A Note on the Regularity of Solutions of Linear Homological Equations

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Abstract

We study the linear homological equation $\mu(t+2\pi\gamma)-\mu(t)=\nu(t)$ on the circle $\mathbf{T}$. We show that if the Fourier coefficients of ν satisfy $\hat{\nu}(n)=O(|n|^{-1-\alpha})$ for some $0<\alpha\le 1$, then for Lebesgue a.e. $\gamma\in(0,1)$ the solution μ belongs to $\mathrm{Lip}_{\alpha'}(\mathbf{T})$ for any $\alpha'<\alpha$.

1 Introduction

Let $\mathbf{T}$ be the circle $\mathbf{R}/2\pi\mathbf{Z}$ and $\nu(t)$ a continuous function on $\mathbf{T}$ satisfying $\int_{\mathbf{T}} \nu(t)\mathrm{d}t = 0$. Let $\gamma \in (0,1)$ be given. A linear homological equation is an equation of the form

$$\mu(t + 2\pi\gamma) - \mu(t) = \nu(t), \tag{1.1}$$

where μ is an unknown function or distribution on $\mathbf{T}$. This equation appears in such diverse fields as theory of dynamical systems, algebraic geometry, classical mechanics, and spectral theory of discrete Schrödinger operators with quasi-periodic potentials (see [A, CFKS]) for references and additional information). Our interest in the subject is motivated by applications to the so-called Maryland model, see [FP, JM, KP, S].

For any function or distribution f on $\mathbf{T}$, we denote by $\hat{f}(n) = (f, \mathrm{e}^{-int})$ its Fourier coefficients. The equation (1.1) is then equivalent to

$$(\mathrm{e}^{2\pi i n\gamma} - 1)\hat{\mu}(n) = \hat{\nu}(n). \tag{1.2}$$

Clearly, the last equation might not have a solution if γ is a rational number. If γ is irrational, then for $n \neq 0$ the solution is formally given by

$$\hat{\mu}(n) = \frac{\hat{\nu}(n)}{\mathrm{e}^{2\pi i n\gamma} - 1}. \tag{1.3}$$

For definiteness, we will always take $\hat{\mu}(0) = 1$. It follows from (1.3) that if γ is well-approximated by rationals, e.g., γ is a Liouville number, then a distributional solution of equation (1.1) may not exist. Thus, at best, one can hope for results which hold for a sufficiently large set of γ's, e.g., of Lebesgue measure 1. To illustrate this point, for given $\delta > 0$ let $\mathcal{D}$ be set of γ's in $(0,1)$ such that for some $C > 0$ and all $n, m \in \mathbf{Z}$, $n \neq 0$,

$$|n\gamma - m| > C|n|^{-1-\delta}. \tag{1.4}$$

The set $\mathcal{D}$ has Lebesgue measure 1 and for $\gamma \in \mathcal{D}$,

$$\left|\mathrm{e}^{2\pi i n\gamma} - 1\right| = 2\,|\sin(\pi n\gamma)| > \frac{\tilde{C}}{|n|^{1+\delta}}. \tag{1.5}$$

(The constants $C, \tilde{C}$ depend on γ.) It follows that for $\gamma \in \mathcal{D}$ the sequence $\hat{\mu}(n)$ is polynomially bounded and that a distributional solution of equation (1.1) exists.

The following general problem arises in many different contexts: given some regularity property of the function ν, to what extent is this property shared by the solution μ? In two important cases, the answer follows directly from (1.3) and (1.5): if ν is analytic or C^∞, then for $\gamma \in \mathcal{D}$ the solution μ is also analytic or C^∞.

We denote by $C^k(\mathbf{T})$ the vector space of all continuous functions on $\mathbf{T}$ whose k-th derivative is a continuous function. For $0 < \alpha < 1$ we denote by $\mathrm{Lip}_\alpha(\mathbf{T})$ the vector space of all continuous functions f such that

$$\sup_{t \neq s} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty.$$

Let $C^{k+\alpha}(\mathbf{T}) \subset C^k(\mathbf{T})$ be the vector space of all continuous functions whose k-th derivative is in $\mathrm{Lip}_\alpha(\mathbf{T})$. Occasionally, we will write $C^\alpha(\mathbf{T})$ for $\mathrm{Lip}_\alpha(\mathbf{T})$.

If $\nu \in C^{k+\alpha}(\mathbf{T})$, $k \geq 2$ and $\gamma \in \mathcal{D}$, then it is well-known that the solution μ of equation (1.1) is in $C^{k-2+\alpha'}(\mathbf{T})$ for any $0 < \alpha' < \alpha$ (see Lemma 2.1 below). In this note we show how this result can be improved.

Before we state our results, we recall some definitions and basic properties of continued fraction expansions (see [Kh1] or any book on Diophantine approximations). Every irrational number $\gamma \in (0, 1)$ has a unique infinite continued fraction expansion

$$\gamma = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}},$$

where the a_k are positive integers. $\gamma = [0; a_1, a_2, \ldots]$ is the customary shorthand for this expansion. The k-th order continued fraction of γ is the rational number

$$\gamma_k = \frac{p_k}{q_k} = [0; a_1, \ldots, a_k]. \tag{1.6}$$

The following relations hold:

$$\begin{aligned} q_{k+1} &= a_k q_k + q_{k-1}, \\ \gamma &= \frac{p_k}{q_k} + \frac{\theta_k}{q_k q_{k+1}}, \end{aligned} \tag{1.7}$$

where $|\theta_k| < 1$. We denote by $\mathcal{P}$ the set of all irrational numbers γ in $(0, 1)$ for which the limit

$$\lim_{k \to \infty} q_k^{\frac{1}{k}} =: \zeta(\gamma) \tag{1.8}$$

exists and is finite. The first relation in (1.7) yields that $\zeta(\gamma) > 1$. It follows from Khinchin's theorem [Kh1, Kh2] that the set $\mathcal{P}$ has Lebesgue measure 1. Our main result is:

Theorem 1.1 *If $\hat{\nu}(n) = O(|n|^{-1-\alpha})$ for some $0 < \alpha \leq 1$, then for $\gamma \in \mathcal{P}$ the solution μ of the homological equation (1.1) belongs to $\mathrm{Lip}_{\alpha'}(\mathbf{T})$ for any $\alpha' < \alpha$.*

Remark 1. The condition $\hat{\nu}(n) = O(|n|^{-1-\alpha})$ implies that $\nu \in \mathrm{Lip}_{\alpha'}(\mathbf{T})$ for any $0 < \alpha' < \alpha$, see Lemma 2.1 below.

Remark 2. If $\hat{\nu}(n) = O(|n|^{-1})$, then the solution of (1.1) might not be a continuous function. For example, if ν is given by the Hardy-Littlewood series,

$$\nu(t) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}n\ln n}}{n} \mathrm{e}^{\mathrm{i}nt},$$

then $\nu \in \mathrm{Lip}_{1/2}(\mathbf{T})$ (see [Z]). However, it is easy to show that for *any* irrational γ $\limsup |\hat{\mu}(n)| \geq 1$, so the solution of (1.1) is certainly not a continuous function.

Our proof of Theorem 1.1 also yields:

Theorem 1.2 *Assume that $\nu \in C^{k+\alpha}(\mathbf{T})$ for some $k \geq 1$ and $0 < \alpha \leq 1$. Then for $\gamma \in \mathcal{P}$ the solution μ of the homological equation (1.1) belongs to $C^{k-1+\alpha'}(\mathbf{T})$ for any $\alpha' < \alpha$.*

Remark. The following example illustrates optimality of Theorem 1.2. Let $\gamma = (\sqrt{5} - 1)/2$ and

$$\nu(t) = \sum_{n=1}^{\infty} F_{2n-1}^{-1-\alpha} \mathrm{e}^{\mathrm{i}F_{2n-1}t},$$

where $\{F_n\}$ is the Fibonacci sequence. Then $\gamma \in \mathcal{P}$, $\nu \in C^{1+\alpha}(\mathbf{T})$ and the solution μ of (1.1) belongs to $C^{\alpha}(\mathbf{T})$, see [Z].

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2 Proofs

We will need:

Lemma 2.1 *Let f be continuous function on $\mathbf{T}$ such that for some integer $k \geq 0$ and $0 < \alpha \leq 1$,*

$$\sum_n |n|^{k+\alpha} |\hat{f}(n)| < \infty. \tag{2.9}$$

Then $f \in C^{k+\alpha}(\mathbf{T})$.

Proof: The case $\alpha = 1$ is exercise 1.4.2 in [Ka]. If $k > 0$, then (2.9) yields that f is k-times continuously differentiable and that $\widehat{f^{(k)}}(n) = (\mathrm{i}n)^k \hat{f}(n)$. Thus it suffices to consider the case $k = 0$.

Set

$$C := (2\pi)^{-1} \sum_n |n|^{\alpha} |\hat{f}(n)|.$$

Let $0 < |h| < 1/2$ and $N(h) = [1/|h|]$ ($[\cdot]$ is the greatest integer part). Then

$$\begin{aligned} 2\pi|f(t+h) - f(t)| &= \left|\sum_n \hat{f}(n)\mathrm{e}^{\mathrm{i}tn}\left(\mathrm{e}^{\mathrm{i}nh} - 1\right)\right| \\ &\le |h| \textstyle\sum_{|n|\le N(h)} |n| \cdot |\hat{f}(n)| + 2\sum_{|n|>N(h)} |\hat{f}(n)|. \end{aligned}$$

In the first sum we rewrite $|n| = |n|^{\alpha}|n|^{1-\alpha}$, and in the second $|\hat{f}(n)| = |\hat{f}(n)| \cdot |n|^{\alpha}/|n|^{\alpha}$. This yields the estimate

$$\begin{aligned} |f(t+h) - f(h)| &\le C|h|N(h)^{1-\alpha} + \frac{2C}{N(h)^{\alpha}} \\ &= O(|h|^{\alpha}). \end{aligned}$$

□

Proof of Theorem 1.1: It follows from Lemma 2.1 that to prove the statement it suffices to show that for $\gamma \in \mathcal{P}$ and $\alpha' < \alpha$,

$$\sum_{n\neq 0} \frac{|n|^{\alpha'}|\hat{\nu}(n)|}{|\sin(\pi n\gamma)|} < \infty. \tag{2.10}$$

In what follows we fix γ and α'. Let $\gamma_k = p_k/q_k$ be given by (1.6) and

$$A_k = \sum_{q_k \le |n| < q_{k+1}} \frac{|n|^{\alpha'}|\hat{\nu}(n)|}{|\sin(\pi n\gamma)|}.$$

Then for some $C > 0$,

$$A_k \le CB_k q_{k+1}^{\alpha'} q_k^{-1-\alpha},$$

where

$$B_k = \sum_{q_k \le n < q_{k+1}} |\sin(\pi n\gamma)|^{-1}.$$

Thus to prove the theorem it suffices to show that

$$\sum_{k>0} B_k q_{k+1}^{\alpha'} q_k^{-1-\alpha} < \infty. \tag{2.11}$$

We proceed to estimate B_k.

It follows from (1.7) that

$$\sin(\pi n\gamma) = \sin\left(\frac{\pi n p_{k+1}}{q_{k+1}} + \frac{\pi n\theta_{k+1}}{q_{k+1}q_{k+2}}\right).$$

Let $\{x\} = x - [x]$ be fractional part of a positive number x. Since p_{k+1} and q_{k+1} are relatively prime, the sequence $(\{np_{k+1}/q_{k+1}\})_{1\le n\le q_{k+1}-1}$ is a permutation of the sequence $(\{n/q_{k+1}\})_{1\le n\le q_{k+1}-1}$. This observation shows that

$$B_k \le \sum_{n=1}^{q_{k+1}-1} \left(\min_{\pm} \left| \sin\left(\frac{\pi n}{q_{k+1}} \pm \frac{\pi}{q_{k+2}} \right) \right| \right)^{-1}.$$

Simple trigonometry and the estimate $\sin x \ge 2x/\pi$, which holds for $x \in [0, \pi/2]$, yield that for some universal constant C,

$$\begin{aligned} B_k \; &\le C \sum_{n=1}^{q_{k+1}} \frac{q_{k+2}q_{k+1}}{q_{k+2}n - q_{k+1}} \\ &\le C \frac{q_{k+2}q_{k+1}}{q_{k+2} - q_{k+1}} + C q_{k+2} \sum_{n=2}^{q_{k+1}} \frac{1}{n-1} \qquad (2.12) \\ &\le C \left(\frac{q_{k+2}q_{k+1}}{q_k} + q_{k+2} \ln q_{k+1} \right). \end{aligned}$$

The estimate (2.12) shows that the sum (2.11) is bounded by

$$C \sum_{k>0} \frac{q_{k+1}^{\alpha'+1}}{q_k^{1+\alpha}} \frac{q_{k+2}}{q_k} + C \sum_{k>0} \frac{q_{k+1}^{\alpha'} q_{k+2} \ln q_{k+1}}{q_k^{1+\alpha}}. \qquad (2.13)$$

It follows from the definition of $\mathcal{P}$ (recall (1.8)) that

$$\lim_{k\to\infty} \left(\frac{q_{k+1}^{\alpha'+1}}{q_k^{1+\alpha}} \frac{q_{k+2}}{q_k} \right)^{1/k} = \lim_{k\to\infty} \left(\frac{q_{k+1}^{\alpha'} q_{k+2} \ln q_{k+1}}{q_k^{1+\alpha}} \right)^{1/k} = \zeta(\gamma)^{\alpha'-\alpha} < 1,$$

and the root test yields that the sum (2.13) is finite. Thus, the sum (2.11) is also finite and Theorem 1.1 follows. □

Proof of Theorem 1.2: If $\nu \in C^{k+\alpha}(\mathbf{T})$, then $\hat{\nu}(n) = O(|n|^{-k-\alpha})$, see, e.g., [Ka]. The proof of Theorem 1.1 shows that for $\gamma \in \mathcal{P}$ and $\alpha' < \alpha$,

$$\sum_n |n|^{k-1+\alpha'} |\hat{\mu}(n)| < \infty,$$

and Lemma 2.1 yields that $\mu \in C^{k-1+\alpha'}(\mathbf{T})$. □

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