum Mechanics depend only on the Jordan triple product rather than on the binary one. In [7] it was shown that the Canonical Anticommutation

Relations could be efficiently represented on the spin factor and the Lorentz group is represented on this factor by spin 1 and spin 1/2 representations.

2 The meaning of the Einstein velocity addition formula

One of the assumptions of Special Relativity is that the velocities of moving objects are bounded by the speed of light c. Thus all possible velocities in any inertial frame form a 3D ball of radius c. If we consider two inertial frames, then for each velocity in one inertial frame there is assigned a unique velocity in the second frame. This assignment generates a smooth transformation of the ball of possible velocities. The transformation is projective, meaning preserving line intervals, but not preserving angles.

The precise formula for this transformation is given by Einstein's velocity addition formula. To understand this formula consider an inertial frame K and another inertial frame K' that moves with velocity u relatively to frame K. Assume that there is an object moving with constant velocity v' in frame K'. Consider two events $A(r'_A, t'_A)$ and $B(r'_B, t'_B)$ expressing that the object was at time t'_A in position r'_A and similarly for B. The space interval between these events in K' is $\triangle r' = r'_B - r'_A$. If the clocks at the points A and B were synchronized in K', the time interval between these events is $\triangle t' = t'_B - t'_A$ and $v' = \triangle r'/\triangle t'$.

In order to be able to calculate the time interval of the above events and the velocity of the object in system K, we must synchronize first the clocks, moving together with K', at the two places r'_A and r'_B with respect of the frame K. Let us denote by $\hat{t}'_B$ the time at clock at r'_B that was synchronized to the clock at r'_A with respect of the K. The difference between the times of occurrence of event B on the two clocks positioned at r'_B is given by the formula for amount of nonsimultaneity at a displacement $\triangle r'$

$$\hat{t}'_B - t'_B = \frac{1}{c^2} < \triangle r' | u >$$

and the time interval between the events measured by clocks in K' but synchronized with respect to K is

$$\triangle \hat{t}' = \hat{t}'_B - t'_A = \frac{1}{c^2} < \triangle r' | u > + \triangle t' = (1 + < v' | u > /c^2) \triangle t'.$$

Thus, the time interval between these events in K will be

$$\triangle t = \frac{\triangle \hat{t}'}{\sqrt{1-|u|^2/c^2}} = \frac{1+ < v'|u > /c^2}{\sqrt{1-|u|^2/c^2}}\triangle t'.$$

Note that the space interval between these events in frame K is

$$\triangle r = u\triangle t + ((\sqrt{1-|u|^2/c^2})P_u + (I-P_u))\triangle r'$$

where P_u denotes the orthogonal projection onto direction of u. This lead to the formula for velocity of the object in K

$$v = \frac{\triangle r}{\triangle t} = u + ((\sqrt{1-|u|^2/c^2})P_u + (I-P_u))\frac{\triangle r'}{\triangle t'}\frac{\sqrt{1-|u|^2/c^2}}{1+ < v'|u > /c^2} =$$

$$u + ((1-|u|^2/c^2)P_u + \sqrt{1-|u|^2/c^2}(I-P_u))\frac{v'}{1+ < v'|u > /c^2}.$$

The above formula defines the transformation φ_u, called boosts, of the ball of possible velocities that, by assuming $c = 1$, is given by

$$v = \varphi_u(v') = u + ((1-|u|^2)P_u + \sqrt{1-|u|^2}(I-P_u))\frac{v'}{1+ < v'|u >}. \quad (6)$$

The set of boosts do not form a group but rather a gyrogroup (see [12] for definitions) but together with rotations they form a group representing the Lorentz group by projective maps on the velocity ball. How do the boosts and rotations interact? If we rotate first the ball by a ϑ and then perform the boost in direction ϑu we get a map $\varphi_{\vartheta u}(\vartheta v')$. From (6) we can see that

$$\varphi_{\vartheta u}(\vartheta v') = \vartheta\varphi_u(u') \quad (7)$$

only if ϑ commutes with $(1-|u|^2)P_u + \sqrt{1-|u|^2}(I-P_u)$. This happen only if the rotation ϑ is about the u direction. The failure of (7) is a result of the space anisotropy under the Lorentz transformations.

3 Symmetric velocity, conformal group and spin triple product

In some dynamical systems the boosts and rotation occur synchronically. Usually in analysis of such systems we separate these two types of operations. Unfortunately, the result of such analysis depends on the order the separations were performed. To avoid the dependence on the order of the above operations we propose a new dynamic variable, called the symmetric velocity. The symmetric velocity was introduced in [3] in order to transform the the transformations φ_u to a conformal one. The symmetric velocity w and velocity v are connected by

$$w = F(v) = \frac{v}{1+\sqrt{1-|v|^2/c^2}}, \quad v = F^{-1} = \frac{2w}{1+|w|^2/c^2}. \tag{8}$$

The symmetric velocity is the relativistic half of the regular velocity. The set of all possible symmetric velocities in any inertial frame form a 3D ball S of radius c. For simplicity we will assume from now $c = 1$. As it was shown the map $\psi = F\varphi F^{-1}$ is a conformal map of the ball. A similar result was obtained by A.Ungar in 1996 (see [12]) in the study of so called Möbius gyrovector space.

An explicit form of the conformal map of the ball was proposed by Ahlfors in1981 and is given by the formula (see [12])for the extended Möbius transformation

$$\psi_u(w) = u \oplus w = \frac{(1+2<u|w>+|w|^2)u+(1-|u|^2)w}{1+2<u|w>+|u|^2|w|^2}, \quad u, w \in S \tag{9}$$

where $\oplus$ will denote the sum of symmetric velocities. Note that in this case if we rotate first the ball by a ϑ and then perform the boost in direction ϑu we get

$$\psi_{\vartheta u}(\vartheta w) = \vartheta \psi_u(w) \tag{10}$$

or

$$\vartheta(u \oplus w) = \vartheta u \oplus \vartheta w. \tag{11}$$

This can be expressed as the commuting of the following diagram, see FIG. 1.

$$\begin{array}{ccc} S \times S & \longrightarrow^{\oplus} & S \\ \vartheta \downarrow & & \vartheta \downarrow \\ S \times S & \longrightarrow^{\oplus} & S \end{array}$$

Figure 1: Commuting of symmetric addition and rotation

Thus, in case the dynamics of the system involve both boosts and rotations the use of symmetric velocities may simplify the description of the evolution and lead to invariant (under the order of decomposition into boosts and rotations) dynamic equations.

If the evolution of the system is described by conformal maps of the symmetric velocities, the dynamic equation will involve the generators of such maps. To obtain the generators of the boosts we have to take a one-parameter family $\psi_{u(\tau)}$ depending on some real parameter τ with $u(0) = 0$. Then the generator is given by

$$\xi_a(w) = \frac{d}{d\tau}\psi_{u(\tau)}(w)|_{\tau=0} = a - 2 < w|a > w + |w|^2 a \tag{12}$$

with $a = \frac{d}{d\tau}u(\tau)|_{\tau=0}$. This is a general formula for generators of translation in the Lie algebra (consisting of the generators of the group) of a bounded symmetric domain [11]. As mentioned above, the generators of the translations are of the form

$$\xi_a(w) = a - \{w, a, w\}, \tag{13}$$

with $\{w, a, w\}$ as the triple product associated with the bounded symmetric domain. In our case the domain is a real domain of type 4 and dimension 3 in Cartan's classification, called also the Spin factor with the triple product defined by (5). The norm is the usual Euclidean norm. The ball of radius one is homogeneous under the group of conformal maps (9).

The Lie algebra of the conformal group consists of generators of boosts described by (12) and (13) in terms of the triple product and of generators of rotations. To describe the generators of rotations in the symmetric velocity ball we chose first an orthonormal basis e_1, e_2, e_3 in R^3 and define

$$\bar{D} = (D(e_2, e_3), D(e_3, e_1), D(e_1, e_2)),$$

where the linear operator $D(a, b)$ is defined by (2). Then the generator of rotation for any symmetric velocity w is expressed in the triple product by

$$\vartheta(w) = (H \cdot \bar{D})(w) = H \times w : \quad H \in R^3.$$

Thus, any element ζ of our Lie algebra is of the form

$$\zeta = \zeta_{a,H}(w) = a + (H \cdot \bar{D})(w) - \{w, a.w\} \;\; a, H \in R^3 \tag{14}$$

and is expressed in terms of the triple product.

Note that also (9) could be obtained in terms of the triple product by exponentiating (13) and using the explicit form for this exponent from [10].

4 Relativistic dynamic equations using symmetric velocity

We are going to describe now how to obtain the relativistic dynamic equation for the symmetric velocities. From the definition of the symmetric velocity

$$\gamma = \frac{1}{\sqrt{1-|v|^2}} = \frac{1+|w|^2}{1-|w|^2}. \tag{15}$$

and thus

$$mv = m_0\gamma v = m_0\frac{2w}{1-|w|^2} \tag{16}$$

with m_0 the rest mass of the object. The relativistic dynamic equation

$$F = \frac{d}{dt}(mv)$$

now becomes

$$F = \frac{d}{dt}m_0\frac{2w}{1-|w|^2} = 2m_0\left(\frac{1}{1-|w|^2}\frac{dw}{dt} + \frac{2w}{(1-|w|^2)^2} < \frac{dw}{dt}|w >\right). \tag{17}$$

By taking the inner product with w we get

$$< F|w >= 2m_0 < \frac{dw}{dt}|w > \frac{1+|w|^2}{(1-|w|^2)^2}. \tag{18}$$

By substituting $< \frac{dw}{dt}|w >$ from (18) into (17) we obtain

$$\frac{2m_0}{1-|w|^2}\frac{dw}{dt} = F - \frac{2w}{1+|w|^2} < F|w > . \tag{19}$$

Multiplying both sides of (19) by $1+|w|^2$ and using that $dt = \gamma d\tau$ we obtain *the relativistic dynamic equation for the symmetric velocities*

$$2m_0 dw/d\tau = F - c^{-2}\{w, F, w\} = \xi_F(w), \tag{20}$$

where τ denote the proper time and the triple product is the spin triple product defined by (5) and ξ by (13).

Let us derive now the relativistic dynamic equation for the electromagnetic field for the symmetric velocities. Let E denotes the electric strength of the field, H denote the magnetic strength. For simplicity we will assume $q = 1$. Then from the formula of the Lorentz force for the electromagnetic field, the dynamic equation becomes

$$\frac{d}{dt}(mv) = E + v \times H.$$

Thus, using equations (15) and (16) we get

$$E + \frac{2w}{1+|w|^2} \times H = \frac{d}{dt} m_0 \frac{2u}{1-|w|^2} = 2m_0 \left(\frac{1}{1-|w|^2}\frac{dw}{dt} + \frac{2w}{(1-|w|^2)^2} < \frac{dw}{dt}|w>\right). \tag{21}$$

By taking the inner product with w we get

$$< E|w >= 2m_0 < \frac{dw}{dt}|w > \frac{1+|w|^2}{(1-|w|^2)^2}. \tag{22}$$

By substituting $< \frac{dw}{dt}|w >$ from (22) into (21) we obtain

$$\frac{2m_0}{1-|w|^2}\frac{dw}{dt} = E + \frac{2w}{1+|w|^2} \times H - \frac{2w}{1+|w|^2} < E|w > . \tag{23}$$

Multiplying both sides of the last equation by $1 + |w|^2$ and using that $dt = \gamma d\tau$ we obtain

$$2m_0 dw/d\tau = q(E - c^{-2}\{w, E, w\} + 2w \times H) \tag{24}$$

that could be considered as the relativistic dynamic equation for the electromagnetic field.

5 The Physical meaning of the Geometric product

In the last decades it was found that the use of Clifford algebra and the Geometric product (see [9]) associated with it simplify the description of different physical phenomena in Classical and Modern physics. But what is the reason that this description is so successful? So far not much is known why this algebraic structure is connected with the description of real phenomena. Often, it is proposed that the success in use of this algebraic structure is connected with the fact that the geometric product contains the dot product and the outer (vector) product.

As it was mentioned in Section 3, for description of dynamical systems in which boost and rotations occur synchronically, it is important to have a good relation between these two operations. Such description must be independent as much as possible on the order in which the operations were performed. We have seen that use of symmetric velocity, associated with the conformal group, satisfies partial commutation, as it is given by (11). A full commutation could not be achieved since during the rotation the direction of the boost was changed.

The symmetric velocity is directly connected to spin triple product described by (5). As it was described in Section 3, the Lie algebra of the conformal group is fully described by this triple product. By using the definition of operator $D(u,v)$, defined above we may rewrite (5) as

$$D(u,v) = < u|v > I + u \wedge v,$$

where I denote the identity operator. This is similar to the geometric product between u and v defined as

$$uv = < u|v > + u \wedge v,$$

where the sum of a scalar $< u|v >$ and bivector $u \wedge v$ makes sense in the Clifford algebra.

6 Conclusions

This paper is an important step in the new mathematical foundation to description of physical phenomena. The proposed mathematical description is based on geometry and symmetry of nature that results in the symmetry of the state space.

Introduction of a new dynamic variable, called symmetric velocity, defined by (8), leads to distributive law (11) for the velocity addition with respect to multiplication with rotations. The Lie algebra of the group generated by the boosts and rotations of symmetric velocities is described in terms of spin triple product (14) associated with a bounded symmetric domain of type 4.

The relativistic dynamic equations (20) and (24) for the symmetric velocity state that its derivative is equal to an element of the Lie algebra. Finally, we show the connection of the spin triple product to the Clifford or Geometric product, explaining why this product simplifies equations in different areas of physics.

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