Hamiltonian equations on $\mathbb{T}^\infty$ and almost-periodic solutions

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Abstract A class of ordinary differential equations exhibiting almost-periodic solutions is studied. The equations are canonical and the hamilton functions are of the type *kinetic energy + potential energy*. The potential-energy part is a multiperiodic function of infinite angles and we find solutions $(A_i(t), \varphi_i(t))$ $A_i \in \mathbb{R}$, $\varphi_i \in \mathbb{T}$, $i \in \mathbb{Z}^d$ that are continuations of $(A_i^o, \varphi_i^o + \omega_i t)$, $t \in \mathbb{R}$ i.e. the solutions when the potential energy is absent. The proof is based on the extension to infinite variables of the KAM theorem in the "configurational version" and the kinetic energy is needed unbounded ($|\omega_i| \to +\infty$ as $|i| \to \infty$)

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1. Introduction

Many physical systems are modelled by hamiltonian conservative equations and most of them have a finite number of equations namely a finite number of degrees of freedom (hereafter d.o.f.). Nevertheless there are problems which are naturally described by equations with an infinite number of variables. The best known arise in the PDE's area, in statistical mechanics and the theory of conduction of energy through a material (typically a crystal).

In this paper we consider equations that represent an infinite set of coupled point particles, usually called rotators, ideally constrained to move circularly and centered on the sites of $\mathbb{Z}^d$. Their equations of motion can have different form depending on the coupling. Our equations are ($h_{A_i} = \frac{\partial h}{\partial A_i}$)

$$\dot{\varphi}_i = h_{A_i}(A_i), \qquad \dot{A}_i = f_i(\varphi_1, \varphi_2, \ldots), \qquad i \in \mathbb{Z}^d, \quad \varphi \in \mathbb{T}^{\mathbb{Z}^d} = \mathcal{T}, \quad A \in \mathbb{R}^{\mathbb{Z}^d} = \mathcal{R} \tag{1.1}$$

whose *formal hamiltonian*, in action-angle coordinates, is $H(A, \varphi) = \sum_{i \in \mathbb{Z}^d} h(A_i) + V(\varphi_1, \varphi_2, \ldots)$; $V(\varphi_1, \varphi_2, \ldots)$ is related in a suitable way (see (3.3) and (3.4)) to $\{f_i\}$. The rotators A_i are coupled via the conservative forces $\{f_i(\varphi)\}$ and we are concerned with the study of the persistence of solutions that without the force would be linear in time $t \to [h_A t] = ([h_{A_1} t], [h_{A_2} t], \ldots)$ (the solutions we would have in case the rotators are *decoupled or free*; $[\,\cdot\,]$ denotes the standard projection of $\mathbb{R}$ onto $\mathbb{T}$).

We find that almost all solutions (in the sense that the excluded subset of solutions have small relative measure) have the form

$$t \to (A(t), \varphi(t)) \equiv (\{A_i\}, \{\varphi_i\})_{i\in\mathbb{Z}^d} \qquad \varphi_i(t) = [\omega_i t + u_i([\omega t])], \quad A_i(t) = v_i([\omega t]) \tag{1.2}$$

u_i and v_i are smooth function for all i and $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ is a "Diophantine sequence" [*i.e.* $\exists$ $\gamma > 0$ and an isomorphism, i_k, from $\mathbb{Z}_+$ onto $\mathbb{Z}^d$ such that, $\forall\, l \geq 1$, $|\sum_{k=1}^l \omega_{i_k} n_{i_k}|^{-1} \leq \gamma(\sum_{k=1}^l |n_{i_k}|)^l$, with $n_{i_k} \in \mathbb{Z}$, $\sum_{k=1}^l |n_{i_k}| > 0$]. Notice that, as a consequence of a classical number theoretical theorem by Liouville, $|\omega_i| \to \infty$ as $|i| \to \infty$.

Clearly this problem falls in the class consisting in the study of solutions which are continuations of infinite-dimensional quasi-periodic solutions. We make the reader note that the solutions (1.2) are really infinite-dimensional being $\omega_i \neq 0$ for all i.

From a physical point of view, the existence of the solutions (1.2) means that for almost all initial data (apart from a set of small relative measure) the solutions are confined in suitable subsets of the phase-space (depending on the initial data) so making impossible the transferring of the energy/information in other portions of the phase-space (see [FSW], [VB], [Pö1]).

The result is obtained through an infinite-dimensional extension of the KAM theorem and the estimates involved leads to $\omega_j \sim j!^a$ and $a \sim 20$.

[CP] is a particular case ($h(A_i) = \frac{1}{2}A_i^2$) among those handled in this paper. The power two of the kinetic energy in the hamiltonian makes (trivially) possible to transform (1.1) into a second order system avoiding the introduction of the $\{v_i\}$ variables.

The PDE area represents a natural context where study models with infinite d.o.f and then almost–periodic solutions. The first results in PDE regarded the non–linear Shrödinger equation and the D'Alembert's equations and interested quasi–periodic solutions i.e. the solution when just a finite number of frequencies are excited. The perturbative techniques have been recently extended to solve the problem of continuations of almost–periodic solutions (see [Wa2], [Ku1], [Ku2], [CW], [Pö3], [Bo], [Pö2]). However the behavior of ω_j with j ($j!^{20}$) is not suitable for the application of our results to (discretized) PDE.

The rest of the paper is organized as follows: In Section 2 we give a precise notion of solutions of (1.1) for Lipschitz vector field (so that existence and uniqueness for all time of the Cauchy problem trivially follow); in Section 3 we introduce the Hamiltonian structure via generalized gradients; in Section 4 we define (maximal) almost–periodic solutions and in Section 5 we define Diophantine sequences and prove their abundance; Section 6 contains the statements of the main results; Section 7 is devoted to a finite dimensional "average theorem", which allows to construct quasi-periodic solutions for a system over $\mathbb{T}^l$ starting from a quasi–periodic solution of a subsystem over $\mathbb{T}^{l-1}$ obtained by averaging the potential associated to the original l–dimensional system: iterating suitably one obtains the proof in infinite dimensions, which is spelled out in Section 8. The average theorem of Section 7 is, in turn, based on tools from perturbation theory (such as Kolmogorov–Arnold–Moser theory, Nash–Moser implicit function theorems, *etc.*). Next there are several Appendices. Appendix 1 contains the proof of the existence (globally in time) of the dynamics. Appendix 2 and 3 some computations necessary to the proof of Proposition 7.1 ("The average theorem"). Appendix 4 contains the proof of Lemma 7.2 which is the KAM theorem. Appendix 5 contains a classical result concerning the full (Lebesgue) measure of Diophantine vectors in $\mathbb{R}^N$. Appendix 6 contains the Cauchy contour estimates (the ever present when working with analytic functions and quantitative results) and a Lemma for Theorem 6.1

2. Setup

Denote by $\mathcal{T}$ the Cartesian product of infinitely many copies of the one dimensional (flat) torus

$$\mathcal{T} \equiv \bigotimes_{i\in\mathbb{Z}^d} \equiv \mathbb{T}^{\mathbb{Z}^d}, \qquad \mathbb{T}_i \equiv \mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z} \tag{2.1}$$

(d being a positive integer) and endow $\mathcal{T}$ with the standard weak topology (see, *e.g.* , [Ke] and [Je]). Such topology is also induced by *metrics*: To any summable positive sequence

$$w: \ \mathbb{Z}^d \to (0,\infty) \quad \text{s.t.} \quad \sum_{i\in\mathbb{Z}^d} w_i < \infty \ , \qquad (w_i > 0 \ \forall i) \tag{2.2}$$

which we shall call a *weight*, we can associate a metric ρ_w by setting, $\forall\ \varphi, \varphi' \in \mathcal{T}$ ($\varphi = \{\varphi_i\}_{i\in\mathbb{Z}^d}$, $\varphi' = \{\varphi'_i\}_{i\in\mathbb{Z}^d}$, $\varphi_i, \varphi'_i \in \mathbb{T}$):

$$\rho_w(\varphi,\varphi') \equiv \sum_{i\in\mathbb{Z}^d} \rho(\varphi_i,\varphi'_i)\ w_i \tag{2.3}$$

where ρ is the standard (flat) metric on $\mathbb{T}\equiv\mathbb{T}_i$:

$$\rho([a],[b]) \equiv \inf_{n\in\mathbb{Z}} |a-b+2\pi n| \tag{2.4}$$

here $a, b\in\mathbb{R}$ and $[\cdot]$ denote equivalence (mod. 2π) class.

We shall denote by $\mathcal{T}_w$ the pair $(\mathcal{T}_w, \rho_w)$ and by $\mathcal{B}_w$ the Banach space formed by the sequences $a \in \mathbb{R}^{\mathbb{Z}^d}$ having finite norm

$$\|a\|_w \equiv \sum_{i\in\mathbb{Z}^d} |a_i|\ w_i < \infty \tag{2.5}$$

$\rho'_w(a,b) \doteq \|a-b\|$

Denote by $\mathcal{R}$ the Cartesian product of infinitely many copies of $\mathbb{R}$

$$\mathcal{R} \equiv \bigotimes_{i\in\mathbb{Z}^d} \equiv \mathbb{R}^{\mathbb{Z}^d} \tag{2.6}$$

and endow $\mathcal{R}$ with the topology induced by the following metric: to the *weight* w we associate the metric $\lambda_w : \forall\ A, A' \in \mathcal{R}$

$$\lambda_w(A, A') \equiv \sum_{i\in\mathbb{Z}^d} w_i \arctan |A_i - A'_i| \tag{2.7}$$

We shall denote by $\mathcal{R}_w$ the pair $(\mathcal{R}_w, \lambda_w)$; $\mathcal{R}_w \times \mathcal{T}_w$ is a complete metric space with metric $\rho_w(\varphi, \varphi') + \lambda_w(A, A')$

We can now give a precise meaning to second order ODE's on $\mathcal{T}_w \times \mathcal{R}_w$: Given a continuous map $(h_A, f) : \mathcal{R}_w \times \mathcal{T}_w \to \mathcal{R}_w \times \mathcal{B}_w$, $(h_{A_i} \doteq \frac{\partial h}{\partial A_i})$ consider the system:

$$\dot\varphi_i = h_{A_i}(A_i), \qquad \dot A_i = f_i(\varphi)\ , \qquad i \in \mathbb{Z}^d \tag{2.8}$$

where $(h_{A_i}, f_i) \equiv (\pi_i^{(A_i)} \circ h_A,\ \pi_i^{(\varphi_i)} \circ f)$ $(\pi_i^{(\varphi_i)} : \mathcal{B}_w \to \mathbb{R}_i,\ \pi_i^{(A_i)} : \mathcal{R}_w \to \mathbb{R}_i$ being the standard projections). A *solution of (2.8)* is just a continuous map $t \in \mathbb{R} \to (A(t), \varphi(t)) \in \mathcal{R}_w \times \mathcal{T}_w$, with $(A_i, \varphi_i) \in C^1(\mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}, \mathbb{T})$, $\forall i$, and satisfying the system (2.8).

Remarks 2.1 (i) In some sense the notion of solution we have just introduced is a "weak notion".

(ii) Let $f \equiv 0$ so that $(A(t), \varphi(t)) = (A^o, [\omega t]) \equiv (\{A_i^o\}, \{[\omega_i t]\})_{i\in\mathbb{Z}^d}$ and $\omega \doteq h_{A^o}$; $(A(t), \varphi(t))$ is a solution for any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ (not necessarily in $\mathcal{B}_w$). Note in particular that $t \in \mathbb{R} \to (A^o, [\omega t]) \in \mathcal{R}_w \times \mathcal{T}_w$ is continuous for any $\omega \in \mathbb{R}^{\mathbb{Z}^d}$. These facts are no longer true if one considers stronger topologies; for example, if $|\omega_i| \to \infty$ as $|i| \to \infty$, $[\omega t]$ is not continuous with respect to the uniform topology [$\rho_{\text{uniform}}(\varphi, \varphi') \equiv \sup_{i\in\mathbb{Z}^d} \rho(\varphi_i, \varphi'_i)$] . Thus, our interest in solutions $(A(t), \varphi(t))$ with $(A_i(t), \varphi_i(t))$ "close" to $(A^o, \omega_i t)$ with $|\omega_i| \to \infty$ as $|i| \to \infty$ explains the choice of the topology induced by $\rho_w + \lambda_w$. Notice that ρ_w induces a compact topology on $\mathcal{T}_w$.

Global existence and uniqueness for the Cauchy problem associated to (2.8) with (h_A, f) Lipschitz, are elementary applications of standard contraction techniques (see Appendix 1). We just stress that the "initial velocities" can be taken to be completely arbitrary (and not necessarily in $\mathcal{B}_w$):

Proposition 2.2 *Let (h_A, f): $\mathcal{R}_w \times \mathcal{T}_w \to \mathcal{R}_w \times \mathcal{B}_w$ be a Lipschitz map (i.e. $\exists\, L > 0$ s.t. $\rho'_w\big(f(\varphi), f(\varphi') + \lambda_w(h_A, h_{A'}) \le L(\rho_w(\varphi, \varphi') + \lambda_w(A, A'))$, $\forall \varphi, \varphi' \in \mathcal{T}_w \quad A, A' \in \mathcal{R}_w$). Given any $\varphi^o \in \mathcal{T}_w$ and any $A^o \in \mathbb{R}^{\mathbb{Z}^d}$, there exists a unique solution, global in time, of the Cauchy problem*

$$\begin{cases} \dot\varphi_i(t) = h_{A_i}(A(t)) & \varphi_i(0) = \varphi_i^o \qquad i \in \mathbb{Z}^d \\ \dot A_i(t) = f_i(\varphi(t)) & A_i(0) = A_i^o. \end{cases} \tag{2.9}$$

Remarks 2.2 In this paper we consider functions $h(A)=\sum_j h(A_j)$

$$0 < a' \le \inf_{A_i\in\mathbb{R}} h_{A_iA_i} \le \sup_{A_i\in\mathbb{R}} h_{A_iA_i} \le a < +\infty \tag{2.10}$$

(2.10) forbids kinetic energies like $h(A_i) = A_i^{2+\delta}$ for any $\delta > 0$ while allows e.g. $h(A_i) = A_i^2 + c\ln(1+A_i^2)$ (c constant) (see Appendix 1 (9.6)).
When $h(A_i) = \frac{1}{2}A_i^2$, (2.9) can be rewritten as a second order system and $a = a' = 1$ ([CP] deals exactly with this case).
Moreover we need h_{A_i} invertible for $A_i's$ sufficiently large i.e.

$$\exists\ h_{A_i}^{(-1)} \text{ for } A_i \ge A_i^o \tag{2.11}$$

and A_i^o big enough (how big depends on the estimates of the theorems we are going to write).

As for f we consider two examples of Lipschitz maps (the force). The first one is so called *short range*; fix $L \ge 1$ and consider, for $j \in \mathbb{Z}^d$, a collection of functions g_j depending on sites of the lattice within (Euclidean) distance L from j; in formulae:

$$g_j \equiv g_j(\varphi^{(L)})\ ,\quad \varphi^{(L)} \equiv \{\varphi_k\}_{k\in B_j(L)}\ ,\quad B_j(L) \equiv \{k : \|k-j\| \le L\} \tag{2.12}$$

We assume that the "localized potentials" g_j are *real–analytic* functions from $\mathbb{T}^{|B_j(L)|}$ $\to \mathbb{R}$ and that, for some positive M:

$$\sup_{j,\varphi^{(L)}\in\mathbb{T}^{|B_j(L)|}} |g_j(\varphi^{(L)})| \le M \tag{2.13}$$

Then we set

$$f_i \equiv \sum_{\|j-i\|\le L} \partial_{\varphi_i} g_j \tag{2.14}$$

The system (1.1) with such f_i is called a finite range system of infinitely many coupled rotators (see also [Wa1] and [VB]). A particular case, often considered, is given in $d = 1$ by $L = 1$, $g_j = \cos(\varphi_j - \varphi_{j-1}) - \cos(\varphi_{j+1} - \varphi_j)$. It can be noted that each rotator interacts with just a finite number of different rotators and this explains the words *short range*

The second example is so called *long range* as each rotator interacts with any other rotator. In $d = 1$ it is given by

$$f_i \equiv \cos\varphi_i \sum_{j\in\mathbb{Z}} a_j \prod_{k\ne 0}(1 + a_{j+k}\sin\varphi_{i+k})\ ,\quad \sum_{j\in\mathbb{Z}} |a_j| < \infty \tag{2.15}$$

The property of being Lipschitz depends of course from the metric. In fact, consider two cases: (1) $a_j \equiv b^{-|j|}$, $b > 1$; (2) $a_j = (1+|j|^p)^{-1}$, $p > 1$. Then, in the first case (1), f is Lipschitz if we take $w_j \equiv c^{-|j|}$ with any $1 < c < b$, while in case (2) we can take $w_j \equiv a_j$.

3. Hamiltonian Structure and Averages

In finite dimensions, the Hamiltonian structure for ODE's, $\dot{A} = f(\varphi)$, $\dot{\varphi} = h_{A_i}$, is expressed by the vector field f being a *gradient*, $f_i = -\partial_{\varphi_i} V$; the hamiltonian function is then $H \equiv h(A) + V(\varphi)$ (see [A] for generalities).

In infinite dimensions, one could require as well that $-f$ be a gradient of a $C^1(\mathcal{T}_w;\mathbb{R})$ function. However this attitude is much too restrictive as it imposes strong decay properties (as $|i| \to \infty$) to the components f_i of the field and even the simple examples (2.14), (2.15) would not fit in the picture. Indeed what one really needs is that f is a "local gradient" (where "local" refers to localized portions of $\mathbb{Z}^d$). For the interested reader, in [Pe] we study the existence and uniqueness problem for some periodic solutions of a sistem like (1.1) and $h(A) = \sum_{i\in\mathbb{Z}^d} \frac{1}{2} A_i^2$. The proof is based on a result of Poincaré and makes use of the implicit function theorem. In that case we need $V(\varphi_1, \varphi_2, \ldots)$ to be a true function belonging to $C^1(\mathcal{T}_2, \mathbb{R})$ where $\mathcal{T}_2 = \{\varphi \in \mathbb{T}^{\mathbb{Z}} \mid \rho_w(\varphi, \varphi') \equiv \left(\sum_{i\in\mathbb{Z}} \rho^2(\varphi_i, \varphi_i')\right)^{\frac{1}{2}}\}$ (an "l_2–metric").

We shall therefore introduce the notion of *generalized gradient* (or "*g–gradient*") bringing in the local structure with the help of finite–dimensional projectors, which just *average out* the dependence upon variables φ_i with i varying in the complementary of a finite subset of $\mathbb{Z}^d$.

To construct such operators, we introduce a *probability measure* on $\mathcal{T}_w$: Consider the σ–algebra, $\mathcal{R}$, generated by the "cylinders" (see, *e.g.* , [Ha])

$$\mathcal{R}_I \equiv \bigotimes_{i\in I} A_i \ \bigotimes_{j\notin I} \mathbb{T}_i \tag{3.1}$$

where A_i is an open subset of $\mathbb{T}_i$ and $I \subset \mathbb{Z}^d$ is a finite subset of $\mathbb{Z}^d$: $|I| < \infty$ ($|\cdot|$ denoting here cardinality). Then there exists (see [Ha], Section 38) a unique (probability) measure μ on $\mathcal{R}$ such that

$$\mu(\mathcal{R}_I) = \prod_{i\in I} \mu_i(A_i) \tag{3.2}$$

where μ_i is the normalized "Lebesgue measure" on $\mathbb{T}_i$.

Now, given any (not necessarily finite) subset I of $\mathbb{Z}^d$, we can construct, as above, a product measure $\mu_I \equiv \bigotimes_{i\in I} d\mu_i$ on $\bigotimes_{i\in I} \mathbb{T}_i$ (endowed with the topology induced by the metric $\rho_I \equiv \sum_{i\in I} \rho_i$). Then if $J \equiv I^c \equiv \mathbb{Z}^d \backslash I$ the product measure $d\mu_I \otimes d\mu_J$ coincide with $d\mu$ and Fubini's theorem holds. Thus, to any bounded measurable function g on $\mathcal{T}$ we can naturally associate a measurable function, $g^{[I]}$, on $\bigotimes_{i\in I}\mathbb{T}_i$, by setting (for $\varphi_I \in \bigotimes_{i\in I}\mathbb{T}_i$, $d\mu_i$–almost everywhere):

$$g^{[I]} \equiv \int g \; d\mu_J \; , \qquad J \equiv \mathbb{Z}^d \backslash I \tag{3.3}$$

If $|I| < \infty$, $g^{[I]}$ is a measurable function on $\mathbb{T}^{|I|}$ and $g^{[I]} \to g$ *a.e.* on $\mathcal{T}$ as $|I| \to \mathbb{Z}^d$. For the examples in (2.14) and (2.15) the convergence is uniform and we shall use this fact in Section 8 after (8.7). We can now define g–gradients.

Definition 3.1 *A continuous function $f: \mathcal{T}_w \to \mathcal{B}_w$ is a* g-gradient *if for any finite $I \subset \mathbb{Z}^d$ there exists a $C^1(\mathbb{T}^{|I|};\mathbb{R})$ function, $V^{(I)}(\varphi)$, so that*

$$f_i^{[I]}(\varphi) = -\partial_{\varphi_i} V^{(I)}(\varphi), \quad \forall\, i \in I \;, \quad \forall\, \varphi \in \mathbb{T}^{|I|} \tag{3.4}$$

It is easy to check that the examples (2.14), (2.15) in Section 2 are g–gradients.

We shall speak of Hamiltonian equations on $\mathcal{T}_w$ whenever f in (2.8) is a g–gradient.

Let us conclude this section by introducing the (strong) *regularity class* we shall work with.

Definition 3.2 *A g-gradient f is called* uniformly weakly real-analytic *if there exists a real number $\sigma > 0$ such that for any finite set $I \subset \mathbb{Z}^d$, $V^{(I)}(\varphi)$ is real-analytic on $\mathbb{T}^{|I|}$ and can be analytically continued to the set $\{z \in \mathbb{C}^{|I|} : \; |\mathrm{Im}\, z_i| < \sigma\}$ and continuous on its closure .*

Remarks 3.3 (i) In fact we could deal with more general classes of vector fields allowing the width of analyticity of $V^{(I)}$ to tend to zero as $|I| \to \infty$ (the allowed rate of decay would then be dictated by the quantitative analysis carried out below).

(ii) Example (2.15) in Section 1 is uniformly weakly analytic and as parameter σ one could take any positive number; example (2.14) is uniformly weakly analytic for some (small enough) $\sigma > 0$.

4. Almost–Periodic Solutions (Definitions)

We start with the definition of maximal almost–periodic functions with "rationally independent" frequency $\omega \in \mathbb{R}^{\mathbb{Z}^d}$.

Definition 4.1 *A sequence $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ is said to be* rationally independent *if for any finite subset I of $\mathbb{Z}^d$ and for any $n_i \in \mathbb{Z}$*

$$\sum_{i\in I} \omega_i n_i \neq 0 \quad \text{unless} \quad n_i = 0 \;\forall\, i \in I \tag{4.1}$$

In other words, ω is rationally independent if any finite vector $\omega^{(I)} \equiv \{\omega_i\}_{i\in I} \in \mathbb{R}^{|I|}$ is rationally independent.

Definition 4.2 *A continuous real function $q(t)$ is called* almost–periodic over $\mathcal{T}_w$ *(with frequency ω) if there exist a rationally independent sequence $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ and a continuous function $Q : \mathcal{T}_w \to \mathbb{R}$ such that $q(t) = Q([\omega t])$. A solution $(A(t), \varphi(t))$ of (2.8) is called* maximal almost–periodic *if $(A_i(t), \varphi_i(t) - [\omega_i t])$ is, for all i, almost–periodic over $\mathcal{T}_w$ with frequency ω.*

Remarks 4.3 (i) Recall [see Remark 2.1, (i)] that $t \to [\omega t]$ is continuous $\forall\, \omega \in \mathbb{R}^{\mathbb{Z}^d}$.

(ii) A function q almost–periodic over $\mathcal{T}_w$ is almost–periodic in the sense of H. Bohr with frequency modulus given by

$$\sigma(q) = \big\{ r \in \mathbb{R} : \ r = \sum_{i\in I} \omega_i n_i \quad \text{for} \quad \text{some} \qquad I \subset \mathbb{Z}^d, \ |I| < \infty, \ n_i \in \mathbb{Z} \big\} \tag{4.2}$$

(see [Ka]).

(iii) The word "maximal" in the above definition refers to the rationally independence of the frequency ω. Indeed, one can consider *quasi–periodic* solutions of (2.8): these are almost–periodic solutions with the associated frequency modulus being generated by a fixed vector $\omega^{(l)} \in \mathbb{R}^l$. The existence of such solutions has been established in a somewhat different context by [Wa2], [Ku1], [Ku2], [Bo], [Pö1]-[Pö3])

5. Diophantine Sequences

Actually the almost–periodic solutions constructed below will have frequencies ω verifying much stronger numerical properties than just being rationally independent: *They will be Diophantine* in the sense of the following definition.

Definition 5.1 *A rationally independent sequence $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ is called* Diophantine *if for any finite set $I \subset \mathbb{Z}^d$, there exist constants $\gamma > 0$ and τ $(\geq |I|)$ such that for any choice of $n_i \in \mathbb{Z}$ with $\sum_{i\in I} |n_i| > 0$ it is:*

$$|\sum_{i\in I} \omega_i n_i| \geq \frac{1}{\gamma \big(\sum_{i\in I} |n_i| \big)^\tau} \tag{5.1}$$

It is well known that Diophantine vectors in $\mathbb{R}^l$ form a set of full Lebesgue measure (for completeness we reproduce this elementary result in Appendix 5), but also in infinite dimensions Diophantine frequencies are rather abundant. One can in fact construct many Diophantine sequences with the help of the following Lemma (compare with Lemma 3 of [CZ]).

We recall that a vector $\omega \in \mathbb{R}^l$ is called (γ, τ)*–Diophantine* if

$$|\omega \cdot n| \geq \frac{1}{\gamma |n|^\tau} \ ; \quad \forall n \in \mathbb{Z}^l \backslash \{0\} \tag{5.2}$$

where $\omega \cdot n$ denotes the standard scalar product in $\mathbb{R}^l$ and $|n| \equiv \sum_{i=1}^l |n_i|$.

Lemma 5.2 *Let $\omega \in \mathbb{R}^{l-1}$ be $(\gamma, l-1)$–Diophantine; let Ω be a positive number satisfying*

$$\Omega \geq 4\sqrt{l-1}|\omega| \ ; \qquad \big(|\omega| \equiv \sum_{i=1}^{l-1} |\omega_i|\big) \tag{5.3}$$

and define the following subset of $[\Omega, \infty)$:

$$\mathcal{A}_l \equiv \mathcal{A}_l(\omega, \Omega) \equiv \{\alpha \geq \Omega : \ |\omega \cdot n + \alpha h| \geq \Omega/[2(|n| + |h|)^l] \ , \ \forall n \in \mathbb{Z}^{l-1} \forall 0 \neq h \in \mathbb{Z}\} \tag{5.4}$$

There exists a universal number $K > 1$ such that:

$$\ell\big([\Omega, \infty) \backslash \mathcal{A}_l\big) < K|\omega| \tag{5.5}$$

where ℓ denotes Lebesgue measure.

Remarks 5.3 (i) Since by definition $\Omega > 2/\gamma$, [as $|\omega| \geq |\omega_i| \geq \gamma^{-1}$ by (5.2)], (ω, α) is (γ, l)–Diophantine whenever $\alpha \in \mathcal{A}_l$.

(ii) Whence, in particular, the above Lemma tells us that given a $(\gamma, l-1)$–Diophantine vector ω in $\mathbb{R}^{l-1}$ (and almost all vector in $\mathbb{R}^{l-1}$ are $(\gamma, l-1)$–Diophantine for some γ: see Appendix 2 we can pick α in $\mathcal{A}_l \subset [\Omega, \infty)$ [whose complementary measure in $[\Omega, \infty)$ is of $O(|\omega|)$] so that the vector (ω, α) is (γ, l)–Diophantine.

(iii) In the above statement and in its proof we could replace $l-1$ with any $\tau > l-2$, however to avoid introducing too many parameters we consider only the case $\tau = l-1$.

(iv) It is now easy to construct many Diophantine sequences. Fix $I_0 \in \mathbb{Z}^d$, $|I_0| = l-1 < \infty$; pick a $(\gamma_0, l-1)$–Diophantine vector $\omega^{(l-1)} \in \mathbb{R}^{l-1}$ (with some $\gamma_0 > 0$); and fix a one–to–one map, j_h, from $\mathbb{Z}_+$ onto $\mathbb{Z}^d \backslash I_0$. Now set $\omega_i \equiv \omega_i^{(l-1)}$ $\forall i \in I_0$ and define ω_{j_h}, $h \geq 1$, inductively as follows. Let $\Omega_1 \equiv 4\sqrt{l}|\omega^{(l-1)}|$ and choose ω_{j_1} in $\mathcal{A}_l(\omega^{(l-1)}, \Omega_1)$ so that, by the above Lemma, $\omega^{(l)} \equiv (\omega^{(l-1)}, \omega_{j_1})$ is (γ_0, l)–Diophantine. Analogously, given $\omega^{(l-1+h)} \equiv (\omega^{(l-1)}, ..., \omega_{j_h}) \in \mathbb{R}^{l-1+h}$, $(\gamma_0, l-1+h)$–Diophantine, we let $\Omega_h \equiv 4\sqrt{l-1+h}\big(|\omega^{(l-1)}| + \sum_{k=1}^h |\omega_{j_k}|\big)$ and pick $\omega_{j_{h+1}} \in \mathcal{A}_{l-1+h}(\omega^{(l-1+h)}, \Omega_h)$. It is clear that in this way one constructs many Diophantine sequences satisfying (5.1) with $\gamma \equiv \gamma_0$ and $\tau = l-1+k$ where $k \equiv 0$ if $I \subset I_0$, and otherwise $k \equiv \max\{h : j_h \in I\}$.

Proof of Lemma 5.2 For vectors in $\mathbb{R}^m$, we denote by $\|\cdot\|$ the Euclidean norm and by $|\cdot|$ the 1–norm (sum of absolute values of components). Set $a \equiv \Omega/(2\|\omega\|)$, $e \equiv \omega/\|\omega\|$, and let σ_{l-1} be the area of the unit-sphere $S^{l-2} \equiv \{x \in \mathbb{R}^{l-1} : \|x\| = 1\}$. Finally denote by n a generic vector in $\mathbb{Z}^{l-1}$ and by h a generic integer number and for a vector $y\in\mathbb{R}^{l-1}$, let $\bar y\ \equiv\ (y_2,...,y_{l-1})$. Now, let first $l-1\ge 2$; then:

$$\begin{aligned}
\ell\big([\Omega,\infty)\backslash\mathcal{A}_l\big) &= \ell\{\alpha\ge\Omega: |\omega\cdot n+\alpha h|<\frac{\Omega}{2(|n|+|h|)^l}\ \text{ for some } h\ne 0\}\\
&\le \sum_{n\ne0,h\ne0}\ell\{\alpha\ge\Omega: |\frac{\omega\cdot n}{h}+\alpha|<\frac{\Omega}{2|h|\ (|n|+|h|)^l}\}\\
&\le 2\Omega\sum_{h\ge1}\frac1h\sum_{|\omega\cdot\frac{n}{h}|\ge\frac{\Omega}{2}}\frac{1}{(|n|+|h|)^l}\\
&= 2\Omega\sum_{h\ge1}\frac1h\sum_{|e\cdot n|\ge ah}\frac{1}{(|n|+h)^l}\\
&\le 2\Omega\sum_{h\ge1}\frac1h\int_{\{x\in\mathbb{R}^{l-1}:\ |e\cdot x|\ge ah-\sqrt{l-1}\}}\frac{dx}{(|x|+h)^l}\\
&\le 2\Omega\sum_{h\ge1}\frac1h\int_{\{y\in\mathbb{R}^{l-1}:\ |y_1|\ge ah-\sqrt{l-1}\}}\frac{dy}{(\|y\|+h)^l}\\
&\le 2\Omega\sum_{h\ge1}\frac1h\ 2^{\frac{l}{2}}\int_{\{y\in\mathbb{R}^{l-1}:\ |y_1|\ge ah-\sqrt{l-1}\}}\frac{dy}{(|y_1|+\|\bar y\|+h)^l}\\
&= 2\Omega\sum_{h\ge1}\frac1h\ \frac{2^{\frac{l+2}{2}}}{l-1}\int_{\mathbb{R}^{l-2}}\frac{d\bar y}{(\|\bar y\|+(a+1)h-\sqrt{l-1})^{l-1}}\\
&\le 2\Omega\sum_{h\ge1}\frac1h\ \frac{2^{\frac{l+2}{2}}}{l-1}\sigma_{l-2}\int_0^\infty\frac{r^{l-3}}{(r+\frac{a}{2}h)^{l-1}}\ dr\\
&\le 2\Omega\sum_{h\ge1}\frac1h\ \frac{2^{\frac{l+2}{2}}}{l-1}\sigma_{l-2}\int_0^\infty\frac{1}{(r+\frac{a}{2}h)^{2}}\ dr\\
&\le K|\omega|
\end{aligned}$$

where: in the sum after the first inequality we have $n\ne0$ as $\Omega>2/\gamma$ [see Remark 5.3, (i)]; in the fourth inequality we set $By=x$ with B a unitary matrix sending the vector $(1,0,...,0)$ to e; in the sixth inequality we used the assumption that $\Omega\ge 4|\omega|\sqrt{l-1}$ and we have taken:

$$K\equiv 2^{9/2}\big(\sum_{h\ge1}h^{-2}\big)\ \sup_l\{\frac{2^{\frac{l-1}{2}}\sigma_{l-2}}{l-1}\} \tag{5.6}$$

The case $l=2$ is just shorter.

6. Main Results

Theorem 6.1 *Let $h_A{:}\,\mathcal{R}_w\to\mathcal{R}_w$ satisfy (2.10) and (2.11) and let $f:\mathcal{T}_w\to\mathcal{B}_w$ be a uniformly weakly analytic g–gradient (Definitions 3.1, 3.2). Then there exist* *uncountably many maximal almost–periodic solutions of (2.8) (Definition 4.2) with Diophantine frequencies (Definition 5.1).*

This result is a simple corollary of the next more detailed theorem and Lemma A6.2 in Appendix 6.

Recall that a *non–degenerate (maximal) quasi–periodic solution* of

$$\dot A = -V_\varphi(\varphi),\ \dot\varphi = h_A(A) \qquad \varphi\in\mathbb{T}^l, \qquad A\in\mathcal{T}^l \tag{6.1}$$

with frequency $\omega\in\mathbb{R}^l$ is a solution of the form

$$\varphi(t) = \big[\omega t + u([\omega t])\big], \qquad A(t) = v([\omega t]) \tag{6.2}$$

for a pair of functions $u:\mathbb{T}^l\to\mathbb{R}^l$ and $v:\mathbb{T}^l\to\mathbb{R}^l$ satisfying

$$\det\big(\mathrm{Id}_l + u_\theta(\theta)\big)\neq 0\ , \qquad \forall\ \theta\in\mathbb{T}^l \tag{6.3}$$

where Id_l is the identity $(l\times l)$–matrix and $(u_\theta)_{hk}\equiv\ \partial_{\theta_k}u_h$, $h,k=1,...,l$; the word maximal refers to the maximal dimension of the frequency ω.

Remarks 6.2 (On quasi–periodic solutions) (i) To any quasi–periodic solution with ω rationally independent (*i.e.* $\omega\cdot n = 0$ for some $n\in\mathbb{Z}^l \Longrightarrow n=0$) one can associate an l–parameter family of solutions obtained by "phase–translation": for each $\theta\in\mathbb{T}^l$,

$$\varphi(t;\theta)\equiv[\theta+\omega t+u([\theta+\omega t])], \qquad A(t;\theta)\equiv v([\theta+\omega t])$$

is still a solution (as the flow $t\to[\omega t]$ is dense in $\mathbb{T}^l$).

(ii) From now on we shall often omit the projection $[\cdot]$ [see (1.2), (2.4) and (ii) of Remark 2.1] in the notation.

(iii) Indeed, the above family corresponds to an *invariant l–torus*, embedded in the phase–space $\mathbb{R}^l\times\mathbb{T}^l$, given by

$$\mathcal{T}^l_\omega\equiv\{(A,\varphi)\in\mathbb{R}^l\times\mathbb{T}^l : (A,\varphi) = \big(v(\theta),\theta+u(\theta)\big)\ ,\ \theta\in\mathbb{T}^l\} \tag{6.4}$$

where $D_\omega\equiv\omega\cdot\partial_\theta\equiv\sum_{i=1}^l\omega_i\partial_{\theta_i}$; the y–component is just the velocity vector corresponding to the point φ (as D_ω corresponds to $\frac{d}{dt}$ along the linear flow $t\to\theta+\omega t$). The non–degeneracy (6.3) of the solution (6.2) allows to see $\mathcal{T}^l_\omega$, as a regular embedded torus in the ambient space $\mathbb{R}^l\times\mathbb{T}^l$.

(iv) In view of the above observations it is clear that to find non–degenerate quasi–periodic solutions of (6.1) *is equivalent to find solutions of the following non–linear PDE on $\mathbb{T}^l$:*

$$D_\omega v(\theta) = -V_\varphi(\theta+u(\theta)), \qquad \omega + D_\omega u(\theta) = h_A(v(\theta)), \qquad \min_{\theta\in\mathbb{T}^l}|\det(\mathrm{Id}_l+u_\theta)|>0 \tag{6.5}$$

(just substitute (6.2) in (6.1) and use the rationally independence of ω to replace ωt with the generic point θ). For investigations on (6.5) see [M1], [M2], [SZ], [CC1], [CZ], [CC2], [CP].

Theorem 6.3 *Let (h_A, f) be as in Theorem 6.1 and assume that for some finite $I_0 \subset \mathbb{Z}^d$ the equation*

$$\dot{A}^{(0)} = -V^{(0)}_{\varphi^{(0)}}(\varphi^{(0)}), \quad \dot{\varphi}^{(0)} = h_{A_i^{(0)}}(A_i^{(0)}) \qquad \varphi^{(0)} \in \mathbb{T}^{l_0}\ , A^{(0)} \in \mathbb{R}^{l_0}, \quad l_0 \equiv |I_0| \tag{6.6}$$

admits a (maximal) non–degenerate quasi–periodic solution with a (γ, l_0)– Diophantine frequency vector $\omega^{(0)} \in \mathbb{R}^{l_0}$ [see (6.1) $\div$ (6.3) and (5.2)]. Then for any $\varepsilon > 0$ there exist uncountably many maximal almost–periodic solutions, $(A(t), \varphi(t))$, of (2.8) with Diophantine frequencies $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\omega_i = \omega_i^{(0)}$ for $i \in I_0$ ($|\omega_i| \to \infty$ as $|i| \to \infty$) and:

$$\begin{aligned}
&\sup_{\substack{t\in\mathbb{R}\\ p=0,1}} \Big|\frac{d^p}{dt^p}[\varphi_i(t) - \varphi_i^{(0)}(t)]\Big| \le \varepsilon \qquad \text{for} \quad i \in I_0\\
&\sup_{\substack{t\in\mathbb{R}\\ p=0,1}} \Big|\frac{d^p}{dt^p}[\varphi_i(t) - \omega_i t]\Big| \le \varepsilon \qquad \text{for} \quad i \notin I_0\\
&\sup_{t\in\mathbb{R}} |A_i(t) - A_i^{(0)}(t)| \le \varepsilon \qquad \text{for} \quad i \in I_0\\
&\sup_{t\in\mathbb{R}} |A_i(t) - h_{A_i}^{(-1)}(\omega_i)| \le \varepsilon \qquad \text{for} \quad i \notin I_0
\end{aligned} \tag{6.7}$$

Remark 6.1 $l_0 = 1$ is allowed.

Theorem 6.3 is a consequence of the next Proposition and will be given in Section 8. Theorem 6.1, which is the object of the paper, is a consequence of Theorem 6.3 and Lemma A6.2 of Appendix 6. In fact, by Lemma A6.2, we can always find the *(maximal) non–degenerate quasi–periodic solution with a (γ, l_0)– Diophantine frequency vector $\omega^{(0)} \in \mathbb{R}^{l_0}$* of the system (6.6)

7. A (Finite–Dimensional) Average Theorem

In this section we discuss a finite–dimensional problem, which may have some interest by itself, and whose solution will constitute the main step of the proof of Theorem 6.3.

Roughly speaking the question is how to construct quasi–periodic solutions for a Hamiltonian system on $\mathbb{T}^l$ if you know the existence of quasi–periodic solutions for a "subsystem" obtained from the original one by averaging out some variables.

More precisely, let $V : \mathbb{T}^l \to \mathbb{R}$, $(l \ge 2)$, be real–analytic and consider the

"Hamiltonian" equations:

$$\begin{cases} \dot\varphi = h_A \\ \dot A = -V_\varphi(\varphi), \ , \qquad\qquad \varphi \in \mathbb{T}^l \qquad A \in \mathbb{R}^l \end{cases} \tag{7.1}$$

To avoid the introduction of too many indices let's indicate with $\varphi = (\hat\varphi, \varphi_l)$ a point of $\mathbb{T}^l$. Analogously we indicate $A = (\hat A, A_l)$ a point of $\mathbb{R}^l$.
Let also denote with $\overline{V}(\hat\varphi)$, $\hat\varphi \in \mathbb{T}^{l-1}$, the "averaged potential"

$$\overline{V}(\hat\varphi) \equiv \frac{1}{2\pi}\int_0^{2\pi} V(\varphi)d\varphi_l \ , \qquad\qquad \varphi \equiv (\hat\varphi, \varphi_l) \in \mathbb{T}^{l-1}\times\mathbb{T} \tag{7.2}$$

and consider the Hamiltonian equations in $\mathbb{T}^{l-1}$ associated to $\overline{V}$:

$$\begin{cases} \dfrac{d\hat\varphi}{dt} = \dfrac{\partial h}{\partial \hat A} \\ \dfrac{d\hat A}{dt} = -\overline{V}_{\hat\varphi}(\hat\varphi) \qquad\qquad \hat\varphi \in \mathbb{T}^{l-1}, \quad \hat A \in \mathbb{R}^{l-1} \end{cases} \tag{7.3}$$

Now *assume that (7.3) admits a non–degenerate (maximal) quasi–periodic solution*

$$\hat\varphi = \hat\omega t + \hat u(\hat\omega t), \qquad \hat A = \hat v(\hat\omega t), \quad (\hat u, \hat v) : \mathbb{T}^{l-1} \to \mathbb{R}^{l-1}\times\mathbb{R}^{l-1}, \tag{7.4}$$

with $\hat\omega \in \mathbb{R}^{l-1}$ a vector $(\gamma, l-1)$–Diophantine and $(\hat u, \hat v)$ real–analytic. The question is: Can one find quasi–periodic solutions for (7.1) "close" (in some sense) to (7.4)?

The answer is positive *provided* the looked after quasi–periodic solutions have frequencies $\omega \equiv (\hat\omega, \alpha)$ with $\alpha \gg |\hat\omega|$ and suitable.

To formulate a precise and quantitative result we need some notations

Given a real-analytic functions $f(x,y)\colon \mathbb{T}^n \times U \to \mathbb{R}$ where $U = \overset{\circ}{U} \subset \mathbb{R}^m$ real analytic over its domain. f can be extended to a holomorphic function (called also $f(y,x)$) and we suppose that there exist two real numbers r and $\xi \in (0,1)$ such that f is holomorphic on

$$\begin{aligned} &W(r;U)\times\Delta^l_\xi = \\ &= \cup_{y\in U}\{z \in \mathbb{C}^n \mid |z-y| < r\} \times \{x \in \mathbb{C}^l \mid \mathrm{Re}\, x_i \in \mathbb{T},\ |Im\, x_i| < \xi,\ 1 \le i \le l\} \end{aligned} \tag{7.5}$$

and continuous on the closure $\overline{W}\times\overline{\Delta}^l_\xi$. The space $C^\omega(W(r;U)\times\Delta^l_\xi;\mathbb{C})\cap C(\overline{W}(r;U)\times\overline{\Delta}^l_\xi;\mathbb{C})$ made of the holomorphic functions on $W(r;U)\times\Delta^n_\xi$ and continuous on the closure, is a Banach space equipped with the norm

$$\|f\|_{r,\xi} \equiv \|f\|_{W(r;U)\times\Delta^n_\xi} = \sup_{x\in\Delta^n_\xi, y\in W(r;U)} |f(x,y)|$$

For a vector valued function $F(x,y)\colon \mathbb{T}^n\times U\to\mathbb{R}^k$ we suppose that each one of the k components of F admits the holomorphic extension with the same parameters r and ξ. In this case we set

$$\|F\|_{r,\xi}\equiv\|F\|_{W(r;U)\times\Delta^n_\xi}=\sum_{j=1}^{k}\|F_i(x,y)\|_{r,\xi} \tag{7.6}$$

For a matrix valued function $\{M(x,y)\}^k_{i,j=1}\colon \mathbb{T}^n\times U\to\mathbb{C}^{2k}$, we set

$$\begin{cases} \|M\|_{r,\xi}\equiv\|M\|_{W(r;U)\times\Delta^n_\xi}=\sup\limits_{x\in\Delta^n_\xi,y\in W(r;U)}\ \sup\limits_{v\in\mathbb{R}^k;|v|=1}|M(x,y)v|= \\ \qquad\qquad =\sup\limits_{x\in\Delta^n_\xi,y\in W(r;U)}\ \sup\limits_{v\in\mathbb{R}^k;|v|=1}\sum\limits_{i=1}^{k}|\sum\limits_{j=1}^{k}M_{ij}(x,y)v_j| \end{cases} \tag{7.7}$$

If $f\colon\mathbb{T}^l\to\mathbb{R}^l$, $\theta\in\mathbb{T}^l$ is a vector–function, f_θ denotes the matrix $\{\frac{\partial f_i}{\partial\theta_j}\}^l_{i,j=1}$ i–rows j–columns. $(\{\frac{\partial f_i}{\partial\theta_j}\}^T)^l_{i,j=1}=\{\frac{\partial f_j}{\partial\theta_i}\}^l_{i,j=1}$ are the elements of the transposed matrix.
We shall also consider also functions $h\colon\mathbb{R}\to\mathbb{R}$ that are analytic over $\mathbb{R}$ and can be extended to an holomorphic function over $D_\rho=\{z\in\mathbb{C}\mid \operatorname{Re}z\in\mathbb{R},\ |\operatorname{Im}z|<\rho\}$ and continuous over the $\overline{D}_\rho$. We call $C^\omega(D_\rho,\mathbb{C})\cap C(\overline{D}_\rho,\mathbb{C})$ this space–functions.
If a function depends only on the $x's$ variables or the $y's$ variables the norm is coherently defined.

Theorem 6.3 is a corollary of the following theorem which is the main part of this work. In the following $B_1\le B_2\le B_3\le\ldots\le B$ represent universal constants.

Proposition 7.1 *Let $V\colon\mathbb{T}^l\to\mathbb{R}$ be a real-analytic function, $\overline{V}$ as in (7.2) such that both V and $\overline{V}$ are in $C^\omega(\Delta^l_\sigma;\mathbb{C})\cap C(\overline{\Delta}^l_\sigma;\mathbb{C})$ $\|V_\varphi\|_\sigma\le V$. Let $h\colon\mathbb{R}\to\mathbb{R}$ be a real–analytic function with $h\in C^\omega(D_\rho;\mathbb{C})\cap C(\overline{D}_\rho;\mathbb{C})$ satisfying (2.10) and assume that (7.3) admits a (maximal) non–degenerate quasi-periodic solution (v^*,u^*) with a $(\gamma,l-1)$–Diophantine frequency vector $\omega=(\omega_1,\ldots,\omega_{l-1})$. (i) u^* and v^* are in $C^\omega(\Delta^{l-1}_{\xi^*};\mathbb{C}^{l-1})\cap C(\overline{\Delta}^{l-1}_{\xi^*};\mathbb{C}^{l-1})$, $\{\theta+u^*(\theta):\theta\in\Delta^{l-1}_{\xi^*}\}\subset\Delta^{l-1}_\sigma$ ii) $\|\operatorname{Im}v^*_i\|_{\xi^*}\le\rho^*$, $\|v^*_\theta\|_{\xi^*}=N^*$, $\|u^*_\theta\|_{\xi^*}=Q^*<\frac12$*

Fix $\xi,\xi',\rho',\overline{\rho},Q',Q$ such that $\xi'<\xi<\xi^*<1$, $Q^*<Q<Q'<\frac12$, $\rho^*<\overline{\rho}<\rho'$ and let Ω such that

$$\Omega\ge\max\{4\sqrt{l-1}|\omega|,\frac{\mathcal{Z}}{\gamma}\}$$

where

$$\begin{aligned}\mathcal{Z}&=B2^{47l}3^{8l}(l!)^7\frac{(1+\gamma aN^*)^4}{Q'-Q}\frac{(1+\gamma^2Va)^3(1+\gamma\rho a)^2}{\gamma a(\rho-\rho')\ \gamma a(\overline{\rho}-\rho^*)}\cdot\\ &\cdot\frac{a}{a'}\Big[\frac{a}{a'}+\frac{1}{\gamma a'(\rho'-\overline{\rho})}+\frac{1}{\gamma a'N^*}\Big](\xi-\xi')^{-(1+8l)}(\xi^*-\xi)^{-(4l+1)}(\sigma-\xi^*)^{-1},\end{aligned} \tag{7.8}$$

where $B > 1$ is a universal constant. Then for any $\alpha \in \mathcal{A}_l(\omega,\Omega)$ [see (5.4)] there exists a (maximal) quasi-periodic solution $(A(t),\varphi(t)) = (v(\omega' t), \omega' t + u(\omega' t))$ of (7.1) i.e.

$$D_{\omega'} v' + V_\varphi(\theta + u') = 0 \qquad D_{\omega'} u' + \omega - h_A(v') = 0 \tag{7.9}$$

$\omega' = (\omega, \alpha)$; $(v', u') \colon \mathbb{T}^l \to \mathbb{R}^l \times \mathbb{R}^l$, are real–analytic, both u' and v' belong to $C^\omega(\Delta^l_{\xi'};\mathbb{C}^l) \cap C(\overline{\Delta}^l_{\xi'};\mathbb{C}^l)$, $\{\theta + u'(\theta) \,:\, \theta \in \Delta^l_{\xi'}\} \subset \Delta^l_\sigma$
The following bounds hold

$$\begin{aligned}
\max\Big\{ &\max_{p=0,1} \|\partial^p_\theta [u' - (u^*,0)]\|_{\xi'}, \quad \gamma \|D_{\omega'} u' - (D_\omega u^*, 0)\|_{\xi'} \\
&\gamma a \|[v' - (v^*,0)]_\theta\|_{\xi'}, \quad \gamma a \|[v' - (v^*, h^{(-1)}_{A_l}(\alpha))]\|_{\xi'} \Big\} \le \qquad (7.10)\\
&\le \frac{\mathcal{Z}}{\gamma\Omega} \frac{1}{(l!)^2} \frac{(\xi^* - \xi)^{1-p}}{2^{20l} 3^{4l}} (Q' - Q) \frac{\gamma a N^*}{(1 + \gamma a N^*)} (\gamma a(\rho - \overline{\rho}))
\end{aligned}$$

$$\sup_{\theta \in \Delta^l_{\xi'}} |\theta + u'(\theta)| < \sigma, \qquad \|u'_\theta\|_{\xi'} < Q'$$

$$\max_{1 \le i \le l} \{\|\mathrm{Im}\, v'_i\|_{\xi'}\} < \rho', \qquad \frac{1}{2} N^* < \|v'_\theta\|_{\xi'} < \frac{3}{2} N^*$$

Remark 7.1 v' and $(v^*, 0)$ are not close as $\gamma\Omega$ goes to infinity.

The proof is based on a result à la Nash–Moser from KAM theoy which guarantees the existence of a solution of the equations (6.5) provided the frequency vector ω' is Diophantine and provided one can find a "good enough *approximate solution*":

Lemma 7.2 *Let $V \colon \mathbb{T}^l \to \mathbb{R}$, be such that $V \in C^\omega(\Delta_\sigma;\mathbb{C}) \cap C(\overline{\Delta}_\sigma;\mathbb{C})$, $\|V_\theta\|_\sigma \le V$. Let $h \colon \mathbb{R}^l \to \mathbb{R}$ be a real functions such that $h \in C^\omega(D_\rho;\mathbb{C}) \cap C(\overline{D}_\rho;\mathbb{C})$ satisfying (2.10). Let (v,u) be two real functions such that: (i) $u \in C^\omega(\Delta_\xi;\mathbb{C}^l) \cap C(\overline{\Delta}_\xi;\mathbb{C}^l)$, $v \in C^\omega(\Delta_\xi;\mathbb{C}^l) \cap C(\overline{\Delta}_\xi;\mathbb{C}^l)$, $\xi < \sigma$ (ii) $\|v_\theta\|_\xi = N$, $\|u_\theta\|_\xi \le Q < \frac{1}{2}$, $\|\mathrm{Im}\, v_i\|_\xi \le \overline{\rho}$ for all $i = (1,\dots,l)$, $\overline{\rho} < \rho$, $\{\theta + u(\theta) \,|\, \theta \in \Delta_\xi\} \subset \Delta_\sigma$ (iii) fix Q', $\overline{\xi}$, ρ' such that $\frac{1}{2} > Q' > Q$, $\xi' < \xi < \sigma < 1$, $\rho > \rho' > \overline{\rho}$ (iv) ω is a (γ, l) Diophantine vector*
Then if ε_1 and ε_2 defined by

$$D_\omega v + V_\varphi(\theta + u) \doteq \varepsilon_1 \qquad D_\omega u + \omega - h_A(v) \doteq \varepsilon_2 \tag{7.11}$$

satisfy

$$\begin{aligned}
&B_6 2^{35l} 3^{8l} (l!)^4 \frac{(1 + a\gamma N)^4 (1 + \gamma^2 a V)(1 + \gamma\rho a)}{\gamma a (\rho - \rho')(\xi - \xi')^{1+8l}} \cdot \\
&\cdot \Big[\frac{a}{a'} \frac{1}{Q' - Q} + \frac{1}{\gamma a'(\rho' - \overline{\rho})} + \frac{1}{\gamma a' N}\Big] \big(\gamma(\|\varepsilon_2\| + \gamma a \|\varepsilon_1\|)_\xi\big) \le 1
\end{aligned} \tag{7.12}$$

there exists two functions (v', u') *such that the equations*

$$D_\omega v' + V_\varphi(\theta + u') = 0 \qquad D_\omega u' + \omega - h_A(v') = 0 \tag{7.13}$$

are verified with $u' \in C^\omega(\Delta_{\xi'}; \mathbb{C}^l) \cap C(\overline{\Delta}_{\xi'}; \mathbb{C}^l)$, *and* $v' \in C^\omega(\Delta_{\xi'}; \mathbb{C}^l) \cap C(\overline{\Delta}_{\xi'}; \mathbb{C}^l)$; $\|u'_\theta\|_{\xi'} \le Q'$, $\|v'_\theta\|_{\xi'} \le 2N$, $\|\mathrm{Im}\, v'_i\|_{\xi'} < \rho'$.
Furthermore the following bounds hold

$$\max\Big\{ \|u' - u\|_{\xi'}, \quad (\xi - \xi')\|(u' - u)_\theta\|_{\xi'}, \quad \gamma\|D_{\omega'}u' - (D_\omega u, 0)\|_{\xi'}, \\ \gamma a\|v' - v\|_{\xi'}, \quad \gamma a(\xi - \xi')\|(v' - v)_\theta\|_{\xi'} \Big\} \le \\ \le \frac{B_3 2^{12l} 3^{4l} (l!)^2}{(\xi - \xi')^{4l}} \frac{a}{a'} (1 + \gamma a N)^2 \big(\gamma(\|\varepsilon_2\| + \gamma a\|\varepsilon_1\|)\big)_\xi \tag{7.14}$$

This lemma is a "version for first order ODE's" of a result which can be called "configurational KAM theorem" (see [M2], [SZ], [CC1], [CC2], [CP]). When $h(A) = \frac{1}{2}A^2$ (6.5) becomes a second order system and the variables $v(\theta)$ disappears. This makes the calculation easier having just one set of variables (those regarding the $\varphi's$) instead of two sets of variables.

Proof of Proposition 7.1. Using (v^*, u^*) we construct the functions (v, u) of (7.11) such that $\gamma(\|\varepsilon_2\| + \gamma a\|\varepsilon_1\|)_\xi$ is small enough (precisely $O(\frac{1}{\gamma\Omega})$). Then we apply Lemma 7.2 getting (v', u'). At the end of this procedure we have the solution with l d.o.f. of (7.13) being started with the solution of (7.3) and $l-1$ d.o.f.
Let (v^*, u^*) be the solution of

$$D_\omega v^*(\hat\theta) = -\overline{V}_{\hat\varphi}(\hat\theta + u^*(\theta)), \quad \omega + D_\omega u^*(\hat\theta) = h_A(v^*(\hat\theta)), \qquad \min_{\hat\theta \in \mathbb{T}^{l-1}} |\det(\mathrm{Id}_l + u_{\hat\theta})| > 0 \tag{7.15}$$

where $\hat\theta = (\hat\theta_1, \dots, \hat\theta_{l-1}) \in \mathbb{T}^{l-1}$, $u^*(\hat\theta) = (u_1^*(\hat\theta), \dots, u_{l-1}^*(\hat\theta)) \colon \mathbb{T}^{l-1} \to \mathbb{R}^{l-1}$, $\overline{V}_{\hat\varphi} = (\overline{V}_{\hat\varphi_1}, \dots, \overline{V}_{\hat\varphi_{l-1}})$, $\overline{V}_{\hat\varphi_j} = \frac{1}{2\pi}\int_{\mathbb{T}} V_{\hat\varphi_j}(\hat\varphi_1, \dots, \hat\varphi_{l-1}, \varphi_l) d\varphi_l$. $\hat\theta \in \Delta_{\xi^*}^{l-1}$, ω is (γ, τ) Diophantine vector.
Let's consider the functions

$$v = -D_{\omega'}^{-1} V_\varphi(\hat\theta + u^*, \theta_l) + b, \qquad u = D_{\omega'}^{-1}(h_A(v) - \omega') \tag{7.16}$$

b is close to $h_{A_l}^{(-1)}(\alpha) \in \mathbb{R}$ and such that $\frac{1}{(2\pi)^l}\int_{\mathbb{T}^l} h_A(v) = \omega$ and $\omega' = (\omega, \alpha)$, $\alpha \in \mathcal{A}(\omega, \Omega)$ (the existence of b is obtained through the the implicit function theorem and will be given in the Appendix 2).
The following bounds hold see Appendix 3 (just write the differences and use the Cauchy theorem in order to estimate the derivatives)

$$\|u - (u^*, 0)\|_\xi \le \frac{V\gamma^2 a}{\gamma\Omega} \frac{a}{a'} \frac{2^{12l+7}(l!)^2}{(\xi^* - \xi)^{4l}} \tag{7.17}$$

$$\|(u-(u^*,0))_\theta\|_\xi \le \frac{V\gamma^2 a}{\gamma\Omega}\frac{a}{a'}\frac{2^{11l+10}(l!)^2}{(\xi^*-\xi)^{4l+1}} \tag{7.18}$$

$$\gamma a\|v-(v^*,h_{A_l}^{(-1)}(\alpha))\|_\xi \le \frac{\gamma^2 Va}{\gamma\Omega}\frac{a}{a'}\frac{2^{6l}l!}{(\xi^*-\xi)^{2l}} \tag{7.19}$$

$$\max\{\gamma\|D_{\omega'}u-(D_\omega u,0)\|_{\xi'}\quad \gamma a\|(v-(v^*,0))_\theta\|_\xi\} \le \frac{\gamma^2 Va}{\gamma\Omega}\frac{2^{5l+1}l!}{(\xi^*-\xi)^{2l+1}} \tag{7.20}$$

$\theta+u=\theta+(u^*,0)+(u-(u^*,0))$ and $\theta\in\Delta^l_\xi$; by (7.8) it is $\|u-(u^*,0)\|_\xi\le(\xi^*-\xi)$ (see the exponent $4l+1$ in (7.8) and $4l$ in (7.17)) so that $\sup_{\theta\in\Delta_{xi}}|\mathrm{Im}\,(\theta+u(\theta))|\le \xi+\|(u^*,0)\|_{\xi^*}+\xi^*-\xi=\xi^*+\|(u^*,0)\|_{\xi^*}$ then $\{\theta+u\ |\theta\in\Delta_\xi\}\subset\Delta_\sigma$ and makes sense $V_\varphi(\theta+u)$;

$$\varepsilon_2\equiv 0 \qquad\qquad \varepsilon_1=D_{\omega'}v+V_\varphi(\theta+u)$$

by (14.1)

$$\|\varepsilon_1\|_\xi \le \frac{V}{\sigma-\xi^*}\frac{\gamma^2 Va}{\gamma\Omega}\frac{a}{a'}\frac{2^{12l+7}(l!)^2}{(\xi^*-\xi)^{4l}} \tag{7.21}$$

The details are explained in Appendix 3. In order to have here an idea we write

$$\begin{aligned}\varepsilon_1(\theta)&=D_{\omega'}v+V_\varphi(\theta+u)=-V_\varphi(\hat\theta+u^*,\theta_l)+V_\varphi(\hat\theta+\hat u,\theta_l+u_l)=\\&=\int_0^1 d\lambda\Big(V_{\varphi\hat\varphi}(\hat\theta+\lambda\hat u+(1-\lambda)u^*,\theta_l+\lambda u_l)(\hat u-u^*)+\\&\qquad\qquad+V_{\varphi\varphi_l}(\hat\theta+\lambda\hat u+(1-\lambda)u^*,\theta_l+\lambda u_l)u_l\Big)\end{aligned} \tag{7.22}$$

The second derivatives of V are bounded by $\frac{V}{\sigma-\xi^*}$ while the rest of (7.21) is due to (7.17). The bound (7.21) is substituted in (7.12) and if $\gamma\Omega$ is big enough Lemma 7.2 gives the solution (v',u') of the Proposition. (v',u') are in $C^\omega(\Delta_{\xi'};\mathbb{C}^l)\cap C(\overline{\Delta}_{\xi'};\mathbb{C}^l)$ as so do (v',u') in the Lemma.
Moreover we have

$$\begin{aligned}\|v_\theta\|_\xi&=N\le\|(v^*,0)_\theta\|_\xi+\|(v-(v^*,0))_\theta\|_\xi\le N^*+\frac{V}{\Omega}\frac{2^{5l+1}l!}{(\xi^*-\xi)^{2l+1}}\le\\&\le N^*+\frac{1}{2^l}N^*\end{aligned}$$

((7.8) has been used)

$$\begin{aligned}&\|(u'-(u^*,0))_\theta\|_{\xi'}\le\|(u'-u)_\theta\|_{\xi'}+\|(u-(u^*,0))_\theta\|_\xi\le\\&\le\frac{V\gamma^2a}{\gamma\Omega}\frac{a}{a'}\frac{2^{11l+10}(l!)^2}{(\xi^*-\xi)^{4l+1}}+\\&+\frac{B_3 2^{12l}3^{4l}(l!)^2}{(\xi-\xi')^{1+4l}}\frac{a}{a'}(1+\gamma aN)^2\frac{\gamma^2aV}{\sigma-\xi^*}\frac{V\gamma^2a}{\gamma\Omega}\frac{a}{a'}\frac{2^{12l+7}(l!)^2}{(\xi^*-\xi)^{4l}}\le\\&\le\frac{\mathcal{Z}}{\gamma\Omega}\frac{1}{(l!)^2}\frac{(\xi^*-\xi)^{1-p}}{2^{20l}3^{4l}}(Q'-Q)\frac{\gamma aN^*}{(1+\gamma aN^*)}(\gamma a(\rho-\overline{\rho}))\end{aligned}$$

The other bounds of (7.10) are analogous to the previous one. From (7.10) it follows

$$\|u'_\theta\|_{\xi'} \le \|(u' - (u^*,0))_\theta\|_{\xi'} + \|(u^*,0)_\theta\|_{\xi'} \le (Q'-Q) + Q^* \le Q + (Q'-Q) = Q'$$

For $\theta \in \Delta^l_{\xi'}$, we have

$$\begin{aligned}|\theta + u'| &= |\theta + (u^*,0) + (u' - (u^*,0))| \le \xi' + \|(u^*,0)\|_{\xi^*} + (\xi^* - \xi) \le \\ &\le \xi' + \|(u^*,0)\|_{\xi^*} + \xi^* - \xi' = \xi^* + \|(u^*,0)\|_{\xi^*} < \sigma\end{aligned}$$

This bound let the functions (v', u') be "new functions (v^*, u^*)" for the next problem obtained "adding one d.o.f." (see Section 7).
Moreover $\|(v - (v^*,0))_\theta\|_\xi \le \frac{1}{2^l} N^*$ and then $\frac{1}{2}N^* \le \|v'_\theta\|_{\xi'} \le \frac{3}{2}N^*$ easily follows.
(7.8) and (7.10) imply that $\max_{1\le i\le l}\{\|\mathrm{Im}\, v'_i\|_\xi\} \le \rho$ as

$$\mathrm{Im}\left(|(v'_i - v^*_i)| + |v^*_i|\right) \le \rho^* + (\rho' - \overline{\rho}) \le \rho^* + (\rho' - \rho^*) = \rho'$$

The insertion of (7.21) into (7.12) gives a bound implied by (7.8).∎
It is clear that, provided a "good choice" in (7.16), the main point is Lemma 7.2. Its proof will be given in Appendix 3.

8. Proofs of Section 6

Proof of Theorem 6.3 The idea is to use iteratively the result of Section 7 to construct quasi-periodic solutions for larger and larger subsystems and to obtain, in the limit, infinite-dimensional quasi-periodic solutions. By hypoteses we are given a real-analytic quasi-periodic solution $(A^{(0)}, \varphi^{(0)}) = (v^{(0)}(\omega^{(0)}t), \omega^{(0)}t + v^{(0)}(\omega^{(0)}t)$ of the system

$$\dot\varphi^{(0)} = h_{A^{(0)}} A^{(0)} \qquad \dot A^{(0)} = -V^{(I_0)}_{\varphi^{(0)}}(\varphi^{(0)}), \qquad \varphi^{(0)} \in \mathbb{T}^{l_0}, \quad A^{(0)} \in \mathbb{R}^{l_0} \tag{8.1}$$

where $\omega^{(0)}$ is a $(\gamma, l_0)-$Diophantine vector. The functions $(A^{(0)}, \varphi^{(0)})$ are analytic in $\Delta^{l_0}_{\xi_{l_0}}$ and $i = 1, \dots, i = l_0$ correspond to a set I_0 of sites of $\mathbb{Z}^d$ whose cardinality is l_0. In other words we choose a subset of $\mathbb{Z}^d$ with l_0 sites and rename them $1, 2, \dots l_0$. Let σ be the analyticity paremeter of the g-gradient f (see Definition 3.2). We suppose that

$$\{\theta^{(0)} + u^{(0)}(\theta^{(0)}) : \theta^{(0)} \in \Delta^{l_0}_{\xi_{l_0}}\} \subset \Delta^{l_0}_{\overline{\xi}_{\varsigma_{l_0}}}$$

$$\{v^{(0)}(\theta^{(0)}) : \theta^{(0)} \in \Delta^{l_0}_{\xi_{l_0}}\} \subset D_{\rho_{l_0}}, \quad \|v^{(0)}\|_{\xi_{l_0}} \le N^{(0)}$$

In the following construction there is quite a bit of freedom (whence comes the uncountability of the solution) as the extra frequencies α_i $i \notin I_0$ will be arbitrarily chosen in sets of positive measure. There is also some more (less substantial) freedom in the order of "invading" $\mathbb{Z}^d$. More precisely, fix (arbitrarily) a one-to-one map, $k \in \mathbb{Z}_+ \to j_k \in \mathbb{Z}^d \backslash I_0$, and set, recursively, for $k \geq 1$,

$$I_k \equiv I_{k-1} \cup \{j_k\} \tag{8.2}$$

so that

$$I_{k-1} \subset I_k, \qquad |I_k| = l_0 + k \doteq l_k, \qquad I_k \nearrow \mathbb{Z}^d \tag{8.3}$$

We shall use the following notations. Denote by $(\varphi_1, ..., \varphi_{l_0})$ coordinates for $\bigotimes_{i \in I_0} \mathbb{T}_i$ and by φ_{l_k} the coordinate associated to $\mathbb{T}_{j_k}$. Then set, for $k \geq 0$:

$$V^{(k)} \equiv V^{(I_k)} \ , \quad \varphi^{(k)} \equiv (\varphi_1, ..., \varphi_{l_k}) \in \mathbb{T}^{l_k} \tag{8.4}$$

The $(k+1)^{\text{th}}$ step will consist in constructing quasi–periodic solutions with $(u^{(k+1)}, v^{(k+1)}) : \mathbb{T}^{l_{k+1}} \to \mathbb{R}^{l_{k+1}} \times \mathbb{R}^{l_{k+1}}$ and frequency $\omega^{(k+1)} \in \mathbb{R}^{l_{k+1}}$ of the form $(\omega^{(k)}, \alpha_k)$ with $\alpha_k \in \mathcal{A}_{l_{k+1}}(\omega^{(k)}, \Omega_k)$ for suitable $\Omega_k \gg \Omega_{k-1}$.

Notice that the functions $V^{(k)}$ are defined up to an additive constant which we shall choose so that, for all $k \geq 0$:

$$V^{(k)} = \frac{1}{2\pi} \int_0^{2\pi} V^{(k+1)} dx_{j_{k+1}} \equiv \overline{V}^{(k+1)} : \ \mathbb{T}^{l_k} \to \mathbb{R} \tag{8.5}$$

Now imagine that for any integer k we are given $\rho^{(k)} < \overline{\rho}^{(k)} < \rho^{(k+1)} < \rho$, $\rho^{(k)} \nearrow \rho$, $\frac{1}{2}\xi_{l_0} < \xi^{(k+1)} < \overline{\xi}^{(k)} < \xi^{(k)} < 1$, $\xi^{(k)} \searrow \frac{1}{2}\xi_{l_0}$, $\quad Q^{(k)} < \overline{Q}^{(k)} < Q^{(k+1)} < \frac{1}{2}$

$$\begin{aligned} \mathcal{Z}_k = B 2^{47 l_{k+1}} 3^{8 l_{k+1}} (l_{k+1}!)^7 \frac{(1+\gamma a N^{(k)})^4}{Q^{(k+1)} - \overline{Q}^{(k)}} \frac{(1+\gamma a V^{(k+1)})^3 (1+\gamma a \rho)^2}{\gamma a(\rho - \rho^{(k+1)}) \gamma a(\overline{\rho}^{(k)} - \rho^{(k)})} \cdot \\ \cdot \frac{a}{a'} \Big[\frac{a}{a'} + \frac{1}{\gamma a' (\rho^{(k+1)} - \overline{\rho}^{(k)})} + \frac{1}{\gamma a' N^{(k)}} \Big] \cdot \\ \cdot (\overline{\xi}^{(k)} - \xi^{(k+1)})^{-(1+8l_{k+1})} (\xi^{(k)} - \overline{\xi}^{(k)})^{-(1+4l_{k+1})} (\sigma - \xi^{(k)})^{-1} \end{aligned} \tag{8.6}$$

Imagine that $(v^{(k)}(\theta^{(k)}), u^{(k)}(\theta^{(k)})) \in C^\omega(\Delta^{l_k}_{\xi_{l_k}}; \mathbb{C}^{l_k}) \cap C(\overline{\Delta}^{l_k}_{\xi_{l_k}}; \mathbb{C}^{l_k})$ are given togheter with $\omega^{(k)} \in \mathbb{R}^{l_k}$; $\theta^{(k)} + u^{(k)} \in \Delta^{l_k}_\sigma$, $v^{(k)}(\theta^{(k)}) \in D_{\rho^{(k)}}$, $\|v^{(k)}_{\theta^{(k)}}\|_{\xi_{l_k}} \leq N^{(k)}$, $\|u^{(k)}_{\theta^{(k)}}\|_{\xi_{l_k}} \leq Q^{(k)}$, $\omega^{(k)}$ is (γ, l_k)–Diophantine. Now for a given ε fix Ω_k such that

$$\Omega_k \equiv \max\{4\sqrt{l_k}|\omega^{(k)}|, \ \frac{\mathcal{Z}_k}{\gamma\varepsilon}\}$$

and choose $\alpha_k \in \mathbb{R}$ such that

$$\alpha_k \in \hat{\mathcal{A}}_{l_{k+1}} \equiv [\Omega_k, \Omega_k + K|\omega^{(k)}|] \cap \mathcal{A}_{l_{k+1}}(\omega^{(k)}, \Omega_k)$$

(recall from (5.5) that $\mathcal{A}_{l_{k+1}}$ has positive Lebesgue measure).
Applying Proposition 7.1 we get the solution corresponding to one more d.o.f. namely $(v^{(k+1)}, u^{(k+1)})$ as functions of $\theta^{(k+1)}$ and the following bounds hold

$$\begin{aligned} &\max\Big\{ \|u^{(k+1)} - (u^{(k)}, 0)\|_{\xi_{l_{k+1}}} , \quad \|(u^{(k+1)} - (u^{(k)}, 0))_{\theta^{(k+1)}}\|_{\xi_{l_{k+1}}}, \\ &\gamma a \|v^{(k+1)} - (v^{(k)}, h^{(-1)}_{A_l}(\alpha_{k+1}))\|_{\xi_{k+1}}, \quad \gamma a\|(v^{(k+1)} - (v^{(k)}, 0))_{\theta^{(k+1)}}\|_{\xi_{k+1}}, \\ &\qquad\qquad \gamma \|D_{\omega^{(k+1)}} u^{(k+1)} - (D_{\omega^{(k)}} u^{(k)}, 0)\|_{\xi_{l_{k+1}}} \Big\} \le \\ &\le 2^{-20(l_0+k+1)}((l_0+k+1)!)^{-2} \end{aligned} \tag{8.7}$$

For $\theta^{(k)} \in \Delta^{l_k}_{\xi_{l_k}}$ we have the sequences ($iii)$ and $iv)$ are matrices)

$i)$ $u^{(k)} = (u^{(k)}_1, u^{(k)}_2, \dots, u^{(k)}_k, 0, 0, \dots)$

$ii)$ $\overline{v}^{(k)} = (v^{(k)}_1, v^{(k)}_2, \dots, v^{(k)}_k, h^{(-1)}_{A_{k+1}}(\alpha_{k+1}), h^{(-1)}_{A_{k+2}}(\alpha_{k+2},), \dots)$

$iii)$ $u^{(k)}_{\theta^{(k)}} = ((u^{(k)}_1)_{\theta^{(k)}}, (u^{(k)}_2)_{\theta^{(k)}}, \dots, (u^{(k)}_k)_{\theta^{(k)}}, 0, 0, \dots)$

$iv)$ $v^{(k)}_{\theta^{(k)}} = ((v^{(k)}_1)_{\theta^{(k)}}, (v^{(k)}_2)_{\theta^{(k)}}, \dots, (v^{(k)}_k)_{\theta^{(k)}}, 0, 0, \dots)$

$v)$ $D_{\omega^{(k)}} u^{(k)} = (D_{\omega^{(k)}} u^{(k)}_1, D_{\omega^{(k)}} u^{(k)}_2, \dots, D_{\omega^{(k)}} u^{(k)}_k, 0 \dots)$

$vi)$ $v^{(k)} = (v^{(k)}_1, v^{(k)}_2, \dots, v^{(k)}_k, 0, 0, \dots)$

$vii)$ $D_{\omega^{(k)}} v^{(k)} = (D_{\omega^{(k)}} v^{(k)}_1, D_{\omega^{(k)}} v^{(k)}_2, \dots, D_{\omega^{(k)}} v^{(k)}_k, 0 \dots)$

By (8.7), if $P^{(k)}$ denotes the generic element of $i) - v)$

$$\lim_{k\to+\infty} \sup_{\theta^{(k)} \in \Delta^{l_k}_{\xi_{l_k}}} P^{(k)}(\theta^{(k)})$$

exists uniformly in $\theta \in \frac{1}{2}\Delta_{l_0}$ $(\Delta_{l_0} \ni \mathcal{T})$ defining an analytic function.
As for $vi)$ we write (see (2.7))

$$\begin{aligned} &\lambda_w\big(v^{(k+N)},\ v^{(k)}\big) \le \sum_{j=k+1}^{k+N} \lambda_w\big(v^{(j)},\ v^{(j-1)}\big) \le \\ &\le \sum_{j=k+1}^{k+N} \Big(\sum_{i=1}^{j-1} w_i \arctan|v^{(j)}_i - v^{(j-1)}_i| + w_j \arctan|v^{(j)}_j| \Big) \le \\ &\le \sum_{j=k+1}^{k+N} \Big\{ (\sum_{i=1}^{\infty} w_i) \|v^{(j)} - (v^{(j-1)}, h^{(-1)}_{A_{l_j}}(\alpha_j))\|_{\xi_{l_j}} + w_j \frac{\pi}{2} \Big\} \le \\ &\le \sum_{j=k+1}^{k+N} (\sum_{i=1}^{\infty} w_i) \frac{1}{2^{l_0+j}} + \frac{\pi}{2} \sum_{j=k+1}^{k+N} w_j \end{aligned}$$

It follows that $\{v^{(k)}\}$ is a Cauchy sequence in $(\mathcal{R}_w, \lambda_w)$ which is complete and then its convergence is guaranteed ((8.7) has been used in the last $\le$).
As for $vii)$ its convergence is induced by the uniform convergence of $f_i^{(k)}(\varphi_1, \dots, \varphi_{l_k})$ to $f_i(\varphi)$ on $\mathcal{T}_w$ (sae Section 3 and (8.5)) [(1)]). In fact $D_{\omega^{(k)}} v_j^{(k)} = -V^{(k)}_{\varphi_j^{(k)}}(\theta^{(k)} + u^{(k)})$ converges on $\mathcal{T}_w$ as the same does $V^{(k)}_{\varphi_j^{(k)}}(\varphi^{(k)})$ and $\theta^{(k)} + u^{(k)}$ spans a dense subset of $\mathbb{T}^k$.
Finally define, for any $\theta \in \mathcal{T}_w$:

$$A(t) \equiv A(t;\theta) = v(\omega t;\theta), \qquad \varphi(t) \equiv \varphi(t;\theta) \equiv [\theta + \omega t + u([\theta + \omega t])] \tag{8.8}$$

where

$$\omega_i \equiv \begin{cases} \omega_i^{(0)} \ , & i \in I_0 \\ \omega_{j_k} \equiv \alpha_k \ , & i \equiv j_k \notin I_0 \end{cases} \tag{8.9}$$

From the above construction the following facts immediately follow:

$$\text{(i)}: \quad \frac{d}{dt}\varphi_i(t) = \omega_i + D_\omega u_i([\theta + \omega t]) \ , \qquad \frac{d}{dt}A_i(t) = D_\omega v_i([\theta + \omega t]) \tag{8.10}$$

(ii): $(A(t;\theta), \varphi(t;\theta))$ is solution of (2.8); (iii): denoting, for $i \in I_0$, $A_i^{(0)}(t;\theta) = v_i^{(0)}(\omega^{(0)}t + \theta^{(0)})$, $\quad \varphi_i^{(0)}(t;\theta) \equiv \theta_i^{(0)} + \omega_i^{(0)}t + u_i^{(0)}(\theta^{(0)} + \omega^{(0)}t)$ see (7.10),

$$\begin{aligned}
&\sup_{\substack{t\in\mathbb{R}\\ p=0,1}} \left|\frac{d^p}{dt^p}\big(\varphi_i(t;\theta) - \varphi_i^{(0)}(t;\theta)\big)\right| \le \\
&\le \max\Big\{\sum_{k=0}^{\infty} \|u^{(k+1)} - u^{(k)}\|_{\frac12 \xi_{l_0}}, \quad \sum_{k=0}^{\infty} \gamma \|D_{\omega^{(k+1)}} u^{(k+1)} - D_{\omega^{(k)}} u^{(k)}\|_{\frac12 \xi_{l_0}}\Big\} \le \\
&\le \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{l_0+k+1}} \le \varepsilon \qquad i \in I_0
\end{aligned} \tag{8.11}$$

$$\begin{aligned}
&\sup_{\substack{t\in\mathbb{R}\\ p=0,1}} \left|\frac{d^p}{dt^p}\big(\varphi_i(t;\theta) - \omega_i t\big)\right| \le \\
&\le \max\Big\{\sum_{k=0}^{\infty} \|u^{(k+1)} - u^{(k)}\|_{\xi_\infty}, \quad \sum_{k=0}^{\infty} \gamma \|D_{\omega^{(k+1)}} u^{(k+1)} - D_{\omega^{(k)}} u^{(k)}\|_{\frac12 \xi_{l_0}}\Big\} \le \\
&\le \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{l_0+k+1}} \le \varepsilon \qquad i \notin I_0
\end{aligned} \tag{8.12}$$

[(1)] The uniform convergence is restricted to our examples (2.14) and (2.15). In fact for a generic continuous measurable functions on $\mathcal{T}_w$ the convergence is just everywhere ([Je] p.276

$$\sup_{t\in\mathbb{R}} |A_i(t;\theta) - A_i^{(0)}(t;\theta)| \le \sum_{k=0}^{\infty} \|v^{(k+1)} - v^{(k)}\|_{\frac{1}{2}\xi_{l_0}} \le$$

$$\le \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{l_0+k+1}} \le \varepsilon \qquad i \in I_0$$

$$\sup_{t\in\mathbb{R}} |A_i(t;\theta) - h_{A_i}^{(-1)}(\omega_i)| \le \sum_{k=0}^{\infty} \|v^{(k+1)} - v^{(k)}\|_{\frac{1}{2}\xi_{l_0}} \le$$

$$\le \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{l_0+k+1}} \le \varepsilon \qquad i \notin I_0 \blacksquare$$

The proof of Theorem 6.3 is completed
Choosing $\xi^{(k)} - \xi^{(k+1)} \sim \rho^{(k+1)} - \rho^{(k)} \sim Q^{(k+1)} - Q^{(k)} \sim \frac{1}{k \ln^2 k}$ it results $\gamma\Omega_k \sim \frac{1}{\left((l_0+k)!\right)^{23}}$

The regularity properties of the almost–periodic solutions, $(A(t), \varphi(t))$ constructed above are much stronger than just being continuous and having $C^1(\mathbb{R})$ components $(A_i(t), \varphi_i(t))$ (such properties are best reflected by the approximations via the real–analytic functions $(v^{(k)}, u^{(k)})$. Notice however that $(A_i(t), \varphi_i(t))$ *is not* C^2. In fact the second derivative respect to time would involve the second power of $\omega^{(k)}$.

Appendix 1 The existence of the dynamic

Proof of Proposition 2.2 Let's consider $C([-T,T], \mathcal{T}_w)$, the space of continuous functions from $[-T,T] \subset \mathcal{R}$ to $\mathcal{T}_w$ and $C([-T,T], \mathcal{R}_w)$, the space of continuous functions from $[-T,T]$ to $\mathcal{R}_w$. Both of them are complete with metric respectively given by $\rho_{w,T}(\varphi, \psi) = \sup_{|t|\le T} \rho_w\big(\varphi(t), \psi(t)\big)$ and $\lambda_{w,T}(A,B) = \sup_{|t|\le T} \lambda_w\big(A(t), B(t)\big)$. Moreover define the subset of $C([-T,T], \mathcal{R}_w) \times C([-T,T], \mathcal{T}_w)$

$$\mathcal{S}^T_{A_o} \times \mathcal{S}^T_{\varphi^o} = \{(B,\psi) \in C([-T,T], \mathcal{R}_w) \times C([-T,T], \mathcal{T}_w) \ :$$
$$: \rho_{w,T}(\psi, \varphi^o) + \lambda_{w,T}(B, A^o) \le 2\pi \sum_{i\in\mathbb{Z}^d} w_i$$

$\sum_{i\in\mathbb{Z}^d} w_i \doteq s_w$. $\mathcal{S}^T_{A^o} \times \mathcal{S}^T_{\varphi^o}$ is a complete metric space respect to $\rho_{w,T} + \lambda_{w,T}$.
We define the operator $F\colon \mathcal{S}^T_{A^o} \times \mathcal{S}^T_{\varphi^o} \to \mathcal{S}^T_{A^o} \times \mathcal{S}^T_{\varphi^o}$ as

$$\begin{aligned}
&\big(F(A,\varphi)\big)(t) = \{F(A,\varphi)\}_i(t) = \\
&= \Big(A_i^o + \int_0^t d\tau\ f_i(\varphi(\tau;\ A^o, \varphi^o)),\ \varphi_i^o + \int_0^t d\tau\ h_{A_i}(A_i(\tau;\ A^o, \varphi^o))\Big) \qquad (9.1)\\
&\doteq (F^{(A)}, F^{(\varphi)})(t) = \big(\{(F^{(A)})\}_i, \{(F^{(\varphi)})\}_i\big)(t)
\end{aligned}$$

$$\rho_{w,T}(F^{(\varphi)},\varphi^o)+\lambda_{w,T}(F^{(A)},A^o)=\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\rho\Big(\int_0^t d\tau\ h_{A_i}(A_i(\tau;\,A^o,\varphi^o)),\,0\Big)w_i+$$
$$+\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\arctan\Big|\int_0^t d\tau\ f_i(\varphi(\tau;\,A^o,\varphi^o))\Big|w_i\le s_w(\pi+\frac{\pi}{2}) \tag{9.2}$$

so F maps $\mathcal{S}^T_{A^o}\times\mathcal{S}^T_{\varphi^o}$ into itself.

$$\rho_{w,T}(F^{(\varphi)},F^{(\psi)})+\lambda_{w,T}(F^{(A)},F^{(B)})\le(\rho_{w,T}(\varphi,\psi)+\lambda_{w,T}(A,B))TL(1+\pi^2) \tag{9.3}$$

(the proof is postponed). If T is such that $TL(1+\pi^2)<1$ the operator F is a contraction on $\mathcal{S}^T_{A^o}\times\mathcal{S}^T_{\varphi^o}$ and there exists a unique solution $(A(t),\varphi(t))$ which solves (2.9) and is defined for $|t|\le T$.

In order to prove the existence for all $t\in\mathbb{R}$ we suppose that there is not a solution defined for $t\in[-T',T']$ with $T'>T$, say $(B(t),\psi(t))$, such that $(B(t),\psi(t))|_{t\in[-T,T]}=(A(t),\varphi(t))$. In other words the solution (A,φ) cannot be extended at right and left. We investigate just what happen "at right" (the "left" case is very similar). Let's consider $\limsup_{t\to T^-}\lambda_w(A(t),A^o)\doteq\eta$ and $\limsup_{t\to T^-}\rho_w(\varphi(t),\varphi^o)\doteq\xi$; observing that $\xi\le\frac{\pi}{2}s_w$, $\eta\le\pi s_w$ it follows that there exists an open interval $(T-\delta,T)$ such that $\sup_{t\in(T-\delta,T)}\rho_w(\varphi(t),\varphi^o)\le\pi s_w$ and $\sup_{t\in(T-\delta,T)}\lambda_w(A(t),A^o)\le\frac{\pi}{2}s_w$. Without loss of generality we can take $0<\delta<T$. Now take the following system for $t_o\in(T-\delta,T)$

$$\begin{cases}\dfrac{d\tilde\varphi_i}{dt}(t)=h_{\tilde A_i}(\tilde A_i(t)), & \tilde\varphi_i(t_o)=\varphi_i(t_o) \qquad i\in\mathbb{Z}^d\\[2mm] \dfrac{d\tilde A_i}{dt}(t)=f_i(\tilde\varphi(t))\ , & \tilde A_i(t_o)=A_i(t_o)\end{cases} \tag{9.4}$$

Repeating the same computations for (9.2) and (9.5) we get the solution $(\tilde A(t),\tilde\varphi(t))$ which is defined for $[t_o-T,t_o+T]$. By uniqueness, it coincides with $(A(t),\varphi(t))$ for $t\in[t_o-T,t_o+T]\cap[-T,T]$ but this would mean that $(A(t),\varphi(t))$ can be extended beyond the value $t=T$ contradicting the hypotheses.

The proof of (9.3) proceeds as follows

$$
\begin{aligned}
&\rho_{w,T}(F^{(\varphi)},F^{(\psi)})+\lambda_{w,T}(F^{(A)},F^{(B)})=\\
&=\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\rho\Big(\int_0^t d\tau\;h_{A_i}(A_i(\tau;\,A^o,\varphi^o)),\;\int_0^t d\tau\;h_{A_i}(B_i(\tau;\,A^o,\varphi^o))\Big)w_i+\\
&+\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}w_i\arctan\Big|\int_0^t d\tau\;f_i(\varphi(\tau;\,A^o,\varphi^o))\;-\;\int_0^t d\tau\;f_i(\psi(\tau;\,A^o,\varphi^o))\Big|\le\\
&(1+\pi^2)\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\arctan\Big|\int_0^t d\tau\;h_{A_i}(A_i(\tau;\,A^o,\varphi^o))-\\
&-\int_0^t d\tau\;h_{A_i}(B_i(\tau;\,A^o,\varphi^o))\Big|w_i+\\
&+\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\Big|\int_0^t d\tau\;f_i(\varphi(\tau;\,A^o,\varphi^o))\;-\;\int_0^t d\tau\;f_i(\psi(\tau;\,A^o,\varphi^o))\Big|w_i\le\\
&\le(1+\pi^2)\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\arctan\int_0^t\Big|d\tau\;h_{A_i}(A_i(\tau;\,A^o,\varphi^o))\;-\;h_{A_i}(B_i(\tau;\,A^o,\varphi^o))\Big|w_i+\\
&+\sup_{|t|\le T}\sum_{i\in\mathbb{Z}^d}\Big|\int_0^t d\tau\;f_i(\varphi(\tau;\,A^o,\varphi^o))\;-\;\int_0^t d\tau\;f_i(\psi(\tau;\,A^o,\varphi^o))\Big|w_i\le\\
&\le(1+\pi^2)\sup_{|t|\le T}\int_0^t d\tau\sum_{i\in\mathbb{Z}^d}w_i\arctan\Big|h_{A_i}(A_i(\tau;\,A^o,\varphi^o))-h_{A_i}(B_i(\tau;\,A^o,\varphi^o))\Big|+\\
&\sup_{|t|\le T}\int_0^t d\tau\sum_{i\in\mathbb{Z}^d}w_i\Big|f_i(\varphi(\tau;\,A^o,\varphi^o))-f_i(\psi(\tau;\,A^o,\varphi^o))\Big|\le\\
&\le(1+\pi^2)\sup_{|t|\le T}\int_0^t d\tau\,L\,(\rho_w(\varphi(t),\psi(t))+\lambda_w(A(t),B(t)))\le\\
&\le(\rho_{w,T}(\varphi,\psi)+\lambda_{w,T}(A,B))TL(1+\pi^2)
\end{aligned}
\tag{9.5}
$$

where $x\in[0,\pi]\Rightarrow x\le(1+\pi^2)\arctan x$ and $\arctan x\le x$ for $x>0$ implies the first $\le$. The second $\le$ is due to a simple property of the Riemann integral. The third $\le$ is due to $\arctan(x+y)\le\arctan x+\arctan y$ for $x,y>0$ and to the uniform convergence of the series $\sum_{i\in\mathbb{Z}^d}w_i\arctan(f_i)$ for any set of functions $\{f_i\}$. The fourth $\le$ is due to the Lipschitz property of the vector field (h_A,f).

Trying to show the Lipschitz property for h_A we are lead to

$$
\begin{aligned}
&\lambda_w(h_A(A),h_A(B)) = \sum_i w_i \arctan|h_{A_i}(A_i) - h_{A_i}(B_i)| \le \\
&= \sum_{i\in\mathbb{Z}^d} w_i \arctan\Big|\int_0^1 dt h_{A_iA_i}(tA_i + (1-t)B_i)(A_i - B_i)\Big| \le \\
&\le \sum_{i\in\mathbb{Z}^d} w_i \arctan(\sup_{A_i\in\mathbb{R}} |h_{A_iA_i}||A_i - B_i|) \le \\
&\le \sum_{i\in\mathbb{Z}^d} w_i \max(1, \sup_{A_i\in\mathbb{R}} |h_{A_iA_i}|) \arctan|A_i - B_i|
\end{aligned}
\tag{9.6}
$$

and the sum might be divergent if each A_i is finite but $|A_i| \to \infty$ as $|i| \to \infty$ (exactly what happens; see (7.8)). Note that this problem would be absent in a finite–dimensional problem.

Appendix 2. Existence of b in (7.16)

Let's consider the system

$$
D_\omega v^* = -\overline{V}_{\hat\varphi}(\hat\theta + u^*(\hat\theta)), \quad \omega + D_\omega u^*(\hat\theta) = h_{\hat A}(v^*(\hat\theta)), \qquad \min_{\hat\theta\in\mathbb{T}^{l-1}} |\det(\mathrm{Id}_l + u^*_{\hat\theta})| > 0
\tag{10.1}
$$

and define

$$
v = -D^{-1}_{\omega'} V_\varphi(\hat\theta + u^*, \theta_l) + b, \qquad u = D^{-1}_{\omega'}(h_A(v) - \omega')
\tag{10.2}
$$

We want to fix b in such a way that

$$
< h_A(v) > \doteq \frac{1}{(2\pi)^l} \int_{\mathbb{T}^l} h_A(v(\theta)) d\theta = \omega'
\tag{10.3}
$$

(otherwise the second of (7.16) and (7.16) cannot be defined); knowing that $< h_A(b + (v^*,0)) >= \omega'$ has solution $b = b^o = (\hat 0, h^{(-1)}_{A_l}(\alpha))$. In fact by the second of (10.1) $\frac{1}{(2\pi)^{l-1}} \int_{\mathbb{T}^{l-1}} h_{\hat A}(v^*(\hat\theta)) d\theta_1 \ldots d\theta_{l-1} = \omega$ and trivially $h_{A_l}(h^{(-1)}_{A_l}(\alpha)) = \alpha$

$$
\begin{aligned}
w(\theta) \doteq v(\theta) - (v^*,0) - b &= -\, D^{-1}_{\omega'} V_\varphi(\hat\theta + u^*, \theta_l) + \Big(D^{-1}_\omega \overline{V}_{\hat\varphi}(\hat\theta + u^*), 0\Big) = \\
&= - \sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\neq 0} e^{i\nu\theta} \frac{W_\nu}{i\omega'\cdot\nu}
\end{aligned}
\tag{10.4}
$$

and $W_\nu = \frac{1}{(2\pi)^l} \int_{\mathbb{T}^l} e^{-i\nu\theta} V_\varphi(\hat\theta + u^*, \theta_l) d\theta_1 \ldots d\theta_l$; setting $\xi^* - \overline{\xi} = \frac{1}{2}(\xi^* - \xi)$ and taking (5.4) into account

$$
\|w\|_{\overline{\xi}} \le \sum_{\nu\in\mathbb{Z}^l\setminus\{0\}} e^{-|\nu|\xi^*} e^{|\nu|\overline{\xi}} V \frac{2}{\Omega} \le s(\xi^* - \overline{\xi}) \frac{2V}{\Omega}.
\tag{10.5}
$$

$s(\xi^*-\overline{\xi}) = 2^{2l}s(\xi^*-\xi)$ where, in Appendix 6 we have $\sum_{n\in\mathbb{Z}^m}|n|^p e^{-r|n|} \le p!(\frac{4}{r})^{p+m} \doteq \Sigma_p^m(r)$ and $s(\xi^*-\xi) = \Sigma_l^l(\xi^*-\xi)$ see (14.2)

$$\|w\|_{\overline{\xi}} \le 2^{2l}s(\xi^*-\xi)\frac{2V}{\Omega}. \tag{10.6}$$

Rewrite (10.3) as

$$< h_A(b+(v^*,0)+w) - h_A(b+(v^*,0)) > + < h_A(b+(v^*,0)) - h_A(b^o+(v^*,0)) >= 0 \tag{10.7}$$

and defining $Q(b) =< h_A(b+(v^*,0)) >$ (10.7) is

$$Q(b) - Q(b^o) - Q_b(b^o)(b-b^o) + Q_b(b^o)(b-b^o) + R(b) = 0 \tag{10.8}$$

$R =< \int_0^1 d\lambda\, h_{AA}(b+(v^*(\hat{\theta}),0)+\lambda w(\theta))w(\theta) >$. $Q_b(b^o) =< h_{AA}(b^o+(v^*,0)) >$ is a linear operator and (2.10) implies that the operator is non–singular. This fact makes (10.8) equivalent to the equation

$$b = b^o + \big(Q_b(b^o)\big)^{(-1)}\Big(Q(b)-Q(b^o)-Q_b(b^o)(b-b^o)\Big) - \big(Q_b(b^o)\big)^{(-1)}R \doteq K(b) \tag{10.9}$$

which is an equation for b. Recalling by hypotheses that $|\mathrm{Im}\ v_i^*| \le \rho^*$ $1 \le i \le l-1$, $\theta \in \Delta_\xi^l$, we have the

Lemma A2.1 *Let be $\overline{\rho}$ and r two parameters such that $\rho^* < \overline{\rho} < \rho$*

$$r \le \min\{\frac{2}{3}(\overline{\rho}-\rho^*), \frac{a'}{4la}(\rho-\overline{\rho})\}, \qquad \frac{V}{\Omega}\frac{a}{a'}\frac{2^{6l+1}l!}{(\xi^*-\xi)^{2l}} \le \frac{r}{2}. \tag{10.10}$$

As a consequence there exists a unique solution of the equation $K(b) = b$ for $|b-b^o| \le r$.

Proof We use the implicit function theorem showing that $K(b)$ is a contraction of a closed metric–space into itself. (10.9) is

$$\begin{aligned} b - b^o = &\big(< h_{AA}(b+(v^*,0)) >\big)^{(-1)}\Big[\sum_{i,j=1}^{l}\int_0^1 d\lambda(1-\lambda) \\ &< h_{AA_iA_j}((v^*,0)+\lambda b+(1-\lambda)b^o) > (b-b^o)_i(b-b^o)_j\Big] - \\ &-\big(< h_{AA}(b+(v^*,0)) >\big)^{(-1)}\Big[\int_0^1 d\lambda < h_{AA}((v^*,0)+\lambda w+b) > w\Big] \end{aligned} \tag{10.11}$$

$B(r;b^o) = \{y \in \mathbb{C}^l \mid |y-b^o| \le r\}$. We impose on b a condition such that

$$|\mathrm{Im}\,\big((v^*,0)+\lambda b+(1-\lambda)b^o\big)| \le \overline{\rho} \tag{10.12}$$

Invariance of $B(r;b^o)$ The Cauchy estimates, Lemma A6.1 (2.10) give

$$|b-b^o| \le \frac{1}{a'}|b-b^o|^2\frac{la}{\rho-\overline{\rho}} + \frac{a}{a'}\|w\|_{\overline{\xi}} \le \frac{la}{a'}\frac{r^2}{\rho-\overline{\rho}} + \frac{a}{a'}\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi) \le r$$

Taking

$$2^{2l}\frac{a}{a'}\frac{2V}{\Omega}s(\xi^*-\xi) \le \frac{r}{2}, \qquad \frac{la}{a'}\frac{r}{\rho-\overline{\rho}} \le \frac{1}{4} \tag{10.13}$$

$B(r;b^o)$ is an invariant metric–space for $K(b)$.

Contraction We take $b, b' \in B(r;b^o)$

$$\begin{aligned} b-b' &= \big(Q_b(b^o)\big)^{(-1)}\Big(Q(b)-Q(b^o)-Q_b(b^o)(b-b')\Big)- \\ &- \big(Q_b(b^o)\big)^{(-1)} < \Big[\int_0^1 d\lambda < h_{AA}\big((v^*,0)+\lambda w+b\big) - h_{AA}\big((v^*,0)+\lambda w+b'\big)w\Big] > \\ &\doteq K(b)-K(b') \end{aligned} \tag{10.14}$$

and then $|K(b)-K(b')| \le l\frac{a}{a'}\frac{|b-b'|^2}{\rho-\overline{\rho}} + \frac{a}{a'}\frac{|b-b'|}{\rho-\overline{\rho}}\frac{2V}{\Omega}2^{2l}s(\xi^*-\xi)$. The following conditions (the first one is due to $b \in B(r;b^o)$ and $b' \in B(r;b^o)$)

$$|b-b'| < 2r < \frac{a'}{a}\frac{\rho-\overline{\rho}}{2l} \qquad \frac{a}{a'}\frac{2^{2l+2}}{\rho-\overline{\rho}}\frac{V}{\Omega}s(\xi^*-\xi) < 1 \tag{10.15}$$

imply that $|K(b)-K(b')| \le 1^-|b-b'|$ (1^- is a number strictly less than 1). (10.12) is implied by $|\mathrm{Im}\, w_i|+|\mathrm{Im}\, b_i|+\rho^* < \overline{\rho}$ and then $\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)+r+\rho^* \le \frac{2^{2l+1}V}{\Omega}\frac{a}{a'}s(\xi^*-\xi)+r+\rho^* \le \frac{3}{2}r+\rho^* \le \overline{\rho}$ which gives $r \le \frac{2}{3}(\overline{\rho}-\rho^*)$ (use has been done of $b^o \in \mathbb{R}^l$). Observe that (10.13) implies the second of (10.15).

The conditions (10.10) are (10.13) and $r \le \frac{2}{3}(\overline{\rho}-\rho^*)$. Taking $r = \frac{V}{\Omega}\frac{a}{a'}\frac{2^{6l+2}l!}{(\xi^*-\xi)^{2l}}$, $r \le \frac{2}{3}(\overline{\rho}-\rho^*)$ is implied by (7.8) (see $\gamma a(\overline{\rho}-\rho^*)$ at denominator).

Appendix 3. Proof of (7.17)–(7.21) and (7.10)

$$\begin{aligned} \varepsilon_1(\theta) = D_{\omega'}v + V_\varphi(\theta+u) &= -V_\varphi(\hat\theta+u^*,\theta_l) + V_\varphi(\hat\theta+\hat u,\theta_l+u_l) = \\ &= \int_0^1 d\lambda\Big(V_{\varphi\hat\varphi}(\hat\theta+\lambda\hat u+(1-\lambda)u^*,\theta_l+\lambda u_l)(\hat u-u^*)+ \\ &+ V_{\varphi\varphi_l}(\hat\theta+\lambda\hat u+(1-\lambda)u^*,\theta_l+\lambda u_l)u_l\Big) \end{aligned} \tag{11.1}$$

(7.17)

$$
\begin{aligned}
u_l = D_{\omega'}^{-1}(h_{A_l}(v_l)-\omega_l) = & \sum_{\nu\in\mathbb{Z}^l\backslash\{0\}} e^{i\nu\theta}\frac{(h_{A_l}(v_l)-\omega_l)_\nu}{i(\omega\cdot\hat\nu+\alpha\nu_l)} = \sum_{\nu\in\mathbb{Z}^l\backslash\{0\}} e^{i\nu\theta}\frac{h_{A_l}(v_l)_\nu}{i\omega'\cdot\nu} = \\
= & \sum_{\nu\in\mathbb{Z}^l\backslash\{0\}} e^{i\nu\theta}\frac{[h_{A_l}(b_l+w_l)-h_{A_l}(b_l)-h_{A_lA_l}(b_l)w_l]_\nu}{i(\omega\cdot\hat\nu+\alpha\nu_l)} + \\
+ & \sum_{\nu\in\mathbb{Z}^l\backslash\{0\};\nu_l\neq0} e^{i\nu\theta}\frac{[h_{A_lA_l}(b_l)w_l]_\nu}{i(\omega\cdot\hat\nu+\alpha\nu_l)} = \\
= & \sum_{\nu\in\mathbb{Z}^l\backslash 0,;\nu_l\neq0} \frac{e^{i\nu\theta}}{i(\omega\cdot\hat\nu+\alpha\nu_l)}\Big(\int_0^1 d\lambda(1-\lambda)h_{A_lA_lA_l}(b_l+\lambda w_l)w_l^2\Big)_\nu + \\
+ & \sum_{\nu\in\mathbb{Z}^l\backslash\{0\};\nu_l\neq0} e^{i\nu\theta}\frac{[h_{A_lA_l}(b_l)w_l]_\nu}{i(\omega\cdot\hat\nu+\alpha\nu_l)}
\end{aligned}
\tag{11.2}
$$

$\nu_l\neq0$ as $\frac{1}{2\pi}\int_{\mathbb{T}} d\theta_l w_l = 0$ being $w_l = -D_{\omega'}^{-1}V_{\varphi_l}(\hat\theta+u^*,\theta_l)$ see (10.4). The analitycity of h and w gives (recall $|\mathrm{Im}\, v_l|\le\overline\rho$)

$$
\begin{aligned}
&\Big|\Big[\int_0^1 d\lambda(1-\lambda)h_{A_lA_lA_l}(b_l+\lambda w_l)w_l^2\Big]_\nu\Big| \le \\
&\Big\|\int_0^1 d\lambda(1-\lambda)h_{A_lA_lA_l}(b_l+\lambda w_l)w_l^2\Big\|_{\overline\xi}e^{-|\nu|\overline\xi} \le \\
&\le e^{-|\nu|\overline\xi}\frac{a}{\rho-\overline\rho}\frac12\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&\Big\|\sum_{\nu\in\mathbb{Z}^l\backslash\{0\}} e^{i\nu\theta}\frac{[h_{A_l}(b_l+w_l)-h_{A_l}(b_l)-h_{A_lA_l}(b_l)w_l]_\nu}{i(\omega\cdot\hat\nu+\alpha\nu_l)}\Big\|_\xi \le \\
&\le \frac{a}{\rho-\overline\rho}\frac12\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)\Big)^2\sum_{\nu\in\mathbb{Z}^l\backslash\{0\}} e^{-|\nu|(\overline\xi-\xi)}\gamma|\nu|^l \le \\
&\le \frac{\gamma a}{\rho-\overline\rho}\frac12\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)\Big)^2 2^{2l}s(\xi^*-\xi)
\end{aligned}
\tag{11.3}
$$

$$
\begin{aligned}
&\Big\|\sum_{\nu\in\mathbb{Z}^l\backslash\{0\};\nu_l\neq0} e^{i\nu\theta}\frac{h_{A_lA_l}(b_l)(w_l)_\nu}{i\omega'\cdot\nu}\Big\|_\xi \le \sum_{\nu\in\mathbb{Z}^l} e^{-|\nu|(\overline\xi-\xi)}\frac{2}{\Omega}|\nu|^l a\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi) \le \\
&\le \frac{2^{4l+2}aV}{\Omega^2}s^2(\xi^*-\xi')
\end{aligned}
\tag{11.4}
$$

(11.3) plus (11.4) is bounded by

$$\begin{aligned}
&\|u_l\|_\xi \le \frac{\gamma a}{\rho-\overline{\rho}}\frac{1}{2}\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)\Big)^2 2^{2l}s(\xi^*-\xi)+\frac{2^{4l+2}aV}{\Omega^2}s^2(\xi^*-\xi')\le\\
&\le 2^{6l+1}\frac{Va}{\Omega^2}s^3(\xi^*-\xi)(1+\frac{V\gamma}{\rho-\overline{\rho}})
\end{aligned}\tag{11.5}$$

$$\begin{aligned}
&\hat u-u^*=D^{-1}_{\omega'}\big(h_{\hat A}(\hat v)-\omega\big)-D^{-1}_{\omega}\big(h_{\hat A}(v^*)-\omega\big)=\\
&=\sum_{\nu\in\mathbb{Z}^l\setminus\{0\}}\frac{e^{i\nu\theta}}{i\omega'\nu}(h_{\hat A}(\hat v))_\nu-\sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}}\frac{e^{ik\hat\theta}}{i\omega k}(h_{\hat A}(v^*))_k=\\
&\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}\frac{e^{i\nu\theta}}{i\omega'\nu}(h_{\hat A}(\hat v))_\nu+\sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}}\frac{e^{ik\hat\theta}}{i\omega k}\Big((h_{\hat A}(\hat v))_{(k,0)}-(h_{\hat A}(v^*))_k\Big)
\end{aligned}\tag{11.6}$$

$$\begin{aligned}
&\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}\frac{e^{i\nu\theta}}{i\omega'\nu}(h_{\hat A}(\hat v))_\nu=\\
&=\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}\frac{e^{i\nu\theta}}{i\omega'\nu}\Big((h_{\hat A}(\hat b+\hat w+v^*)-h_{\hat A}(v^*)-h_{\hat A\hat A}(v^*)(\hat b+\hat w)\Big)_\nu+\\
&+\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}\frac{e^{i\nu\theta}}{i\omega'\nu}\Big(h_{\hat A\hat A}(v^*)(\hat b+\hat w)\Big)_\nu
\end{aligned}\tag{11.7}$$

$$\begin{aligned}
&\Big\|\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}\frac{e^{i\nu\theta}}{i\omega'\nu}(h_{\hat A}(\hat v))_\nu\Big\|_\xi\le\\
&\le\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}e^{-|\nu|(\overline{\xi}-\xi)}\frac{2}{\Omega}|\nu|^l\frac{al}{\rho-\overline{\rho}}\|\hat b+\hat w\|^2_{\overline{\xi}}+\\
&+\sum_{\nu\in\mathbb{Z}^l\setminus\{0\};\nu_l\ne0}e^{-|\nu|(\overline{\xi}-\xi)}\frac{2}{\Omega}|\nu|^l a\|\hat b+\hat w\|_{\overline{\xi}}\le\\
&\le\frac{2^{2l+1}}{\Omega}s(\xi^*-\xi)\frac{al}{\rho-\overline{\rho}}\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)+\frac{2^{2l+2}V}{\Omega}\frac{a}{a'}s(\xi^*-\overline{\xi})\Big)^2+\\
&+\frac{2^{2l+1}}{\Omega}s(\xi^*-\xi)a\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi)+\frac{2^{2l+2}V}{\Omega}\frac{a}{a'}s(\xi^*-\xi)\Big)\le\\
&\le\frac{2^{6l+7}}{\Omega}s^3(\xi^*-\xi)\frac{al}{\rho-\overline{\rho}}\frac{V^2}{\Omega^2}\frac{a}{a'}+\frac{2^{4l+4}}{\Omega}s^2(\xi^*-\xi)a\frac{V}{\Omega}\frac{a}{a'}
\end{aligned}\tag{11.8}$$

The second sum of (11.6) is estimated as follows

$$\sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}} \frac{e^{ik\hat\theta}}{i\omega k}\Big[h_{\hat A}(\hat v) - h_{\hat A}(v^*)\Big]_k =$$

$$= \sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}} \frac{e^{ik\hat\theta}}{i\omega k}\Big[h_{\hat A}(v^* + \hat b + \hat w)) - h_{\hat A}(v^*) - h_{\hat A\hat A}(v^*)(\hat b + \hat w)\Big]_k + \tag{11.9}$$

$$+ \sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}} \frac{e^{ik\hat\theta}}{i\omega k}\Big[h_{\hat A\hat A}(v^*)(\hat b + \hat w)\Big]_k$$

and then

$$\Big\|\sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}} \frac{e^{ik\hat\theta}}{i\omega k}\Big[h_{\hat A}(\hat v)) - h_{\hat A}(v^*)\Big]_k\Big\|_\xi \le$$

$$\le \sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}} \gamma|k|^{l-1}e^{-|k|(\overline\xi-\xi)}\frac{la}{\rho-\overline\rho}\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi) + \frac{2^{2l+2}V}{\Omega}\frac{a}{a'}s(\xi^*-\overline\xi)\Big)^2 +$$

$$+ \sum_{k\in\mathbb{Z}^{l-1}\setminus\{0\}} \gamma|k|^{l-1}e^{-|k|(\overline\xi-\xi)}a\Big(\frac{2^{2l+1}V}{\Omega}s(\xi^*-\xi) + \frac{2^{2l+2}V}{\Omega}\frac{a}{a'}s(\xi^*-\overline\xi)\Big) \le$$

$$\le \frac{2^{6l+6}}{\Omega}s^3(\xi^*-\xi)\frac{al}{\rho-\overline\rho}\frac{V^2}{\Omega}\frac{a}{a'} + \frac{2^{4l+4}}{\Omega}s^2(\xi^*-\xi)aV\frac{a}{a'} \tag{11.10}$$

It then follows

$$\|\hat u - u^*\|_\xi \le$$

$$\le \Big(\frac{2^{6l+6}}{\Omega^2}s^3(\xi^*-\xi)\frac{al}{\rho-\overline\rho}V^2\frac{a}{a'} + \frac{2^{4l+3}}{\Omega}s^2(\xi^*-\xi)aV\frac{a}{a'}\Big) \tag{11.11}$$

$$\le (\frac{2}{\Omega}+\gamma)\frac{2^{4l+3}Va}{\Omega}\frac{a}{a'}s^2(\xi^*-\xi)\Big(1 + \frac{2^{2l+3}V}{\Omega(\rho-\overline\rho)}\frac{a}{a'}s(\xi^*-\xi)\Big)$$

Imposing on Ω that (see Remark 5.3, (7.8) and (10.15))

$$\frac{2}{\Omega} \le \gamma, \qquad \frac{2^{2l+1}V}{\Omega(\rho-\overline\rho)}\frac{a}{a'}s(\xi^*-\xi) < 1 \tag{11.12}$$

(11.11) is bounded by

$$2^{4l+7}\frac{V\gamma a}{\Omega}\frac{a}{a'}s^2(\xi^*-\xi). \tag{11.13}$$

By (11.5)

$$\|u_l\|_\xi \le \frac{2^{6l+1}Va}{\Omega^2}s^3(\xi^*-\xi)\Big(1+\frac{\gamma V}{\rho-\overline\rho}\Big) \tag{11.14}$$

and added with (11.14) give

$$\frac{2^{6l+7}V\gamma a}{\Omega}\frac{a}{a'}s^2(\xi^*-\xi)+\frac{2^{6l+1}Va}{\Omega^2}s^3(\xi^*-\xi)\Big(1+\frac{\gamma V}{\rho-\overline{\rho}})\leq$$
$$\leq\frac{2^{4l+7}V\gamma a}{\Omega}\frac{a}{a'}s^2(\xi^*-\xi)\Big[1+\frac{2^{2l+1}s(\xi^*-\xi)}{\gamma\Omega}\big(1+\frac{V\gamma}{\rho-\rho'})\Big]\leq \tag{11.15}$$
$$2^{6l+5}3\frac{V\gamma^2 a}{\gamma\Omega}\frac{a}{a'}s^3(\xi^*-\xi)$$

where in the last $\leq$ (11.12) and (7.8) have been used.

(7.18)

$$\|(u_l)_\theta\|_\xi=\|\partial_\theta D^{(-1)}_{\omega'}(h_{A_l}(v_l)-\omega_l)\|_\xi=\|\sum_{\nu\in\mathbb{Z}^l\setminus\{0\}}e^{i\nu\theta}\frac{i\nu(h_{A_l}(v_l)-\omega_l)_\nu}{i(\omega\cdot\hat\nu+\alpha\nu_l)}\|_\xi=$$
$$=\|\sum_{\nu\in\mathbb{Z}^l;\nu\neq 0}e^{i\nu\theta}i\nu\frac{h_{A_l}(v_l)_\nu}{i\omega'\cdot\nu}\|_\xi=$$

The estimate is the same of (11.2) except the presence of $|\nu|^{l+1}$ instead of $|k|^l$; so one $s(\cdot)$ is substituted by $\Sigma^l_{l+1}(\cdot)$ see Lemma A6.1.

$$\|(u_l)_\theta\|_\xi\leq\frac{2^{6l+1}aV}{\Omega^2}s^2(\xi^*-\xi)\Sigma^l_{l+1}(\xi^*-\xi)(1+\frac{\gamma V}{\rho-\overline{\rho}})$$
$$\|(\hat u-u^*)_\theta\|_\xi\leq\frac{2^{4l+8}\gamma^2 aV}{\gamma\Omega^2}\frac{a}{a'}s(\xi^*-\xi)\Sigma^l_{l+1}(\xi^*-\xi) \tag{11.16}$$

and then

$$\|(\hat u-(u^*,0))_\theta\|_\xi\leq 2^{6l+5}\frac{Va}{\Omega^2}s^3(\xi^*-\xi)\Sigma^l_{l+1}(\xi^*-\xi)(1+\frac{V\gamma}{\rho-\overline{\rho}})$$

which is (7.18).

(7.19)
By (10.4) and Lemma A2.1 we have $v=(v^*,0)+b+w$ and $|b-b^o|\leq r$ with $r=\frac{2^{2l+2}V}{\Omega}\frac{a}{a'}s(\xi^*-\xi)$. $\|v-(v^*,h^{(-1)}_{A_l}(\alpha))\|_\xi=\|v-(v^*,b^o_l)\|_\xi\leq|b-b^o_l|+\|w\|_{\overline{\xi}}$ and

$$\|v-(v^*,h^{(-1)}_{A_l}(\alpha))\|_\xi\leq\frac{6V}{\Omega}\frac{a}{a'}s(\xi^*-\xi)2^{2l}$$

(7.20)
In the same manner

$$\|[v-(v^*,h^{(-1)}_{A_l}(\alpha))]_\theta\|_{\overline{\xi}}\leq\frac{V}{\Omega}\Sigma^l_{l+1}(\xi^*-\xi)2^{2l+2}$$

$$\gamma\|D_{\omega'}u - D_\omega(u^*,0)\|_\xi =$$
$$= \gamma\big\|\big(h_A(v) - \omega' - D_\omega\big[D_\omega^{-1}(h_{\hat A}(v^*) - \omega), 0\big]\big\|_\xi$$
$$= \gamma\|h_A(v) - \omega' - (h_{\hat A}(v^*) - \omega, 0)\|_\xi$$
$$= \gamma\|(h_{\hat A}(\hat v) - h_{\hat A}(v^*)), h_{A_l}(v_l) - \omega'_l\|_\xi$$
$$\Big(\gamma\|\int_0^1 d\lambda h_{\hat A\hat A}(\lambda\hat v + (1-\lambda)v^*)(\hat v - v^*)\|_\xi,$$
$$\|\int_0^1 d\lambda h_{\hat A_l\hat A_l}(\lambda v_l + (1-\lambda)h_{A_l}^{(-1)}(\alpha))(v_l - h_{A_l}^{(-1)}(\alpha))\|_\xi\Big) \leq$$
$$\leq \gamma a\|(v - (v^*, h_{A_l}^{(-1)}(\alpha_l)))\|_\xi \leq \frac{\gamma^2 Va}{\gamma\Omega}\Sigma_{l+1}^l(\xi^* - \xi)2^{2l+2}$$

Now inserting (7.17) in (11.1) and taking into account that $\{\theta + u(\theta) : \theta \in \Delta_{\frac{\xi}{\xi}}^l\} \subset \{\theta + u(\theta) : \theta \in \Delta_\xi^l\} \subset \Delta_{\xi^*}^l$ we get (7.21).

Appendix 4. Proof of Lemma 7.2

Here we write the proof of Lemma 7.2 which is the main part of the paper togheter with Proposition 7.1. It consists of two parts: the first one, Lemma A4.1, is essentially algebraic while the second one, Lemma A4.2), consists of an iteration of the first one and quantitative considerations. The first part is

Lemma A4.1 *Let's consider*

$$\big(D_\omega p + V_x(\theta + q), \qquad \omega + D_\omega q - h_y(p)\big) \doteq (\varepsilon_1, \varepsilon_2) \tag{12.1}$$

where $(p,q)\colon \mathbb{T}^l \to \mathbb{R}^l \times \mathbb{R}^l)$ such that $M \doteq I + q_\theta > 0\,\forall\,\theta \in \mathbb{T}^l$, $h_{yy}(y) = h_{yy}^T(y) > 0$ for all $y \in \mathbb{R}$. Define

$$\begin{aligned}
&\Delta q = \\
&= MD_\omega^{-1}\big[M^{-1}(h_{yy}(p)M^{-T}D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1) - \varepsilon_2 - h_{yy}(p)M^{-T}m)\big] + Mm' \\
&\Delta p = M^{-T}\big(m + D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1)\big) + (p_\theta)M^{-1}\Delta q
\end{aligned} \tag{12.2}$$

with $m = < M^{-1}h_{yy}(p)M^{-T} >^{-1} < M^{-1}h_{yy}(p)M^{-T}D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1) - M^{-1}\varepsilon_2 >$. Then $q + \Delta q$ and $p + \Delta p$ satisfy the relations

$$(D_\omega(p+\Delta p) + V_x(\theta + q + \Delta q),\ \omega + D_\omega(q + \Delta q) - h_y(p + \Delta p)) \doteq (\varepsilon'_1, \varepsilon'_2) \tag{12.3}$$

with

$$\begin{aligned}
&\varepsilon'_2 = (\varepsilon_2)_\theta M^{-1}\Delta q - \big(h_y(p+\Delta p) - h_y(p) - h_{yy}(p)\Delta p\big) \\
&\varepsilon'_1 = -M^T\Big(V_x(\theta+q+\Delta q) - V_x(\theta+q) - V_{xx}(\theta+q)\Delta q\Big) - \\
&- (\varepsilon_2)_\theta^T\Delta p + \big(M^T(\varepsilon_1)_\theta + (\varepsilon_2)_\theta^T p_\theta\big)M^{-1}\Delta q + \\
&+ D_\omega^{-1}\Big((\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta + M^T(\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M\Big)D_\omega(M^{-1}\Delta q)
\end{aligned} \tag{12.4}$$

$$< (\varepsilon_2)_\theta^T p_\theta - p_\theta^T (\varepsilon_2)_\theta + M^T (\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M >= 0, \quad < p_\theta^T \varepsilon_2 - M^T \varepsilon_1 >= 0;$$

Remark 12.1 The meaning of this algebraic lemma is quite clear. Δp and Δq are of order one respect to ε_1 and ε_2 and then ε_1', ε_2' are of order two. In the next Lemma we repeat infinite times this procedure getting the solution of (7.13).

Proof By (12.1) and (12.3) we get

$$\begin{aligned}
\varepsilon_1' &= \varepsilon_1 - V_x(\theta + q) + D_\omega \Delta p + V_x(\theta + q + \Delta q) \\
\varepsilon_2' &= \varepsilon_2 + h_y(p) + D_\omega \Delta q - h_y(p + \Delta p)
\end{aligned}$$

and consequently

$$\begin{aligned}
\varepsilon_1 + V_{xx}(\theta + q)\Delta q + D_\omega \Delta p &= \varepsilon_1' + \Delta V \\
\varepsilon_2 - h_{yy}(p)\Delta p + D_\omega \Delta q &= \varepsilon_2' + \Delta h
\end{aligned} \tag{12.5}$$

where

$$\begin{aligned}
\Delta V &= -(V_x(\theta + q + \Delta q) - V_x(\theta + q) - V_{xx}(\theta + q + \Delta q)) \\
\Delta h &= -(h_y(p + \Delta p) - h_y(p) - h_{yy}(p)\Delta p).
\end{aligned}$$

Multiply the first by M^T, the second by $-p_\theta^T$ (at left) and add

$$\begin{aligned}
M^T D_\omega \Delta p - p_\theta^T D_\omega \Delta q = -M^T V_{xx}(\theta + q)\Delta q - p_\theta^T h_{yy}(p)\Delta p - \\
- M^T \varepsilon_1 + M^T \varepsilon_1' + M^T \Delta V - p_\theta^T \varepsilon_2' + p_\theta^T \varepsilon_1 - p_\theta^T \Delta h.
\end{aligned} \tag{12.6}$$

Now derive (12.1) respect to θ

$$D_\omega p_\theta + V_{xx}(\theta + q)M = (\varepsilon_1)_\theta, \qquad D_\omega q_\theta - h_{yy}(p)p_\theta = (\varepsilon_2)_\theta \tag{12.7}$$

which gives

$$V_{xx}(\theta + q) = ((\varepsilon_1)_\theta - D_\omega p_\theta)M^{-1}, \quad h_{yy}(p)p_\theta = D_\omega q_\theta - (\varepsilon_2)_\theta = D_\omega M - (\varepsilon_2)_\theta \tag{12.8}$$

and substitute in (12.6)

$$\begin{aligned}
& D_\omega(M^T \Delta p) - p_\theta^T D_\omega \Delta q - M^T (D_\omega p_\theta) M^{-1} \Delta q = \\
& = -M^T \varepsilon_1 + p_\theta^T \varepsilon_2 - M^T (\varepsilon_1)_\theta M^{-1} \Delta q + (\varepsilon_2)_\theta^T \Delta p + \\
& + M^T \varepsilon_1' - p_\theta^T \varepsilon_2' + M^T \Delta V - p_\theta^T \Delta h
\end{aligned} \tag{12.9}$$

whose left hand side can be rewritten as

$$\begin{aligned}
& D_\omega(M^T \Delta p) - p_\theta^T D_\omega \Delta q - M^T (D_\omega p_\theta) M^{-1} \Delta q = \\
& = D_\omega(M^T \Delta p) - p_\theta^T \Delta_\omega (M M^{-1} \Delta q) - (D_\omega (M^T p_\theta)) M^{-1} \Delta q + \\
& + (D_\omega M^T) p_\theta M^{-1} \Delta q = \\
& = D_\omega(M^T \Delta p) - D_\omega(M^T p_\theta M^{-1} \Delta q) - p_\theta^T (D_\omega M) M^{-1} \Delta q - p_\theta^T M D_\omega (M^{-1} \Delta q) + \\
& + M^T p_\theta D_\omega (M^{-1} \Delta q) + (D_\omega M^T) p_\theta M^{-1} \Delta q = \\
& D_\omega (M^T \Delta p - M^T p_\theta M^{-1} \Delta q) + (M^T p_\theta - p_\theta^T M) D_\omega (M^{-1} \Delta q) + \\
& + \big((D_\omega M^T) p_\theta - p_\theta^T D_\omega M)\big) M^{-1} \Delta q.
\end{aligned} \tag{12.10}$$

$$D_\omega(M^T p_\theta - p_\theta^T M) = (D_\omega M^T)p_\theta + M^T D_\omega(p_\theta) - (D_\omega p_\theta^T)M - p_\theta^T D_\omega M$$

so that

$$M^T p_\theta - p_\theta^T M = D_\omega^{-1}\big((D_\omega M^T)p_\theta + M^T D_\omega(p_\theta) - (D_\omega p_\theta^T)M - p_\theta^T D_\omega M\big)$$

By $(D_\omega M)^T = p_\theta^T h_{yy} + (\varepsilon_2)_\theta^T$ and its transposed, $D_\omega p_\theta + V_{xx}(\theta+q)M = (\varepsilon_1)_\theta$ and its transposed we have

$$\begin{aligned}
&(D_\omega M^T)p_\theta - p_\theta^T(D_\omega M) = (\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta\\
&M^T D_\omega(p_\theta) - (D_\omega p_\theta^T)M = M^T(\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M\\
&M^T p_\theta - p_\theta^T M = D_\omega^{-1}\Big[(\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta + M^T(\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M\Big]
\end{aligned}$$

and (12.10) becomes

$$\begin{aligned}
&D_\omega(M^T\Delta p - M^T p_\theta M^{-1}\Delta q) + D_\omega^{-1}\Big[(\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta + M^T(\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M\Big] +\\
&+ \big((\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta\big)M^{-1}\Delta q
\end{aligned}$$

Consequently (12.9) becomes

$$\begin{aligned}
&D_\omega\big(M^T(\Delta p - p_\theta^T M^{-1}\Delta q)\big) + (M^T\varepsilon_1 - p_\theta^T\varepsilon_2) +\\
&+ D_\omega^{-1}\Big[(\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta + M^T(\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M\Big] D_\omega(M^{-1}\Delta q) +\\
&+ \Big((\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta\Big)M^{-1}\Delta q =\\
&= -M^T(\varepsilon_1)_\theta M^{-1}\Delta q + (\varepsilon_2)_\theta^T\Delta p + M^T\varepsilon_1' - p_\theta^T\varepsilon_2' + M^T\Delta V - p_\theta^T\Delta h
\end{aligned}\tag{12.11}$$

Now $< -M^T\varepsilon_1 + p_\theta^T\varepsilon_2) >= 0$ allows us to solve the linear equation

$$D_\omega\big(M^T(\Delta p - p_\theta^T M^{-1}\Delta q)\big) = (-M^T\varepsilon_1 + p_\theta^T\varepsilon_2)\tag{12.12}$$

(so eliminating the terms of first order in Δq or Δp or ε_1 or ε_2) and obtain the second of (12.2).

$$\begin{aligned}
&< p_\theta^T\varepsilon_2 - M^T\varepsilon_1 >=< p_\theta^T(\omega + D_\omega q - h_y(p)) - (I+q_\theta)^T(D_\omega p + V_x(\theta+q) >=\\
&=< -\big(h(p) + V_x(\theta+p)\big)_\theta + (p_\theta^T D_\omega q - q_\theta^T D_\omega p) - D_\omega p >
\end{aligned}$$

The average over $\mathbb{T}^l$ of the first and third terms are trivially zero being derivatives of periodic functions. As for the second term we write the k^{th} component

$$< (p_\theta^T D_\omega q - q_\theta^T D_\omega p)_k >=< \sum_{i,j=1}^{l}\omega_j(\partial_{\theta_k}p_i)(\partial_{\theta_j}q_i) - \sum_{r,n=1}^{l}\omega_r(\partial_{\theta_k}q_n)(\partial_{\theta_r}p_n) >=$$

$$< \sum_{i,j=1}^{l}\omega_j\partial_{\theta_i}(q_i\partial_{\theta_k}p_i) - \sum_{i,j=1}^{l}\omega_j q_i\partial_{\theta_j\theta_k}p_i - \sum_{r,n=1}^{l}\omega_r\partial_{\theta_r}(p_n\partial_{\theta_k}q_n) > +$$

$$< \sum_{r,n=1}^{l}\omega_r p_n\partial_{\theta_r\theta_k}q_n >=< \sum_{i,j=1}^{l}\omega_j q_i\partial_{\theta_j\theta_k}p_i - \sum_{r,n=1}^{l}\omega_r p_n\partial_{\theta_r\theta_k}q_n >= 0$$

Moving the second derivatives on p_n in the second sum and using the periodicity of p and q.
Inserting the second of (12.2) (or (12.12)) in the second of (12.5) we get

$$\begin{aligned}&\varepsilon_2 + D_\omega(MM^{-1}\Delta q) - h_{yy}\big(M^{-T}m - M^{-T}D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1) + p_\theta M^{-1}\Delta q\big) =\\ &+\Delta h + \varepsilon_2'\end{aligned} \tag{12.13}$$

and using (12.8)

$$\begin{aligned}&\varepsilon_2 + MD_\omega(M^{-1}\Delta q) - h_{yy}M^{-T}m - h_{yy}M^{-T}D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1) +\\ &+(\varepsilon_2)_\theta M^{-1}\Delta q - \Delta h - \varepsilon_2' = 0\end{aligned} \tag{12.14}$$

Choosing m as after (12.2), the equation

$$\varepsilon_2 + MD_\omega(M^{-1}\Delta q) - h_{yy}M^{-T}m - h_{yy}M^{-T}D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1) = 0$$

can be solved giving the first of (12.2) and

$$\varepsilon_2' = (\varepsilon_2)_\theta M^{-1}\Delta q - \Delta h \tag{12.15}$$

i.e. the first of (12.4). Finally insert (12.2) and the first of (12.4) in

$$\begin{aligned}&D_\omega^{-1}\Big[(\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta + M^T(\varepsilon_1)_\theta - (\varepsilon_1)_\theta^T M\Big]D_\omega(M^{-1}\Delta q) +\\ &+\Big((\varepsilon_2)_\theta^T p_\theta - p_\theta^T(\varepsilon_2)_\theta\Big)M^{-1}\Delta q =\\ &= -M^T(\varepsilon_1)_\theta M^{-1}\Delta q + (\varepsilon_2)_\theta^T\Delta p + M^T\varepsilon_1' - p_\theta^T\varepsilon_2' + M^T\Delta V - p_\theta^T\Delta h\end{aligned} \tag{12.16}$$

which is (12.11) without the first three terms (of order one) and so obtain the second of (12.4).∎
Let be

$$\|V_\varphi\|_\sigma = V, \qquad \|\varepsilon_1\|_\xi = E_1, \qquad \|\varepsilon_2\|_\xi = E_2$$

$$\|p_\theta\|_\xi = N, \quad \|p\|_\xi \le \rho, \quad \|h_{yy}\|_{D_\rho} \le a, \qquad \|h_{yy}^{-1}\|_{D_\rho} \le \frac{1}{a'}$$

$$\|q_\theta\|_\xi \le Q \le \frac{1}{2}, \qquad \|M\|_\xi = \|M^T\|_\xi \le \frac{3}{2}, \qquad \|M^{-1}\|_\xi = \|M^{-T}\|_\xi \le 2$$

and using the positivity properties of h_{yy} and the simmetry of $M^T h_{yy}^{-1} M$ the following bound holds $\| < (M^T h_{yy}^{-1}(p)M)^{-1} >^{-1} \| \le \|M^T h_{yy}^{-1}(p)M\|_{D_\rho\times\Delta_\xi}$ (see [CC2]

Appendix 4 Lemma 3.5)

$$\begin{aligned}
&|m| \le \| < (M^T h_{yy}^{-1}(p) M)^{-1} >^{-1} \| \cdot \\
&\cdot \| < M^{-1} h_{yy}(p) M^{-T} D_\omega^{-1}(p_\theta^T \varepsilon_2 - M^T \varepsilon_1) - M^{-1}\varepsilon_2 > \| \le \\
&\le \|M^T h_{yy}^{-1}(p) M\|_{D_\rho \times \Delta_\xi} \big(\|M^{-1} h_{yy}(p) M^{-T}\|_{D_\rho \times \Delta_\xi} \cdot \\
&\cdot \|D_\omega^{-1}(p_\theta^T \varepsilon_2 - M^T \varepsilon_1)\|_{\xi-\delta} + \|M^{-1}\varepsilon_2\|_\xi \big) \le \\
&\le (\frac{3}{2})^2 \frac{1}{a'} 4a\gamma s(\delta) \|p_\theta^T \varepsilon_2 - M^T \varepsilon_1\|_\xi + 2E_2 (\frac{3}{2})^2 \frac{1}{a'} \le \\
&\le 9 \frac{a}{a'} \gamma s(\delta)(NE_2 + E_1) + 2\frac{9}{4}\frac{E_2}{a'} \ ;
\end{aligned}$$

$$\begin{aligned}
&\|\Delta q\|_{\xi-2\delta} \le \\
&\le \|M\|_\xi \|D_\omega^{-1}\Big[M^{-1}\big(h_{yy} M^{-T} D_\omega^{-1}(p_\theta^T \varepsilon_2 - M^T\varepsilon_1) - \varepsilon_2 - h_{yy} M^{-T} m\big)\Big]\|_{\xi-2\delta} \le \\
&\frac{3}{2}\gamma s(\delta) \|M^{-1}\big(h_{yy}(p) M^{-T} D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1) - \varepsilon_2 - h_{yy}(p) M^{-T} m\big)\|_{\xi-\delta} \le \\
&\le \frac{3}{2}\gamma s(\delta)\big(4a\gamma s(\delta)\|p_\theta^T\varepsilon_2 - M^T\varepsilon_1\|_\xi + 2E_2 + 4a|m|\big) \le \\
&\le B_1 \gamma s^2(\delta) \frac{a}{a'} [E_2(1+\gamma a N) + a\gamma E_1] \le B_1 \gamma s^2(\delta)\frac{a}{a'}(1+\gamma a N)[E_2 + a\gamma E_1] \ ;
\end{aligned} \tag{12.17}$$

$$\begin{aligned}
\|\Delta p\|_{\xi-2\delta} \le & \|M^{-T}\|_\xi |m| + \|M^{-T}\|_\xi \cdot \\
& \cdot \|\big(D_\omega^{-1}(p_\theta^T\varepsilon_2 - M^T\varepsilon_1)\big) + (p_\theta) M^{-1}\Delta q\|_{\xi-2\delta} \le \\
& \le 2|m| + 2\gamma s(\delta)\|p_\theta^T\varepsilon_2 - M^T\varepsilon_1\|_{\xi-\delta} + \|p_\theta\|_\xi 2 \|\Delta q\|_{\xi-2\delta} \le \\
& \le B_2 \frac{s^2(\delta)}{a'}(1+\gamma a N)^2 [\gamma a E_1 + E_2]
\end{aligned} \tag{12.18}$$

$$\varepsilon_2' = (\varepsilon_2)_\theta M^{-1}\Delta q - \sum_{i,j=1}^{l} \int_0^1 d\lambda \ (1-\lambda)(\Delta p)_i (\Delta p)_j h_{zz_iz_j}(p + \lambda\Delta p)$$

If $\|\Delta p\|_{\xi-2\delta} \le \Delta\rho$ and $\overline{\rho} + \Delta\rho < \rho$ we get

$$\begin{aligned}
&|\int_0^1 d\lambda \ (1-\lambda)(\Delta p)_i(\Delta p)_j h_{zz_iz_j}(p+\lambda\Delta p)| \le \frac{1}{2}\sum_{k=1}^{l}\sum_{i,j=1}^{l} |\Delta p_j||\Delta p_i| \frac{a}{\rho - \overline{\rho} - \Delta\rho} \le \\
&\le \frac{la\|\Delta p\|_{\xi-2\delta}^2}{2(\rho - \overline{\rho} - \Delta\rho)}
\end{aligned}$$

hence

$$\begin{aligned}
&\|\varepsilon_2'\|_{\xi-2\delta} \le 2\|(\varepsilon_2)_\theta\|_{\xi-2\delta}\|\Delta q\|_{\xi-2\delta} + \frac{la\|\Delta p\|_{\xi-2\delta}^2}{2(\rho-\overline{\rho}-\Delta\rho)} \le \\
&\le B_3 (E_2 + \gamma a E_1)^2 (1+\gamma a N)^4 \Big(\frac{\gamma a}{\delta a'} + \frac{las^2(\delta)}{(a')^2(\rho - \overline{\rho} - \Delta\rho)} \Big)
\end{aligned}$$

we impose $\|\Delta q\|_{\xi-2\delta}\le\delta$ so that $\sup_{\theta\in\Delta_{\xi-2\delta}}|\mathrm{Im}\,(\theta+q+\Delta q|)\le\sup_{\theta\in\Delta_\xi}|\mathrm{Im}\,(\theta+q)|-\delta$

$$
\begin{aligned}
&|V_x(\theta+q+\Delta q)-V_x(\theta+q)-V_{xx}(\theta+q)\Delta q|=\\
&=|\sum_{i,j=1}^{l}\int_0^1 d\lambda(1-\lambda)(\Delta q)_i(\Delta q)_j V_{xx_ix_j}(\theta+q+\lambda\Delta q)|
\end{aligned}
$$

and then

$$
\begin{aligned}
&\|V_x(\theta+q+\Delta q)-V_x(\theta+q)-V_{xx}(\theta+q)\Delta q\|_{\xi-2\delta}\le\\
&\le\sum_{i,j=1}^{l}\frac12\|\Delta q_i\|_{\xi-2\delta}\|\Delta q_j\|_{\xi-2\delta}\sup_{\theta\in\Delta_{\xi-2\delta}}|V_{xx_ix_j}(\theta+q+\Delta q)|\le\frac12\|\Delta q\|^2_{\xi-2\delta}\frac{2V}{\delta^2}
\end{aligned}
$$

$$
\begin{aligned}
&\|\varepsilon_1'\|_{\xi-2\delta}\le\|M^T\|_\xi\|\Delta q\|^2_{\xi-2\delta}\frac{V}{\delta^2}+\frac1\delta\|\varepsilon_2\|_{\xi-\delta}\|\Delta p\|_{\xi-2\delta}+\\
&+\big(\|M^T\|_\xi\frac1\delta\|\varepsilon_1\|_{\xi-\delta}+\frac1\delta\|\varepsilon_2\|_{\xi-\delta}N\big)\|M^{-1}\|_\xi\|\Delta q\|_{\xi-2\delta}+\\
&+\gamma s(\delta)\big(\frac2\delta\|\varepsilon_2\|_{\xi-\delta}N+\frac3\delta\|\varepsilon_1\|_{\xi-\delta}\big)\|D_\omega^{-1}(M^{-1}\Delta q)\|_{\xi-\delta}\le\\
&\le B_4(E_2(1+\gamma aN)+\gamma aE_1)^2\frac{s^4(\delta)}{\delta^2}\frac{a}{(a')^2}(1+V\gamma^2a)
\end{aligned}
$$

$$
\begin{aligned}
&\|q_\theta+\Delta q_\theta\|_{\xi-3\delta}\le\|q_\theta\|_\xi+\frac1\delta\|\Delta q\|_{\xi-2\delta}\le\\
&\qquad\le Q+B_1\gamma s^2(\delta)\frac{a}{a'}[E_2(1+\gamma aN)+a\gamma E_1]=Q'\le\frac12
\end{aligned}
$$

$$
\begin{aligned}
&\|p_\theta+\Delta p_\theta\|_{\xi-3\delta}\le\|p_\theta\|_\xi+\frac1\delta\|\Delta p\|_{\xi-2\delta}\le\\
&\qquad\le N+B_2\frac{s^2(\delta)}{a'}(1+\gamma aN)(\gamma aE_1+E_2(1+\gamma aN))
\end{aligned}
$$

$$
\|\varepsilon_2'+\gamma a\varepsilon_1'\|_{\xi-2\delta}\le(E_2+\gamma aE_1)^2(1+\gamma aN)^4(1+\gamma aV)(1+\gamma\rho a)\frac{B_4s^4(\delta)}{\delta^2a(\rho-\overline{\rho}-\Delta\rho)}
$$

Moreover we have to impose respectively that $\|\Delta q\|_{\xi-2\delta}\le\delta$ and $\|\Delta p\|_{\xi-2\delta}\le\Delta\rho$ i.e.

$$
\frac{B_1}{\delta}\gamma s^2(\delta)\frac{a}{a'}(1+\gamma aN)[E_2+a\gamma E_1]\le 1 \tag{12.19}
$$

and

$$
\frac{B_2}{\Delta\rho}\frac{s^2(\delta)}{a'}(1+\gamma aN)^2[E_2+a\gamma E_1]\le 1 \tag{12.20}
$$

Suppose now to repeat the previous procedure j_o times getting, for any integer $0 \le j \le j_o$,

$$\begin{aligned}
&D_\omega p^{(j)} + V_\varphi(\theta + q^{(j)}) = \varepsilon_1^{(j)}, \quad \omega + D_\omega q^{(j)} - h_y(p^{(j)}) = \varepsilon_2^{(j)} \\
&p^{(j)} = p^{(0)} + \sum_{k=0}^{j-1} \Delta p^{(k)}, \quad q^{(j)} = q^{(0)} + \sum_{k=0}^{j-1} \Delta q^{(k)} \\
&\|\Delta p^{(k)}\|_{\xi_k - 2\delta_k} \le \Delta\rho_k, \quad \|\Delta q^{(k)}\|_{\xi_k - 2\delta_k} \le \delta_k, \quad \Delta\rho_k = \frac{1}{2}\frac{\rho' - \overline{\rho}}{2^{k+1}} \\
&\|\mathrm{Im}\, p_i^{(j)}\|_{\xi_j} \le \rho'_j, \quad \|q_\theta^{(j)}\|_{\xi_j} \le Q_j = Q' - \frac{(Q' - Q)}{2^j} \\
&\|p_\theta^{(j)}\|_{\xi_j} \le N_j = 2N(1 - \frac{1}{2^{j+1}}), \quad \theta \in \Delta_{\xi_j}, \quad \|\varepsilon_1^{(j)}\|_{\xi_j} = E_1^{(j)}, \quad \|\varepsilon_2^{(j)}\|_{\xi_j} = E_2^{(j)} \\
&\rho'_j = \overline{\rho} + \sum_{k=0}^{j-1} \Delta\rho_k, \quad \xi_j = \xi - 3\sum_{k=0}^{j-1} \delta_k, \quad \delta_k = \frac{1}{6}\frac{\xi - \xi'}{2^k}
\end{aligned} \tag{12.21}$$

when $j = 0$ we have the quantities of the Lemma 7.2 i.e. $\rho'_0 = \overline{\rho}$, $\xi_0 = \xi$, $N_0 = N$, $Q_0 = Q$.

Lemma A4.2 *Suppose (12.21) are verified for $0 \le j \le j_0$ (see (7.8). If*

$$\begin{aligned}
&B_6 2^{35l} 3^{8l} (l!)^4 \frac{(1 + a\gamma N)^4 (1 + \gamma^2 a V)(1 + \gamma\rho a)}{\gamma a(\rho - \rho')(\xi - \xi')^{1+8l}} \\
&\qquad \Big[\frac{a}{a'}\frac{1}{Q' - Q} + \frac{1}{\gamma a'(\overline{\rho} - \rho')} + \frac{1}{\gamma a' N}\Big]\big(\gamma(\|\varepsilon_2\| + \gamma a\|\varepsilon_1\|)_\xi\big) \le 1
\end{aligned} \tag{12.22}$$

there exists $(\Delta p^{(j_0)}, \Delta q^{(j_0)})$ such that

$$\begin{aligned}
&\|\Delta q^{(j_0)}\|_{\xi_{j_0} - 2\delta_{j_0}} \le B_1 \gamma s^2(\delta_{j_0}) \frac{a}{a'}(1 + \gamma a N_{j_0})(E_2^{(j_0)} + \gamma a E_1^{(j_0)}) \le \delta_{j_0} \\
&\|\Delta p^{(j_0)}\|_{\xi_{j_0} - 2\delta_{j_0}} \le B_2 \frac{s^2(\delta_{j_0})}{a'}(1 + \gamma a N_{j_0})^2 (E_2^{(j_0)} + \gamma a E_1^{(j_0)}) \le \Delta\rho_{j_0} \\
&Q^{(j_0)} + \|\Delta q_\theta^{(j_0)}\|_{\xi_{j_0} - 3\delta_{j_0}} \le Q^{(j_0+1)} \le \frac{1}{2}, \quad N_{j_0} + \|\Delta p^{(j_0)}\|_{\xi_{j_0} - 3\delta_{j_0}} \le N_{j_0+1} \\
&N_{j_0} \le 2N
\end{aligned} \tag{12.23}$$

The functions $p^{(j_0+1)} = p^{(j_0)} + \Delta p^{(j_0)}$, $q^{(j_0+1)} = q^{(j_0)} + \Delta q^{(j_0)}$ satisfy

$$D_\omega p^{(j_0+1)} + V_\varphi(\theta + q^{(j_0+1)}) = \varepsilon_1^{(j_0+1)}, \quad \omega + D_\omega q^{(j_0+1)} - h_y(p^{(j_0+1)}) = \varepsilon_2^{(j_0+1)} \tag{12.24}$$

and

$$\|\varepsilon_2^{(j_0+1)}+\gamma a\varepsilon_1^{(j_0+1)}\|_{\xi_{j_0}-2\delta_{j_0}}\le (E_2^{(j_0)}+\gamma aE_1^{(j_0)})^2(1+\gamma aN_{j_0})^4(1+\gamma aV)(1+\gamma\rho a)\cdot \\ \cdot\frac{B_4s^4(\delta_{j_0})}{\delta_{j_0}a(\rho-\rho'_{j_0}-\Delta\rho_{j_0})} \tag{12.25}$$

Proof It is quite long but straightforward. The functions $\Delta q^{(j_0)}$ and $\Delta p^{(j_0)}$ with their bounds are given by the estimates after Lemma A4.1 ((12.17) and (12.18), when the quantities there present are substituted by those one with the index j_0). The bounds $\|\Delta q^{(j_0)}\|_{\xi_{j_0}-2\delta_{j_0}}\le\delta_{j_0}$, $\|\Delta p^{(j_0)}\|_{\xi_{j_0}-2\delta_{j_0}}\le\Delta\rho_{j_0}$, $Q^{(j_0)}+\|\Delta q_\theta^{(j_0)}\|_{\xi_{j_0}-3\delta_{j_0}}\le Q^{(j_0+1)}$, $N_{j_0}+\|\Delta p^{(j_0)}\|_{\xi_{j_0}-3\delta_{j_0}}\le N_{j_0+1}$, taking into account the values of δ_j, $\Delta\rho_j$ and N_j, Q_j, $N_j\le 2N$ are implied by

$$4B_2(1+\gamma aN)^2s^2(\delta_{j_0})\Big[\frac{2^{j_0+1}}{\delta_{j_0}(Q'-Q)}\frac{a}{a'}+\frac{1}{\gamma a'\Delta\rho_{j_0}}+\frac{2^{j_0}}{N\delta_{j_0}\gamma a'}\Big]\cdot \\ \cdot\gamma[E_2^{(j_0)}+\gamma aE_1^{(j_0)}]\le 1 \tag{12.26}$$

i.e.

$$B_5(1+\gamma aN)^2\frac{(2^{4l}l!)^26^{4l}}{(\xi-\xi')^{4l+1}}2^{(2+4l)j_0}\Big[\frac{1}{(Q'-Q)}\frac{a}{a'}+\frac{1}{\gamma a'(\rho'-\overline{\rho})}+\frac{1}{N\gamma a'}\Big]\cdot \\ \cdot\gamma[E_2^{(j_0)}+\gamma aE_1^{(j_0)}]\le 1 \tag{12.27}$$

As $\rho-\rho'_{j_0}-\Delta\rho_{j_0}\ge\rho-\rho'$ (12.25) implies

$$\begin{aligned}E_2^{(j_0+1)}+\gamma aE_1^{(j_0+1)}&\le\frac{1}{a}(E_2^{(j_0)}+\gamma aE_1^{(j_0)})^22^4(1+\gamma aN)^4(1+\gamma aV)(1+\gamma a\rho)\cdot\\ &\quad\cdot\frac{B_4}{a(\rho-\rho')}\frac{2^{25l}3^{8l+1}(l!)^42^{j_0(8l+1)}}{(\xi-\xi')^{8l+1}}=\\ &=\frac{1}{a}(E_2^{(j_0)}+\gamma aE_1^{(j_0)})^2(1+\gamma aN)^4(1+\gamma aV)(1+\gamma a\rho)\cdot\\ &\quad\cdot\frac{B_5}{a(\rho-\rho')}\frac{2^{25l}3^{8l}(l!)^42^{j_0(8l+1)}}{(\xi-\xi')^{8l+1}}\end{aligned}$$

which we rewrite as

$$\gamma[E_2^{(j_0+1)}+\gamma aE_1^{(j_0+1)}]\le[\gamma(E_2^{(j_0)}+\gamma aE_1^{(j_0)})]^22^{j_0(1+8l)}\frac{S}{\gamma} \tag{12.28}$$

and

$$S = \frac{B_5}{a(\rho-\rho')}(1+\gamma aN)^4(1+\gamma aV)(1+\gamma a\rho)\frac{2^{25l}3^{8l+1}(l!)^4}{(\xi-\xi')^{8l}}$$

Iterating, (12.28) becomes

$$\gamma[E_2^{(j_0)}+\gamma aE_1^{(j_0)}]\le[\gamma(E_2^{(0)}+\gamma aE_1^{(0)})]^{2^{j_0}}\big(\frac{S}{\gamma}2^{1+8l}\big)^{2^{j_0}-1} \tag{12.29}$$

and inserting it in (12.27) we get

$$\begin{aligned}B_5(1+\gamma aN)^2\frac{(2^{4l}l!)^26^{4l}}{(\xi-\xi')^{4l+1}}2^{(2+4l)j_0}\Big[\frac{1}{(Q'-Q)}\frac{a}{a'}+\frac{1}{\gamma a'(\rho'-\overline{\rho})}+\frac{1}{N\gamma a'}\Big]\cdot\\ [\gamma(E_2^{(0)}+\gamma aE_1^{(0)})]^{2^{j_0}}\big(\frac{S}{\gamma}2^{1+8l}\big)^{2^{j_0}-1}\le 1\end{aligned} \tag{12.30}$$

Rewrite (12.30) as $R2^{j_0\alpha}(\gamma A)^{2^{j_0}}q^{2^{j_0}-1}\le 1$ (with obvious meaning of the letters) which is implied by

$$R2^{j_0\alpha}(\gamma A)^{2^{j_0}}\tilde q^{2^{j_0}-1}\le 1 \tag{12.31}$$

$\tilde q\doteq q\Big[\frac{1}{(Q'-Q)}\frac{a}{a'}+\frac{1}{\gamma a'(\rho'-\overline{\rho})}+\frac{1}{N\gamma a'}\Big]$ ($\frac{\tilde q}{R}>1$). By induction $A\le A_o=\frac{1}{\gamma\tilde q}2^{-\alpha}$ implies (12.31) for any $j_0\ge 1$ i.e. (12.22).

By (12.22) the bounds on the four sequences $\{q^{(j)}\},\{p^{(j)}\}\{q_\theta^{(j)}\},\{q_\theta^{(j)}\}$ follow

$$\sum_{k=M'}^{M}\|\Delta q^{(k)}\|_{\xi_k-2\delta_k}\le\sum_{k=M'}^{M}\delta_k,\quad \sum_{k=M'}^{M}\|\Delta p^{(k)}\|_{\xi_k-2\delta_k}\le\sum_{k=M'}^{M}\Delta\rho_k$$

$$\sum_{k=M'}^{M}\|\Delta q_\theta^{(k)}\|_{\xi_k-3\delta_k}\le\sum_{k=M'}^{M}B_1\frac{a}{a'}\frac{s^2(\delta_k)}{\delta_k}(1+\gamma aN_k)\gamma(E_2^{(k)}+\gamma aE_1^{(k)})\le$$

$$\le\sum_{k=M'}^{M}(Q_{k+1}-Q_k)\le\sum_{k=M'}^{M}2^{-k}$$

$$\gamma a\sum_{k=M'}^{M}\|\Delta p_\theta^{(k)}\|_{\xi_k-3\delta_k}\le\sum_{k=M'}^{M}B_2\frac{a}{a'}\frac{s^2(\delta_k)}{\delta_k}(1+\gamma aN_k)^2\gamma(E_2^{(k)}+\gamma aE_1^{(k)})\le$$

$$\le\sum_{k=M'}^{M}(N_{k+1}-N_k)\le\sum_{k=M'}^{M}2^{-k}$$

They are Cauchy sequences in $C^\omega(\Delta_{\xi'};\mathbb{C}^l)\cap C(\overline{\Delta}_{\xi'};\mathbb{C}^l)$ which is a complete space so the sequences converge to the functions $\{q^{(\infty)}\},\{p^{(\infty)}\},\{q_\theta^{(\infty)}\},\{p_\theta^{(\infty)}\}$ and being

$$\gamma\|\varepsilon_2^{(j_0+1)}+\gamma a\varepsilon_1^{(j_0+1)}\|_{\xi_{j_0}-2\delta_{j_0}}\le 2^{-j_0}$$

$\{q^{(\infty)}\}$, $\{p^{(\infty)}\}$ verify the equation

$$D_\omega p^{(\infty)} + V_x(\theta + q^{(\infty)}) = 0, \quad \omega + D_\omega q^{(\infty)} - h_y(p^{(\infty)}) = 0$$

Next we show (7.14).

$$\begin{aligned}
&\gamma a \|p^{(\infty)} - p^{(0)}\|_{\xi'} \le \gamma a \sum_{k=0}^{\infty} \|\Delta p_\theta^{(k)}\|_{\xi_k - 3\delta_k} \le \\
&\le 4\frac{a}{a'} B_2 (1+\gamma a N)^2 \Big\{ \frac{s^2(\delta_0)}{\delta_0} \gamma(\gamma a E_1^{(0)} + E_2^{(0)}) + \gamma \sum_{k=1}^{\infty} \frac{s^2(\delta_k)}{\delta_k} (\gamma a E_1^{(k)} + E_2^{(k)}) \Big\}
\end{aligned} \tag{12.32}$$

Using the definition of δ_k, (12.29) becomes

$$\begin{aligned}
&\gamma \sum_{k=1}^{\infty} \frac{s^2(\delta_k)}{\delta_k} (\gamma a E_1^{(k)} + E_2^{(k)}) \le \\
&\le \sum_{k=1}^{\infty} \frac{2^{8l} (l!)^2 6^{4l+1}}{(\xi - \xi')^{1+4l}} 2^{(4l+1)k} [\gamma(\gamma a E_1^{(0)} + E_2^{(0)})]^{2^k} \big(\frac{S}{\gamma} 2^{1+8l}\big)^{2^k - 1}
\end{aligned}$$

By induction, if

$$[\gamma(\gamma a E_1^{(0)} + E_2^{(0)})] \big(\frac{S}{\gamma} 2^{1+8l}\big) 2^{4l+1} \le 1$$

(a condition implied by (12.22)) the following holds for any $k \ge 1$

$$2^{(4l+1)k} [\gamma(\gamma a E_1^{(0)} + E_2^{(0)})]^{2^k} \big(\frac{S}{\gamma} 2^{1+8l}\big)^{2^k - 1} \le [\gamma(\gamma a E_1^{(0)} + E_2^{(0)})]^k$$

As a consequence

$$\begin{aligned}
&\gamma a \|p_\theta^{(\infty)} - p_\theta^{(0)}\|_{\xi'} \le \gamma a \sum_{k=0}^{\infty} \|\Delta p_\theta^{(k)}\|_{\xi_k - 3\delta_k} \le \\
&\le 4\frac{a}{a'} B_2 (1+\gamma a N)^2 \Big\{ \frac{s^2(\delta_0)}{\delta_0} \gamma(\gamma a E_1^{(0)} + E_2^{(0)}) + \sum_{k=1}^{\infty} [\gamma(\gamma a E_1^{(0)} + E_2^{(0)})]^k \Big\} \le \\
&\le 4\frac{a}{a'} B_2 (1+\gamma a N)^2 \frac{2^{8l} (l!)^2 6^{4l+1}}{(\xi - \xi')^{4l+1}} 3[\gamma(\gamma a E_1^{(0)} + E_2^{(0)})]
\end{aligned} \tag{12.33}$$

An identical computation is needed for

$$\|q_\theta^{(\infty)} - q_\theta^{(0)}\|_{\xi'}, \quad \gamma a \|p^{(\infty)} - p^{(0)}\|_{\xi'}, \quad \|q^{(\infty)} - q^{(0)}\|_{\xi'}$$

Moreover

$$D_\omega q^{(\infty)} - D_\omega q^{(0)} = h_A(p^{(\infty)}) - \omega - \varepsilon_2 + \omega - h_A(p^{(0)}) =$$
$$= -\varepsilon_2 + \int_0^1 d\lambda h_{AA}(\lambda p^{(\infty)} + (1-\lambda)p^{(0)})(p^{(\infty)} - p^{(0)})$$

and then

$$\gamma \| D_\omega q^{(\infty)} - D_\omega q^{(0)} \|_{\xi'} \le \gamma \|\varepsilon_2\|_{\xi'} + \gamma a \|p^{(\infty)} - p^{(0)}\|_{\xi'} \; \blacksquare$$

Lemma A4.1 and A4.2 lead to Lemma 7.2.

Appendix 5. Full Measure of Diophantine vectors in $\mathbb{R}^N$

Let $\mathcal{D}_\tau$ denote the set of vectors ω in $\mathbb{R}^N$ which are (γ,τ)–Diophantine [see (5.3)] for some $\gamma > 0$ and let ℓ be Lebesgue measure.

Proposition A5.1 *$\ell(\mathbb{R}^N \backslash \mathcal{D}_\tau)$=0 provided $\tau > N-1$.*

Proof It is enough to check that $\overline{\mathcal{D}}_{R,\tau} \equiv \{\omega : \|\omega\| \le R,\ \omega \notin \mathcal{D}_\tau\}$ has Lebesgue measure zero for any $R > 0$. Now, if

$$\mathcal{C}_{R,\gamma} \equiv \{\omega : \|\omega\| \le R \ \text{ and } \ \exists\ 0 \ne n \in \mathbb{Z}^N :\ |\omega \cdot n| < \frac{1}{\gamma |n|^\tau}\} \tag{13.1}$$

(recall that $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$ while $|\cdot|$ is the sum of absolute values), then $\overline{\mathcal{D}}_{R,\tau} = \cap_{\gamma>0} \mathcal{C}_{R,\gamma}$ and the claim follows from the following estimate:

$$\begin{aligned} \ell(\mathcal{C}_{R,\gamma}) &\le \sum_{n\ne 0} \ell(\{\omega : \|\omega\| \le R\ ,\ |\omega\cdot n| < \frac{1}{\gamma|n|^\tau}\}) \\ &\le \sum_{n\ne 0} \frac{b_1 R^{N-1}}{\gamma |n|^\tau \|n\|} < \frac{b_2 R^{N-1}}{\gamma} \end{aligned} \tag{13.2}$$

where b_1, b_2 are suitable (N,τ–dependent) positive constants. $\blacksquare$

Appendix 6. Cauchy estimates on holomorphic functions

We start with a basic (elementary) tool (a “Cauchy estimate”):

Lemma A6.1 *Let g be an analytic function from $\mathcal{D} \subset \mathbb{C}^N \to \mathbb{C}$ ($\partial\mathcal{D}$ smooth). Then for any subdomain $\mathcal{D}' \subset \mathcal{D}$ with $\delta \equiv$ dist.$(\mathcal{D}', \partial\mathcal{D}) > 0$ and any $n \in \mathbb{N}^N$, one has:*

$$\|\partial_z^n g\|_{\mathcal{D}'} \equiv \sup_{\mathcal{D}'} |\frac{\partial^{|n|} g}{\partial_{z_1}^{n_1} \cdots \partial_{z_N}^{n_N}}| \le |n|!\ \delta^{-|n|}\ \|g\|_{\mathcal{D}} \tag{14.1}$$

The proof is a straightforward exercise based upon Cauchy's integral formula using the contour $|\zeta_h - z_h| = \delta$, $h = 1, ..., N$ ($z \in \mathcal{D}'$ fixed and $\zeta \in \mathcal{D}$ variable of integration. The exercise is carried out, *e.g.* , in [CC2]).

As an application of Lemma 4.1 we prove a useful bound. If $r \in (0,1)$, $p \in \mathbb{N}$ and $N \in \mathbb{Z}_+$, one has:

$$\Sigma_p^N(r) \equiv \sum_{n\in\mathbb{Z}^N} |n|^p e^{-r|n|} \ \le\ p!\ \big(\frac{4}{r}\big)^{p+N} \tag{14.2}$$

Proof of (14.2) For any $r > 0$

$$\Sigma_p^N = (-1)^p \partial_r^p \sum_{n\in\mathbb{Z}^N} e^{-r|n|} = (-1)^p \partial_r^p \big(\frac{e^r+1}{e^r-1}\big)^N \equiv (-1)^p \partial_r^p E(r)^N \tag{14.3}$$

Now, the function $g(x) \equiv (-1)^p E(x)^N$ is analytic and bounded in $B_{r-s}(r) \equiv \{x \in \mathbb{C}:\ |x-r| < r-s\}$, for any $0 < s < r$. Applying Lemma A2.1with $\mathcal{D}' \equiv B_\varepsilon(r)$ ($0 < \varepsilon < r-s$ arbitrary), $\mathcal{D} \equiv B_{r-s}(r)$ and noting that $\|E\|_{\mathcal{D}} = E(s) < 1 + \frac{2}{s}$, one gets:

$$\Sigma_p^N < \|\partial_x^p g\|_{B_\varepsilon(r)} \le \frac{p!}{(r-s-\varepsilon)^p}\big(1+\frac{2}{s}\big)^N \tag{14.4}$$

Inequality (14.2) follows by taking $s = \frac{3}{4}r$ and using the arbitrariness of ε. ∎

In the hypotheses of Theorem 6.3 there is the assumption of the existence of $(A^{(0)}, \varphi^{(0)})$ and $\omega^{(0)}$ Diophantine. In the next Lemma this fact is proved. Pick a (γ, l_0) Diophantine vector ω and let's define the vector $\beta\omega$ with $\beta \in \mathbb{R}$ (it will be the vector $\omega^{(0)}$ of the theorem 6.3.)

Lemma A6.2 *If β is a real number sufficiently great there exists the solution $(v^{(0)}, u^{(0)})$ of the system*

$$\beta D_\omega v^{(0)} + V^{(I_0)}_{\varphi^{(0)}}(\theta^{(0)} + u^{(0)}) = 0, \qquad \beta\omega + \beta D_\omega u^{(0)} - h_{A^{(0)}}(v^{(0)}) = 0$$

close to $(\tilde v^{(0)}, \tilde u^{(0)}) \doteq \big(0, h^{-1}_{A^{(0)}}(\beta\omega) - \frac{1}{\beta} D_\omega^{-1} V^{(I_0)}_{\varphi^{(0)}}(\theta^{(0)})\big)$. $(v^{(0)}, u^{(0)})$ *is analytic in* $\Delta^{l_0}_{\frac{1}{2}\sigma}$

Proof Apply Lemma 7.2 observing that

$$\varepsilon_1 \equiv 0, \qquad \varepsilon_2 = \beta\omega - h_{A^{(0)}}\big(h^{-1}_{A^{(0)}}(\beta\omega) - \beta^{-1} D_\omega^{-1} V^{(I_0)}_{\varphi^{(0)}}(\theta^{(0)})\big)$$

i.e.

$$-\int_0^1 d\lambda h_{A^{(0)}A^{(0)}}\big(h^{(-1)}_{A^{(0)}}(\beta\omega) - \lambda\beta^{-1} D_\omega^{-1} V^{(I_0)}_{\varphi^{(0)}}(\theta^{(0)})\big)\beta^{-1} D_\omega^{-1} V^{(I_0)}_{\varphi^{(0)}}(\theta^{(0)}) \tag{14.5}$$

and due to (2.10) it can be arbitrarily small as we need with β great satisfying (7.12) adapted to this case.
The relations $\|u_\theta\|_\xi \le Q < \frac{1}{2}$ is of course verified being $\tilde{u}^{(0)} \equiv 0$ while the analogous of $\|\mathrm{Im}\, v_i\|_\xi \le \overline{\rho} < \rho$ is $\|\mathrm{Im}\, \tilde{v}_i^{(0)}\|_\xi \le \overline{\rho}^{(0)} < \rho$ and then, for a $\sigma' < \sigma$, $\frac{1}{\beta} V^{(0)} \gamma s(\sigma - \sigma') < \rho$.
The solution $(v^{(0)}, u^{(0)})$ is analytic in $\Delta^{l_0}_{\sigma''}$, $\sigma'' < \sigma'$.∎
Remark 14.1 i) σ'' becomes ξ_{l_0} ii) As a corollary of Lemma A6.2 and Theorem 6.3 we have Theorem 6.1

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