Delocalization in polymer models

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Abstract

A polymer model is a one-dimensional Schrödinger operator composed of two finite building blocks. If the two associated transfer matrices commute, the corresponding energy is called critical. Such critical energies appear in physical models, an example being the widely studied random dimer model. Although the random models are known to have pure-point spectrum with exponentially localized eigenstates for almost every configuration of the polymers, the spreading of an initially localized wave packet is here proven to be at least diffusive for every configuration.

1 Introduction

Until quite recently, common wisdom was that one-dimensional random Schrödinger operators are in a strong localization phase and that there is nothing else of any interest to be discovered. In 1990, Dunlap, Wu and Phillips [DWP] studied the random dimer model. It is similar to the conventional Bernoulli-Anderson model with a Hamiltonian given by the sum of the discrete Laplacian and a random potential taking only two values, except that these potential values now always come in neighboring pairs (dimers). For suitable values of the parameters, this model has so-called critical energies at which the two transfer matrices across the dimers commute. This leads to a vanishing of the Lyapunov exponent and a divergence of the localization length at these energies. They then argued and showed numerically that the second moment of the position operator X on the lattice grows superdiffusively under the dynamics like $\langle\psi|X^2(t)|\psi\rangle \approx C\,t^{3/2}$ for a localized initial state ψ and any typical dimer configuration. This is likely to be responsible for the high conductivities of certain organic polymer chains [PW] and quasi-$1D$ semiconductor superlattices [PTB].

This work considers Jacobi matrices H arbitrarily composed of two building blocks modeling two different polymers. The associated polymer transfer matrices are supposed to commute at an energy lying within the spectra of the periodic operators containing only one of the polymers. It is proven that there exists a constant C such that for any such one-dimensional polymer Hamiltonian

$$\int_0^T \frac{dt}{T}\,\langle 0|e^{\imath Ht}|X|^q e^{-\imath Ht}|0\rangle \;\geq\; C\,T^{q-1}\;, \tag{1}$$

where $|0\rangle$ is the state localized at the origin. This rigorously confirms the findings of [DWP] that there is quantum transport in this model.

This should be confronted with the fact that the spectrum of a randomly composed polymer Hamiltonian is almost surely pure-point with exponentially localized eigenfunctions. For the related Bernoulli-Anderson model, such *spectral* localization results were first proven in [CKM], later on in [SVW]. More recently, the random dimer model and the continuous Bernoulli-Anderson model were treated in [BG] and [DSS] respectively. The fact that such spectral localization can in principle coexist with quantum transport (even almost ballistic; note that ballistic is impossible with pure point spectrum [Sim]) was demonstrated by an example in [RJLS] (see also [BT]). However, those examples were rather artificial and much research since then was devoted to the program of proving dynamical localization (i.e. boundedness in time of the left-hand side of (1)) in models of physical interest with previously established spectral localization, by means of upgrading the proof of pure point spectrum (see [GK] and references therein, also [BJ]). The success of this program may have raised doubts as to the validity of the distinction between spectral and dynamical localization in physically relevant contexts. This paper demonstrates that the distinction should indeed be made as it shows that exponential localization and quantum transport coexist also in certain very natural random polymer models.

The bound (1) is deterministic albeit independent of the polymer configuration. To deduce what we believe to be almost surely the correct scaling behavior for the random polymer model, let us sketch the heuristics of [DWP]. It is known [Bov] and will be proven in [JSS] that the Lyapunov exponent generically vanishes quadratically like $\gamma(E_c+\epsilon) = c\epsilon^2+\mathcal{O}(\epsilon^3)$ in the vicinity of the critical energy E_c. The extension of the eigenstates in an ϵ-neighborhood of E_c is given by their localization length equal to the inverse of the Lyapunov exponent. Therefore the portion of the initial wave packet lying energetically in this ϵ-neighborhood spreads out ballistically up to time scales $T \approx \epsilon^{-2}$. Because it can be shown that the density of states is positive at E_c (see Proposition 2 below), this portion of states is proportional to ϵ. Consequently, the qth moment of the position operator should grow like T^q multiplied by this factor $\epsilon \approx T^{-1/2}$, showing that almost surely

$$\int_0^T \frac{dt}{T}\,\langle \psi|e^{\imath Ht}|X|^q e^{-\imath Ht}|\psi\rangle \;\approx\; C\,T^{q-\frac{1}{2}}\;,$$

should hold. This overdiffusive spreading is faster than in (1) and was numerically verified in [DWP] for $q=2$.

The main technical tools for the eigenvalue estimates are adequate action-angle variables, also called modified Prüfer variables in the mathematical literature. A perturbative expansion around the critical energy then allows to obtain estimates on the density of states in Chapter 3. These are complemented with deterministic upper and lower bounds on the corresponding eigenfunctions in Chapter 4. With this information, splitting of the initial wave packet as in [GSB] allows to conclude the proof of (1) in Chapter 5. The methods also allow to show quantitatively how large the moments of the position operator have to be if the commutator of the transfer matrices is small, but does not vanish.

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2 Model and main result

Let $\hat{t}_\pm = (\hat{t}_\pm(0), \ldots, \hat{t}_\pm(L_\pm - 1))$ and $\hat{v}_\pm = (\hat{v}_\pm(0), \ldots, \hat{v}_\pm(L_\pm - 1))$ be two pairs of finite sequences of real numbers, satisfying $\hat{t}_\pm(l) > 0$ for all $l = 0, \ldots, L_\pm - 1$. These numbers are the hopping and potential terms of two different polymers. A family of random Jacobi matrices is now constructed by random juxtaposition of these polymers. More precisely, to any sequence $\omega = (\omega_l)_{l\in\mathbb{Z}}$ of signs $+$ and $-$ one associates sequences $t_\omega = (t_\omega(n))_{n\in\mathbb{Z}}$ and $v_\omega = (v_\omega(n))_{n\in\mathbb{Z}}$ by means of $t_\omega = (\ldots, \hat{t}_{\omega_0}, \hat{t}_{\omega_1}, \ldots)$ and $v_\omega = (\ldots, \hat{v}_{\omega_0}, \hat{v}_{\omega_1}, \ldots)$. The polymer Hamiltonian H_ω of the configuration ω is then defined by

$$(H_\omega\psi)(n) \;=\; -t_\omega(n+1)\psi(n+1) + v_\omega(n)\psi(n) - t_\omega(n)\psi(n-1) \;, \qquad \psi \in \ell^2(\mathbb{Z}) \;. \tag{2}$$

The polymer transfer matrices $T^E_\pm$ at energy $E \in \mathbb{R}$ are introduced by

$$T^E_\pm \;=\; \prod_{l=L_\pm-1}^{0} T_{\hat{v}_\pm(l)-E,\hat{t}_\pm(l)} \;, \qquad \text{where} \qquad T_{v,t} \;=\; \frac{1}{t}\begin{pmatrix} v & -t^2 \\ 1 & 0 \end{pmatrix} \;. \tag{3}$$

The transfer matrices over several polymers are then

$$T^E_\omega(m,k) \;=\; T^E_{\omega_{k-1}} \cdot T^E_{\omega_{k-2}} \cdot \ldots \cdot T^E_{\omega_m} \;, \qquad k > m \;, \tag{4}$$

and $T^E_\omega(m,k) = (T^E_\omega(k,m))^{-1}$ if $k < m$, $T(m,m) = I$. For a random polymer model, namely if the ω_l's are independent random variables, the Lyapunov exponent $\gamma(E)$ gives the almost sure rate of exponential growth of the norms of the transfer matrices, that is $\|T^E_\omega(1,k)\| \sim \exp(\gamma(E)k)$ (see *e.g.* [PF] for a precise definition). The inverse of the Lyapunov exponent is also called the localization length. Hence vanishing of the Lyapunov exponent, that is slower than exponential growth of the transfer matrices $T^E_\omega(1,k)$, is considered an indicator for possible delocalization. For a polymer chain this happens in the following situation:

Definition 1 *An energy $E_c \in \mathbb{R}$ is called critical for the family $(H_\omega)_{\omega\in\Omega}$ of polymer Hamiltonians if the polymer transfer matrices $T^{E_c}_\pm$ are elliptic (i.e. $|\mathrm{Tr}(T^{E_c}_\pm)| < 2$) or equal to $\pm\mathbf{1}$ and commute*

$$[T^{E_c}_-, T^{E_c}_+] \;=\; 0 \;. \tag{5}$$

Remark 1 The definition does *not* allow the critical energy to be at the band edges of one of the periodic operators (constructed from $(\hat{t}_+, \hat{v}_+)$ and $(\hat{t}_-, \hat{v}_-)$ respectively) except for points of band touching.

Remark 2 The condition (5) contains 4 equations. Given a model, one can only vary the energy. Hence, in the space of polymer models existence of critical energies is a non-generic property. On the other hand, given an energy E_c, it is always possible to construct polymer models that have E_c as a critical energy.

Example The most prominent [DWP, Bov, BG] example is the random dimer model for which $L_+ = L_- = 2$ and $\hat{v}_+(0) = \hat{v}_+(1) = \lambda$ and $\hat{v}_-(0) = \hat{v}_-(1) = -\lambda$ ($\lambda \in \mathbb{R}$), and $t(l) = 1$ for all $l \in \mathbb{Z}$. This model has two critical energies $E_c = \lambda$ and $E_c = -\lambda$ as long as $\lambda < 1$.

The definition of the critical energy assures that there exists a real invertible matrix M transforming $T_-^{E_c}$ and $T_+^{E_c}$ simultaneously into rotations by angles η_- and η_+ respectively:

$$M T_\pm^{E_c} M^{-1} \;=\; \begin{pmatrix} \cos(\eta_\pm) & -\sin(\eta_\pm) \\ \sin(\eta_\pm) & \cos(\eta_\pm) \end{pmatrix} . \tag{6}$$

This implies that $\|T_\omega^{E_c}(1,k)\| = \mathcal{O}(1)$ independently of ω. Consequently $\gamma(E_c) = 0$. Because $T_\pm^E$ are polynomials of degree $L_\pm$ in E, one can expand $T_\pm^{E_c+\epsilon}$ into powers of ϵ. Since $M T_\pm^{E_c} M^{-1}$ are rotations, this implies that $\|M T_\pm^{E_c+\epsilon} M^{-1}\| \leq 1 + c|\epsilon|$ for $|\epsilon| \leq \epsilon_0$ and one deduces the following:

Proposition 1 *For every $\epsilon_0 > 0$ there exists a constant $C < \infty$ such that for all $|\epsilon| \leq \epsilon_0$ and $m, k \in \mathbb{Z}$,*

$$\|T_\omega^{E_c+\epsilon}(m,k)\| \;\leq\; C\, e^{C|\epsilon|\,|k-m|} . \tag{7}$$

Remark 3 In terms of the Lyapunov exponent, this shows $\gamma(E_c + \epsilon) \leq C|\epsilon|$, an estimate that also follows from the techniques of [KS]. For the random dimer model it was previously (non-rigorously) known that $\gamma(E_c + \epsilon) = \mathcal{O}(\epsilon^2)$ [Bov]. A proof based on a modified Prüfer variable analysis similar to that of Pastur and Figotin [PF] is given in [JSS].

Proposition 1 shows that the localization length diverges at the critical energy. The next proposition gives estimates on the number of the eigenvalues of finite size approximations of H_ω which are close to the critical energy.

Proposition 2 *For $N \in \mathbb{N}$, let Π_N be the projection from $\ell^2(\mathbb{Z})$ onto $\ell^2(\{0, \ldots, N-1\})$. For given $\epsilon_0 > 0$, there exist positive constants C_1 and C_2 such that for all $|\epsilon| \leq \epsilon_0$ and $N \in \mathbb{N}$,*

$$C_1\,(N\epsilon - 1) \;\leq\; \#\,\{\text{eigenvalues of } \Pi_N H_\omega \Pi_N \text{ in } [E_c - \epsilon, E_c + \epsilon]\,\} \;\leq\; C_2\,(N\epsilon + 1) \; . \tag{8}$$

This shows in particular that if only $N\epsilon \geq 1 + 1/C_1$ there is an eigenstate in the ϵ-neighborhood of E_c. The estimate (7) allows to show that this state is spread out over the whole sample of length N (see Proposition 3 below). In order to analyse whether this extended state leads to any quantum transport, let us consider the time averaged moments of the position operator X on $\ell^2(\mathbb{Z})$:

$$M_{\omega,q,\Delta}(T) \;=\; \int_0^T \frac{dt}{T}\,\langle 0|\chi_\Delta(H_\omega)e^{\imath H_\omega t}|X|^q e^{-\imath H_\omega t}\chi_\Delta(H_\omega)|0\rangle\;, \qquad q > 0\;, \tag{9}$$

where χ_Δ is the characteristic function of the Borel set $\Delta \subset \mathbb{R}$. Their growth exponents are called the diffusion exponents:

$$\beta^\pm_{\omega,q}(\Delta) \;=\; \lim_{T\to\infty}{}^{\pm}\,\frac{\log(M_{\omega,q,\Delta}(T))}{\log(T^q)}\;. \tag{10}$$

Here $\lim^\pm$ denote the superior and inferior limit respectively. If $\Delta = \mathbb{R}$, the argument Δ will be dropped. For the case of the random dimer model discussed above, DeBievre and Germinet [BG] have shown that almost surely $M_{\omega,q,\Delta}(T) < \infty$ uniformly in T as long as Δ is a compact set that does not contain the critical energies $E_c = \pm\lambda$ and $\lambda \neq 1/\sqrt{2}$. The methods developed in [DSS] to prove dynamical localization for continuum Bernoulli-Anderson models carry over to discrete random polymer models (2), where they prove that $\int d\mathbf{P}(\omega) M_{\omega,q,\Delta}(T) < \infty$ if a compact Δ does not contain critical energies. Hence the diffusion exponents vanish as long as E_c is not in Δ. It turns out that a *single* critical energy leads to nontrivial quantum transport. Namely, the main result gives a lower bound on the diffusion exponents in this situation. It is deterninistic in the sense that it holds for every configuration $\omega \in \Omega$.

Main Theorem *Suppose that a polymer Hamiltonian H_ω has a critical energy E_c. Then*

$$\beta^-_{\omega,q} \;\geq\; 1 - \frac{1}{q}\;.$$

Remark 4 The Main Theorem also extends to the situation where the initial state $|0\rangle$ in (9) is replaced by a compactly supported one. It does, however, not hold for an arbitrary inital state $\psi \in \ell^2(\mathbb{Z})$. Indeed, choosing ψ to be an eigenvector of H_ω obviously does not lead to any wave packet spreading.

The remainder of the paper contains the proofs of the above results as well as complementary ones. As everything will hold for all configurations with uniform constants, the index ω is considered fixed. Except for the argument in Section 3.3 it is therefore dropped.

3 Prüfer variables and bounds on eigenvalues

The first two Sections of this Chapter give general information on free and modified Prüfer variables of Jacobi matrices. These are used in Section 3.3 in order to prove Proposition 2.

3.1 Free Prüfer variables

Let $(t(n))_{n\in\mathbb{Z}}$ be a sequence of positive numbers and $(v(n))_{n\in\mathbb{Z}}$ a sequence of real numbers. As in (2) they define a Jacobi matrix H acting on $\ell^2(\mathbb{Z})$. Given an initial angle $\theta^0 \in [0,2\pi)$ and an energy $E\in\mathbb{R}$, let us construct the formal solution $(u_E(n))_{n\in\mathbb{Z}}$ by

$$-t(n+1)u_E(n+1)+v(n)u_E(n)-t(n)u_E(n-1) \;=\; E u_E(n)\;, \tag{11}$$

and the initial conditions $u_E(-1)=\sin(\theta^0)$ and $t(0)u_E(0)=\cos(\theta^0)$. Using the definition (3) of the transfer matrices, the solution satisfies

$$\begin{pmatrix} t(n+1)\,u_E(n+1) \\ u_E(n)\end{pmatrix} \;=\; T_{v(n)-E,t(n)} \begin{pmatrix} t(n)\,u_E(n) \\ u_E(n-1)\end{pmatrix}\;.$$

The (free) Prüfer phase $\theta_E(n)$ and amplitude $R_E(n)>0$ are now defined by

$$\begin{pmatrix} R_E(n)\cos(\theta_E(n)) \\ R_E(n)\sin(\theta_E(n))\end{pmatrix} \;=\; \begin{pmatrix} t(n)u_E(n) \\ u_E(n-1)\end{pmatrix}\;, \tag{12}$$

as well as

$$-\frac{\pi}{2} < \theta_E(n+1)-\theta_E(n) < \frac{3\pi}{2}\;, \qquad \theta_E(0)\;=\;\theta^0\;.$$

Lemma 1

$$R_E^2(n)\,\frac{\partial}{\partial E}\,\theta_E(n) \;=\; \begin{cases} \sum_{l=0}^{n-1} u_E^2(l) & \text{if} \quad n>0\;, \\ \\ -\sum_{l=n}^{-1} u_E^2(l) & \text{if} \quad n<0\;. \end{cases} \tag{13}$$

Proof: From the recurrence relation (11) and the definition of $\theta_E(n)$ one gets

$$\cot(\theta_E(n)) \;=\; -t^2(n-1)\tan(\theta_E(n-1))+v(n-1)-E\;.$$

Differentiation leads to

$$\frac{\partial}{\partial E}\theta_E(n) \;=\; \frac{t^2(n-1)\sin^2(\theta_E(n))}{\cos^2(\theta_E(n-1))}\,\frac{\partial}{\partial E}\theta_E(n-1)+\sin^2(\theta_E(n))\;.$$

Multiplying with $R_E^2(n)$ and using the definition of $R_E(n)$ and $\theta_E(n)$ gives

$$R_E^2(n)\frac{\partial}{\partial E}\theta_E(n) \;=\; R_E^2(n-1)\frac{\partial}{\partial E}\theta_E(n-1)+u_E^2(n-1). \tag{14}$$

The above deduction of (14) has used that $u(n-1)\neq 0$. If $u(n-1)=0$, then one may deduce (14) in a similar way from

$$\tan(\theta_E(n)) \;=\; \frac{\cot(\theta_E(n-1))}{-t^2(n-1)+(v(n-1)-E)\cot(\theta_E(n-1))}\;.$$

The lemma now follows by iterating (14). $\Box$

Note in particular that (13) implies that $\partial\theta_E(n)/\partial E$ is strictly positive for $n \geq 2$ and strictly negative for $n \leq -2$. Furthermore, it follows from elementary considerations for transfer matrices that there are constants C_1 and C_2 such that

$$0 \;<\; C_1 \;\leq\; \left|\frac{\partial}{\partial E}\,\theta_E(n)\right| \;\leq\; C_2 \;<\; \infty\;, \tag{15}$$

where C_1 and C_2 can be uniformly bounded away from 0 and ∞ as long as $|n| \geq 2$ and the quantities $|n|$, E and $\max_{|k|\leq|n|}\{|v(k)|, t(k), 1/t(k)\}$ remain bounded (where only the lower bound requires $|n| \geq 2$).

Let Π_N be the projection on $\ell^2(\{0,\ldots,N-1\})$ and denote the associated finite-size Jacobi matrix by $H_N = \Pi_N H \Pi_N$. As H_N has Dirichlet boundary conditions, let us choose the $u_E(-1) = 0$ and $t(0)u_E(0) = u_E(0) = 1$ as initial conditions in the recurrence relation (11). This corresponds to an initial Prüfer phase $\theta^0 = 0$. The formal solution u_E then gives an eigenvector (and E is an eigenvalue of H_N) if and only if $t(N)u_E(N) = R_E(N)\cos(\theta_E(N)) = 0$, that is $\theta_E(N) = \frac{\pi}{2} \mod \pi$ (note herefore that $u_E(0) \neq 0$ for any eigenvector of H_N).

For E sufficiently close to $-\infty$, one checks iteratively in n that $u(n) > 0$ for all $n = 0,\ldots,N-1$. Moreover, $\lim_{E\to-\infty} u_E(n)/u_E(n-1) = \infty$ for such n. Hence $\lim_{E\to-\infty}\theta_E(N) = 0$. As $\theta_E(N)$ is monotone increasing in E, it follows that the jth eigenvalue E_j of H_N (counted from below $E_1 < E_2 < \ldots < E_N$) satisfies

$$\theta_{E_j}(N) \;=\; \frac{\pi}{2} + \pi(j-1)\;, \qquad \theta^0 \;=\; 0\;. \tag{16}$$

This Oscillation Theorem implies immediately (if we allow arbitrary initial phase $\theta^0 \in [0, 2\pi)$):

$$\left|\frac{1}{\pi}\,\theta_E(N) \;-\; \#\;\{\text{negative eigenvalues of }\;(H_N - E)\,\}\right| \;\leq\; 3\;. \tag{17}$$

3.2 Modified Prüfer variables

Let M be a real 2×2 matrix with $\det(M) = 1$. Define a smooth function $m : \mathbb{R} \to \mathbb{R}$ with $m(\theta+\pi) = m(\theta) + \pi$, $0 < C_1 \leq m' \leq C_2 < \infty$, by

$$\begin{pmatrix} r\cos(m(\theta)) \\ r\sin(m(\theta)) \end{pmatrix} \;=\; M \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}\;, \qquad r > 0\;, \qquad m(0) \in [0, 2\pi)\;.$$

Then the M-modified Prüfer variables $(R_E^M(n), \theta_E^M(n)) \in \mathbb{R}_+ \times \mathbb{R}$ for initial condition $\theta^{M,0} = m(\theta^0)$ are given by

$$\theta_E^M(n) \;=\; m(\theta_E(n))\;, \tag{18}$$

and

$$\begin{pmatrix} R^M_E(n)\cos(\theta^M_E(n)) \\ R^M_E(n)\sin(\theta^M_E(n)) \end{pmatrix} \;=\; M \begin{pmatrix} t(n)\,u_E(n) \\ u_E(n-1) \end{pmatrix} . \tag{19}$$

Bounds of the form (15) also hold for an M-modified Prüfer phase $\theta^M_E(n)$ because $\theta^M_E(n) = m(\theta_E(n))$ leads to $(\min m')|\partial_E\theta_E| \le |\partial_E\theta^M_E| \le (\max m')|\partial_E\theta_E|$. Furthermore, as $|\theta^M_E(n) - \theta_E(n)| \le 3\pi$, (17) implies that

$$\left| \frac{1}{\pi}\,\theta^M_E(N) \;-\; \#\,\{\text{negative eigenvalues of } \;\; (H_N - E)\,\} \right| \;\le\; 6\ . \tag{20}$$

3.3 Bounds on modified Prüfer phases of polymer models

This section contains the proof of Proposition 2. The main tool will be the polymer phase shifts defined next. The polymer phase shifts $\varphi_{\epsilon,\pm} : \mathbb{R} \to \mathbb{R}$ at energy $E_c + \epsilon$ (and of the M-modified Prüfer phases) are defined by

$$\varphi_{\epsilon,\pm}(\theta) \;=\; \theta^M_{E_c+\epsilon}(L_\pm) - \theta\ , \qquad \theta^{M,0} \;=\; \theta\ .$$

Associated to the configuration $\omega = (\omega_k)_{k\in\mathbb{Z}}$ are the lower polymer nodes denoted by $(n_k)_{k\in\mathbb{Z}}$. For simplicity, let us assume $n_0 = 0$. Then

$$\theta^M_{E_c+\epsilon}(n_k) \;=\; \varphi_{\epsilon,\omega_k}(\theta^M_{E_c+\epsilon}(n_{k-1})) + \theta^M_{E_c+\epsilon}(n_{k-1})\ . \tag{21}$$

A key observation following from (6) and (19) is that the polymer phase shifts at the critical energy are independent of the incoming phase:

$$\varphi_{0,\pm}(\theta^M) \;=\; \eta_\pm\ , \qquad \forall\;\, \theta^M \in \mathbb{R}\ . \tag{22}$$

Another property of the polymer phase shifts due to the equivalent of (15) for M-modified Prüfer variables is (under hypothesis that $L_\pm \ge 2$)

$$0 \;<\; C_1 \;\le\; \frac{\partial}{\partial\epsilon}\,\varphi_{\epsilon,\pm}(\theta^M) \;\le\; C_2 \;<\; \infty\ , \tag{23}$$

with constant $C_{1,2}$ which are uniform in θ^M. These properties now allow to prove the following estimates on the M-modified Prüfer phases:

Lemma 2 *For every $\epsilon_0 > 0$ there exist positive constants C_1 and C_2 such that for all $0 \le \epsilon \le \epsilon_0$, initial conditions θ, and $k \ge 2$,*

$$C_1\,\epsilon k \;\le\; \theta^M_{E_c+\epsilon}(n_k) - \theta^M_{E_c}(n_k) \;\le\; C_2\,\epsilon k.$$

The same bounds hold for $\theta^M_{E_c}(n_k) - \theta^M_{E_c-\epsilon}(n_k)$.

Proof: Using (21) in order to develop into telescopic sums and subsequently (22),

$$\begin{aligned}
\theta^M_{E_c+\epsilon}(n_k) - \theta^M_{E_c}(n_k) &= \sum_{l=1}^{k} \varphi_{\epsilon,\omega_l}(\theta^M_{E_c+\epsilon}(n_{l-1})) - \varphi_{0,\omega_l}(\theta^M_{E_c}(n_{l-1})) \\
&= \sum_{l=1}^{k} \varphi_{\epsilon,\omega_l}(\theta^M_{E_c+\epsilon}(n_{l-1})) - \varphi_{0,\omega_l}(\theta^M_{E_c+\epsilon}(n_{l-1})) \\
&= \sum_{l=1}^{k} \int_0^{\epsilon} de \, \frac{\partial}{\partial e} \, \varphi_{e,\omega_l}(\theta^M_{E_c+\epsilon}(n_{l-1}))
\end{aligned}$$

If $L_\pm \geq 2$, then the lemma follows from (23). If $\min\{L_\pm\} = 1$, a similar argument can be done where the k polymers in the string are grouped in pairs and a corresponding telescopic sum is used. The bounds for $\theta^M_{E_c}(n_k) - \theta^M_{E_c-\epsilon}(n_k)$ are proven in the same way. □

Proof of Proposition 2: If 0 and N are lower polymer nodes, then (8) is a direct consequence of Lemma 2 and (20), noting that the number of polymers in $\{0, \ldots, N-1\}$ grows linearly in N. It then follows for the general case by applying Lemma 2 and (20) to the interval from the left most to the right most polymer node in $\{0, \ldots, N\}$ and by noting that the Prüfer phase changes by no more than $2\pi L_\pm$ over the lattice sites of the boundaries. □

4 Bounds on eigenfunctions

4.1 Upper bound on eigenfunctions

Proposition 3 *To every $\epsilon_0 > 0$ there is a constant $C < \infty$ such that the following holds: for any $N \in \mathbb{N}$ and any normalized eigenstate ϕ_E of H_N with energy E satisfying $|E - E_c| = \epsilon \leq \epsilon_0$,*

$$\max_{0 \leq n \leq N-1} |\phi_E(n)|^2 \;\leq\; C \, \max\left\{\epsilon, \frac{1}{N}\right\} .$$

Proof: For given n, let n_k be the nearest polymer node. The definition of the transfer matrices gives

$$\begin{pmatrix} t(n_l)\phi_E(n_l) \\ \phi_E(n_l-1) \end{pmatrix} \;=\; T^E(k,l) \begin{pmatrix} t(n_k)\phi_E(n_k) \\ \phi_E(n_k-1) \end{pmatrix} .$$

As $\|T^E(k,l)^{-1}\| = \|T^E(k,l)\|$, this implies

$$\big(t(n_k)|\phi_E(n_k)|^2 + |\phi_E(n_k-1)|^2\big) \, \frac{1}{\|T^E(k,l)\|^2} \;\leq\; \big(t(n_l)|\phi_E(n_l)|^2 + |\phi_E(n_l-1)|^2\big) . \tag{24}$$

Assume now that there are K polymer nodes between 0 and N. Summing (24) over these nodes and using $c < t(n) < C$ as well as the fact that ϕ_E is normalized shows that

$$|\phi_E(n_k)|^2+|\phi_E(n_k-1)|^2 \;\leq\; C\,\left(\sum_{l=1}^{K}\frac{1}{\|T^E(k,l)\|^2}\right)^{-1}\;. \tag{25}$$

By Proposition 1 and $K\sim N$,

$$\sum_{l=1}^{K}\frac{1}{\|T^E(k,l)\|^2}\;\geq\;\sum_{l=1}^{K}Ce^{-C|\epsilon||n_l-n_k|}\;\geq\;\frac{C'}{\epsilon}\,(1-e^{-C|\epsilon|N})\;.$$

As $e^{-x}\leq 1-x/2$ for $0\leq x\leq 1$, this implies that the l.h.s. of (25) is smaller or equal than $C\,\max\{\epsilon,1/N\}$. Since $|n-n_k|<\max L_\pm$, an elementary bound on transfer matrices completes the proof of the proposition. □

4.2 A local lower bound on eigenfunctions

If $\epsilon N\sim 1$, then the argument in Section 4.1 also gives a lower bound $|\phi_E(n)|^2+|\phi_E(n-1)|^2\geq C/N$, but due to the oscillatory nature of solutions there is no such lower bound for a single term $|\phi_E(n)|^2$. However, one can get a lower bound at a single point near the center of the chain if one sums over two adjacent eigenvalues:

Proposition 4 *Let $\epsilon_0>0$ and $C>0$. Then there exists a constant $C_1>0$ such that for all $N\geq 4$ and $|\epsilon|\leq\epsilon_0$ with $N\epsilon\leq C$ the following holds: if ϕ_j and ϕ_{j+1} are normalized eigenfunctions for H_N to two adjacent eigenvalues $E_j,E_{j+1}\in[E_c-\epsilon,E_c+\epsilon]$ and if N_c satisfies $|N_c-N/2|\leq 1/2$, then*

$$|\phi_j(N_c)|^2+|\phi_{j+1}(N_c)|^2\;\geq\;\frac{C_1}{N}\;. \tag{26}$$

The proof is based on the following lemma holding for arbitrary Jacobi matrices associated to a sequence $(t(n),v(n))_{n\in\mathbb{Z}}$. Herefore let us introduce not only the polymer transfer matrices, but also arbitrary ones by $\mathcal{T}^E(k,l)=\prod_{n=l-1}^{k}T_{v(n)-E,t(n)}$ where $T_{v,t}$ is defined in (3).

Lemma 3 *Let H be a Jacobi matrix with uniformly bounded matrix elements. Let $N\geq 4$, $E_j<E_{j+1}$ be subsequent eigenvalues of H_N with ϕ_j and ϕ_{j+1} being the corresponding normalized eigenfunctions. Then there exists a constant $C>0$ only depending on*

$$\max_{E_j\leq E\leq E_{j+1}}\;\max_{0\leq k,l\leq N}\;\|\mathcal{T}^E(k,l)\|\qquad\textit{and}\qquad\max\left\{\frac{m}{N-m},\frac{N-m}{m}\right\}\;, \tag{27}$$

such that for $m\geq 2$ and $N-m\geq 2$

$$|\phi_j(m)|^2+|\phi_{j+1}(m)|^2\;\geq\;\frac{C}{N}\;. \tag{28}$$

Proof: For the entire proof, C denotes varying positive and finite constants, which only depend on the quantities in (27), i.e. which can be uniformly bounded away from 0 and ∞ if these quantities satisfy uniform upper bounds.

Dirichlet boundary conditions are used so that (16) holds. As technical tool, let us also consider the formal solution $\hat{u}_E$ starting from the right boundary, notably satisfying the initial conditions $\hat{u}_E(N-1) = 1$ and $\hat{u}_E(N) = 0$. The associated (free Dirichlet) Prüfer variables $(\hat{\theta}_E(n), \hat{R}_E(n))$ starting from the right boundary also satisfy (12), but the (right-side) initial condition $\hat{\theta}_E(N) = \pi/2$ is being imposed.

By additivity of Prüfer phases and (16),

$$\theta_{E_j}(m) + \left(\frac{\pi}{2} - \hat{\theta}_{E_j}(m)\right) \;=\; \theta_{E_j}(N) \;=\; \frac{\pi}{2} + \pi(j-1)\;.$$

Using the same equation for E_{j+1}, this implies $\alpha + \hat{\alpha} = \pi$ if one sets $\alpha = \theta_{E_{j+1}}(m) - \theta_{E_j}(m)$ and $\hat{\alpha} = \hat{\theta}_{E_j}(m) - \hat{\theta}_{E_{j+1}}(m)$. From Lemma 1 and a shifted version of it follows that

$$\alpha \;=\; \int_{E_j}^{E_{j+1}} dE\, \frac{\partial}{\partial E}\,\theta_E(m) \;=\; \int_{E_j}^{E_{j+1}} dE\; R_E^{-2}(m) \sum_{n=0}^{m-1} u_E^2(n)\;, \tag{29}$$

and

$$\hat{\alpha} \;=\; -\int_{E_j}^{E_{j+1}} dE\, \frac{\partial}{\partial E}\,\hat{\theta}_E(m) \;=\; \int_{E_j}^{E_{j+1}} dE\; \hat{R}_E^{-2}(m) \sum_{n=m}^{N-1} \hat{u}_E^2(n)\;, \tag{30}$$

If $\|\mathcal{T}^E(k,l)\|^2 \leq C$ for all $k, l = 0, \ldots, N$ and $E_j \leq E \leq E_{j+1}$, then it follows that, for all $n = 0, \ldots, N$,

$$\frac{1}{C} \;\leq\; \frac{1}{\|\mathcal{T}^E(0,n)\|^2} \;\leq\; R_E^2(n) \;=\; \left\|\mathcal{T}^E(0,n)\begin{pmatrix}1\\0\end{pmatrix}\right\|^2 \;\leq\; \|\mathcal{T}^E(0,n)\|^2 \;\leq\; C\;,$$

This yields $0 < \frac{1}{C} \leq R_E^2(n) \leq C < \infty$ and $u_E^2(n) \leq C < \infty$ for $E_j \leq E \leq E_{j+1}$ and $n = 0, \ldots, N$. Furthermore $u_E^2(n-1) + u_E^2(n) \geq C R_E^2(n)$ which implies for $m \geq 2$ that $m/C \leq \sum_{n=0}^m u_E^2(n) \leq Cm$. Similar estimates hold for $\hat{R}_E$ and $\hat{u}_E$ leading to $(N-m)/C \leq \sum_{n=m}^N \hat{u}_E^2(n) \leq C(N-m)$ for $N-m \geq 2$. Inserting these inequalities into (29) and (30) shows that

$$\max\{\alpha,\hat{\alpha}\} \leq C(E_{j+1}-E_j)\max\{m, N-m\}\;, \qquad \min\{\alpha,\hat{\alpha}\} \geq C(E_{j+1}-E_j)\min\{m, N-m\}\;.$$

Using now the uniform bound on the second term in (27), these estimates lead to $0 < 1/C \leq \alpha/\hat{\alpha} \leq C < \infty$. Recalling that $\alpha + \hat{\alpha} = \pi$ this yields the existence of $\eta > 0$ such that $\eta \leq \frac{\alpha}{\pi} \leq 1 - \eta$.

As $\alpha = \theta_{E_{j+1}}(m) - \theta_{E_j}(m)$, one infers that $\cos^2(\theta_{E_{j+1}}(m)) + \cos^2(\theta_{E_j}(m)) \geq \eta^2 > 0$. For the normalized eigenfunctions ϕ_j and ϕ_{j+1} this implies

$$
\begin{aligned}
|\phi_j(m)|^2 + |\phi_{j+1}(m)|^2 &= \frac{u^2_{E_j}(m)}{\sum_{n=0}^{N-1} u^2_{E_j}(n)} + \frac{u^2_{E_{j+1}}(m)}{\sum_{n=0}^{N-1} u^2_{E_{j+1}}(n)} \\
&\geq \frac{1}{CN}\left(u^2_{E_j}(m) + u^2_{E_{j+1}}(m)\right) \\
&= \frac{1}{CNt^2(m)}\left(R^2_{E_j}(m)\cos^2(\theta_{E_j}(m) + R^2_{E_{j+1}}(m)\cos^2(\theta_{E_{j+1}}(m)\right) \\
&\geq \frac{\eta^2}{CN} ,
\end{aligned}
$$

which completes the proof. □

Proof of Proposition 4: If n_l, n_k are polymer nodes, then $\mathcal{T}^E(n_k, n_l) = T^E(k,l)$. Since $N\epsilon \leq C$, Proposition 1 implies $\|\mathcal{T}^E(n_k, n_l)\| \leq C$. By elementary bounds on transfer matrices, this allows to deduce a uniform bound on the first quantity in (27). For $m = N_c$ also the second quantity in (27) remains bounded. Thus Proposition 4 follows from Lemma 3. □

5 Lower bound on quantum diffusion

Proof of the Main Theorem: Let us fix q and ϵ_0. To given $T > 0$ choose K to be the smallest integer such that $K \geq T/c$, where c is the constant from Proposition 5. Let H^K and X^K be the restrictions of the polymer Hamiltonian H and the position operator X to $\ell^2(-K, \ldots, K)$. Here the upper index is used to designate the restrictions centered at the origin. Clearly all the results obtained in the preceding sections carry over from the right-sided restriction H_N to the centered one H^K. If $M_q(T)$ is defined as in (9) with $H = H_\omega$ and $\Delta = \mathbb{R}$, and

$$
M_q^K(T) \;=\; \int_0^T \frac{dt}{T}\, \langle 0| e^{\imath H^K t} |X^K|^q e^{-\imath H^K t} |0\rangle \;, \tag{31}
$$

then it follows from Proposition 5 in Appendix A that

$$
|M_q(T) - M_q^K(T)| \;\leq\; c'\, K^{q+2}\, e^{-c'' K} \;. \tag{32}
$$

Let g be a positive C^∞-function supported by $[0,1]$, uniformly bounded by 1 and satisfying $\int dx\, g(x) = 1/2$. Then for any $N \leq K$

$$
M_q^K(T) \;\geq\; N^q \int \frac{dt}{T}\, g\left(\frac{t}{T}\right) \left(1 - \sum_{|n|\leq N} |\langle n| e^{-\imath H^K t} |0\rangle|^2\right) \;.
$$

Choose $\delta = \min\{(1+2/C_1)/(2K+1), \epsilon_0\}$ with C_1 from (8) such that Proposition 2 guarantees that H^K has at least two eigenvalues in $[E_c - \delta, E_c + \delta]$. Let J be the largest integer such that

$J \leq \epsilon_0/\delta$ and set $I_j = \{E \in \mathbb{R} | (j-1)\delta \leq |E - E_c| \leq j\delta\}$ for $j = 1, \ldots, J$ as well as $I_\infty = \mathbb{R} \setminus \bigcup_{j=1}^J I_j$. As usual, $I^c = \mathbb{R} \setminus I$ denotes the complementary set. Splitting of the wave packet as in [GSB] leads to

$$\begin{aligned} M_q^K(T) \ \geq \ N^q \int \frac{dt}{T}\, g\left(\frac{t}{T}\right) \Bigg(\langle 0|\chi_{I_1}(H^K)|0\rangle - \sum_{|n|\leq N} |\langle n|e^{-\imath H^K t}\chi_{I_1}(H^K)|0\rangle|^2 \\ - 2\,\Re e \sum_{|n|\leq N} \langle 0|e^{\imath H^K t}\chi_{I_1}(H^K)|n\rangle\langle n|e^{-\imath H^K t}\chi_{I_1^c}(H^K)|0\rangle \Bigg) \ . \end{aligned}$$

Next write $\chi_{I_1^c} = \sum_{j=2}^J \chi_{I_j} + \chi_{I_\infty}$. With the definition

$$A_{T,N}(I,I') \ = \ \int \frac{dt}{T}\, g\left(\frac{t}{T}\right) \sum_{|n|\leq N} \langle 0|e^{\imath H^K t}\chi_I(H^K)|n\rangle\langle n|e^{-\imath H^K t}\chi_{I'}(H^K)|0\rangle \ ,$$

one then has

$$M_q^K(T) \ \geq \ \frac{1}{2}\, N^q \left(\langle 0|\chi_{I_1}(H^K)|0\rangle - 4 \sum_{j=1}^J |A_{T,N}(I_1,I_j)| - 4\,|A_{T,N}(I_1,I_\infty)| \right) \ . \tag{33}$$

Since $K\delta$ is bounded and $I_1 = [E_c - \delta, E_c + \delta]$ contains at least two eigenvalues, Proposition 4 with $\epsilon = \delta$ implies that there is a constant $C' > 0$ such that

$$\langle 0|\chi_{I_1}(H^K)|0\rangle \ = \ \sum_{E\in I_1} |\phi_E(0)|^2 \ \geq \ \frac{C'}{K} \ , \tag{34}$$

where the sum is taken over all eigenvalues E of H^K in I_1 and ϕ_E are the corresponding normalized eigenfunctions.

Lemma 4 *Let*

$$B_N(I,I') \ = \ \sum_{|n|\leq N} \langle 0|\chi_I(H^K)|0\rangle\, \langle n|\chi_{I'}(H^K)|n\rangle \ .$$

Denoting the pth derivative of g by $g^{(p)}$, one has

$$|A_{T,N}(I,I')| \ \leq \ \min\left\{\frac{1}{2}, (T\cdot \mathrm{dist}(I,I'))^{-p}\|g^{(p)}\|_\infty\right\}\ B_N(I,I')^{1/2}\, B_N(I',I)^{1/2} \ . \tag{35}$$

Proof: Define the complex spectral measures $\mu_{l,k}$ by $\int d\mu_{l,k}(E)\, f(E) = \langle l|f(H^K)|k\rangle$, $f \in C_0(\mathbb{R})$, and write μ_k for $\mu_{k,k}$. Then

$$A_{T,N}(I,I') \ = \ \sum_{|n|\leq N} \int_I d\mu_{0,n}(E) \int_{I'} d\mu_{n,0}(E') \int_0^T \frac{dt}{T}\, g\left(\frac{t}{T}\right)\ e^{\imath(E-E')t} \ . \tag{36}$$

Integrating by parts p times gives

$$\int_0^T \frac{dt}{T}\, g\left(\frac{t}{T}\right)\, e^{\imath(E-E')t} \;=\; (-\imath T(E-E'))^{-p}\int_0^T \frac{dt}{T}\, g^{(p)}\left(\frac{t}{T}\right)\, e^{\imath(E-E')t}\ ,$$

and thus

$$|A_{T,N}(I,I')| \;\leq\; \min\left\{\frac{1}{2}, (T\cdot \mathrm{dist}(I,I'))^{-p}\|g^{(p)}\|_\infty\right\}\sum_{|n|\leq N}|\mu_{0,n}|(I)|\mu_{n,0}|(I')\ , \tag{37}$$

where $|\mu_{j,k}|$ is the total variation of $\mu_{j,k}$. For every Borel set Δ, $|\mu_{j,k}|(\Delta)\leq \mu_j(\Delta)^{1/2}\mu_k(\Delta)^{1/2}$. Thus the Cauchy-Schwarz inequality implies

$$\begin{aligned}\sum_{|n|\leq N}|\mu_{0,n}|(I)|\mu_{n,0}|(I') \;&\leq\; \left(\sum_{|n|\leq N}\mu_0(I)\mu_n(I')\right)^{1/2}\left(\sum_{|n|\leq N}\mu_0(I')\mu_n(I)\right)^{1/2}\\ &=\; B_N(I,I')^{1/2}B_N(I',I)^{1/2}\ .\end{aligned}$$

Therefore (35) follows from (37). □

Proposition 2 gives a bound $C_2((2K+1)j\delta+1)$ on the number of eigenvalues in I_j, while Proposition 3 gives a bound $C\max(j\delta,1/K)$ on the values of squares of corresponding normalized eigenfunctions. This yields that $\sup_{|n|\leq K}\langle n|\chi_{I_j}(H^K)|n\rangle \leq CKj^2\delta^2$. Using also the trivial bound $\langle n|\chi_{I_\infty}(H^K)|n\rangle\leq 1$, one finds

$$B_N(I_1,I_j) \;\leq\; \begin{cases} CNj^2\delta^2 & \text{if } j=1,\dots,J\ ,\\ CN\delta & \text{if } j=\infty\ ,\end{cases}$$

and the same bounds for $B_N(I_j,I_1)$. Inserting this into (35) with $p=4$ gives the bounds

$$|A_{T,N}(I_1,I_j)| \leq \begin{cases} CN\delta^2 & \text{if } j=1,2,\\ CNT^{-4}(j\delta)^{-2} & \text{if } j=3,\dots,J\ ,\\ CNT^{-4}\delta & \text{if } j=\infty\ .\end{cases}$$

This can now be used to bound the negative terms on the r.h.s. of (33):

$$\sum_{j=1}^{J}|A_{T,N}(I_1,I_j)|+|A_{T,N}(I_1,I_\infty)| \;\leq\; CN\delta^2+CNT^{-4}\delta+CT^{-4}N\delta^{-2}\sum_{j=3}^{J}j^{-2} \;\leq\; C''N/K^2\ ,$$

since $T\sim K\sim 1/\delta$. Inserting this and (34) into (33) leads to

$$M_q^K(T) \;\geq\; \frac{1}{2}N^q \left(\frac{C'}{K} - \frac{4C''N}{K^2}\right)\,.$$

Now choose N to be the largest integer such that $N \leq K\cdot \min\{1, C'/8C''\}$. Thus

$$M_q^K(T) \;\geq\; \frac{C'N^q}{4K} \;\geq\; C\,T^{q-1}\,,$$

for some constant $C > 0$. Combined with (32) this gives the desired lower bound for the diffusion exponent.

A Comparing dynamics at finite and infinite volume

A metric on $\mathbb{Z}^d$ is given by $|n| = \max_{j=1\ldots d}|n_j|$ where $n = (n_1 \ldots n_d) \in \mathbb{Z}^d$. Given $K > 0$, a finite volume projection in $\ell^2(\mathbb{Z}^d)$ is defined by $\Pi^K = \sum_{|n|\leq K}|n\rangle\langle n|$. Let us introduce the finite volume restriction of an operator A on $\ell^2(\mathbb{Z}^d)$ by $A^K = \Pi^K A \Pi^K$. Finally, $\vec{X} = (X_1, \ldots, X_d)$ denotes the standard unbounded position operator on $\ell^2(\mathbb{Z}^d)$ with self-adjoint domain.

Proposition 5 *Let $H = \sum_{n,k\in\mathbb{Z}^d} h_{n,k}\,|n\rangle\langle k|$ be a self-adjoint Hamiltonian satisfying $|h_{n,k}| \leq C$ for some $C < \infty$ and $|h_{n,k}| = 0$ if $|n-k| > D$ for some $D > 0$. Let $q > 0$. Then there exist positive constants c, c', c'' depending only on C, D, q and d such that for all K and $t \leq cK$*

$$\left|\langle 0|e^{\imath Ht}|\vec{X}|^q e^{-\imath Ht}|0\rangle - \langle 0|e^{\imath H^K t}|\vec{X}^K|^q e^{-\imath H^K t}|0\rangle\right| \;\leq\; c' K^{q+d+1} e^{-c''K}\,. \tag{38}$$

Proof: Following the strategy of proving ballistic upper bounds given in [Gua], let us fix $\alpha > 0$ and define a Banach subspace B_α of $\ell^2(\mathbb{Z}^d)$ by asking its elements to have a finite norm $\|\psi\|_\alpha = \sup_{k\in\mathbb{Z}^d} e^{\alpha|k|}|\langle k|\psi\rangle| < \infty$. It can be checked from the hypothesis that H is a bounded operator on B_α. This implies that

$$|\langle n|e^{-\imath Ht}|0\rangle| \;\leq\; \exp\left(-n\alpha + t\,\|H\|_\alpha\right)\,. \tag{39}$$

The same bound holds for H^K. Using (39), one immediately verifies that

$$\left|\langle 0|e^{\imath Ht}|\vec{X}|^q e^{-\imath Ht}|0\rangle - \langle 0|e^{\imath Ht}|\vec{X}^K|^q e^{-\imath Ht}|0\rangle\right| \;\leq\; cK^q \exp\left(-K\left(\alpha - \frac{2t}{K}\,\|H\|_\alpha\right)\right)\,.$$

Hence, for all times $t \leq K\alpha/(4\|H\|_\alpha)$, one may use $\vec{X}^K$ instead of $\vec{X}$ with an error of the order of the r.h.s. of (38). Furthermore, it follows from DuHamel's formula that

$$\begin{aligned}
&\langle 0|e^{\imath Ht}|\vec{X}^K|^q e^{-\imath Ht}|0\rangle - \langle 0|e^{\imath H^K t}|\vec{X}^K|^q e^{-\imath H^K t}|0\rangle \\
&\quad = -\imath \int_0^t ds\ \langle 0|e^{\imath Ht}|\vec{X}^K|^q e^{-\imath H(t-s)}(H-H^K)e^{-\imath H^K s}|0\rangle \\
&\qquad + \imath \int_0^t ds\ \langle 0|e^{\imath H(t-s)}(H-H^K)e^{-\imath H^K s}|\vec{X}^K|^q e^{-\imath H^K t}|0\rangle\ .
\end{aligned}$$

Now $H-H^K$ is an operator localized outside of Π_K, except for hopping terms within a distance D from the boundary of Π_K. Inserting unity $\mathbf{1}=\sum_{n\in\mathbb{Z}^d}|n\rangle\langle n|$ three times and using (39) on all terms, one can verify that the r.h.s. is of the order $K^{q+d}t\exp(-K(\alpha/2-2t\|H\|_\alpha/K))$, which allows to conclude. □

Remark 5 Instead of the finite size operator, one can also use the periodic operator with period K in (38).

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