Parabolic resonances in near integrable Hamiltonian systems

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Abstract

When an integrable Hamiltonian system, possessing an m-resonant lower dimensional normally parabolic torus is perturbed, a parabolic m-resonance occurs. If, in addition, the isoenergetic nondegeneracy condition for the integrable system fails, the near integrable Hamiltonian exhibits a flat parabolic m-resonance. It is established that most kinds of parabolic resonances are persistent in n ($n \geq 3$) d.o.f. near integrable Hamiltonians, without the use of external parameters. Analytical and numerical study of a phenomenological model of a 3 degrees of freedom (d.o.f.) near integrable Hamiltonian system reveals that in 3 d.o.f. systems new types of parabolic resonances appear. Numerical study suggests that some of them cause instabilities in several directions of the phase space and a new type of complicated chaotic behavior. A model describing weather balloons motion exhibits the same dynamical behavior as the phenomenological model.

1. Introduction

Hamiltonian systems which exhibit *parabolic resonances (PR)* appear in many applications. In fact, PR typically appears in n degrees of freedom (d.o.f.) near integrable Hamiltonian systems depending on p parameters provided $n \geq 2$ and

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$n+p \geq 3$, see section 4. For example, PR arises in a model describing the motion of weather balloons on geopotential surfaces of the earth atmosphere, see sections 2 and 3. Here we extend the 2 d.o.f. model of [8, 11, 12] to 3 d.o.f. by considering the vertical oscillations of the weather balloon. Other examples where PR appear are the Duffing equation with a k-torus attached to it (the tori represent the effect of neutral modes), and a reduction of the nonlinear Srödinger equation ([4] and references therein) to a six dimensional Hamiltonian ODE.

What is a parabolic resonance? Let us first briefly review the relevant results regarding the phase space structure of near integrable Hamiltonians. In the completely integrable case the phase space is foliated by invariant tori on which quasi-periodic/periodic motion occurs. Although this is not a typical scenario for nonlinear systems with $n \geq 2$, invariant tori may still exist. Indeed, the subject of their persistence in near integrable Hamiltonian systems was extensively studied in this century; The KAM theory [1] assures that the majority of the maximal dimensional invariant tori, satisfying some nondegeneracy and Diophantine conditions, persist under small Hamiltonian perturbations (hereafter perturbation will always mean Hamiltonian perturbations). These invariant tori play a major role in 1 and 2 d.o.f. systems: The solutions of such perturbed systems are either confined on KAM tori or trapped between pairs of them, and hence cannot wander arbitrarily far in the phase space. The KAM theory is true for higher dimensional systems as well, but in 3 or more d.o.f., the invariant tori do not bound the motion in phase space.

A lower dimensional torus is a torus with a vector of inner frequencies ω, of dimension smaller than the number of d.o.f., and a normal frequencies vector Ω, which determines its stability. The normal frequencies are the imaginary parts of the multipliers [2] of the linearization of the system at the lower dimensional torus. Such lower dimensional tori persist under small perturbations if ω satisfies certain Diophantine conditions and if, additionally, either the tori are normally hyperbolic[1] [3] or they are normally elliptic[2] and (ω,Ω) are Diophantine [9]. A lower dimensional torus is said to be *normally parabolic* if it has at least one zero normal frequency. Maximal lower dimensional *normally parabolic* tori [6] (together with a saddle-center bifurcation scenario) survive small Hamiltonian perturbations under Diophantine conditions on their inner frequencies (for other lower dimensional normally parabolic tori, additional Diophantine conditions are

[1]Hyperbolic lower dimensional invariant torus is a torus such that its multipliers have a non-zero real part.

[2]Elliptic low dimensional tori are those with non-zero purely imaginary multipliers.

required on the elliptic part of Ω [6]). Summarizing, in the typical near integrable case, normally elliptic, hyperbolic or parabolic lower dimensional tori persist on Cantor sets of action values consisting of the Diophantine frequency vectors. In the integrable case, resonant[3] tori reside in the gaps between the nonresonant tori. Resonant tori of any dimension occupy a set of measure zero, yet they are dense in the phase space. In 3 or more d.o.f. integrable Hamiltonian systems, resonant tori are connected, creating a resonance web. Under small Hamiltonian perturbations the resonant tori break into resonant zones and orbits may drift along them far from their initial values (this possible scenario, called Arnold Diffusion, is still not well understood, see [1], and references therein). In previous works it has been shown that orbits which start in the vicinity of low dimensional resonant tori, exhibit under small perturbations intricate chaotic motion. See, for example, [4, 5] and references therein.

Here we study a special class of resonant tori, the *parabolic resonances*. A *parabolic m-resonance* occurs when a small Hamiltonian perturbation is applied to an integrable n d.o.f. Hamiltonian system, which possesses a *normally parabolic m-resonant $(n-1)$-torus* (namely, on the $n-1$ dimensional torus m independent resonant conditions are satisfied and the normal frequency of the torus vanishes). Numerical experiments suggest that slightly perturbed orbits, starting near the parabolic resonant torus of the integrable system, exhibit complicated behavior and instabilities. If in addition to the existence of an m-resonant parabolic torus in the integrable system, the isoenergetic nondegeneracy condition fails so that a whole family of resonant tori exist on the same energy surface, the near integrable system exhibits a *flat parabolic m-resonance*. In such cases, order one instabilities of slightly perturbed orbits are observed in numerical simulations even in the 2 d.o.f. case [10].

The term persistent is used throughout this article for parabolic resonances. By persistent we mean physically that the phenomena is robust - small changes in the form of the integrable system (or the family of systems for phenomena of positive co-dimension) will not alter the results. Mathematically, this means that the systems are persistent on a C^1-open set under C^r $(r \geq 1)$ perturbations of the class of integrable Hamiltonian systems of the form $H_0(x, y, I; \mu)$. In what follows we describe the 2 d.o.f. case [10], the 3 d.o.f. case [7], compare between these cases, and state some of our results for the n d.o.f. case [7].

[3] A torus is said to be resonant if its frequencies are rationally dependent.

2. The 2 d.o.f. case

The Hamiltonian (2.1) exhibits a typical behavior of a 2 d.o.f. near integrable system. It corresponds to a model describing the motion of particles (weather balloons) on a rotating sphere (geopotential surfaces of the earth atmosphere) which are subject to a conservative force (traveling wave pressure term) which breaks the angular symmetry. In [11] the global phase space structure of this system has been studied by a new method in which the motion in phase space is described by the structure of the energy surfaces and by shaded energy momentum bifurcation diagrams (bifurcation diagrams in the space of constants of motion). In [10, 12] the phenomena of PR was discovered, has been shown to exist in this model, and it has been demonstrated that this PR is a source of strong, yet rare phase space instability, with qualitative agreement to the observations of motion of weather balloons in field experiments.

Let ϕ be the latitude on the sphere and v its conjugate northward velocity. Let Λ_1 be the longitude in the traveling wave frame, and D_1 its conjugate angular momentum. The corresponding Hamiltonian in these canonical coordinates is:

$$\begin{aligned} H(\phi, v, \Lambda_1, D_1) &= \frac{v^2}{2} + \frac{1}{8}(\frac{2D_1}{\cos(\phi)} - \cos(\phi))^2 - cD_1 + B(\phi) + \varepsilon A(\phi)\sin(k\Lambda_1) \\ &= H_0(\phi, v, D_1; c) + H_1(\phi, v, \Lambda_1, D_1; k, \varepsilon), \end{aligned} \tag{2.1}$$

where:

$$\begin{aligned} &A(\phi) = \cos(\phi), \quad B(\phi) = \beta \tanh(\frac{\phi^2 - \phi_0^2}{\alpha}), \quad \phi_0 = 30^o \approx 0.5236, \quad \alpha = 0.05, \\ &(\phi, v, \Lambda_1, D_1) \in U \subseteq \mathbb{R}^2 \times \mathbb{T} \times \mathbb{R}, \quad k \in \mathbb{Z}, \beta, \varepsilon \in \mathbb{R}, \varepsilon \ll 1. \end{aligned} \tag{2.2}$$

$A(\phi)$ and $B(\phi)$ represent the latitude-dependent amplitudes of the traveling pressure wave and the constant pressure term, respectively. When $B(\phi)$ is taken in the form of (2.2), with $\beta = -0.001$, it represents a steady pressure term that is monotonically decreasing pole-ward, allowing strong zonal jets to appear near ϕ_0. The Hamiltonian (2.1) exhibits a parabolic resonance for $D_1 = 0.5$, $c = 0$, and a flat (degenerate) parabolic resonance if in addition $B(\phi) = 0$ (near flat parabolic resonance for $|B(\phi)| \ll 1$). These parabolic resonances correspond to two simultaneous phenomena - the change in the linear stability in latitude direction of particles motion about the equator (about $(\phi, v) = (0, 0)$) and the coincidence of the traveling wave speed with the west-ward speed along the equator ($\dot{\Lambda}_1|_{(\phi,v,D_1)=(0,0,0.5)} \equiv 0$).

More generally, parabolic resonance occurs in 2 d.o.f. systems when the integrable Hamiltonian possesses a parabolic circle of fixed points. Such a circle is persistent in a one parameter family of 2 d.o.f. integrable Hamiltonian systems. Numerical simulations of systems exhibiting parabolic resonance, such as the atmospheric model (2.1), show that perturbed orbits, starting in the vicinity of a parabolic circle of fixed points, exhibit chaotic behavior which combines hyperbolic homoclinic chaos and elliptic resonance type localizations, while the orbit slowly slides through bifurcations. Although these orbits are chaotic, they stay relatively close to their initial values (though the instability here is stronger than near a hyperbolic resonance or near an elliptic resonance). In the flat parabolic resonance case, order one instabilities of perturbed orbits were observed. The large instabilities that were observed for the atmospheric model with realistic parameters value, correspond mathematically to a nearly flat parabolic resonance, and result in a relatively fast traveling of the weather balloons from the equator to the poles, see [12].

The *tangential parabolic resonance* models the near flat case, and it was proved in [10] that it is a co-dimension two phenomenon for 2 d.o.f. near integrable Hamiltonians. Furthermore, in [10] it was shown that the parabolic resonance, and the near flat parabolic resonance, appear in some common 2 d.o.f. physical models.

3. Parabolic Resonance in 3 d.o.f. Systems

We add to the atmospheric model in (2.1) a third degree of freedom, which describes the small height oscillations of the particles. Let Λ_2 denote the normalized vertical oscillation amplitude and D_2 its conjugate momenta. The new 3 d.o.f. Hamiltonian is given by:

$$\begin{aligned} H(\phi, v, \Lambda, D; \mu, k, \varepsilon) &= \frac{v^2}{2} + \frac{1}{8}(\frac{2D_1}{\cos(\phi)} - \cos(\phi))^2 + \frac{D_2^2}{2} - cD_1 + \\ &+ aD_2(D_1 - \frac{1}{2}) + bD_2 + B(\phi) + A(\phi)(\varepsilon_1 \sin(k_1\Lambda_1) + \varepsilon_2 \sin(k_2\Lambda_2)) \\ &= H_0(\phi, v, D; \mu) + H_1(\phi, v, \Lambda, D; \mu, k, \varepsilon), \end{aligned} \tag{3.1}$$

where: $\mu = (a, b, c, k_1, k_2, \beta)$ with: $a, b, c, \beta \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}$, $\varepsilon_1, \varepsilon_2 \ll 1$. Mathematically, the second constant of motion, D_2, replaces the role of the parameter c in the 2 d.o.f. model (2.1), and when it is set to be zero, a parabolic 1-resonance occurs for $D_1 = 0.5$. The parameter c was left here for compatibility, the parameter b is responsible for creation of a parabolic 2-resonance, and the parameter a is for the generality of the mixed term of the constants of motion.

The simplest model exhibiting parabolic resonances (and near flat parabolic resonances) in near integrable 3 d.o.f. Hamiltonian systems is constructed using normal form techniques [7]; The conditions for existence of parabolic 1 and 2-resonant 2-tori are formulated for the integrable Hamiltonian, and an expansion near these conditions is made. The Hamiltonian of this model, $H(x, y, \theta, I; \mu, \varepsilon) = H_0(x, y, I; \mu) + \varepsilon H_1(x, y, \theta, I; \mu, \varepsilon)$, consists of:

$$\begin{aligned} H_0(x, y, I; \mu) &= \frac{y^2}{2} - I_1\frac{x^2}{2} + \eta\frac{x^3}{3} + \frac{x^4}{4} + \alpha_2 I_2 + (\frac{1}{2} + \alpha_1)\frac{I_1^2}{2} + \frac{I_2^2}{2} + \alpha_3 I_1 I_2, \\ H_1(x, y, \theta, I; \mu, \varepsilon) &= \sum_{i=1}^{2} \varepsilon_i f_i(x, I_1, I_2, k_i\theta_i), \end{aligned} \tag{3.2}$$

where f_i $(i = 1, 2)$ are polynomials in x, I_1, I_2, $\mu = (\alpha_1, \alpha_2, \alpha_3, \eta, k_1, k_2)$, with $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3, \eta \geq 0$ (due to phase space symmetries), $k_i \in \mathbb{Z}_+$, and $0 \leq \varepsilon_i \ll 1$; $i = 1, 2$. H_0 is the integrable part of the Hamiltonian, and H_1 is the Hamiltonian perturbation, confined to be small by ε_i $(i = 1, 2)$. By construction, the unperturbed system $(\varepsilon_i = 0)$ possesses a two dimensional normally parabolic resonant torus at the origin, O_{xyI} $((x, y, I_1, I_2) = (0, 0, 0, 0))$, and a four dimensional normally hyperbolic surface (consisting of a family of 2-tori) at $O_{xy}((x, y, I_1, I_2) = (0, 0, I_1, I_2))$. For the atmospheric model the same statements apply when replacing (I_1, I_2) with $(-D_1 + 0.5, D_2)$. We assume that the phenomenological model is in the standard form for mechanical Hamiltonian systems which are integrable to quartic order, that the energy surfaces are compact in the (x, y) plane, and that the system undergoes a pitchfork bifurcation. By simplectic change of variables (for the integrable part of the Hamiltonian) only the first action, I_1, governs the stability type of the invariant 2-tori, while the other action, I_2, together with the external parameter, α_2, governs the existence of resonances. For $I_1 < 0$ the 2-torus at O_{xy} is elliptic, for $I_1 > 0$ it is hyperbolic (and additional two elliptic tori are created), and for $I_1 = 0$ it is parabolic. When the parameter α_1 is small or proportional to α_3^2, the system (3.2) exhibits near flat parabolic 1 and 2-resonances. The parameter η governs small asymmetric perturbations. By simplectic change of variables the condition $(k, \omega) = 0$; $k \in \mathbb{Z}^{n-1}\backslash\{0\}$ for one resonance is replaced by vanishing of one of the frequencies. In the integrable case the phenomenological model is studied analytically. We generalize to 3 d.o.f. systems the new method of bifurcation diagrams in the space of constants of motion, and the analysis of the structures of the unperturbed energy surfaces.

Unlike the 2 d.o.f. case, for 3 d.o.f. near integrable systems there are 3 types of parabolic resonance:

1. Parabolic 1-resonance in I_1 direction ($\dot{\theta}_1 = 0$ in the integrable system).

2. Parabolic 1-resonance in I_2 direction ($\dot{\theta}_2 = 0$ in the integrable system).

3. Parabolic 2-resonance ($\dot{\theta}_1 = \dot{\theta}_2 = 0$ in the integrable system).

In cases 1 and 2 the parabolic 2-torus is a torus of closed invariant periodic orbits, and in the third case it is a parabolic 2-torus of fixed points.

In the atmospheric model (3.1), when no pressure wave is applied, case 1 corresponds to simultaneous change of linear stability about the equator ($(\phi, v) = (0,0)$) in the latitude direction and the lack of motion in the relative longitudinal direction along the equator ($\dot{\Lambda}_1|_{(\phi,v,D_1,D_2)=(0,0,0.5,\bar{D}_2)} \equiv 0$). Case 2 corresponds to the same change of stability occuring simultaneously with the lack of vertical oscillations at the equator ($\dot{\Lambda}_2|_{(\phi,v,D_1,D_2)=(0,0,0.5,\hat{D}_2)} \equiv 0$). Case 3 corresponds to the same change of stability combined with having no motion for all initial conditions starting at the equator with zero northward velocity.

We examine numerically the effect of small Hamiltonian perturbation on each of the above cases. The figures and descriptions below correspond to the following perturbation:

$$H_1 = \varepsilon_1 (1 - \frac{x^2}{2}) \cos(k_1 \theta_1) + \varepsilon_2 \cos(k_2 \theta_2). \tag{3.3}$$

Other forms of Hamiltonian perturbations seems to produce qualitatively the same results. Case 1 is equivalent to the 2 d.o.f. parabolic resonance. In figure 3.1, the chaotic behavior of the orbit, and the trapping in the parabolic resonance zone, is seen in the (x, y, I_1) space and in the (θ_1, I_1) plane, together with the small oscillations of the orbit in I_2 direction. The behavior is very similar to the observed 2 d.o.f. orbits in [10]. In case 2 (figure 3.2), although the orbit is trapped in the parabolic resonance zone in the I_2 direction, its behavior inside the resonance zone is quasi-periodic rather than chaotic (see zoom in on the right plot of figure 3.2). In figure 3.3 a parabolic 1-resonance in I_1 direction is shown on the left, where the orbit is seen to exhibit a homoclinic hyperbolic chaos, and a parabolic 1-resonance in I_2 direction is shown on the right, where the orbit creates a complicated, possibly self similar structure. This difference between the two 1-resonance cases is due to the different roles of the actions in the system. Case 3 corresponds to a parabolic 2-torus of fixed points in the integrable system. In this case resonance zones are created in both actions directions, creating instabilities in all directions of the phase space, see for example figure 3.4. This behavior should be contrasted with the 2 d.o.f. case, in which such instabilities in the action variable are possible only in degenerate cases, when the necessary conditions for KAM theory fail. We prove in [7] that cases 1 and 2 are persistent in 3 d.o.f. near integrable Hamiltonian systems without the use of external parameters (where the actions serve as internal parameters), and that case 3 is persistent in a one parameter family of near integrable 3 d.o.f. Hamiltonian systems.

The single case of degenerate (flat) parabolic resonance in the 2 d.o.f. system has again several generalizations in the 3 d.o.f. setting. The near flat parabolic resonance corresponding to case 1 is similar to the one observed in the 2 d.o.f. model. Yet, the order one instabilities in I_1 (respectively, D_1 in the atmospheric model) direction develop *slower* than in the 2 d.o.f. case, probably due to the small oscillations in the I_2 (D_2) direction. Such behavior of perturbed orbits corresponds, for example, to a slower time scale for the travel of the weather balloons from the equator to the poles, when the height oscillations of the balloons are taken under consideration. This kind of tangential parabolic resonance is a

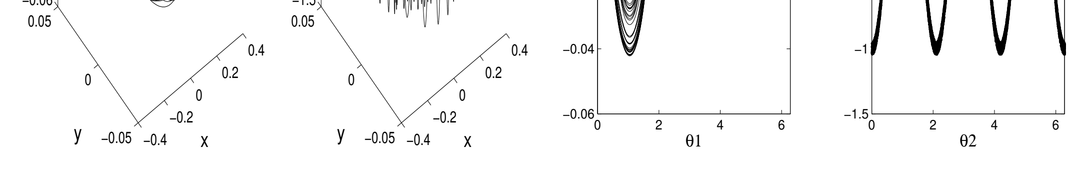

Figure 3.1: A parabolic 1-resonance in I_1 direction. Initial conditions and parameters: $(x, y, \theta_1, I_1, \theta_2, I_2) = (0.2, 0, 1.57, 0, 1.57, 0)$, $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1$, $\varepsilon_1 = \varepsilon_2 = 1e - 3$, $H_1 \approx -4.73E - 6$. Notice the different scales of I_1 and I_2.

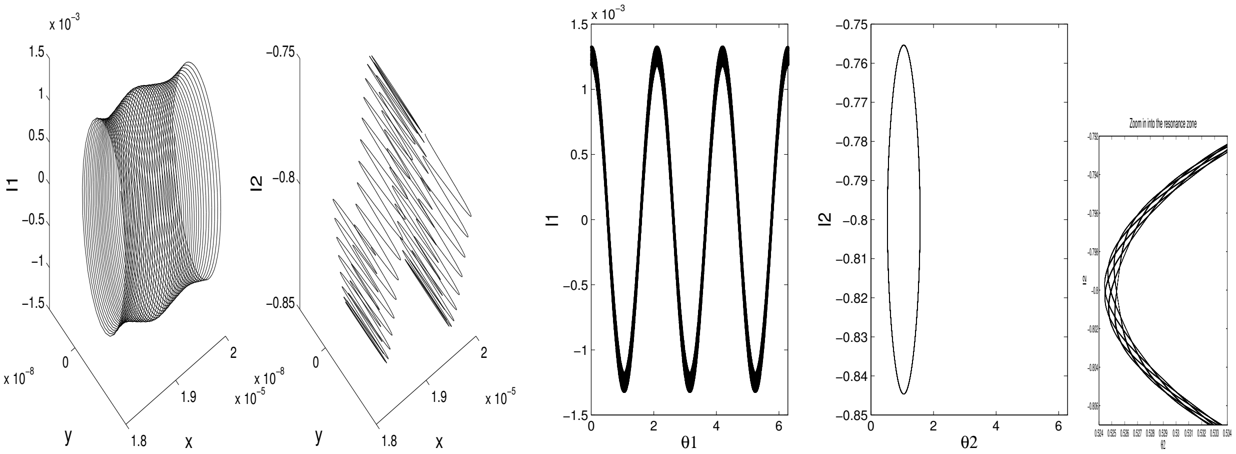

Figure 3.2: A parabolic 1-resonance in I_2 direction. Initial conditions and parameters: $(x, y, \theta_1, I_1, \theta_2, I_2) = (2E-5, 0, 1.57, 0, 1.57, -0.8)$, $\alpha_1 = 1, \alpha_2 = 0.8, \alpha_3 = 1$, $\varepsilon_1 = \varepsilon_2 = 1e-3$, $H_1 \approx -4.78E-6$. Notice the different scales of I_1 and I_2.

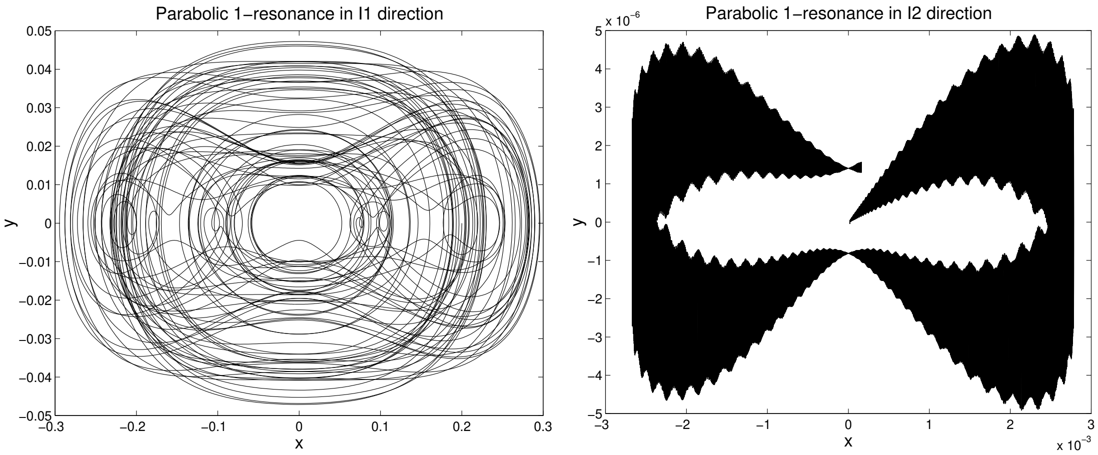

Figure 3.3: Parabolic 1-resonances projected on the (x, y) plane.

co-dimension one phenomenon in 3 d.o.f. near integrable Hamiltonians (see [7] for the proof). In *all* other cases of near flat parabolic resonances order one instabilities in all phase space directions are observed in numerical experiments. Such perturbed orbits have a complicated structure and they wander through

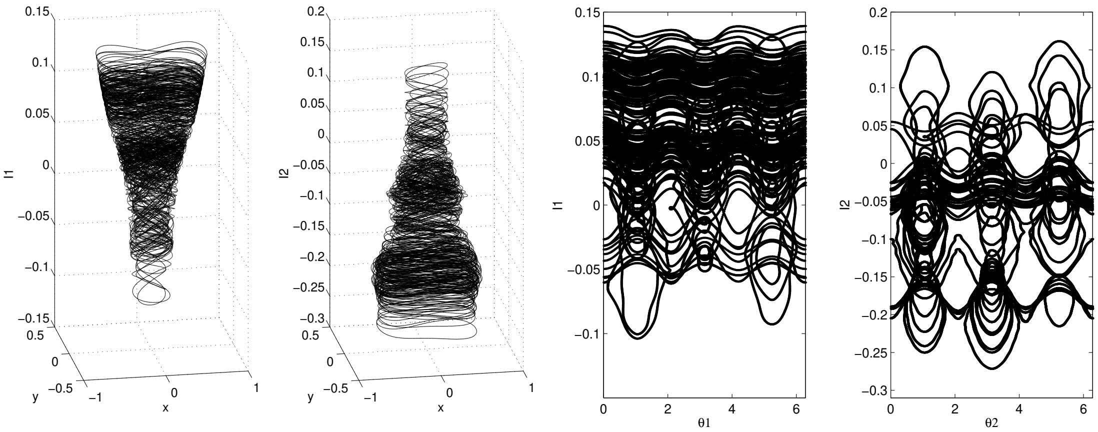

Figure 3.4: A parabolic 2-resonance. Initial conditions and parameters: $(x, y, \theta_1, I_1, \theta_2, I_2) = (0.2, 0, 1.57, 0, 1.57, 0)$, $\alpha_1 = 1.1, \alpha_2 = 0, \alpha_3 = 2$, $\varepsilon_1 = \varepsilon_2 = 1e-3$, $H_1 \approx -4.73E-6$, $t = 8,000$.

successive resonance zones that are created in both actions. These tangential parabolic multi-resonances are a co-dimension two phenomena in 3 d.o.f. near integrable Hamiltonians. See [7] for details.

Experimenting with the atmospheric model, (3.1), shows the same scenarios in phase space as described for the phenomenological model, (3.2). More details regarding the implications on the weather balloons motion will be addressed elsewhere.

4. The n d.o.f. case

Parabolic resonances appear in n d.o.f. near integrable Hamiltonian systems. Bellow is a partial list of the theorems we prove regarding their persistence [7]. We find that some values of n are critical, such as $n = 5$. For $n \geq 5$ d.o.f. the appearance of a whole family of parabolic resonant tori on the same energy surface of an integrable Hamiltonian, is persistent.

Consider the integrable n d.o.f. Hamiltonian system, $H_0(p, q)$; $(p, q) \in M \subseteq \mathbb{R}^n \times \mathbb{R}^n$, with n independent integrals pairwise in involution: $H_0 = F_1, F_2, ..., F_n : M \rightarrow \mathbb{R}$, where the F_i are smooth, M is a simplectic manifold, and $\dim M = 2n$. In most of the phase space all the dF_i are independent, hence most of the phase space is foliated with invariant tori. In particular, if the Hamiltonian level sets,

$M_f = \{(p,q) \in \mathbb{R}^{2n}, \quad F_i = f_i; \quad i = 1,...,n\}$, are compact, and the system is nondegenerate, most of the phase space is foliated with invariant, non-resonant n - tori. These maximal tori are the regular fibers of the ramified torus bundle defined by the integrable system. Consider the Hamiltonian level sets on which one of the dF_i, say, e.g., dF_n, is depending on the other dF_i; $i = 1,...,n-1$. This corresponds to singular leaves of the ramified n - tori bundle defined by the integrable Hamiltonian. Hence, on these level sets, in the vicinity where F_n is singular (assuming that the other $(n-1)$ differentials, dF_i, are independent in this vicinity) the Hamiltonian system $H_0(p,q)$ may be transformed into a Hamiltonian system on the space (see [1]): $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$, with the simplectic structure $dx \wedge dy + \sum_{i=1}^{n-1} d\theta_i \wedge dI_i$. Consider the domain U where this transformation is nonsingular. To each point in the (x,y) plane an $(n-1)$-torus is attached, where if the point in the (x,y) plane is an equilibrium, then it is an invariant torus. The (x,y) plane is the normal plane of the $(n-1)$ tori (see [9]). Integrability implies that the new Hamiltonian, $H_0(x,y,I)$, does not depend on θ. The actions I serve as inner bifurcation parameters for the stability type of the invariant $(n-1)$-tori. Consider a small Hamiltonian perturbation of the form $\varepsilon H_1(x,y,\theta,I;\varepsilon)$. The near integrable Hamiltonian systems discussed here are of the form: $H(x,y,\theta,I;\mu,\varepsilon) = H_0(x,y,I;\mu) + \varepsilon H_1(x,y,\theta,I;\mu,\varepsilon)$ in the domain U, where μ is a real parameter, or a set of real parameters, and ε is small. Let us recall from the Introduction that by persistent in the following statements we mean persistence on a C^1-open set under C^r $(r \geq 1)$ perturbations of the class of integrable Hamiltonian systems.

Let $n, m \in \mathbb{Z}_+$, where $n \geq 3$; $1 \leq m \leq n-2$, then:

Theorem 4.1. *The existence of a $(n-2-m)$-dimensional family of normally parabolic, m-resonant, $(n-1)$-tori is persistent in integrable n d.o.f. Hamiltonian systems, without the use of external parameters. Each such torus consists of a m-dimensional family of non-resonant, $(n-1-m)$-tori.*

Corollary 4.2. *The occurrence of a parabolic m-resonance, for all $1 \leq m \leq n-2$, is persistent in near integrable n d.o.f. Hamiltonian systems.*

Theorem 4.3. *The existence of a normally parabolic, $(n-1)$-resonant, $(n-1)$-torus (i.e. a normally parabolic $(n-1)$-torus of fixed points) is persistent in a one parameter family of integrable n d.o.f. Hamiltonian systems.*

Corollary 4.4. *The occurrence of a parabolic $(n-1)$-resonance is persistent in a one parameter family of near integrable n d.o.f. Hamiltonian systems.*

Remark 1. *Not all the parabolic resonant $(n-1)$-tori are necessarily on the same energy surface. For example, for $n \leq 4$, the existence of only one parabolic resonant $(n-1)$-torus on a given energy surface is persistent.*

In [7] a detailed description and proofs of the persistence of all the cases of parabolic and tangential parabolic resonances is given for n d.o.f. Hamiltonian systems, where tori of dimension less than $(n-1)$ are considered as well.

5. Conclusions

Parabolic and tangential parabolic resonances which exist in near integrable 2 d.o.f. systems are inherited by their extension to 3 d.o.f. systems, with their co-dimension being reduced by one. In addition, other parabolic and tangential parabolic resonances exist in 3 d.o.f. systems, causing new complicated structures and instabilities in new directions of the phase space. A study of an example from the atmospheric sciences emphasizes the differences regarding appearance of parabolic resonances in 2 versus 3 d.o.f. near integrable Hamiltonians. For $n \geq 3$ all kinds of parabolic resonances, except for the kind corresponding to a parabolic torus of fixed points in the integrable Hamiltonian, are persistent without the need of external parameters, and the actions (or the constants of motion) serve as the internal parameters. As in the transition from 2 d.o.f. to 3 d.o.f., it is plausible that while the parabolic and tangential parabolic resonances are inherited from the lower d.o.f. systems to the higher d.o.f. systems, the new d.o.f. cause new scenarios for the additional parabolic resonances. These may involve even more complicated behavior of perturbed orbits and greater instabilities.

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