Traveling Waves in Lattice Dynamical Systems

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Abstract

For a class of 1-dimensional lattice dynamical systems we prove the existence of periodic traveling waves with prescribed speed and arbitrary period. Then we study asymptotic behaviour of such waves for big values of period and show that they converge, in an appropriate topology, to a solitary traveling wave.

1 Introduction

This paper concerns traveling waves for lattices of particles with nearest neighbour interaction. Corresponding equations of motion are of the form

$$\ddot{q}_n = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1}), \quad n \in \mathbb{Z}, \tag{1}$$

where q_n denotes the coordinate of the nth particle and $V(r)$ is the potential of interaction. Equation (1) can be written as an infinite dimensional Hamiltonian system with Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} (\frac{1}{2}p_n^2 + V(q_{n+1} - q_n)),$$

where $p_n = \dot{q}_n$ is the momentum.

After the famous work by E. Fermi, J. Pasta and S. Ulam [9], such systems are of common interest due to their unexpected properties. First of all, the well-known Toda lattice defined by

$$V(r) = ab^{-1}(\exp(-br) + br - 1)$$

is a completely integrable system and its dynamics is understood very well [18]. There is also a number of other potentials, like the power potential, the Lennard-Jones potential, etc., which are of interest for many applications. However, until the beginning of the 90's dynamical properties of equation (1) with potentials of such kind were studied mainly by means of numerical simulations (see, e.g., [7, 8]).

In the 90's, due to the implementation of variational methods, the situation was changed and a number of rigorous results was obtained in the case of the general equation (1). In [1, 2, 3, 4, 5, 16] periodic motions for lattice systems were studied (even for time dependent and spatially inhomogeneous cases).

In the case of a general superquadratic potential, the first rigorous study of traveling waves, i.e. solutions of the form $q_n(t) = u(n - ct)$, was carried out in [11]. Making use of a constrained minimization approach, those authors obtained an existence result for solitary traveling waves with prescribed averaged potential energy. Then the wave speed c is determined in terms of the corresponding Lagrange multiplier.

A more direct approach to the existence problem for solitary waves has been suggested in a recent paper [17]. Here the Mountain Pass Theorem [20] was applied to prove the existence of solitary waves with prescribed wave speed c lying in a natural range.

We have also to point out a recent paper [10] containing, among other things, a number of qualitative properties of lattice solitary waves, mainly in the case of near sonic speed in the terminology of that paper (in our notations $c - c_0 > 0$ is small enough, c_0^2 is the coefficient in the linear term of the right-hand side of (1)).

In the present paper we study both solitary and time periodic traveling waves. Our starting point is the following observation. For the Toda lattice, the explicit expression for k-periodic waves contains, as the limit case $k = \infty$, the Toda soliton, i.e. periodic waves give rise to solitary ones in the large wave length limit [18]. For general potentials we first prove the existence of k-periodic traveling waves, i.e. waves whose profile function $u(t)$ has k-periodic derivative. Of course, for the corresponding lattice solution $\dot{q}_n(t) = -cu'(n - ct)$ is k/c-periodic in time, but only almost periodic with respect to the spatial variable n in general. Such a solution is periodic in n only in the case of rational k. Then we examine the behavior of such waves in the large wave length limit, $k \to \infty$, and prove that (after translations and passage to a subsequence) they converge to a solitary traveling wave in an appropriate local topology. Remark that in the case of Lennard-Jones type potentials an existence result for periodic traveling waves was obtained in [19]. Under some additional, not so restrictive, assumptions we study so-called ground waves, i.e. waves with minimal possible averaged action. Here we use subsequently a constrained minimization approach known as the Nehari variational principle, or the Nehari manifold approach. We prove the existence of periodic ground waves and improve for such waves the above mentioned result on large wave length limit. Namely, in this case the limit is a solitary ground wave and convergence takes place on the entire period intervals, depending on k, i.e. in a sense globally. Roughly speaking, this means that a single bump of a periodic wave has almost the same shape as a solitary one whenever the wave length is large enough. Let us remark also that here we exploit the same ideas as in our previous papers and, therefore, the present paper may be considered as a natural continuation of [13, 14].

Throughout this paper we consider only increasing waves, i.e. waves with monotone profile function $u(t)$. However, similar results hold for decreasing waves as well.

2 Variational framework

Solitary waves of equation (1), i. e. solutions of the form $q_n(t) = u(n - ct)$ are solutions of the equation

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1))\,. \tag{2}$$

In addition to this infinite lattice, we also consider the corresponding problem on a periodic lattice consisting of $2k$ particles, denoted by $-k, -(k-1), \dots, k$, where $-k$ and k are treated as the same point. This leads to the same equation as (1) (resp. (2)), but with an additional periodicity condition.

Equation (2) is equivalent to the following variational problem: Let X be the Hilbert space

$$X = \{u \in H^1_{loc} \mid u' \in L^2(\mathbb{R}),\ u(0) = 0\}$$

with norm $\|u\|^2 = \int_{\mathbb{R}} |u'(t)|^2 dt$. On X we define the operator A by

$$A(u)(t) = u(t+1) - u(t) = \int_t^{t+1} u'(s)\, ds\,.$$

Then a critical point of the functional

$$\Phi(u) = \int_{\mathbb{R}} [\frac{c^2}{2} u'(t)^2 - V(Au(t))]\, dt$$

on X is a solution of (1). The corresponding periodic problem is given by

$$\Phi_k(u) = \int_{-k}^{k} [\frac{c^2}{2} u'(t)^2 - V(Au(t))]\, dt,$$

on the Hilbert space of periodic functions

$$X_k = \{u \in H^1_{loc}(\mathbb{R}) \mid u'(t+2k) = u'(t),\ u(0) = 0\}$$

with norm $\|u\|_k^2 = \int_{-k}^{k} |u'(t)|^2 dt$.

We denote by $\|\cdot\|_*$ and $\|\cdot\|_{*,k}$ the norms in dual spaces X^* and X_k^* respectively.

Note that Φ and Φ_k are Hamiltonian actions for u.

In the following it is important that increasing waves are fixed points of the operator $P : X_k \to X_k$, defined by $P(u)(t) = \int_0^t |u'(s)|ds$.

First we recall the following estimates for A which are proved in [17], Proposition 1, in the case $k = +\infty$ (the case $k < +\infty$ is similar) :

Lemma 1 *A is a bounded operator from X to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (resp. from X_k to $L^2(-k,k) \cap L^\infty(-k,k)$) satisfying $\|Au\|_\infty \le \|u\|$, $\|Au\|_{L^2} \le \|u\|$ for all $u \in X$ (resp. $\|Au\|_\infty \le \|u\|_k$, $\|Au\|_{L^2} \le \|u\|_k$ for all $u \in X_k$).* $\square$

In the sequel, we assume that the potential V satisfies the following:

(i) $V(r) = \frac{c_0^2}{2} r^2 + W(r)$, where $c > c_0 \ge 0$, $W \in C^1$, $W(0) = 0$, $W'(r) = o(|r|)$ as $r \to 0$.

(ii) $W(r_0) > 0$ for some $r_0 > 0$, and there exists some $\mu > 2$ such that $0 \le \mu W(r) \le W'(r)r$ for all $r \ge 0$.

Since we only consider monotone waves (i. e. $Au(t) \ge 0$), we may also assume that $W(r) = 0$ if $r \le 0$, or we may modify W to be symmetric around 0. We note that Assumption (ii) can be written as a differential inequality

$$r^{\mu+1} \frac{d}{dr}(r^{-\mu} W(r)) \ge 0, \quad r > 0\,.$$

Integration of this inequality shows $W(r) \ge a_0 r^\mu$ for $r > r_0$ and $a_0 = r_0^{-\mu} W(r_0)$. Together with Assumption (i) this implies

$$W(r) \ge a_1 (r^\mu - r^2) \quad \text{for } \ r \ge 0 \tag{3}$$

with some constant $a_1 > 0$. It follows from these assumptions and from Lemma 1, that Φ and Φ_k are differentiable; the derivative of Φ_k is given by

$$\langle \Phi_k'(u), v\rangle = \int_{-k}^{k} [c^2 u'(t) v'(t) - V'(Au(t)) Av(t)]\, dt.$$

Remark 1 *Assuming that $W(r_0) > 0$ for some $r_0 < 0$ and $0 \le \mu W(r) \le W'(r)r$ for $r \le 0$, one can treat the case of decreasing waves along the same lines as it is done here for increasing waves.*

3 Periodic lattices

In this section we prove the existence of a sequence of periodic traveling waves by showing that Φ_k satisfies the assumptions of the Mountain Pass Theorem. Moreover, we derive several a priori estimates which enable us to pass to the limit $k \to \infty$ in the next section.

Lemma 2 *There exist $\delta, \rho > 0$ independent of k such that $|\Phi_k(u)| \geq \delta$ if $\|u\|_k = \rho$. Furthermore, there are functions $e_k \in P(X_k)$ satisfying $\Phi_k(e_k) = \Phi_1(e_1) \leq 0$ and $\|e_k\|_k > \rho$.*

Proof: Assumption (i) implies that, given $\varepsilon > 0$, there exists $\rho > 0$ such that $|W(r)| \leq \varepsilon |r|^2$ if $|r| \leq \rho$. If $\|Au\|_\infty \leq \rho$, this and Lemma 1 imply

$$\begin{aligned}\Phi_k(u) &\geq \int_{-k}^{k} [\frac{c^2}{2}|u'|^2 - \frac{c_0^2}{2}|Au|^2 - \varepsilon |Au|^2] dt \\ &\geq \frac{c^2 - c_0^2 - 2\varepsilon}{2} \|u\|_k^2,\end{aligned}$$

and ε, ρ can be choosen such that the right hand side is strictly positive for $\|u\|_k = \rho$.

To construct the functions e_k, we first choose a function $v \in P(X_1)$ satisfying

$$v(t) = 0 \;\text{ for }\; 0 \leq t \leq 1, \quad v'(-1) = 0$$

(v' is nonzero only on intervals $[2l-1, 2l]$, $l \in \mathbb{Z}$) and $V(Av(t_0)) > 0$ for some t_0. Since $Av \geq 0$, we see from (3) that

$$\begin{aligned}\Phi_1(\tau v) &\leq \tau^2 \int_{-1}^{1} [\frac{c^2}{2}|v'|^2 + (a_1 - \frac{c_0^2}{2})|Av|^2] dt - \tau^\mu a_1 \int_{-1}^{1} |Av|^\mu dt \\ &\leq \tau^2 \left(\frac{c^2}{2}\|v\|_1^2 + a_1 \|Av\|_{L^2}^2 \right) - \tau^\mu a_1 \|Av\|_{L^\mu}^\mu .\end{aligned}$$

This shows $\Phi_1(\tau v) \to -\infty$ as $\tau \to \infty$, so we can fix $e_1 = \tau_0 v$ satisfying $\Phi_1(e_1) \leq 0$ and $\|e_1\|_k > \rho$. Then we define $e_k \in X_k$ by

$$e_k(t) = e_1(t) \quad \text{for} \quad |t| \leq 1, \quad e_k'(t) = 0 \quad \text{for} \quad 1 \leq |t| \leq k$$

and extend e_k' periodically to $\mathbb{R}$ (then e_k' is nonzero only on intervals $[2kl - 1, 2kl], l \in \mathbb{Z}$).

It follows that $\|e_k\|_k = \|e_1\|_1$, and we have

$$e_k(t+1) - e_k(t) = \begin{cases} e_1(t+1) - e_1(t) & , t \in [-2, 0] \\ 0 & , t \in [-k, k] \setminus [-2, 0] \end{cases}.$$

Consequently

$$\begin{aligned} \int_{-k}^{k} V(Ae_k(t))dt &= \int_{-2}^{-1} V(e_1(t+1) - e_1(t))dt + \int_{-1}^{0} V(e_1(t+1) - e_1(t))dt \\ &= \int_{0}^{1} V(e_1(t-1) - e_1(t-2))dt + \int_{-1}^{0} V(e_1(t+1) - e_1(t))dt \\ &= \int_{-1}^{1} V(e_1(t+1) - e_1(t))dt, \end{aligned}$$

since the difference $e_1(t+1) - e_1(t)$ is 2-periodic. In particular, we have

$$\Phi_k(e_k) = \Phi_1(e_1) \leq 0 \quad \text{and} \quad \Phi_k(se_k) = \Phi_1(se_1) \quad \text{for } s \in \mathbb{R}. \tag{4}$$

□

Now we can prove the following theorem:

Theorem 1 *There exists a sequence $u_k \in P(X_k)$ of nontrivial critical points of Φ_k such that the critical values $\alpha_k = \Phi_k(u_k)$ satisfy*

$$0 < \delta \leq \alpha_k \leq M \tag{5}$$

with constants δ and M independent of k.

Proof: Since the embedding $X_k \to L^p(-k, k)$ is compact, Lemma 1 and standard arguments (see Proposition B.35 and the proof of Theorem 2.15) in [15] show that Φ_k satisfies the Palais–Smale condition. Lemma 2 shows that Φ_k has the mountain pass geometry and $\Phi_k(P_k(u)) \leq \Phi_k(u)$, $P_k(0) = 0$, $P_k(e_k) = e_k$. Consequently, the Brezis–Nirenberg version of the Mountain Pass Lemma (cf. [6], Theorem 10) shows that

$$\begin{aligned} \alpha_k &= \inf_{\gamma \in \Gamma_k} \sup_{s \in [0,1]} \Phi_k(\gamma(s)), \quad \text{with} \\ \Gamma_k &= \{\gamma \in C([0,1], X_k) \mid \gamma(0) = 0, \gamma(1) \leq 0, \|\gamma(1)\|_k > \rho\} \end{aligned}$$

are critical values of Φ_k with corresponding critical points $u_k \in \overline{P(X_k)}$. Since $P(X_k)$ is closed, $u_k \in P(X_k)$. Moreover, from (4) we get

$$\alpha_k \leq \max_{s\in[0,1]} \Phi_k(se_k) = \max_{s\in[0,1]} \Phi_1(se_1) = M$$

and M is independent of k. $\square$

4 Passage to the limit

To pass to the limit, we need a cut-off operator $T_k : X_k \to X$. which is defined as follows:

$$T_k u(t) = \begin{cases} u(t) & , |t| \leq k+1 \\ u(\pm(k+1)) & , |t| \geq k+1 \end{cases} , \qquad u \in X_k.$$

Obviously, we have $(T_k u)' \in L^2(\mathbb{R})$ and

$$\|T_k u\|_X^2 = \int_{-\infty}^{\infty} |(T_k u)'|^2 dt = \int_{-k-1}^{k+1} |u'|^2 dt \leq 2\|u\|_k^2 ,$$

so T_k is a bounded linear operator with norm independent of k. Remark that

$$(T_k A u)(t) = (A T_k u)(t) = Au(t) \quad \text{ if } \quad |t| \leq k, \tag{6}$$

and for increasing waves we have

$$0 \leq (AT_k(u))(t) \leq A(u(t)) \quad \text{ for all } \quad t. \tag{7}$$

Lemma 3 *Any nontrivial critical point u of Φ_k satisfies $0 < \varepsilon_1 \leq \|u\|_k \leq C_k$, and $0 < \varepsilon_2 \leq \Phi_k(u)$ with ε_1, ε_2 independent of k and C_k only depending on the critical value.*
The same assertions hold for critical points $u \in X$ of Φ.

Proof. To show the upper bounds for $\|u\|_k$, note that

$$\begin{aligned} c_k &= \Phi_k(u) - \frac{1}{2}\langle \Phi_k'(u), u\rangle \\ &= \int_{-k}^{k} [\frac{1}{2} W'(Au)Au - W(Au)]dt \\ &\geq \frac{\mu-2}{2} \int_{-k}^{k} W(Au)dt, \end{aligned}$$

by Assumption (ii). Therefore

$$\begin{aligned} \frac{c^2 - c_0^2}{2}\,\|u\|_k^2 &\leq \Phi_k(u) + \int_{-k}^{k} W(Au)dt \\ &\leq \frac{\mu}{\mu - 2}\,c_k. \end{aligned}$$

For the lower bounds, we assume on the contrary that there exists a sequence of nontrivial critical points $u_{k_n} \in X_{k_n}$ such that $\|u_{k_n}\|_{k_n} \leq 1/n$. Since $\|Au_{k_n}\|_\infty \leq 1/n$ (by Lemma 1), Assumption (i) shows $|W'(Au_{k_n})Au_{k_n}| \leq \varepsilon_n |Au_{k_n}|^2$, $\varepsilon_n \to 0$. Hence

$$\begin{aligned} c^2\|u_{k_n}\|_{k_n}^2 &= \int_{-k_n}^{k_n} [c_0^2|Au_{k_n}|^2 + W'(Au_{k_n})Au_{k_n}]dt \\ &\leq (c_0^2 + \varepsilon_n)\,\|u_{k_n}\|_{k_n}^2 \,. \end{aligned}$$

Since $c^2 > c_0^2$, this is a contradiction.

Finally we obtain the lower bounds for $\Phi_k(u)$ as follows. First, from Assumption (ii) we have

$$\Phi_k(u) \geq \frac{c^2}{2}\|u\|_k^2 - \frac{c_0^2}{2}\|Au\|_{L^2}^2 - \frac{1}{\mu}\int_{-k}^{k} W'(Au)Au\,dt\,.$$

Then, by Lemma 1, we get

$$\begin{aligned} \Phi_k(u) &= \Phi_k(u) - \frac{1}{\mu}\langle \Phi_k'(u), u\rangle \\ &\geq (c^2 - c_0^2)(\frac{1}{2} - \frac{1}{\mu})\,\|u\|_k^2 \\ &\geq (c^2 - c_0^2)(\frac{1}{2} - \frac{1}{\mu})\,\varepsilon_1\,. \end{aligned}$$

Obviously, these lower bounds are independent of k, since they only depend on the geometry of the functional. Moreover, all the calculations above can be repeated for critical points $u \in X$ of Φ. $\square$

Remark. Actually in the proof we have only used the fact that $\langle \Phi_k'(u), u\rangle = 0$, so the statements of this Lemma remain true under this weaker assumption.

For $\zeta \in \mathbb{R}$, $r > 0$ we denote by $K_r(\zeta)$ the interval with center ζ and length r.

Lemma 4 *Let $u_k \in X_k$ be a sequence such that $(\Phi_k'(u_k), u_k) \to 0$ and $\|u_k\|_k \leq C$. Then either $\|u_k\|_k \to 0$, or there exist $r, \eta > 0$ and a sequence $\zeta_k \in \mathbb{R}$ such that, along a subsequence,*

$$\int_{K_r(\zeta_k)} |A(T_k u_k)|^2 dt \geq \eta.$$

Proof. First note that Lemma 1 implies that $A(T_k u_k)$ is a bounded sequence in $H^1(\mathbb{R})$. Now assume that

$$\lim_{k\to\infty} \sup_{\zeta\in\mathbb{R}} \int_{K_r(\zeta)} |A(T_k u_k)|^2 dt = 0.$$

Then the concentration lemma of Lions ([12], Lemma 1.1) shows that $\|A(T_k u_k)\|_{L^s(\mathbb{R})} \to 0$ for all $s > 2$. Since $\|\Phi_k'(u_k)\|_k \to 0$, there is a sequence $\varepsilon_k \to 0$ such that

$$\int_{-k}^{k} [c^2(u_k')^2 - c_0^2|Au_k|^2 - W'(Au_k)Au_k]\, dt = \varepsilon_k \|u_k\|_k\,.$$

Again Lemma 1 shows $\|Au_k\|_\infty \leq C$ and from Assumption (i) we deduce that for $|r| \leq C$, $\varepsilon > 0$ there exist $C_\varepsilon > 0$, $\alpha > 2$ such that $|W'(r)r| \leq \varepsilon r^2 + C_\varepsilon |r|^\alpha$. We get

$$\begin{aligned}
c^2\|u_k\|_k^2 &\leq \int_{-k}^{k} [(c_0^2+\varepsilon)|Au_k|^2 + |Au_k|^\alpha] dt + \varepsilon_k\|u_k\|_k \\
&= (c_0^2+\varepsilon)\|Au_k\|_{L^2}^2 + \int_{-k}^{k} |A(T_k u_k)|^\alpha dt + \varepsilon_k\|u_k\|_k \\
&\leq (c_0^2+\varepsilon)\|u_k\|_k^2 + \|A(T_k u_k)\|_{L^\alpha(\mathbb{R})}^\alpha + \varepsilon_k\|u_k\|_k\,.
\end{aligned}$$

Choosing ε small enough, this shows $\|u_k\|_k \to 0$ as desired. □

In what follows, we frequently consider convergence of some subsequences of a sequence without explicit change of indices.

Theorem 2 *Let $u_k \in P(X_k)$ be a sequence such that the values $\Phi_k(u_k)$ is uniformly bounded and $\|\Phi_k'(u_k)\|_k \to 0$. Then there exists a sequence $\zeta_k \in \mathbb{R}$ such that a subsequence of $T_k(u_k(\cdot+\zeta_k) - u_k(\zeta_k))$ converges weakly and locally in X to a nontrivial solution $u \in P(X)$ of (2). In particular, it is so if $u_k \in P(X_k)$ is a sequence of periodic traveling waves such that $\alpha_k = \Phi_k(u_k)$ satisfy the estimate (5).*

Proof. From (5) and Lemma 3 we conclude that $\|u_k\|_k$ is bounded both from above and below. Hence Lemma 4 and (7) imply that there exist r, η and ζ_k such that

$$\int_{K_r(\zeta_k)} |Au_k|^2 dt \geq \eta$$

for a subsequence. Now set $\tilde{u}_k(t) = u_k(t+\zeta_k) - u_k(\zeta_k)$. Then $\tilde{u}_k \in P(X_k)$, $\|\tilde{u}_k\|_k = \|u_k\|_k$, and since Φ_k is invariant under shifts and addition of constants, $\Phi_k(\tilde{u}_k) = \Phi_k(u_k)$, and $\Phi_k'(\tilde{u}_k) = 0$. Moreover, $T_k\tilde{u}_k$ converges weakly in X to some function u, and since $AT_k\tilde{u}_k$ is bounded in $H^1(\mathbb{R})$, we see that $V'(AT_k\tilde{u}_k) \to V'(Au)$ strongly in $L^\infty_{loc}(\mathbb{R})$. Now for any function $\varphi \in C^\infty(\mathbb{R})$ with $\varphi(0) = 0$, $\varphi' \in C_0^\infty(\mathbb{R})$ (these functions are dense in X), we have

$$\begin{aligned}
\langle \Phi'(u), \varphi \rangle &= \lim_{k\to\infty} \int_{\mathbb{R}} [c^2 (T_k\tilde{u}_k)'\varphi' - V'(AT_k\tilde{u}_k)A\varphi]\, dt \\
&= \lim_{k\to\infty} \int_{-k}^{k} [c^2 \tilde{u}_k'\varphi' - V'(A\tilde{u}_k)A\varphi]\, dt \\
&= 0.
\end{aligned}$$

Hence u is a solution of (2) and clearly belongs to $P(X)$.

Since u_k is a solution of (2) with right hand side converging in L^∞_{loc}, we also have $u_k'' \to u''$ in L^∞_{loc}, and $u_k \to u$ locally in X. □

Theorem 3 *Let $u_k \in X_k$ be a uniformly bounded sequence satisfying $\|\Phi_k'(u_k)\|_k \to 0$ and $\alpha_k = \Phi_k(u_k) \to \alpha > 0$. Assume that W is in C^2. Then there exist critical points $u^i \in X$ of Φ, $i = 1, \ldots, \ell$, and $\zeta_k^i \in \mathbb{R}$ such that*

$$\|u_k - \sum_{i=1}^{\ell} u^i(\cdot + \zeta_k^i)\|_k \to 0, \quad \sum_{i=1}^{\ell} \Phi(u^i) = \alpha\,.$$

Proof. By Theorem 2, there exist $\zeta_k \in \mathbb{R}$ such that $\tilde{u}_k(t) = u_k(t+\zeta_k) - u_k(\zeta_k)$ satisfies $T_k\tilde{u}_k \to u$ locally and weakly in X, where a solution of (2).

Now let $\varphi_k \in C^\infty(\mathbb{R})$ be a sequence with the following properties: $\varphi_k(0) = 0$, $\varphi_k' \in C_0^\infty(\mathbb{R})$, $\mathrm{supp}(\varphi_k') \subset [-k+1, k-1]$, and $\varphi_k \to u$ strongly in X. Let $\bar{\varphi}_k \in X_k$ be the extention of $\varphi_{k|[-k,k]}$. Since Φ is C^1, we get for an arbitrary $\eta \in X_k$:

$$\begin{aligned}
\langle \Phi_k'(\bar{\varphi}_k), \eta \rangle &= \langle \Phi'(\varphi_k), T_k\eta \rangle \\
&\leq \varepsilon_k \|T_k\eta\| \;\leq\; 2\varepsilon_k \|\eta\|_k\,,
\end{aligned}$$

with $\varepsilon_k \to 0$, i. e. $\|\Phi_k'(\bar{\varphi}_k)\|_{*,k} \to 0$.

Set $v_k = \tilde{u}_k - \bar{\varphi}_k$. We want to show that $\|\Phi_k'(v_k)\|_{*,k} \to 0$. First note that

$$\begin{aligned}\langle \Phi_k'(v_k), \eta \rangle &= \langle \Phi_k'(\tilde{u}_k) - \Phi_k'(\bar{\varphi}_k), \eta \rangle \\ &\quad + \int_{-k}^{k} \left[W'(A\tilde{u}_k) - W'(A\varphi_k) - W'(A\tilde{u}_k - A\varphi_k) \right] A\eta dt\,. \end{aligned} \tag{8}$$

Let $B = [-k_0, k_0]$ and $B_k = [-k, k] \setminus B$. Since $\varphi_k \to u$ in X, for any $\varepsilon > 0$ we can fix $k_0 > 0$ such that $|(A\varphi_k)(t)| \leq \varepsilon$ whenever $k \geq k_0$ and $t \notin B$. Hence,

$$\Big| \int_{B_k} W'(A\varphi_k) A\eta dt \Big| \leq \varepsilon \|\eta\|_k\,.$$

Since $W \in C^2$, we have

$$\begin{aligned}\left| \int_{B_k} [W'(A\tilde{u}_k - A\varphi_k) - W'(A\tilde{u}_k)] A\eta dt \right| &\leq \int_{B_k} |A\varphi_k|\, |1{+}W''(A\tilde{u}_k)|\, |A\eta| dt \\ &\leq \|A\eta\|_{L^2} \left(\|A\varphi_k\|_{L^2(B_k)} + \|W''(A\tilde{u}_k)\|_{L^\infty(B_k)} \|A\varphi_k\|_{L^2(B_k)} \right) \\ &\leq C \cdot \varepsilon \|\eta\|_k\,.\end{aligned}$$

Since $A\tilde{u}_k \to Au$ in L^∞_{loc} we see that for B fixed above

$$\left| \int_B \left[W'(A\tilde{u}_k) - W'(A\varphi_k) - W'(A\tilde{u}_k - A\varphi_k) \right] A\eta dt \right| \leq C \cdot \varepsilon \|\eta\|_k,$$

provided k is large enough. Therefore, due to (8), $\|\Phi_k'(v_k)\|_k \to 0$.

Now consider

$$\begin{aligned}\Phi_k(\tilde{u}_k) &= \Phi_k(v_k) + \Phi_k(\bar{\varphi}_k) + (v_k, \bar{\varphi}_k)_k \\ &\quad - \int_{-k}^{k} [V(Av_k - A\varphi_k) - V(Av_k) - V(A\varphi_k)]\, dt. \end{aligned} \tag{9}$$

Since $v_k = \tilde{u}_k - \bar{\varphi}_k$ tends locally to zero and $\varphi_k \to u$ in X, we see that $(v_k, \bar{\varphi}_k)_k \to 0$ as $k \to \infty$. The integral in (9) can be estimated by similar arguments as above (using L^∞_{loc} estimates), so that we finally get

$$\alpha = \lim_{k\to\infty} \Phi_k(\tilde{u}_k) = \Phi(u) + \lim_{k\to\infty} \Phi_k(v_k), \quad \text{i. e. } \Phi_k(v_k) \to \alpha - \Phi(u)\,. \tag{10}$$

Since $\|\Phi_k'(v_k)\|_k \to 0$, either $\Phi_k(v_k) \to 0$ and $\|v_k\|_k \to 0$, or $\Phi_k(v_k) \geq \varepsilon_2 > 0$ and $\|v_k\|_k \geq \varepsilon_1 > 0$ (cf. Remark following Lemma 3). In the first case we are done, with $l = 1$ and $u^1 = u$. In the second case we have $0 < \varepsilon_2 \leq \Phi(u) < \alpha$. Now we can repeat the arguments above, with u_k replaced by v_k and α replaced by $\alpha - \Phi(u)$, to get u^2. After a finite number of such steps (not greater then α/ε_2) we conclude. □

5 Ground Waves

In this section we present another approach to construct travelling wave solutions of (1), which in addition have minimal energy among all nontrivial solutions. Here we use Nehari's variational principle and instead of Assumption (ii) we assume the following:

(N) $W \in C^2$, $W(r_0) > 0$ for some $r_0 > 0$, and there exists $\theta \in (0, 1)$ such that $0 \leq r^{-1}W'(r) \leq \theta W''(r)$ holds for all $r > 0$.

Remark 2 *In the case of decreasing waves one has to assume instead of (N) that $W(r_0) > 0$ for some $r_0 < 0$ and $0 \leq r^{-1}W'(r) \leq \theta W''(r)$ for $r \leq 0$, with some $\theta \in (0, 1)$.*

As before, since we only consider increasing waves, we may assume that $W(r) \equiv 0$ for $r \leq 0$, or that W is symmetric around 0. We note that Assumption (N) implies (ii) with $\mu = (1 + \theta)/\theta$. Now let

$$I_k(u) = \langle \Phi_k'(u), u \rangle = c^2 \|u\|_k^2 - \int_{-k}^{k} V'(Au)Au \, dt$$

and define the Nehari manifold by

$$S_k = \{u \in P(X_k) \setminus \{0\} \mid I_k(u) = 0\} .$$

Note that Assumption (N) implies that $W'(r)/r$ increases monotonically to $+\infty$ on $(0, +\infty)$. Hence, for any $u \in P(X_k) \setminus \{0\}$ there exists a unique $\tau > 0$ such that $\tau u \in S_k$. Indeed, we have obviously

$$I_k(tu) = t^2(c^2\|u\|_k^2 - c_0^2\|Au\|_{L^2}^2 - t^{-1} \int_{-k}^{k} W'(tAu)Au \, dt .$$

Due to Lemma 1, $c^2\|u\|_k^2 - c_0^2\|Au\|_{L^2}^2 > 0$ and from monotonicity of $W'(r)/r$ we get the required immediately.

Now we want to solve the periodic minimization problem

$$m_k = \inf\{\Phi_k(u) \mid u \in S_k\} \tag{11}$$

and the corresponding problem for the infinite lattice

$$m = \inf\{\Phi(u) \mid u \in S\} \tag{12}$$

(where $S \subset P(X)$ is defined in the same way as S_k above).

Note that on S_k the functional has the form

$$\Phi_k(u) = \int_{-k}^{k} [\frac{1}{2} W'(Au)Au - W(Au)]\, dt\,. \tag{13}$$

First we make the following useful observation:

Lemma 5 *Under Assumptions (i) and (N), $\alpha_k = m_k$.*

Proof. Since α_k is a critical value with critical point $u_k \in P(X_k)$, u_k belongs to S_k, hence $m_k \le \alpha_k$.

If $v \in P(X_k)$, $v \not\equiv 0$, then as in the proof of Lemma 2 we see that $\Phi_k(\tau v) \le 0$ for τ large enough. This shows $\alpha_k \le \alpha'_k$, where α'_k is defined as

$$\alpha'_k = \inf_{v \in P(X_k)\setminus\{0\}} \sup_{\tau > 0} \Phi_k(\tau v)\,.$$

Now observe that for $v \in P(X_k)$, $v \not\equiv 0$, there exists $\tau > 0$ such that $\Phi_k(\tau v) \ge \delta > 0$ (cf. Lemma 3). Since $\Phi_k(\tau v) \to -\infty$ as $\tau \to \infty$, the maximum of $\Phi_k(\tau v)$ is obtained for some $\tau_0 > 0$ and we have

$$\begin{aligned} 0 = \frac{d}{d\tau}\,\phi_k(\tau v)_{|\tau_0} &= \int_{-k}^{k} [c^2\tau_0 |v'|^2 - V'(\tau_0 Av)Av]\, dt \\ &= \frac{1}{\tau_0}\, I_k(\tau_0 v)\,, \end{aligned}$$

hence $\tau_0 v \in S_k$. This shows $\alpha'_k = \inf_{v\in S_k} \Phi_k(v) = m_k$. □

Since Assumption (N) implies (ii), from this Lemma and Theorem 1 we conclude the following:

Theorem 4 *Under Assumptions (i) and (N), for every k there exists a nontrivial minimizer u_k of (11) which is a solution of (2). Moreover, the sequence m_k is uniformly bounded from above and below.* □

Remark that any minimizer of (11) is a critical point of Φ_k (this follows from the Lagrange multiplier theorem and Assumption (N)). Moreover, the existence of a minimizer can be proved directly by standard variational methods, since Φ_k has the necessary compactness properties.

For these ground waves we obtain the following stronger convergence result:

Theorem 5 *Under Assumptions (i) and (N), there exists a sequence $\zeta_k \in \mathbb{R}$ and a ground wave $u \in X$ such that, along a subsequence, the functions $\tilde{u}_k = u_k(\cdot + \zeta_k) - u_k(\zeta_k)$ satisfy*

$$\lim_{k\to\infty} \|\tilde{u}_k - u\|_k = 0\,.$$

Proof. The existence of a nontrivial limit which is a solution of (2) follows by the same arguments as in the proof of Theorem 2. Note that Lemmas 5 and 3 imply the uniform boundedness of u_k.

To prove that the limit is actually a ground state, first note that for any $v \in S$, $\varepsilon > 0$, there exist $v_k \in S_k$ such that $\Phi_k(v_k) \leq \Phi(v)+\varepsilon$ if k is sufficiently large. To show this, take a sequence $\varphi_k \in X_k$, $\varphi_k' \in C_0^\infty(-k+1, k-1)$, $\varphi_k \to v$ in X, and let $\tau_k > 0$ be the unique minimal number such that $\tau_k\varphi_k \in S_k$. ¿From $\Phi(\varphi_k) \to \Phi(v)$, $I(\varphi_k) \to I(v)$, we see that $\tau_k \to 1$ and $v_k = \tau_k\varphi_k$ satisfies $\Phi_k(v_k) = \Phi(v_k) \to \Phi(v)$.

Since $m_k \leq \Phi_k(v_k)$, this implies $\limsup_{k\to\infty} m_k \leq \Phi(v) + \varepsilon$ for all $v \in S$, $\varepsilon > 0$, hence

$$\limsup_{k\to\infty} m_k \leq m.$$

On the other hand consider the function $g(r) = \frac{1}{2}W'(r)r - W(r)$. Due to Assumption (N), $g(Au) \geq 0$. Since $g(A\tilde{u}_k) \to g(Au)$ locally in L^∞, and

$$m_k = \int_{-k}^{k} g(A\tilde{u}_k)dt \geq \int_B g(A\tilde{u}_k)dt$$

for any bounded interval B and k large enough, we see

$$\liminf_{k\to\infty} m_k \geq m.$$

Thus, $m_k \to m = \Phi(u)$ as desired.

The last statement of Theorem 5 now follows directly from Theorem 3 (set $\alpha = m$, $\alpha_k = m_k$, and $\ell = 1$). □

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