On the cohomology of discrete abelian group actions

Oliver Knill

Abstract

Generalizing a result of Depauw, we prove that the geometric cohomology groups for a $\mathbf{Z}^d$ action are in any dimension isomorphic to the Feldman-Moore cohomology groups rsp. the algebraic group cohomology groups. This result holds in the smooth, topological or measurable category and for general Polish groups as coefficient groups. The groups discussed here lead to algebraic invariants of dynamical systems like tiling dynamical systems and play a role in classes of higher dimensional discrete random difference operators.

1 Introduction

The topic of this paper are cohomological groups for dynamical systems. These groups are defined, when an abelian group $G = \mathbf{Z}^d$ acts as a group of transformations on a probability space, a topological space or a smooth manifold.

Our main result is that if the group action satisfies an aperiodicity condition, then the geometric cohomology coincides with the algebraic group cohomology of Eckmann-Eilenberg-McLane. The algebraic cohomology has earlier been proven equivalent to the orbit cohomology, so that there are three cohomologies which agree.

Despite the equivalence of the cohomologies, each has its advantages and generalizes to situations where the others don't. The algebraic group cohomology can be extended to groups acting on nonabelian groups and is formally related to simplicial cohomology in differential topology. The orbit cohomology of Feldman-Moore can be defined more generally for equivalence relations. It is useful in the ergodic theory of aperiodic tilings which are in general not Cayley graphs of a finitely presented group and can serve as an invariant for tiling dynamical systems in the topological situation as we will see later. The third geometric cohomology has been defined in two dimensions in the theses of Depauw [14] and in arbitrary dimensions in our theses [28]. For smooth $\mathbf{Z}^d$ actions and real coefficients, the cohomology has been investigated in [23]. It is formally related to de Rham cohomology in differential topology and is useful in mathematical physics, where these cohomology groups arise as obstructions to the construction of random fields.

Let us consider an example of a geometric cohomological group. Assume that $T_1, T_2, \dots T_d$ generate an ergodic $\mathbf{Z}^d$ action on the probability space (X, m). This $\mathbf{Z}^d$ action operates as automorphisms on the additive group of the Banach algebra $\mathcal{U} = L^\infty(X)$. Consider a "one-form" $A = (A_1, A_2, \dots, A_d) \in \mathcal{U}^d$. Its "curl" is $F_{ij} = (dA)_{ij} = (A_i(T_j) - A_i - (A_j(T_i) - A_j)) \in L^\infty(X, \mathbf{R})$, where i, j runs over all pairs $i < j$. For $G \in L^\infty(X)$ the "gradient" $(G(T_1) - G, G(T_2) - G, \dots, G(T_d) - G)$, is a "one-form". One sees immediately, that the 'curl' of a 'gradient' vanishes. An obvious cohomology problem is to determine the vector space of one-forms with zero rotation modulo the vector space of gradients. For any constant vector field $v = (c_1, \dots, c_d)$ with $c_i \in \mathbf{R}$, the curl vanishes but v is not a gradients if $(c_1, \dots, c_d) \neq (0, \dots, 0)$. So, the first cohomology group $H^1(\mathbf{R})$ with real coefficients contains at least the group $\mathbf{R}^d$. If $H^1(\mathbf{R})$ isomorphic to $\mathbf{R}^d$, then the cohomology group is called trivial. In the C^∞ case, when T_j are hyperbolic automorphisms of the torus $X = \mathbf{T}^N$ and $\mathcal{U} = L^\infty(X, U)$, where U is a unitary group, then the first cohomology group and indeed all higher cohomology groups except the highest one are trivial [23]. As a second example, consider the $(d-1)$-form $A = (A_1, \dots, A_d) \in \mathcal{U}^d$. Its divergence is $dA = \sum_{j=1}^d (A_j(T_j) - A_j)$. The d'th cohomology group is the vector space $H^d(\mathbf{R}) = \mathcal{U}/\{dA\}$. Again, this vector space contains at least $\mathbf{R}$ because $\int_X dA(x)\ dm(x) = 0$. However, if U is compact abelian like $U = \mathbf{T}^1$, then $H^d(U) = \{0\}$ if the $\mathbf{Z}^2$ action is free. This has been noticed by Feldman and Moore: for $d = 2$, it implies for example that for any circle-valued measurable map f and any two rationally independent rational numbers α, β, we can find two circle-valued measurable maps g, h such that $f(x) = g(x+\alpha) - g(x) + h(x+\beta) - h(x)$. Lind has shown that in the case $d = 2$, this result is also true for nonabelian U [33].

In the case of group actions by measure preserving transformations, only the first cohomology group $H^1(U)$ is nontrivial. It depends only on the coefficient group U. This has applications in mathematical physics because it allows to define new classes of Schrödinger operators [29, 30]. We will see in an appendix that $H^1(U)$ is uncountable.

While equivalence of the cohomologies still holds, the triviality of the higher cohomologies is no more expected to hold if X is a topological space and $\mathcal{L}(X,U)$ consists of continuous functions or when X is a smooth manifold and $\mathcal{U}$ is a group of smooth maps from X to a Lie group U. In those cases, they lead to invariants of the corresponding categorie of dynamical system.

2 Comparison of the three cohomologies

We consider three cohomological constructions for a pair $(G,\mathcal{U})$, where $G = \mathbf{Z}^d$ is an abelian group of Borel automorphisms (or homeomorphisms or diffeomorphisms) which acts on the abelian group $\mathcal{U} = \mathcal{L}(X,U) = \{X \to U$, measurable continuous smooth $\}$ and where U is an abelian Polish group. We write the group operation in G and U additively. Let $T_1, \dots, T_d$ be d commuting automorphism on the probability space $(X, \mathcal{F}, m)$ which generate the $\mathbf{Z}^d$-action. In the topological setup, we assume T_i to be homeomorphisms on a topological space $(X, \mathcal{F})$, in a geometric setup, we assume the T_i to be diffeomorphisms on a smooth manifold X. We use the notation $T^g = \prod_{i=1}^d T_i^{g_i}$ for $g = (g_1, g_2, \dots, g_d) \in \mathbf{Z}^d$.

I) Algebraic group cohomology (Eckmann-Eilenberg-McLane) (see [17])
The group $G = \mathbf{Z}^d$ acting on X induces a G-action on the abelian group $\mathcal{U} = \mathcal{L}(X,U)$ of all measurable (continuous, smooth) maps from X to U. Define for $0 \le p \le d$ the set $\mathcal{C}^p$ of maps $a : G^{p+1} \to \mathcal{U}$ which satisfies $T^g a(g_0, g_1, \dots, g_p) = a(g+g_0, g+g_1, \dots, g+g_p)$. Define the map $d_p : \mathcal{C}^p \to \mathcal{C}^{p+1}$

$$(d_p a)(g_0, \dots, g_{p+1}) = \sum_{j=0}^{p+1} (-1)^j a(g_0, \dots, \hat{g}_j, \dots, g_p) .$$

We use a notation in which the entry $\hat{g}_j$ has been deleted. Elements in the kernel of d_p are *algebraic cocycles of degree* p, elements in the image of d_{p-1} are the *algebraic coboundaries of degree* n. Since $d_{p+1} \circ d_p = 0$, this gives the *algebraic group cohomology groups* $\mathcal{H}_{alg}(G,\mathcal{U}) = \ker(d_p)/\mathrm{im}(d_{p-1})$.

II) Orbit cohomology (Feldman-Moore) (see [18])
Define $\mathcal{R}^0 = X$ and $\mathcal{R}^p = \{(x_0, \dots, x_p) \in X^{p+1} \mid \exists\, g_i \in G,\ x_i = T^{g_i} x_0\}$. Define the set $\mathcal{C}^p(\mathcal{R}_G, U)$ of all measurable (continuous, smooth) maps $a : \mathcal{R}^p \to U$. Consider the map $d_p : \mathcal{C}^p \to \mathcal{C}^{p+1}$ defined by

$$(d_p a)(x_0, \dots, x_{p+1}) = \sum_{j=0}^{p+1} (-1)^j a(x_0, \dots, \hat{x}_j, \dots, x_p) .$$

$\ker(d_p)$ consist of *orbit cocycles of degree* p, while $\mathrm{im}(d_{p-1})$ are *orbit coboundaries of degree* p. Since $d_{p+1} \circ d_p = 0$, one obtains the *orbit cohomology groups* $\mathcal{H}^p_{orb}(\mathcal{R}, U) = \ker(d_p)/\mathrm{im}(d_{p-1})$. This cohomology can be defined more generally for (hyperfinite) equivalence relations (see [18]).

III) Geometric group cohomology (see [23, 14, 28]).
Let $I = \{1, \dots, d\}$ and let $\mathcal{I}_p$ be the set of sets $J = \{j_1 < j_2 < \dots < j_p\} \subset I$. Let $\mathcal{C}^p$ be the set of maps $A : \mathcal{I}_p \to \mathcal{U}$ which becomes a group by pointwise addition. Extend this map to the set of all p-tuples $J = (j_1, j_2 \dots, j_p)$ with $j_k \in I$ by requiring $A_{\pi(J)} = \mathrm{sign}(\pi) A_J$ for any permutation π of J. We write $A = \sum_J A_J \tau_J$. Define $d_p : \mathcal{C}^p \to \mathcal{C}^{p+1}$ by

$$d_p A = \sum_{i,J} (A_J(T_i) - A_J) \tau_{iJ} \ .$$

The kernel of d_p contains *cocycles of degree* p, whereas the image of d_{p-1} consists of *coboundaries of degree* p. Because for $A = \sum_J A_J \tau_J$

$$d_p \circ d_{p-1} A = \sum_{i,j,J} [A_J(T_i T_j) - A_J(T_i) - A_J(T_j) + A_J] \tau_{ijJ}$$

is both symmetric and antisymmetric in i, j, it must vanish and $d_p \circ d_{p-1} = 0$ gives rise to *geometric cohomology groups* $\mathcal{H}^p_{geom}(G, \mathcal{U}) = \ker(d_p)/\mathrm{im}(d_{p-1})$.

Remark. In [23], geometric cohomology was considered for $\mathcal{U} = C^\infty(M, \mathbf{R}^l)$, where T_j are hyperbolic automorphisms of the torus $M = \mathbf{T}^d$. The definition in [23] is slightly different but equivalent to the definition given here.

The purpose of our article is to show that the cohomology in III) is isomorphic to the other two.

3 Isomorphism of Feldman-Moore and group cohomology

In this section, we review the equivalence of the Feldman-Moore and group cohomology.

Definition. A group G of measure preserving transformations acts *freely* on the probability space X if $m(\{x \in X \mid T^n x = x\}) = 0$ for all $n \in G \setminus \{0\}$. If G is a group of homeomorphisms or diffeomorphisms, we ask that the set $\{x \in X \mid T^n x = x\}$ is empty for all $n \in G \setminus \{0\}$.

Theorem 3.1 (Feldman-Moore) *If G acts freely, then $\mathcal{H}^p_{orb}(\mathcal{R}_G, U) \cong \mathcal{H}^p_{alg}(G, \mathcal{U})$ for all $p = 0, \dots, d$.*

Proof. The map $\phi : \mathcal{C}^p(G, \mathcal{U}) \to \mathcal{C}^p(\mathcal{R}_G, U)$, $a \mapsto \overline{a}$

$$\overline{a}(x_0, \dots, x_p) = a(0, g_1, \dots, g_p)(x_0), \ x_i = T^{g_i} x_0$$

is defined if the action of G is free: for (almost) every $(x_i, x_0) \in \mathcal{R}_G$ there exists a unique g_i with $T^{g_i}(x_0) = x_i$. It is invertible:

$$a(g_0, \dots, g_p)(x) = \overline{a}(T^{g_0}(x), T^{g_1}(x), \dots, T^{g_p}(x)) \ .$$

Because ϕ commutes with d_p:

$$\begin{aligned} d_p \overline{a}(x_0, \dots, x_{p+1}) &= \sum_{j=0}^{p+1} (-1)^j \overline{a}(x_0, \dots, \hat{x}_j, \dots, x_p) \\ &= \sum_{j=0}^{p+1} (-1)^j a(0, g_1, \dots, \hat{g}_j, \dots, g_p) \\ &= \overline{d_p a} \ , \end{aligned}$$

cocycles are mapped into cocycles and coboundaries into coboundaries. □

Note that this equivalence also holds for example, when G is a finite abelian group.

4 Equivalence of algebraic and geometric cohomology

In this section, we prove our main result. It generalizes the result in [14], where the equivalence of geometric and algebraic cohomology was shown in the special case $d = 2$ and $U = \mathbf{R}$.

Theorem 4.1 (Main theorem) *If $G = \mathbf{Z}^d$ acts freely on $\mathcal{U}$, then $\mathcal{H}^p_{geom}(\mathbf{Z}^d,\mathcal{U}) \cong \mathcal{H}^p_{alg}(\mathbf{Z}^d,\mathcal{U})$ for $0 \leq p \leq d$.*

Proof. For $J \subset \{1,\dots,d\}$ with $|J| = p$, consider the cube S_J in $G = \mathbf{Z}^d$ spanned by $\{e_j\}_{j\in J}$. The set S_j is a disjoint union of simplices $\Delta_i = \Delta(g_0^{(i)}, g_1^{(i)}, \dots, g_p^{(i)})$. For a simplex $\Delta(g_0,\dots,g_p)$ and an algebraic cocycle $a \in \mathcal{C}^p$, write $\int_{\Delta(g_0,\dots,g_p)} a = a(g_0,\dots,g_p)$. If S is a finite union of disjoint simplices Δ_i, we set

$$\int_S a = \sum_i \int_{\Delta_i} a \ .$$

Define a map $\phi : \mathcal{C}^p_{alg} \to \mathcal{C}^p_{geom}$ by

$$\phi(a)_J = A_J = \int_{S_J} a \ .$$

(i) ϕ commutes with d.
Let $\sigma_{iJ,K}$ be the signature of the permutation $iJ \mapsto K$.

$$\begin{aligned} d\phi(a)_K = (dA)_K &= \sum_{i\cup J=K} (A_J(T_i) - A_J)\tau_{iJ} = \sum_{i\cup J=K} \sigma_{iJ,K}(A_J(T_i) - A_J)\tau_K \\ &= \sum_{i\cup J=K} \sigma_{iJ,K} \int_{S_J} (a(T_i) - a)\tau_K = \int_{\delta S_K} a = \int_{S_K} da = \phi(da)_K \ . \end{aligned}$$

(ii) ϕ maps cocycles into cocycles and coboundaries into coboundaries.
Proof. From (i), we know that $da = 0$ implies $dA = d\alpha = \phi(da) = \phi(0) = 0$. Also $a = dc$ implies $A = \phi(a) = \phi(dc) = d\phi(c) =: dC$.

(iii) Some definitions: a p-dimensional simplex and a $(d-p)$-dimensional simplex are called *transversal*, if their intersection is a point which is in the interior of both simplices. There are only countably many simplices $\Delta(g_0,\dots,g_p), g_i \in \mathbf{Z}^d$. To each p-dimensional oriented cube $S_J(g) = \{g + e_i \cdot [0,1] \mid i \in J\}$ in $\mathbf{Z}^d$ is attached a $(d-p)$-dimensional oriented dual cube $S_{J^*}(g) = \{g + e_i \cdot [0,1] \mid i \in J^* = \{1,\dots,d\} \setminus J\}$. One has $S_K(g) \cap S_{J^*}(g) = \{g\}$ and their Cartesian product is a positively oriented d-dimensional cube. Denote by W_p the union of all p-dimensional cubes in $\mathbf{Z}^d$ and call it the *p-dimensional web* in $\mathbf{Z}^d$. A p-dimensional simplex Δ is called transversal to a $(p-d)$-dimensional web W_{d-p}, if Δ intersects every cube in W_{d-p} transversely. To every transversal intersection of two cubes is attached a sign, defined by the orientation of the cube generated the union of the generators of the individual cubes.

(iv) For each $1 \leq p \leq d$, there exists a $\mathbf{Z}^d-$ periodic homeomorphic deformation $\tilde{W_p}$ of the $(d-p)$-dimensional web W_p in $\mathbf{R}^d$, such that every simplex Δ has only transversal intersections with the $(d-p)$-dimensional deformed dual cubes $\tilde{S} \subset \tilde{W_p}$.
Proof. The set $\mathcal{W}$ of $\mathbf{Z}^d$-periodic homeomorphisms of $\mathbf{R}^d$ is a complete metric space with metric $d(f,g) = \sup_{x\in\mathbf{R}^d/\mathbf{Z}^d} |f(x) - g(x)|$. The set $\mathcal{W}(\Delta)$ of homeomorphisms in $\mathcal{W}$ for which a given

simplex Δ intersects every cube in $\tilde{W}_{d-p}$ transversely, is an open dense set. By Bair's theorem, the intersection of all $\mathcal{W}(\Delta)$, where Δ runs over the countable set of simplices is nonempty and dense.

(iv) Surjectivity of ϕ. If A is a geometric p-cocycle, there exists an algebraic p-cocycle $a = \psi(A)$ such that $\phi(a) = A$.
Proof. Given a geometric cocycle A, define the algebraic cocycle

$$a(g_0, \dots, g_p) = \sum_{\tilde{g}:\tilde{S}_J(\tilde{g}) \cap \Delta(g_0,\dots,g_p) \neq \emptyset} = \psi(A_J)(g) \; .$$

If $S_J(g)$ is a p-dimensional cube, it is a finite union of p-simplices Δ_i. Only one of these simplices has an intersection with the deformed dual cube $\tilde{S}_{J^*}(g)$ since $S_J(g) \cap S_{J^*}(g)$ and so $S_J(g) \cap \tilde{S}_{J^*}(g)$ is a point. This intersection point is by construction of the homeomorphisms in (iv) in the interior of exactly one of the simplices. Therefore

$$\phi(a)_J(g) = \int_{S_J(g)} a = A_J(g) \; .$$

(v) $\mathcal{H}^p_{geom}(\mathbf{Z}^d, \mathcal{U}) \cong \mathcal{H}^p_{alg}(\mathbf{Z}^d, \mathcal{U})$.
Proof. The group homomorphism on cocycles induces the surjective map $\phi : \mathcal{H}^p_{alg}(\mathbf{Z}^d, \mathcal{U}) \to \mathcal{H}^p_{geom}(\mathbf{Z}^d, \mathcal{U})$. We have seen that $\phi \circ \psi(a) = a$.
To see $\psi \circ \phi(A) = A$ and so to establish the bijectivity of ϕ, we can by additivity restrict ourself to the case, when $A = A_J$. Now $\phi(A_J)(g) = A_J(\tilde{g})$ if the dual cube $\tilde{S}_J(\tilde{g})$ intersects $\Delta(g)$ and $\phi(A_J)(g) = 0$ else. Fill the cube S_J with simplices Δ. Only one of these simplices has an intersection with the dual cube $\tilde{S}_J(\tilde{g})$, so that $\psi \circ \phi(A_J) = \sum_\Delta \int_\Delta \phi(A_J)(\tilde{g}) = A_J(\tilde{g})$. $\square$

5 Remarks and illustration of the proof

Depauw [14] has shown the relation of geometric and algebraic cohomology in the special case $d = 2$ and $U = \mathbf{R}$, where the construction of the inverse of ϕ is more direct. Our construction of the deformed web is necessary since we do not assume that one can divide in the group U. In finite groups U for example, not every element g can be written as $g = h + h$.

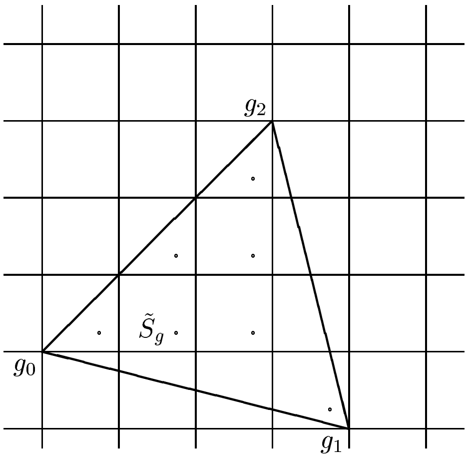

Fig. 1a). Reconstruction of the algebraic cocycle a from the geometric cocycle A in the case $d = p = 2$. The deformed web $\tilde{W}_0$ is zero-dimensional and is disjoint from any line connecting two points in $\mathbf{Z}^2$. The value of $a(g_0, g_1, g_2)$ is the flux through the triangle $\Delta(g_0, g_1, g_2)$ and given by

the sum of the fluxes $A(g)$ through plaquettes $P(g)$ of all the $g \in \mathbf{Z}^2$ for which $\tilde{S}(g)$ is inside the simplex $\Delta(g_0, g_1, g_2)$.

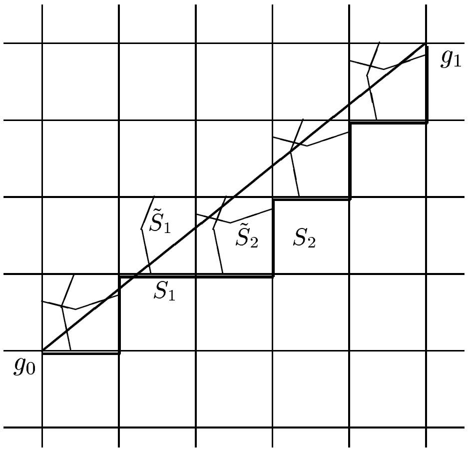

Fig. 1b). Reconstruction of the algebraic cocycle a from the geometric cocycle A in the case $d = 2$, $p = 1$. The deformed web is one dimensional and the figure shows only the part of this web which intersects the line $\Delta(g_0, g_1)$. The value of $a(g_0, g_1)$ is the flux through the line $\Delta(g_0, g_1)$ and given by the sum of the fluxes $A(g)$ through the edges $S_i(g)$ of all the $g \in \mathbf{Z}^2, i = 1, 2$ for which $\tilde{S}_i(g)$ intersects $\Delta(g_0, g_1)$.

Let us illustrate why the case $U = \mathbf{R}$, the proof can also simplify in higher dimensions. In the case $d = 3$ and $p = 2$, we can construct the algebraic cocycle a from a geometric cocycle A by taking a smooth divergence free vector field in $\mathbf{R}^3$ which has the flux A_P through each plaquette P. The flux of this vector field through a triangle $\Delta(g_0, g_1, g_2)$ is then the correct value for $a(g_0, g_1, g_2)$.

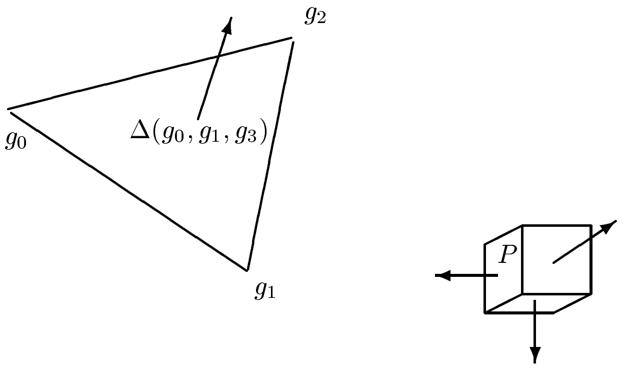

Fig. 2. The problem of constructing an algebraic 2-cocycle from the given geometric 2−cocycle in dimension $d = 3$ is the task to determine the flux through a simplex $\Delta(g_0, g_1, g_2)$, if the flux through all plaquettes P is known.

6 Triviality of cohomology groups in the case $p > 1$

This section is a review. It contains Depauw's proof of the triviality of the $p = 2$ algebraic cohomology group. It implies the triviality of the higher geometric cohomology groups in the measurable case. Furthermore, Eilenberg-MacLane's reduction theorem allows to extend the results for $p \geq 2$.

Proposition 6.1 (Eilenberg-McLane) $\mathcal{H}^2_{alg}(\mathbf{Z},\mathcal{U}) \cong \{0\}$.

Proof. (Proof due to Depauw)
(i) Every algebraic cocycle $a \in \mathcal{C}^2(G,U)$ is cohomologous to an alternating cocycle: $a(g_{\pi(0)},\dots,g_{\pi(2)}) = (-1)^{\text{sign}\pi} a(g_0,\dots,g_2)$.
Proof. Define the 1-cocycle b by $b(g,h) = 0$ if $g \neq h$ and $b(g,g) = a(g,g,g)$. A first normalization $a_1 = a - db$ gives $a_1(g,g,h) = a_1(g,h,h) = 0$. Proof. Use $db(g,g,h) = b(g,g) = a(g,g,g)$ and $0 = da(g,g,g,h) = a(g,g,h) - a(g,g,g)$ to get

$$a_1(g,g,h) = a(g,g,h) - db(g,g,h) = 0$$

and similarly

$$a_1(g,h,h) = a(g,h,h) - db(g,h,h) = 0 \; .$$

Define the cocycle $b_1(g,h) = 0$ if $h \geq g$ in the lexicographic ordering on $G = \mathbf{Z}^d$ and $b_1(g,h) = a_1(g,h,g)$ if $h < g$. The second normalization $a_2 = a_1 - db_1$ gives an alternating cocycle a_2. Proof. If $h < g$, then $b_1(g,h) = a_1(g,h,g)$ and $a_2(g,h,g) = 0$. ¿From (i) we have $0 = da_1(g,h,g,h) = a_1(g,h,g) - a_1(h,g,h)$. If $h \geq g$ we have so $db_1(g,h,g) = a_1(h,g,h) = a_1(g,h,g)$ giving again $a_2(h,g,h) = 0$. The skew-symmetry follows now from

$$0 = da_2(h_0,h_1,h_0,h_2) = a_2(h_0,h_1,h_2) + a(h_1,h_0,h_2) \; ,$$

$$0 = da_2(h_0,h_1,h_2,h_1) = a_2(h_0,h_1,h_2) + a(h_0,h_2,h_1) \; .$$

(ii) Given a 2-cocycle a. By (i), one can assume that it is alternating. The function $b(k,l) = -\sum_{i=k}^{l-1} a(k,i,i+1)$ is a degree two cochain since

$$\begin{aligned} T^m b(k,l) &= -T^m \sum_{i=k}^{l-1} a(k,i,i+1) = -\sum_{i=k}^{l-1} a(m+k,m+i,m+i+1) \\ &= -\sum_{i=k+m}^{l+m-1} a(k,i,i+1) = b(m+k,m+l) \; . \end{aligned}$$

Since $0 = da(k,l,i,i+1) = a(l,i,i+1) - a(k,i,i+1) + a(k,l,i+1) - a(k,l,i)$, we have

$$\begin{aligned} db(k,l,m) &= b(l,m) - b(k,m) + b(k,l) \\ &= -\sum_{i=l}^{m-1} a(l,i,i+1) + \sum_{i=k}^{m-1} a(k,i,i+1) - \sum_{i=k}^{l-1} a(k,i,i+1) \\ &= \sum_{i=l}^{m-1} -a(l,i,i+1) + a(k,i,i+1) = \sum_{i=l}^{m-1} a(k,k,i+1) - a(k,l,i) \\ &= a(k,l,m) - a(k,l,l) = a(k,l,m) \; . \end{aligned}$$

□

Theorem 6.2 (Dye,Krieger,Connes,Feldman,Weiss) *If G is a group of automorphisms of a probability space, then $\mathcal{H}^p_{orb}(\mathcal{R}_{\mathbf{Z}^d},U) \cong \mathcal{H}^p_{orb}(\mathcal{R}_{\mathbf{Z}},U)$ for all $p \geq 1$.*

Proof. Dye-Krieger-Connes-Feldman-Weiss theorem [10] assures the existence of a measurable invertible map $\psi : X^{p+1} \to X^{p+1}$ mapping $\mathcal{R}^p_G$ bijectively onto $\mathcal{R}^p_{\mathbf{Z}}$. Given a cochain $a \in \mathcal{C}^p(\mathcal{R}_G,U)$, then $\overline{a}(x) = a(\psi(x))$ is in $\mathcal{C}^p(\mathcal{R}_{\mathbf{Z}},U)$. Because ψ commutes with d, cocycles are mapped bijectively onto cocycles and coboundaries onto coboundaries. □

Corollary 6.3 $\mathcal{H}^2_{alg}(\mathbf{Z}^d,\mathcal{U}) \cong \{0\}$ *in the measurable case.*

Proof. By Proposition 6.1 and Proposition 6.2

$$\mathcal{H}^2_{alg}(\mathbf{Z}^d,\mathcal{U}) \cong \mathcal{H}^2_{orb}(\mathcal{R}_{\mathbf{Z}^d},U) \cong \mathcal{H}^2_{orb}(\mathcal{R}_{\mathbf{Z}},U) \cong \mathcal{H}^2_{alg}(\mathbf{Z},\mathcal{U}) \cong \{0\} .$$

□

To see that the higher cohomology groups $\mathcal{H}^p$ with $p \geq 2$ are trivial, one reduces higher dimensional cohomology groups to $\mathcal{H}^2$ with the help of a reduction theorem of Eilenberg-McLane for algebraic cohomology [17, 18]. To any algebraic cochain a is assigned a *non homogeneous cochain* h by the formulas

$$\begin{aligned} a(g_0, g_1, \dots, g_n) &= g_0 h(g_1 - g_0, \dots, g_n - g_{n-1}) \\ h(g_1, \dots, g_n) &= a(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdot \dots \cdot g_n) . \end{aligned}$$

The coboundary operator is in the non homogeneous approach given by

$$\begin{aligned} dh(g_1, \dots g_{p+1}) &= g_1 h(g_2, \dots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i h(g_1, \dots, g_i g_{i+1}, \dots, g_{p+1}) \\ &+ (-1)^{p+1} h(g_1, \dots, g_p) . \end{aligned}$$

A non-homogeneous cocycle is called *normalized* if $h(g_1, \dots, g_p) = 1$ if at least one $g_i = 1$. Denote by $\mathcal{C}(\mathbf{Z}^d, U)$ the set of all normalized cocycles h from $\mathbf{Z}^d$ to U, where the action of $\mathbf{Z}^d$ on $\mathcal{C}(\mathbf{Z}^d, U)$ is given by $(xh)(y) = h(x + y) - yh(x)$.

Theorem 6.4 (Reduction theorem) $\mathcal{H}^{p+1}_{alg}(\mathbf{Z}^d, U) \cong \mathcal{H}^p_{alg}(\mathbf{Z}^d, \mathcal{C}(\mathbf{Z}^d, U))$, $p > 0$.

Proof. (Eilenberg-McLane [17]) Define the homomorphism $\phi : \mathcal{C}^p(\mathbf{Z}^d, \mathcal{C}(\mathbf{Z}^d, U)) \to \mathcal{C}^{p+1}(\mathbf{Z}^d, U)$ by

$$\phi h(g_1, \dots, g_{p+1}) = (-1)^p h(g_2, \dots, g_{p+1})(g_1)$$

which is an isomorphisms of groups by the homogeneity assumption. A computation shows that ϕ commutes with d: Compute

$$\begin{aligned} (\phi \circ d) h(g_1, \dots, g_{p+2}) &= (-1)^{p+1} dh(g_2, \dots, g_{p+2})(g_1) \\ &= (-1)^{p+1} g_2 h(g_3, \dots, g_{p+2})(g_1) + \\ & \quad (-1)^{p+1} \sum_{i=2}^{p+1} (-1)^{i-1} h(g_2, \dots, g_i g_{i+1}, \dots, g_{p+2})(g_1) \\ & \quad +(-1)^{p+1}(-1)^{p+2} h(g_2, \dots . g_{p+1})(g_1) \end{aligned}$$

and

$$\begin{aligned} (d \circ \phi) h(g_1, \dots, g_{p+2}) &= g_1 (\phi h)(g_2, \dots, g_{p+2}) + (-1)(\phi h)(g_1 g_2, g_3, \dots, g_{p+2}) \\ &+ \sum_{i=2}^{p+2} (-1)^i (\phi h)(g_2, \dots, g_i g_{i+1}, \dots, g_{p+2}) \\ &+ (-1)^{p+3} (\phi h)(g_1, \dots, g_{p+1}) \\ &= (-1)^p g_1 h(g_3, \dots, g_{p+2})(g_2) + (-1)^{p+1} h(g_3, \dots, g_{p+2})(g_1 g_2) \\ & \quad +(-1)^p \sum_{i=1}^{p+1} (-1)^i h(g_2, \dots, g_i g_{i+1}, \dots, g_{p+2})(g_1) \\ & \quad +(-1)^p (-1)^{p+3} h(g_2, \dots, g_{p+1})(g_1) . \end{aligned}$$

Equality follows since by definition of the $\mathbf{Z}^d$-action on $\mathcal{C}^1$

$$\begin{aligned}(-1)^{p+1} g_2 h(g_3,\dots,g_{p+2})(g_1) &= (-1)^p g_1 h(g_3,\dots,g_{p+2})(g_2) \\ &\quad + (-1)^{p+1} h(g_3,\dots,g_{p+2})(g_1 g_2) .\end{aligned}$$

□

Corollary 6.5 *If the $\mathbf{Z}^d$ action is free, then $\mathcal{H}^p_{geom}(\mathbf{Z}^d,\mathcal{U}) \cong 0$ for $p \geq 2$ in the measurable case.*

Remark. For $d = 2$, this is the theorem proven in [18]. An alternative proof which generalizes to nonabelian U has been given in [33], an other proof by Depauw [14]. The case $p \geq 2$ in Theorem 6.4 is a new result.

7 Cohomology of aperiodic tilings

This section is motivated by the following question about aperiodic tilings in $\mathbf{R}^2$.

Question. Is it possible to assign $U(1)$-valued random variables to the edges of the graph of the tiling such that the product of the group elements along the boundary of each tile are independent, identically distributed $U(1)$-valued random variables?

This question is nontrivial if the distribution of these random variables is not the uniform Lebesgue distribution on the circle.

We mean here with a tiling a graph on $\mathbf{R}^d$ which has the property that the closure X of the set of all its translates is compact in the tiling topology. The compact metric space X is equipped with a natural $\mathbf{R}^d$ action. It is furthermore required that this action is uniquely ergodic. The ergodic theory of a tiling dynamical system is then determined without having to choose an invariant measure of the $\mathbf{R}^d$ action.

For simplicity, we consider tilings, where every tile is a simplex. This assumption is not a big loss of generality because in general, every tile can be decomposed into simplices.

Two tilings are called topologically isomorphic if the corresponding $\mathbf{R}^d$ actions are equivalent as topological dynamical systems, i.e. if there exists a homeomorphism from one tiling space X to the other Y which conjugates the $\mathbf{R}^d$ actions. The tilings are called measure theoretically isomorphic, if the corresponding dynamical systems are isomorphic as measure-preserving dynamical systems.

Proposition 7.1 *Measurable and topological cohomology groups exist for a tiling dynamical system. The topological cohomology groups are invariants under topological equivalence of tiling dynamical systems, the measurable cohomology groups are invariants of the measurable equivalence of tiling dynamical systems.*

Proof. Let X be a compact metric space of tilings obtained from one tile x by closing all translates of x. Let V be the set of all vectors which occur as possible difference of two vertices in x. Define the countable equivalence relation

$$\mathcal{R} = \{(x,y) \in X^2 \mid x = y + v, \text{ where } v \in V\}.$$

This equivalence relation defines Feldman-Moore cohomology groups both in the measurable as well as in the topological setup. □

Measure theoretical cohomology groups $\mathcal{H}^p(\mathcal{R},U)$ for $p \geq 2$ are not suitable as invariants for tilings with abelian U because of the following theorem.

Theorem 7.2 *In the measurable case, all higher cohomology groups $\mathcal{H}^p(\mathcal{R}, U)$ with $p \geq 2$ of a tiling dynamical system are trivial.*

Proof. The equivalence relation $\mathcal{R}$ is defined by a countable group action (see [18] Theorem 1) and defines a hyperfinite von Neumann algebra. The hyperfiniteness follows from the fact that $\mathcal{R}$ is a subset of a countable equivalence relation $\mathcal{R}_G$, where $G \subset \mathbf{R}^2$ is the abelian countable group generated by

$$\{v \in \mathbf{R}^2 \mid v \text{ occurs as a difference of two vertices in a tiling } \} .$$

Since $\mathcal{R}_G$ is hyperfinite, also $\mathcal{R}$ is hyperfinite ([18] Proposition 4.2.c).

By Proposition 6.2, $\mathcal{H}^2(\mathcal{R}, U)$ is isomorphic to the group $\mathcal{H}^2(\mathcal{R}_{\mathbf{Z}}, U)$ which is trivial. $\square$

But with this result, one can answer the above flux-phase problem for tiling dynamical systems at least in an ergodic setup and for some classes of magnetic fluxes.

Proposition 7.3 (Application) *Given a tiling dynamical system in the plane $\mathbf{R}^2$ with triangular tiles. Assume that $U(1)$-random variables are attached to the triangular tiles of the tiling such that the map $x \mapsto B_0(x)$ giving the flux of the triangular tile Δ in the tiling x which intersects the origin in the plane is measurable, we can attach $U(1)$-random variables to the edges such that the product of the random variables at the edges is the random variable of the tile.*

Proof. Attaching $U(1)$-valued random variables $a(\Delta_i)$ to the triangular tiles Δ_i defines an orbit cocycle $\mathcal{C}^3$. This is a measurable map $B : \mathcal{R}^3 = \{(x, y, z) \in X^3 \mid (x, y), (y, z), (x, z) \in \mathcal{R} \} \to \mathbf{R}$ such that for each tile $\Delta = (x_0, x_1, x_2)$ with vertices x_i, one has $B(\Delta) = B(x_0, x_1, x_2)$.

The theorem 7.2 gives a coboundary B in $\mathcal{C}^2$ such that $dA = B$. The coboundary A defines random variables $A(x_0, x_1)$ which are attached to the edges of the tiles and which solve the flux phase problem. $\square$

There exists in general no cocycle B from $\mathcal{R}^3$ to U such that $x \in X \mapsto B_0(x) \in U$ has the property that $x \mapsto B_0(T_g x), x \mapsto B_0(T_h x)$ are independent if $g \in G$ and $h \in G$ differ. However, we can find a second Lebesgue space Y with a G-action and a measureable map $\tilde{B}_0$ on $\tilde{X} = X \times Y$ such that $\tilde{x} \mapsto \tilde{B}_0(\tilde{T}_g(\tilde{x}))$ and $\tilde{x} \mapsto \tilde{B}_0(\tilde{T}_h(\tilde{x}))$ are independent whenever $g \neq h$.

This answers the question posed at the beginning of the section. Given any distribution on $U(1)$, we can attach (in general correlated) random variables at the edges of a triangular tile such that the magnetic fluxes on these tiles are independent random variables with that distribution. More generally, the distribution of the flux $B_0(x)$ through the triangle having the origin in its interior could be required to depend continuously on the tiling x.

Appendix: On the first cohomology group

The first cohomology group $\mathcal{H}^1(U)$ is the only nontrivial cohomology group in the measurable case. In this appendix, we review some facts about this group and will see that the cardinality of $\mathcal{H}^1(U)$ is uncountable if U is nontrivial.

Assume U is abelian and that it contains more than one element. Because the first cohomology group $\mathcal{H}^1(T, U)$ is the same for all aperiodic ergodic abstract dynamical systems (X, T, m), one can investigate it for one specific aperiodic system. The most convenient choice is an irrational rotation on the circle $X = \mathbf{R}/\mathbf{Z}$, $T : x \mapsto x + \alpha$ which leaves the Lebesgue measure m invariant. The following results follows from a theorem of Derrien [15].

Proposition 7.4 *There exists a countable subgroup $K(U)$ of $\mathcal{H}^1(U)$ which is isomorphic to the subgroup of $U^{\mathbf{Z}}$ given by sequences for which only finitely many elements are different from* 1.

Proof. Let P be a partition of $X = \mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ into finitely many intervals $P_i = [t_{i+1} - t_i)$ with rational t_i and let ϕ be a measurable map $X \to U$ which is constant on the intervals of P but not constant on X. There exists a bijective relation between $U^{\mathbf{Q}\cap\mathbf{T}}$ for which only finitely many elements are different from 1 and the set of so defined cocycles: given ϕ define $\tilde{\phi} \in U^{\mathbf{Q}\cap\mathbf{T}}$ by $\tilde{\phi}_t = \phi_{t+0} - \phi_{t-0}$ which is different from 1 only at finitely many places. The inverse of this map defines for each $\tilde{\phi} \in A^{\mathbf{Q}\cap\mathbf{T}}$ a cocycle ϕ. The set of such cocycles together with the unit cocycle forms a subgroup of all cocycles. By [15], all of these cocycles except the unit element are not coboundaries. Therefore, they are pairwise not cohomologous. $\square$

Remark.
There are situations, where the first cohomology group for *continuous* functions is known: for Markov chains (X, T, m) with infinite X and $U = \mathbf{R}$ and $d = 1$, the cohomology group $C(X)/\{f \in C(X) \mid f = g(T) - g\}$ is the free abelian group with a countable infinite number of generators ([38] p.62).

We want to illustrate now in the simplest situation in a self-contained way, why the cohomology group $\mathcal{H}^1(U)$ is infinite if U is $U(1)$. According to [26], the nontriviality of $\mathcal{H}^1(U)$ seems first have been noticed by Gleason (unpublished). We consider $\mathbf{Z}_2 = \{-1, 1\}$ as a discrete subgroup of $U(1)$.

Lemma 7.5 *Assume T is an ergodic automorphism of the probability space $(X, \mathcal{F}, m)$. Given $B \in \mathcal{L}(X, \mathbf{Z}_2)$. If $B = A(T)A^{-1}$ with $A \in \mathcal{L}(X, U(1))$, then there exists $C \in \mathcal{L}(X, \mathbf{Z}_2)$, such that $B = C(T)C^{-1}$.*

Proof. ¿From $B = A(T)A^{-1}$, we get $A(Tx) = \pm A(x)$ so that $\arg(A(x)) \pmod{\pi}$ is T-invariant. The ergodicity of T implies that $\beta = \arg(A(x)) (\mathrm{mod}\ \pi)$ is constant almost everywhere so that $A(x) = C(x)e^{i\beta}$ with $C(x) \in \mathbf{Z}_2 = \{1, -1\}$. Now $A(x) = C(Tx)C^{-1}(x)$. $\square$

Remark. This lemma shows that $\mathcal{H}^1(\mathbf{Z}_2)$ is a subgroup of $\mathcal{H}^1(U)$. Similarly, one can show that if U is a subgroup of V, then $\mathcal{H}^1(U)$ is a subgroup of $\mathcal{H}^1(V)$.

A result of [26] illustrates in the simplest possible situation, how cohomological constraints emerge in ergodic theory:

Corollary 7.6 *If both T and T^2 are ergodic, then $B(x) = -1$ can not be written as $B(x) = A(Tx)A(x)^{-1}$ with $A \in \mathcal{U}$.*

Proof. ¿From the above lemma, we have only to show that B can not be written as $B = A(T)A^{-1}$ with $A \in \mathcal{L}(X, \mathbf{Z}_2)$. Assume, there exists a solution A of this equation. Then, since $A(Tx) = -A(x)$ also $A(T^2x) = A(x)$, so that by the ergodicity of T^2, A must be constant. There are only two possibilities: $A = 1$ or $A = -1$ and both do not solve $-1 = A(T)A^{-1}$. $\square$

This leads to a special case of Derrien's result 7.4 in the case $U = \mathbf{Z}_2$, where the group of cocycles is isomorphic to the σ-algebra of the probability space $\mathcal{F}$ with addition given by the symmetric difference Δ and where the coboundaries form the additive subgroup of $\mathcal{F}$ consisting of elements $Z \Delta T(Z)$.

Corollary 7.7 *If T is the irrational rotation of the circle X, then every dyadic interval $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$ is not a coboundary and two dyadic intervals of length 2^{-n} are pairwise not cohomologous.*

Proof. Corollary 7.6 shows that X, the dyadic interval of length 2^{-0} is not a coboundary. If a dyadic interval of length 2^{-n} was a coboundary then all of them would be coboundaries and so also a disjoint union of 2^n such intervals. This would contradict that X is not a coboundary. Inductively, assume the claim is true for coboundaries of length 2^{-n} and assume that the union $I = I_1 \Delta I_2$ of two disjoint dyadic intervals of length $2^{-(n+1)}$ is a coboundary. If I_1 and I_2 are adjacent, then I is a dyadic interval of length 2^{-n} and can not be a coboundary. Also in the other case, I is not a coboundary because else $K = I\Delta(I + 2^{-(n+1)} \bmod 1)$ is a coboundary which is the union of two dyadic intervals of length 2^{-n} which is not a coboundary by induction. □

After I had asked Greg Hjorth at Caltech on the cardinality of the group $\mathcal{H}^1(U)$, he showed me on the blackboard a proof of the following result. The proof needs some descriptive set theory.

Proposition 7.8 (Hjorth) *The group $\mathcal{H}^1(U)$ is uncountable if U is abelian.*

Proof. [Proof due to [22]] The set $\mathcal{U}$ becomes a Polish group with the metric $d(A, B) = P[A \neq B]$. The automorphism T induces a continuous map on the group $\mathcal{U}$. The set of coboundaries $\mathcal{C} = \{A(T)A^{-1}\}$ is a continuous image of the group homomorphism $f_T(A) = A(T)A^{-1}$ of the Polish space $\mathcal{U}$. It is therefore a injective continuous image of the Polish group $\mathcal{U}/(\ker f_T)$ where $\ker f_T = \{1, -1\}$ if T is ergodic. The set $\mathcal{C}$ is therefore a Borel set by a theorem of Lusin-Souslin (Theorem 15.1 in [25]) and has so the Baire property.

By Derrien's result **??**, the set $\mathcal{C}$ is not open. So, it is meager by a result of Pettis (Theorem 9.9 in [25]). The equivalence relation on $\mathcal{U}$ given by $A \sim B$ if $A\mathcal{C} = B\mathcal{C}$ is meager. (This uses a theorem of Kuratovsky-Ulam [25] which states that if a subset A of $X \times X$ has the Baire property, then $A_x = \{y \mid (x, y) \in A\}$ has the Baire property.) By a theorem of Mycielski (Theorem 19.1 in [25]), there are uncountably many equivalence classes. □

Remarks.
1) In the topology $d(Y, Z) = m(Y \Delta Z)$ on the σ-algebra $\mathcal{F}$, both the set $\{Y \in \mathcal{F} \mid Y = Z(T)\Delta Z\}$ of coboundaries as well as its complement are dense in $\mathcal{F}$ ([27]).
2) If X is a finite set and T is a permutation of X, then $\mathcal{H}^1(U) = U$ for any group U.

Appendix: On cohomology in ergodic theory

In this appendix we trace some of the developments in the topic of cohomologies in dynamical systems.

Cohomology in dynamical systems was used early without using that name. The first appearance of the cohomological equation we could locate is in Hilbert's list of problems [21], where the cohomological equation $f(T) - f = g$ appears as a side remark in Problem 5. Hilbert mentioned it as an example of an analytic functional equation which has only a continuous non-differentiable solution.

In von Neumann's 1932 paper [44] (p.641), the cohomology equation appears in the research on spectra of ergodic flows on the torus. The nontriviality of the cohomology group manifests itself in the existence of ergodic flows on the torus which have continuous spectrum (a topic with much research later [31, 42, 24, 3, 11]).

Wintner looked at the cohomological equation in the realanalytic case [47].

Group cohomology came up independently by Eckman and Eilenberg-McLane [16, 17] (see [32] p. 137).

Abstract algebraic cohomology goes also under the name 'Galois cohomology' which is used to analyze how a group G acts on an other group $\mathcal{U}$.

In Anzai's 1951 paper [4], the name "equivalent" is used instead of "cohomologuous".

In Hamiltonian dynamics, the cohomological equation appears in Mosers twist map theorem [37].

Influencal for "cohomology of dynamical systems" in ergodic theory is [26], where interesting questions are asked for this cohomology and especially for $H^1(T, \mathbf{Z}_2)$, the cohomology of measurable sets. The paper [34] was appearantly unaware of [26].

In [19], the notion of "generalized eigenvalues" is introduced as an invariant for dynamical systems. This cohomology can be defined in a more general way: an automorphism T of an arbitrary abelian group $\mathcal{G}$ gives a discrete "differentiation" $f \mapsto dg = g(T) \circ g^{-1}$ on the group $\mathcal{G}$. If $\mathcal{E}$ is the trivial group $\{1\}$ in $\mathcal{G}$, then $d^{-n}\mathcal{E}$ is a subgroup of $\mathcal{G}$. (One can think of them as "polynomials" of degree $\leq (n-1)$ because n times "differentiation" annilihates them.) Halmos defined the groups $\mathcal{H}_n(\mathcal{R}, \mathcal{G}) = d^{-n}\mathcal{E}/d^{-(n-1)}\mathcal{E}$ called generalized eigenvalues in the case $\mathcal{G} = L^\infty(X)$, where the group automorphism is coming from a measure theoretical dynamical system $T : X \to X$. Then, $\mathcal{H}_1(T, \mathbf{R}) = d^{-1}\mathcal{E}$ is the space of functions f left-invariant under T and the first Halmos homology group $\mathcal{H}_1$ measures how many ergodic components T has. The next group $\mathcal{H}_2(T, \mathbf{R}) = d^{-2}\mathcal{E}/d^{-1}\mathcal{E}$ measures the point spectrum of the unitary Koopman-operator belonging to T and Halmos called the other groups $\mathcal{H}_n$ generalized eigenvalues. There is a dual construction. Start with $\mathcal{G}$ and form the groups $\mathcal{H}^n = d^{n-1}\mathcal{G}/d^n\mathcal{G}$. In [28], we called these groups Halmos cohomology groups. These groups are already interesting for finite groups:

Example 1 [28].
Let $\mathcal{G}$ be the multiplicative group of $\mathbf{Z}_p$, where p is prime. Consider the automorphisms $T : A \mapsto A^{-1}$, which exists on all groups. We have $\mathcal{H}^1 = \mathcal{G}/\{A \in \mathcal{G} \mid A = B(T)B^{-1} = B^2\}$ and since the quotient consists of quadratic residues and contains so $(p-1)/2$ elements, $\mathcal{H}^1(\mathcal{G}) = \mathbf{Z}_2$. Also $\mathcal{H}_1(\mathcal{G}) = \{g | g(T) = g\} = \{1, -1\} = \mathbf{Z}_2$.

We had asked in [28], whether a duality $\mathcal{H}^p = \mathcal{H}_p$ would hold in the case of finite groups. This is not the case without further restrictions:

Example 2. (M. Aschbacher [6]). Consider the group $\mathcal{U} = \mathbf{Z}_{p^2} \oplus \mathbf{Z}_p$ written multiplicatively, where $\mathbf{Z}_{p^2}$ is generated by x and $\mathbf{Z}_p$ is generated by y. Let T be the automorphism which fixes x and maps y into $y + x$. $\mathcal{H}_1$, the fixed point set of T is $\mathbf{Z}_{p^2}$. We have $d(x^n y^m) = x^n y^m x^m x^{-n} y^{(-m)} = x^m$ so that $d\mathcal{U} =< px >$ and $\mathcal{H}^1 = \mathcal{U}/d\mathcal{U} = \mathbf{Z}_p \oplus \mathbf{Z}_p$.

Work on finding Halmos invariants in ergodic theory continued for example in [2] and [43].

The link of cohomology of dynamical systems with algebraic group cohomology was established in [18]. It was motivated by cohomology constructions of Westman for virtual groups [46] who generalized a construction of C.C. Moore in 1964. The representation theory of Mackey's virtual groups [35] was also a motivation for [26], because Mackey had proposed to call elements of $H^1(G, U(N))$ 'unitary representations of a virtual subgroup' of G. The term "virtual subgroup" has now been soaked up by "groupoids" and is a central in noncommutative mathematics" (see [45] for a review).

For more information on orbit cohomology, we refer to the second chapter in [40].

The first cohomolocal group appears at different other places in other function spaces. They are especially important for the structural stability of dynamical systems: when conjugating two dynamical systems $T, \tilde{T} = T + f$ with a conjugating map $\phi = Id + h$, the unknown function h has to satisfy the functional equation

$$T(x) + h(T(x)) = T(x + h(x)) + f(x + h(x)) \; .$$

In the case $T : \mathbf{T}^1 \to \mathbf{T}^1, x \mapsto x + \alpha$, this equation can be written as $h(x+\alpha) - h(x) = f(x + h(x))$ which has the linearisation the homological equation $h(x+\alpha) - h(x) = f(x)$. Solving this equation in a space of analytic, periodic functions is important in Arnold's proof [5] of the theorem, that an analytic diffeomorphism of the circle $\tilde{T}$ with Diophantine rotation number can be conjugated to the rigid rotation T if the two diffeomorphisms $T, \tilde{T}$ are close enough to each other (see [20] for the global solution).

For Anosov $\mathbf{Z}$ actions, cohomology questions can be reduced to the cohomology on the periodic orbits. See [34] in the Hölder and C^1 case, [13] in the C^∞ case following work of Guillemin and Kazhdan and in the realanalytic case [12].

The cohomology in the Hölder category is also relevant in statistical mechanics [9, 39].

Because in the measurable category, the cohomology groups are independent of the dynamical system, one usually takes an ergodic group translation on the torus. The topic has applications for representation theory of noncompact nonabelian Lie groups [36, 8, 7], especially when the coefficient group is the circle $U(1)$ or other unitary groups $U(n)$. When the coefficient group is $\mathbf{Z}_2$, one deals with the cohomology of measurable sets which first appeared in [26]. The group is relevant for Lyapunov exponent of $\mathrm{SL}(2,\mathbf{R})$-cocycles [27] because cohomology determines whether switching of stable and unstable manifolds of a cocycle makes the Lyapunov exponent vanish.

Research on the first cohomology group for $\mathbf{Z}^d$ Anosov actions started in [1] and is an active research topic (see [41] section 30 for some review on cohomological rigidity).

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