C^*-Algebras of Energy Observables: I. General Theory and Bumps Algebras

by

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Abstract

To a quantum system subject to a qualitatively specified interaction we associate a C^*-algebra acting on the state space of the system: the C^*-algebra generated by the operators which can be interpreted as hamiltonians of the system. Our purpose is to show the relevance of this algebra in the study of the spectral properties of the hamiltonians and to give concrete methods of construction of such algebras. The main tool we use is the crossed product of C^*-algebras by actions of groups. Applications include systems with anisotropic behaviour at infinity, generalized N-body problems, quantum field models. We study in detail the algebra associated to a system subject to a Klaus type interaction (infinitely many bumps).

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1 Introduction

1.1. The C^*-algebra generated by the "admissible" hamiltonians of a quantum system often has a remarkable algebraic structure which allows one to compute explicitly its quotient with respect to the ideal of compact operators (what we mean by admissible will become clear later on). This statement should be considered as an "experimental" observation: our aim is to justify it by examples and to show that it has interesting consequences in the spectral theory of the hamiltonians of physical systems with a rich internal structure (like many body or quantum field systems) or a high degree of anisotropy in configuration or phase space. We have in mind here only spectral properties which are stable under (certain classes of) compact perturbations: for the moment we are concerned only with the description of the essential spectrum and the proof of the Mourre estimate.

Instead of focusing on the study of one self-adjoint operator H (*the* hamiltonian), we consider as primary object of the theory the richer mathematical structure $(\mathscr{H}, \mathfrak{C})$ consisting of a Hilbert space $\mathscr{H}$ (the state space of the quantum system) and a C^*-subalgebra $\mathfrak{C}$ of $B(\mathscr{H})$, the *algebra of energy observables*, or *algebra of hamiltonians*. In our approach $\mathfrak{C}$ is considered as an a priori given object; practically, its choice is dictated by the qualitative features of the physical system. The self-adjoint operators affiliated to $\mathfrak{C}$ are the "admissible" hamiltonians: they become hamiltonians of the system once some external or internal parameters have been fixed.

Two restrictions are imposed on $\mathfrak{C}$. First, $\mathfrak{C}$ must contain the C^*-algebra $K(\mathscr{H})$ of compact operators on $\mathscr{H}$ (in particular, we are not in a purely algebraic framework, an initial Hilbert space $\mathscr{H}$ has to be given). Thus $K(\mathscr{H})$ is an ideal in $\mathfrak{C}$ and the quotient $\widehat{\mathfrak{C}} := \mathfrak{C}/K(\mathscr{H})$ is a C^*-algebra. The second restriction is that one must have a rather explicit description of $\widehat{\mathfrak{C}}$ or, vaguely speaking, $\widehat{\mathfrak{C}}$ must be computable. We shall discuss this in detail below.

If $T \in \mathfrak{C}$ we denote $\widehat{T}$ its image in $\widehat{\mathfrak{C}}$. It is well known that $\boldsymbol{\sigma}_{\text{ess}}(T) = \boldsymbol{\sigma}(\widehat{T})$. The definition of $\widehat{T}$ for an unbounded self-adjoint operator affiliated to $\mathfrak{C}$ and a discussion of the preceding relation can be found in §2.2. The only point which matters here is that if one is able to compute $\widehat{T}$, then one can determine the essential spectrum of T. In Chapter 8 from [ABG] it is explained how to deduce the Mourre estimate for T from a similar estimate for $\widehat{T}$.

The exact meaning of "computable" depends on the situation one is studying; one of the main points of our paper is to explain this in several concrete situations. In any case, $\widehat{\mathfrak{C}}$ must be in some sense simpler than $\mathfrak{C}$ (and $\widehat{T}$ simpler than T). This is essentially a condition on the size of $\mathfrak{C}$. If $\mathfrak{C}$ is small, then the quotient is easy to compute, but this will not be of much interest, standard methods of spectral analysis would give as easily the same results on the spectral properties of the hamiltonian (like in the first example below, a two body problem). If $\mathfrak{C}$ is too large, there is no point in computing the quotient, it will probably be more complicated than $\mathfrak{C}$ itself.

However, we must stress the fact that we shall meet situations when $\widehat{\mathfrak{C}}$ is not computable in any simple sense and is not "simpler" than $\mathfrak{C}$, but interesting information can be deduced from its form. So the condition of "computability" of $\widehat{\mathfrak{C}}$ should not be interpreted in a too strict sense. Moreover, the hypothesis $K(\mathscr{H}) \subset \mathfrak{C}$ is too strong in some cases. Then one should take $\widehat{\mathfrak{C}} = \mathfrak{C}/\left[\mathfrak{C} \cap K(\mathscr{H})\right]$; of course, if $\mathfrak{C} \cap K(\mathscr{H}) = \{0\}$ then one will not get any interesting result.

We shall give now three abstract examples (of increasing order of generality) of "computable" quotients. More specific examples and quotients of a very different nature will be presented in later sections. Some of the less standard notions and notations that we use are explained at the end of this Introduction and in Sections 2 and 3.

1.2. Let $\mathfrak{C}_0$ be a C^*-algebra of operators on $\mathscr{H}$ such that $\mathfrak{C}_0 \cap K(\mathscr{H}) = \{0\}$ (this means that $\mathfrak{C}_0$ has no nonzero finite rank projections) and let $\mathfrak{C} = \mathfrak{C}_0 + K(\mathscr{H})$. Then $\mathfrak{C}$ is a C^*-algebra (see [Mur]) and the projection $\mathfrak{C} \to \mathfrak{C}_0$ associated to the linear direct sum decomposition which defines $\mathfrak{C}$ is a morphism which gives a canonical identification $\widehat{\mathfrak{C}} \cong \mathfrak{C}_0$. This is the simplest case of "computability" of $\widehat{\mathfrak{C}}$ and it is the only case when $\widehat{\mathfrak{C}}$ is naturally realized on the same Hilbert space as $\mathfrak{C}$.

The quantum mechanical two-body problem is a typical particular case of this example: one may take $\mathscr{H} = L^2(\mathbb{R}^n)$ and $\mathfrak{C}_0 = \{\varphi(P) \mid \varphi \in C_0(\mathbb{R}^n)\}$ ($P = -i\nabla$ is the momentum observable). The operators affiliated to $\mathfrak{C}$ are of the form $H = H_0 + V$, where $H_0 = h(P)$ for some continuous function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(k) \to \infty$ if $|k| \to \infty$ and V is an "operator" which tends to zero as $|x| \to \infty$ (in a very general sense).

Our approach is not really useful in such a simple situation. Indeed, a bounded operator H on $\mathscr{H}$ belongs to $\mathfrak{C}$ if and only if it can be written as $H = H_0 + V$ with $H_0 \in \mathfrak{C}_0$ and $V \in K(\mathscr{H})$. Then this decomposition is unique and we have $\widehat{H} = H_0$. An arbitrary self-adjoint operator H on $\mathscr{H}$ is affiliated to $\mathfrak{C}$ (i.e. is an admissible hamiltonian) if and only if there is a self-adjoint operator H_0 affiliated to $\mathfrak{C}_0$ such that $(H+i)^{-1} - (H_0+i)^{-1}$ is compact, and then $\widehat{H} = H_0$. Now the fact that $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(H_0)$ is a standard fact of Hilbert space nature and the algebraic approach we propose is, at this level, irrelevant.

1.3. The second example reveals, on the contrary, the force of the algebraic formalism. Here we assume that the quotient is embedded in a direct product of C^*-algebras, more precisely: there is a family $\{\mathfrak{C}_i\}_{i\in I}$ of C^*-algebras and an embedding

$$\mathfrak{C}/_{K(\mathscr{H})} \hookrightarrow \textstyle\prod_{i\in I} \mathfrak{C}_i. \tag{1.1}$$

In other terms, assume that a family of morphisms $\mathcal{P}_i : \mathfrak{C} \to \mathfrak{C}_i$ has been given such that $\bigcap_{i\in I} \ker \mathcal{P}_i = K(\mathscr{H})$. If we set $\mathcal{P}[S] = \prod_{i\in I} \mathcal{P}_i[S]$ we shall get a morphism $\mathcal{P} : \mathfrak{C} \to \prod_{i\in I} \mathfrak{C}_i$ with kernel $K(\mathscr{H})$, which induces the required embedding. Now let, for simplicity, H be a bounded admissible hamiltonian and set $H_i = \mathcal{P}_i[H]$. Then, as a consequence of (1.1), we shall have an identification $\widehat{H} = \prod_{i\in I} H_i$ and thus:

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \overline{\textstyle\bigcup_{i\in I} \boldsymbol{\sigma}(H_i)}. \tag{1.2}$$

This covers the theorems of HVZ type known in the usual N-body problem and other anisotropic situations.

A particularly interesting abstract class of algebras whose quotient is of the preceding form consists of C^*-algebras graded by finite semi-lattices. Such algebras appear naturally in the study of N-body systems and are presented in detail in [ABG]. In [DaG1, DaG2] the theory is extended to the case of arbitrary (infinite) semi-lattices; we shall recall some of these results in Section 4 because they are needed in one of our applications.

As a third example we consider one of the worse situations, when the quotient has the following structure: there is a family $\{(\mathfrak{C}_i, \mathfrak{J}_i)\}_{i\in I}$ of C^*-algebras $\mathfrak{C}_i$ equipped with ideals $\mathfrak{J}_i$ and there is a natural embedding

$$\mathfrak{C}/_{K(\mathscr{H})} \hookrightarrow \textstyle\prod_{i\in I} \mathfrak{C}_i \Big/ \textstyle\bigoplus_{i\in I} \mathfrak{J}_i \,. \tag{1.3}$$

Note that in the right hand side we have both a direct product $\prod_{i\in I}$ and a direct sum $\bigoplus_{i\in I}$ of C^*-algebras. The preceding case is obtained when $\mathfrak{J}_i = \{0\}$ for all $i \in I$. Similar but more complicated formulas of type (1.2) will be obtained for the general case (1.3) in Theorem 2.2. Assume, more specifically, that each $\mathfrak{C}_i$ is realized on a Hilbert space $\mathscr{H}_i$ and that

$\mathfrak{J}_i = K(\mathscr{H}_i)$. The representation (1.3) of the quotient $\widehat{\mathfrak{C}}$ does not seem to be very simple. However, it is not difficult to show that if H is a self-adjoint operator affiliated to $\mathfrak{C}$ and $\widehat{H}$ is, in the representation given by the right hand side of (1.3), the quotient of some family $\{H_i\}_{i\in I}$ of operators H_i affiliated to $\mathfrak{C}_i$, then

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcap_{\substack{F\subset I\\ F\,\mathrm{finite}}} \left\{ \left[\textstyle\bigcup_{i\in F} \boldsymbol{\sigma}_{\mathrm{ess}}(H_i)\right] \cup \left[\overline{\textstyle\bigcup_{i\in I\setminus F} \boldsymbol{\sigma}(H_i)}\right] \right\}. \tag{1.4}$$

Another interesting version of this formula, involving ultrafilters on the set I, will also be pointed out. Concrete hamiltonians which are covered by this example are the Schrödinger operators with potentials having "widely separated bumps".

1.4. The conventional approach to the problems treated in this paper is of a quite different nature. The starting point is a couple $(\mathscr{H}, H)$ consisting of the Hilbert space $\mathscr{H}$ of the states of the quantum system and a self-adjoint operator H acting on $\mathscr{H}$, the hamiltonian. H is considered as a fixed object and, in the simplest cases, its spectral analysis is performed in a perturbative manner: a second operator $\widehat{H}$ (the "free hamiltonian", usually denoted H_0) with known spectral properties is given such that $(H-z)^{-1} - (\widehat{H}-z)^{-1}$ is compact. From this one deduces that H has spectral properties similar to those of $\widehat{H}$. In other terms, one obtains a simpler operator $\widehat{H}$ by subtracting (in a generalized sense) from H a compact operator.

Unfortunately, in more complicated situations (e.g. in the three body problem) H does not become significantly simpler after the subtraction of a compact operator. So among the operators acting in $\mathscr{H}$ there is no natural candidate for $\widehat{H}$. In order to bypass this problem in concrete models of the N-body problem, quantum field theory, or solid state physics, several methods have been invented. For example, the partitions of unity in configuration space introduced by B. Simon became a very powerful tool in the spectral analysis of N-body systems and some quantum field models, see [DeG1, DeG2, DeG3, FrHe, Ger, PSS, Sim2] and references therein. Other methods consist in modifying the second resolvent equation (cf. Weinberg, Fadeev, Yakubovsky, and others). Notice that in these methods the notion of free hamiltonian associated to H plays no role or only a secondary one: indeed, in the absence of interactions H becomes an operator H_0 often called free hamiltonian, but H is far from being a small or weakly compact perturbation of H_0.

On the contrary, our approach emphasizes the role of the (suitably defined) concept of free hamiltonian: intuitively speaking, this is the self-adjoint operator closest to H modulo the compact operators, so one should expect a kind of "maximal simplification" when computing $\widehat{H}$. In mathematical terms $\widehat{H}$ is defined as the image of the observable H in the quotient algebra $B(\mathscr{H})/K(\mathscr{H})$. In order to have a precise terminology, if the hamiltonian of the system is H, we call $\widehat{H}$ its *co-hamiltonian*. $\widehat{H}$ has no meaning in a purely hilbertian setting and only in very special situations it acts in a natural way as a self-adjoint operator in the Hilbert space $\mathscr{H}$. Thus the assertion "$(H-z)^{-1} - (\widehat{H}-z)^{-1}$ is compact" does not make sense in general (in fact in the most interesting cases), but the preceding definition of $\widehat{H}$ confers it an algebraic interpretation. The Calkin algebra $B(\mathscr{H})/K(\mathscr{H})$ is a very complicated object, so it does not seem that the computation of $\widehat{H}$ could be done. Our proposal is to construct a C^*-algebra $\mathfrak{C}$ to which H is affiliated and whose quotient with respect to $K(\mathscr{H})$ can be computed in rather explicit terms. Of course, the difficulty now is in isolating the right algebra $\mathfrak{C}$ and in computing the quotient.

We think, however, that the approach developed in this paper is less ad hoc and has certain conceptual and technical advantages with respect to other methods. Its main feature is that it allows one to go beyond the usual perturbative techniques (even in their refined forms involving various kinds of partitions of unity) in a simple and natural way. The co-hamiltonian $\widehat{H}$, canonically associated to the hamiltonian of the system, plays an important role not only in the spectral theory of H, but also in its scattering theory.

There is a very large literature on the applications of the abstract theory of C^*-algebras to the study of the spectral properties of various classes of operators. One may find references on this question in [Dav, Dou, Mur, We-O] for example (in particular, the theory of extensions of $K(\mathscr{H})$ is devoted to the classification of the algebras having certain quotients with respect to $K(\mathscr{H})$). But our purposes and techniques here are quite different. Closer to ours is the point of view of H. O. Cordes [Cor]; but his goal, the class of examples he considers, and his methods are very far from ours. The approach to the N-body problem exposed in the chapters 8 and 9 from [ABG] (see also references therein) should be considered as an example of the techniques developed in the present paper.

1.5. The first problem one has to solve in the algebraic approach that we use is the choice of the C^*-algebra of energy observables of a given system: one has to isolate the class of self-adjoint operators which are natural candidates as hamiltonians of the system in a given physical situation. Then $\mathfrak{C}$ will be the C^*-algebra generated by the resolvents of these operators. A priori this does not seem to give a nice mathematical object. The core of our paper is to show that, on the contrary, one often gets in this way interesting classes of C^*-algebras. This question will be discussed in detail in Section 4; here we shall only point out the role of C^*-crossed products in this context (crossed products and other notions which are not standard for people working in spectral theory will be introduced in the body of the paper).

Let $\mathscr{A}$ be a closed $*$-subalgebra of the algebra $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R}^n)$ of bounded uniformly continuous functions on $\mathbb{R}^n$. Assume that $\mathscr{A}$ contains the constants and is stable under translations and let us denote $\mathscr{A}^\infty$ the set of $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi^{(\alpha)} \in \mathscr{A}$ for all α. Let $\Delta = P^2$ be the (positive) Laplace operator and let us consider self-adjoint operators of the form $H = \Delta + V$, where V is a first order differential operator with coefficients in $\mathscr{A}^\infty$. Then the smallest C^*-algebra of operators on $L^2(\mathbb{R}^n)$ which contains the resolvents of the operators H is equal to the norm closed subspace generated by operators of the form $\varphi(Q)\psi(P)$, where $\varphi \in \mathscr{A}$ and $\psi \in C_0(\mathbb{R}^n)$ (space of continuous, convergent to zero at infinity functions). Now this algebra is just the crossed product $\mathscr{A} \rtimes \mathbb{R}^n$ of $\mathscr{A}$ by the natural action of the translation group on it. This is, in fact, a particular case (see [DaG1]) of Proposition 3.16, which justifies the interpretation of a general class of crossed products $\mathscr{A} \rtimes X$ as algebras of energy observables.

Physically speaking, $\mathscr{A}$ is the algebra of potential energies that we consider as natural for the given system. In the two body case we take $\mathscr{A} = C_\infty(\mathbb{R}^n)$ (continuous functions which have a limit at infinity). For the simplest type of anisotropy in one dimension we take $\mathscr{A} = C(\overline{\mathbb{R}})$, the algebra of continuous functions which have limits (distinct in general) at $+\infty$ and $-\infty$. $\mathscr{A}$ could be an algebra of periodic or almost periodic functions, etc. A treatment of the N-body problem from this point of view can be found in [DaG1].

It is important to notice that above we started with a rather small algebra of continuous functions $\mathscr{A}$ and we considered only Δ as kinetic energy operator. But the class of operators H affiliated to $\mathscr{A} \rtimes \mathbb{R}^n$ is much larger than one would expect at first sight. This follows from the affiliation criterion proved in [DaG2]. In the two body case for example, a self-adjoint operator H on $L^2(\mathbb{R}^n)$ is strictly affiliated to $\mathscr{A} \rtimes \mathbb{R}^n$ if and only if there is a real continuous divergent function h on $\mathbb{R}^n$ such that $(H+i)^{-1} - (h(P)+i)^{-1}$ is a compact operator.

1.6. We give now a short description of the content of the paper. We have chosen to study (except for examples) systems having as configuration space an arbitrary abelian locally compact group X. This is not only natural, but also shows that the algebraic methods work without difficulty in a very general context. The basic examples one should have in mind are $X = \mathbb{R}^n$ or $\mathbb{Z}^n$; however, the case of finite dimensional vector spaces over local fields (e.g. p-adic numbers) is very interesting and also of some importance (see [Tai, Sa-C]). Many other nontrivial examples can be given, like the "p-adic torus" (the dual of the compact, totally disconnected, non-discrete group of p-adic integers); see [Fol2, Gu, Wei]. But we stress the fact that even in the simplest situations ($X = \mathbb{R}$ or $\mathbb{Z}$) the algebraic techniques give, rather easily, results which do not seem to be covered by other means.

In Section 2 we discuss several questions concerning the self-adjoint operators affiliated to C^*-algebras, their essential spectra and the connection with the problem of computing quotients of C^*-algebras with respect to some ideals. Theorem 2.2 will be particularly useful in a later section.

Section 3 is devoted to a short presentation of the theory of crossed products of C^*-algebras by actions of abelian groups with emphasis on some results that we need and which we have not been able to find in the literature (at least in a sufficiently explicit form). Especially important for us are Theorem 3.3 and Theorem 3.12; for the proof of this last theorem we benefited from a discussion with G. Skandalis. In writing this section (and also §6.1, where several standard definitions and results on enveloping C^*-algebras are gathered), we had in mind mainly people working in spectral theory which are not familiar with such topics. We mention that the version of the Riesz-Kolmogorov compacity criterion stated in §3.6 is important not only for technical reasons, but also because it is the prototype of several other theorems which give a "geometric" (or "phase space") description of crossed products and other C^*-algebras of pseudo-differential operators (compare Propositions 3.14 and 3.13 with the Definition 5.4 and the Theorem 5.11; other characterizations of the same nature appear in the second part [GI2] of this paper).

Section 4 is the most important one: it contains examples of C^*-algebra of energy observables which are both natural and physically important. Some of these examples are only sketched either because they are treated in detail in other publications or because, although we find the algebras natural and physically interesting, we have not been able to give a reasonably simple description of their quotients with respect to the algebra of compact operators. We devote Section 5 to a detailed study of one of the most interesting of these algebras: that suggested by the work of Klauss [Kla] on potentials with infinitely many "bumps". This algebra has the most interesting quotient with respect the ideal of compact operators, being at the same time explicit and highly nontrivial. One of the main technical points of our paper is Theorem 5.11 (and its corollary) which gives a description of this algebra in the spirit of the Riesz-Kolmogorov compacity criterion (i.e. in terms of phase space properties of the operators).

1.7. We recall several notations and conventions which are usual in the theory of C^*-algebras. A $*$-homomorphism between two C^*-algebras will be called *morphism*. $\mathfrak{A} \cong \mathfrak{B}$ means that the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are canonically isomorphic; in such a situation the canonical morphism is either obvious from the context or we give it explicitly. An algebra with unit element is also called unital; a morphism between two unital C^*-algebras which sends the unit of the first in the unit of the second is called *unital*. By *ideal* we mean a closed bilateral (hence self-adjoint) ideal. We make the same conventions for the more general case of Banach $*$-algebras.

If $\mathscr{H}$ is a Hilbert space then $B(\mathscr{H})$ is the space of bounded linear operators on $\mathscr{H}$ and $K(\mathscr{H})$ the subspace of compact operators. If $\mathscr{A}$, $\mathscr{B}$ are subalgebras of $B(\mathscr{H})$ then $\mathscr{A} \cdot \mathscr{B}$ is the set of finite sums of the form $A_1 B_1 + \cdots + A_n B_n$ with $A_i \in \mathscr{A}, B_i \in \mathscr{B}$; this is a linear subspace of $B(\mathscr{H})$ and we denote by $[\![\mathscr{A} \cdot \mathscr{B}]\!]$ its norm closure.

If the symbol $T^{(*)}$ appears in a relation involving an operator T, this means that the relation is satisfied (or has to be satisfied) both by T and by T^*.

If X is a locally compact topological space then $C_{\mathrm{b}}(X)$ is the C^*-algebra of complex continuous bounded functions on X, $C_\infty(X)$ the C^*-subalgebra of functions which have a limit at infinity, and $C_0(X)$ the C^*-subalgebra of functions convergent to zero at infinity. If the topology of X is associated to a given uniform structure (e.g. if X is a locally compact abelian group) then $C_{\mathrm{b}}^{\mathrm{u}}(X)$ denotes the C^*-algebra of bounded uniformly continuous functions.

The notion of filter appears to be quite useful for our purposes. The reader has to know only the simplest facts concerning their theory (the deepest thing we use is that an ultrafilter on a compact space is convergent); see [Bou2] for a very clear presentation. If $\mathcal{F}$ is a filter on a set X, Y is a topological space, and $\phi : X \to Y$ has a limit y along $\mathcal{F}$ (i.e. one has $\phi^{-1}(V) \in \mathcal{F}$ for each neighbourhood V of y), then we write $\lim_{\mathcal{F}} \phi = y$ or $\lim_{x,\mathcal{F}} \phi(x) = y$.

One of our conventions is not quite standard: if X is a noncompact locally compact topological space, then we call *Fréchet filter* on X the family of subsets V such that $X \setminus V$ is relatively compact (these are the traces on X of the neighbourhoods of infinity in the Alexandroff compactification of X). The limit along the Fréchet filter is denoted $\lim_{x\to\infty} \phi(x)$. We say that a property holds for large x, or for $x \to \infty$, if it holds for x outside some compact set. Note that if X is equipped with the discrete topology, then "compact" means "finite".

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2 Observables and their Essential Spectrum

2.1. If $\mathfrak{C}$ is a C^*-algebra then an *observable H affiliated to* $\mathfrak{C}$ is a morphism $H : C_0(\mathbb{R}) \to \mathfrak{C}$. In order to keep close to standard notations we shall denote by $\varphi(H)$ (*not* $H(\varphi)$) the image of $\varphi \in C_0(\mathbb{R})$ through this morphism. We say that H is $\mathfrak{C}$-*nondegenerate* if the linear subspace generated by $\{\varphi(H)S \mid \varphi \in C_0(\mathbb{R}),\ S \in \mathfrak{C}\}$ is dense in $\mathfrak{C}$. In this case H is also called *strictly affiliated* to $\mathfrak{C}$. The *spectrum* of the observable H is defined by:

$$\boldsymbol{\sigma}(H) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Longrightarrow \varphi(H) \neq 0\}. \tag{2.1}$$

We define the image of an observable through a morphism as follows. If $\mathfrak{C}_1$, $\mathfrak{C}_2$ are C^*-algebras, H_1 is an observable affiliated to $\mathfrak{C}_1$ and $\mathcal{P} : \mathfrak{C}_1 \to \mathfrak{C}_2$ is a morphism, then $\varphi \mapsto \mathcal{P}[\varphi(H_1)]$ is a morphism $C_0(\mathbb{R}) \to \mathfrak{C}_2$. Thus we get an observable affiliated to $\mathfrak{C}_2$ that we shall denote by $H_2 = \mathcal{P}[H_1]$. Obviously $\boldsymbol{\sigma}(H_2) \subset \boldsymbol{\sigma}(H_1)$.

In particular, if $\mathfrak{J}$ is an ideal in $\mathfrak{C}$, $\widehat{\mathfrak{C}} = \mathfrak{C}/\mathfrak{J}$ is the quotient algebra and H is an observable affiliated to $\mathfrak{C}$, we may define the quotient $\widehat{H}$ as the observable affiliated to $\widehat{\mathfrak{C}}$ given by $\widehat{H} = \pi(H)$, where π is the canonical morphism $\mathfrak{C} \to \widehat{\mathfrak{C}}$. In this context, it is useful to remark the similarity between:

$$\boldsymbol{\sigma}(\widehat{H}) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Longrightarrow \varphi(H) \notin \mathfrak{J}\}, \tag{2.2}$$

and one of the characterizations of the usual notion of essential spectrum in a Hilbert space setting (see §2.2). It is thus natural to call this set the essential spectrum of H with respect to the ideal $\mathfrak{J}$, and denote it by $\mathfrak{J}\text{-}\boldsymbol{\sigma}_{\text{ess}}(H)$.

2.2. We consider now the case where the C^*-algebra $\mathfrak{C}$ is realized on a Hilbert space $\mathscr{H}$, i.e. $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathscr{H})$. A self-adjoint operator

H on $\mathscr{H}$ is called *affiliated to* $\mathfrak{C}$ if $(H-z)^{-1} \in \mathfrak{C}$ for some $z \in \mathbb{C} \setminus \boldsymbol{\sigma}(H)$. This implies $\varphi(H) \in \mathfrak{C}$ for all $\varphi \in C_0(\mathbb{R})$, so each self-adjoint operator on $\mathscr{H}$ affiliated to $\mathfrak{C}$ defines an observable affiliated to $\mathfrak{C}$. The other observables affiliated to $\mathfrak{C}$ can be realized as *non-densely* defined operators on $\mathscr{H}$.

If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathfrak{C}$ and if the corresponding observable is $\mathfrak{C}$-nondegenerate we say that H *is strictly affiliated to* $\mathfrak{C}$. In this case, the Cohen-Hewitt theorem (see V.9.2 in [FeD]) implies that for each $S \in \mathfrak{C}$ there are $\varphi_1,\ \varphi_2 \in C_0(\mathbb{R})$ and $T \in \mathfrak{C}$ such that $S = \varphi_1(H)T\varphi_2(H)$. As a consequence, the operators $\varphi(H)S$ and $S\varphi(H)$ belong to $\mathfrak{C}$ for all $\varphi \in C_{\mathrm{b}}(\mathbb{R})$.

We give an example which clarifies the distinction between affiliation and strict affiliation. Consider the C^*-algebra $C_0(\mathbb{R})$ realized as usual on $L^2(\mathbb{R})$ and let h be the real function given by $h(x) = x + x^{-1}$ for $x \neq 0$ and $h(0) = 0$. Then the operator H of multiplication by h is affiliated to $C_0(\mathbb{R})$, because $(h+i)^{-1}$ is equal outside zero (so almost everywhere) to a function from $C_0(\mathbb{R})$. But if $\varphi \in C_0(\mathbb{R})$ then $\varphi(H)$ is the operator of multiplication by a function from $C_0(\mathbb{R})$ which vanishes at zero, so H cannot be strictly affiliated to $C_0(\mathbb{R})$. Moreover, if φ is a continuous function equal to zero near $-\infty$ and to 1 near $+\infty$, then $\varphi(H)$ is multiplication by a function discontinuous at zero, so $\varphi(H)S$ does not belong to $C_0(\mathbb{R})$ if $S \in C_0(\mathbb{R})$ does not vanishes at zero.

One says that $\mathfrak{C}$ is *nondegenerate on* $\mathscr{H}$ if the linear subspace generated by the elements of the form Sf, with $S \in \mathfrak{C}$ and $f \in \mathscr{H}$, is dense in $\mathscr{H}$. It can be shown that *if* $\mathfrak{C}$ *is nondegenerate on* $\mathscr{H}$*, then the correspondence between self-adjoint operators on* $\mathscr{H}$ *strictly affiliated to* $\mathfrak{C}$ *and observables strictly affiliated to* $\mathfrak{C}$ *defined above is bijective.*

We stress the fact that if $\mathfrak{J}$ is an ideal in $\mathfrak{C}$ and H is a self-adjoint operator affiliated to $\mathfrak{C}$, then the quotient $\widehat{H}$ is a well defined observable affiliated to $\widehat{\mathfrak{C}} = \mathfrak{C}/\mathfrak{J}$. This operation has no meaning from the strictly hilbertian point of view, and this is the main advantage of the algebraic framework that we use. Indeed, in most cases $\widehat{H}$ has no meaning as operator on $\mathscr{H}$, because $\widehat{\mathfrak{C}}$ has no natural realization on $\mathscr{H}$. But if H is strictly affiliated to $\mathfrak{C}$, then one can realize $\widehat{H}$ as a self-adjoint operator in each nondegenerate representation of $\mathfrak{C}$.

Now assume that $K(\mathscr{H}) \subset \mathfrak{C}$ and let us take above $\mathfrak{J} = K(\mathscr{H})$. We recall that a real number λ does not belong to the essential spectrum of a self-adjoint operator H if and only if $\varphi(H) \in K(\mathscr{H})$ for some $\varphi \in C_0(\mathbb{R})$ such that $\varphi(\lambda) \neq 0$. Hence if H is affiliated to $\mathfrak{C}$ we get $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \boldsymbol{\sigma}(\widehat{H})$.

2.3. The algebras that we consider have to be rather small, such that the quotient with respect to the ideal of compact operators be computable. On the other hand we would like that the class of self-adjoint operators affiliated to it be large. So we are interested in having efficient affiliation criteria. We recall two such criteria below.

We remain in the setting of §2.2 and consider a self-adjoint operator H_0 on $\mathscr{H}$. We say that V is a *standard form perturbation* of H_0 if V a continuous symmetric sesquilinear form on $\mathscr{G} = D(|H_0|^{\frac{1}{2}})$ and if there are numbers $\mu \in [0,1)$ and $\delta \in \mathbb{R}$ such that either $\pm V \le \mu|H_0| + \delta$ as forms on $\mathscr{G}$ or H_0 is bounded from below and $V \ge -\mu H_0 - \delta$ as forms on $\mathscr{G}$. Then the form sum $H = H_0 + V$ is a self-adjoint operator on $\mathscr{H}$ with the same form domain as H_0. We give a criterion which ensures the affiliation of H to $\mathfrak{C}$ if H_0 is affiliated to $\mathfrak{C}$. We introduce

$$\mathcal{M}(\mathfrak{C}) \;=\; \{S \in B(\mathscr{H}) \,|\, T \in \mathfrak{C} \Longrightarrow ST,\; TS \in \mathfrak{C}\}. \tag{2.3}$$

Clearly $\mathcal{M}(\mathfrak{C})$ is a C^*-algebra. It can be identified with the usual multiplier algebra (as defined e.g. in [Lan]) if and only if $\mathfrak{C}$ is nondegenerate on $\mathscr{H}$. The next result is from [DaG2].

Theorem 2.1 (i) *Let H_0 be bounded from below and affiliated to $\mathfrak{C}$ and assume that the operator $U \equiv (|H_0| + 1)^{-1/2} V (|H_0| + 1)^{-1/2}$ belongs to $\mathcal{M}(\mathfrak{C})$. Then H is affiliated to $\mathfrak{C}$.*
(ii) *If H_0 is strictly affiliated to $\mathfrak{C}$ then $U \in \mathcal{M}(\mathfrak{C})$ if and only if one has $\varphi(H_0)V(|H_0|+1)^{-1/2} \in \mathfrak{C}$ for all $\varphi \in C_{\mathrm{c}}(\mathbb{R})$ and then H is strictly affiliated to $\mathfrak{C}$.*

We mention a second method for checking the affiliation to $\mathfrak{C}$ of a formal sum $H = H_0 + V$; this is useful when V is in no sense dominated by H_0 (in the preceding situation we have $\pm V \le C(|H_0| + 1)$ for some number C). Assume that H_0 and V are self-adjoint operators bounded from below and that their operator sum is essentially self-adjoint; denote H the closure of $H_0 + V$. If $\mathrm{e}^{-tH_0}\mathrm{e}^{-2tV}\mathrm{e}^{-tH_0} \in \mathfrak{C}$ for all $t > 0$, then H is affiliated to $\mathfrak{C}$ (see Chapter 8 in [ABG] and [ITZ] for a recent improvement of the result of Rogova used in [ABG] which implies the preceding result).

2.4. Let $\{\mathfrak{C}_i\}_{i\in I}$ be an arbitrary family of C^*-algebras. We recall the definition of their direct product $\prod_{i\in I}\mathfrak{C}_i$ and their direct sum $\bigoplus_{i\in I}\mathfrak{C}_i$:

$$\begin{aligned}
\textstyle\prod_{i\in I}\mathfrak{C}_i &= \{S = (S_i)_{i\in I} \mid S_i \in \mathfrak{C}_i \text{ and } \|S\| := \sup_{i\in I}\|S_i\| < \infty\}\\
\textstyle\bigoplus_{i\in I}\mathfrak{C}_i &= \{S = (S_i)_{i\in I} \mid S_i \in \mathfrak{C}_i \text{ and } \|S_i\| \to 0 \text{ as } i \to \infty\}.
\end{aligned}$$

These are C^*-algebras for the usual operations and $\bigoplus_{i\in I}\mathfrak{C}_i$ is an ideal in $\prod_{i\in I}\mathfrak{C}_i$. We denote by $\prod_{i\in I}S_i$ and $\bigoplus_{i\in I}S_i$ an element of $\prod_{i\in I}\mathfrak{C}_i$ and $\bigoplus_{i\in I}\mathfrak{C}_i$ respectively.

If for each $i\in I$ an observable H_i affiliated to $\mathfrak{C}_i$ is given, we may associate to it an observable $H=\prod_{i\in I}H_i$ affiliated to $\mathfrak{C}=\prod_{i\in I}\mathfrak{C}_i$ by setting $\varphi(H)=\prod_{i\in I}\varphi(H_i)$ for each $\varphi\in C_0(\mathbb{R})$. It is easily shown that H is affiliated to the subalgebra $\bigoplus_{i\in I}\mathfrak{C}_i$ if and only if $H_i\to\infty$ as $i\to\infty$ in I in the following sense: for each compact real set K there is a finite subset $F\subset I$ such that $\boldsymbol{\sigma}(H_i)\cap K=\emptyset$ if $i\in I\setminus F$. One has

$$\boldsymbol{\sigma}(H)=\overline{\bigcup_{i\in I}\boldsymbol{\sigma}(H_i)}$$

and if H is affiliated to $\bigoplus_{i\in I}\mathfrak{C}_i$ then the union is already closed. The following generalization of this relation will be important for us.

Theorem 2.2 *For each $i\in I$ let $\mathfrak{J}_i$ be an ideal in $\mathfrak{C}_i$ and let $\mathfrak{J}=\bigoplus_{i\in I}\mathfrak{J}_i$, so that $\mathfrak{J}$ is an ideal in $\mathfrak{C}=\prod_{i\in I}\mathfrak{C}_i$. Denote by $\widehat{H_i}$ the quotient of H_i in $\mathfrak{C}_i/\mathfrak{J}_i$ and let $\widehat{H}$ be the quotient of H in $\mathfrak{C}/\mathfrak{J}$. Then*

$$\boldsymbol{\sigma}(\widehat{H})=\bigcap_{\substack{F\subset I\\ F\text{finite}}}\left\{\left(\bigcup_{i\in F}\boldsymbol{\sigma}(\widehat{H_i})\right)\cup\left(\overline{\bigcup_{j\in I\setminus F}\boldsymbol{\sigma}(H_j)}\right)\right\}. \tag{2.4}$$

Remark: With a notation introduced above, observe that (2.4) may be written as:

$$\mathfrak{J}\text{-}\boldsymbol{\sigma}_{\text{ess}}(H)=\bigcap_{\substack{F\subset I\\ F\text{finite}}}\left\{\left(\bigcup_{i\in F}\mathfrak{J}_i\text{-}\boldsymbol{\sigma}_{\text{ess}}(H_i)\right)\cup\left(\overline{\bigcup_{j\in I\setminus F}\boldsymbol{\sigma}(H_j)}\right)\right\}.$$

Proof: Let $\lambda\notin\boldsymbol{\sigma}(\widehat{H})$. By (2.2) there exists $\varphi\in C_0(\mathbb{R})$ such that $\varphi(\mu)=1$ on a neighborhood J of λ and $\varphi(H)\in\mathfrak{J}$. Thus for all $i\in I$ one has $\varphi(H_i)\in\mathfrak{J}_i$ and $\|\varphi(H_i)\|\to 0$ as $i\to\infty$ in I. The first assertion shows (again by (2.2)) that $\lambda\notin\bigcup_{i\in I}\boldsymbol{\sigma}(\widehat{H_i})$ and the second one ensures the existence of a finite set $F\subset I$ such that $\|\varphi(H_i)\|<1$ if $i\notin F$. But $\sup\{|\varphi(x)|\mid x\in\boldsymbol{\sigma}(H_i)\}=\|\varphi(H_i)\|<1$, so $J\cap\boldsymbol{\sigma}(H_i)=\emptyset$ for all $i\in I\setminus F$, hence $\lambda\notin\overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)}$. Since $\boldsymbol{\sigma}(\widehat{H_i})\subset\boldsymbol{\sigma}(H_i)$ for all i, we get $\lambda\notin\bigcup_{i\in I}\boldsymbol{\sigma}(\widehat{H_i})\cup\overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)}=\bigcup_{i\in F}\boldsymbol{\sigma}(\widehat{H_i})\cup\overline{\bigcup_{i\in I\setminus F}\boldsymbol{\sigma}(H_i)}$ for some finite F.

Reciprocally, if λ does not belong to the r.h.s. of (2.4) (which is a closed set of the form $\bigcap_{F\subset I}\Sigma_F$) there is a compact neighborhood J of λ disjoint from it. Since the upper directed family of open sets $\mathbb{R}\setminus\Sigma_F$ covers J, there is F_0 (finite subset of I) such that $J\subset\mathbb{R}\setminus\Sigma_{F_0}$. Thus $J\cap\bigcup_{i\in F_0}\boldsymbol{\sigma}(\widehat{H_i})$ and

$J \cap \bigcup_{i\in I\setminus F_0} \boldsymbol{\sigma}(H_i)$ are empty sets. This means that there is a $\varphi \in C_c(J)$, with $\varphi = 1$ on a neighborhood of λ, such that $\varphi(H_i) \in \mathfrak{J}_i$ for all $i \in F_0$ and $\varphi(H_i) = 0$ for all $i \notin F_0$. In particular one has $\varphi(H_i) \in \mathfrak{J}_i$ for all $i \in I$ and $\|\varphi(H_i)\| \to 0$ as $i \to \infty$ in I. Thus $\varphi(\widehat{H}) \in \mathfrak{J}$, i.e. $\lambda \notin \boldsymbol{\sigma}(\widehat{H})$. ∎

2.5. We consider here the case when $\mathfrak{C}_i \equiv \mathfrak{A}$ is an algebra independent of $i \in I$. Then we denote by $\mathfrak{A}^{(I)}$ the C^*-algebra $\bigoplus_{i\in I} \mathfrak{A}$. Besides the direct product and the direct sum one can introduce now a third C^*-algebra, namely

$$\mathfrak{A}^{[I]} := \{(S_i)_{i\in I} \in \textstyle\prod_{i\in I} \mathfrak{A} \mid \{S_i \mid i \in I\} \text{ is relatively compact in } \mathfrak{A}\}. \tag{2.5}$$

This is a C^*-subalgebra of $\prod_{i\in I} \mathfrak{A}$ and $\mathfrak{A}^{(I)}$ is an ideal in $\mathfrak{A}^{[I]}$. The set I being equipped with the discrete topology we denote by $c_0(I;\mathfrak{A})$ and $l^\infty(I;\mathfrak{A})$ the C^*-algebras consisting of $\mathfrak{A}$ valued maps on I which converge to zero at infinity or are bounded, respectively. Then $\mathfrak{A}^{(I)} = c_0(I;\mathfrak{A})$ and $\prod_{i\in I} \mathfrak{A} = l^\infty(I;\mathfrak{A})$. Moreover,

$$\mathfrak{A}^{[I]} = l^{\mathrm{rc}}(I;\mathfrak{A}) := \{S : I \to \mathfrak{A} \mid S \text{ has relatively compact range in } \mathfrak{A}\}.$$

Lemma 2.3 *If $\mathfrak{A}$ is a C^*-algebra, then*

$$\mathfrak{A}^{(I)} \cong c_0(I) \otimes \mathfrak{A} \quad \text{and} \quad \mathfrak{A}^{[I]} \cong l^\infty(I) \otimes \mathfrak{A}. \tag{2.6}$$

Proof: The first relation is obvious. To prove the second one, assume that $\mathfrak{A} \subset B(\mathscr{H})$ and realize $l^\infty(I)$ as a C^*-algebra on $l^2(I)$ in the standard way. Then $l^\infty(I) \otimes_{\mathrm{alg}} \mathfrak{A}$ is realized on $l^2(I) \otimes \mathscr{H} = l^2(I;\mathscr{H})$ as the set $l^\infty_{\mathrm{fin}}(I;\mathfrak{A})$ of operators of multiplication by functions $F : I \to \mathfrak{A}$ such that the range of F is included in a finite dimensional subspace of $\mathfrak{A}$. Now it suffices to prove that the closure of $l^\infty_{\mathrm{fin}}(I;\mathfrak{A})$ in $l^\infty(I;\mathfrak{A})$ is equal to $l^{\mathrm{rc}}(I;\mathfrak{A})$, which is easy. ∎

More generally, if $\{\mathfrak{C}_i\}_{i\in I}$ is an arbitrary family of C^*-subalgebras of a C^*-algebra $\mathfrak{C}$ one can define their *restricted product* $\prod^{\mathrm{rc}}_{i\in I} \mathfrak{C}_i$ as in (2.5).

2.6. One more object will appear naturally in our later investigations: the *I-asymptotic algebra* of $\mathfrak{A}$. This is the quotient algebra:

$$\mathfrak{A}^{\langle I\rangle} := \mathfrak{A}^{[I]} \big/ \mathfrak{A}^{(I)} \tag{2.7}$$

The following description of $\mathfrak{A}^{\langle I\rangle}$ explains the name we gave it. Let $\widetilde{I}$ be the set of ultrafilters $\varkappa$ on I finer than the Fréchet filter (since I is a discrete space, this is the family of sets having finite complementaries). $\widetilde{I}$ should be thought as the "boundary" of I. Then for each $S=(S_i)_{i\in I}\in\mathfrak{A}^{[I]}$ and each $\varkappa\in\widetilde{I}$ the limit $\lim_{i,\varkappa} S_i := S_\varkappa$ along $\varkappa\in\widetilde{I}$ exists (because the range of the map $S: I\to\mathfrak{A}$ is included in a compact subset of $\mathfrak{A}$, see page TG I.59 in [Bou2]). We thus get a map

$$\mathfrak{A}^{[I]}\ni S\mapsto (S_\varkappa)_{\varkappa\in\widetilde{I}}\in l^\infty(\widetilde{I};\mathfrak{A})\equiv \textstyle\prod_{\varkappa\in\widetilde{I}}\mathfrak{A} \tag{2.8}$$

which is clearly a morphism. Let us prove that its kernel is precisely $\mathfrak{A}^{(I)}$. Indeed, if $S\notin\mathfrak{A}^{(I)}$ there is an infinite subset $J\subset I$ such that $\|S_i\|\ge$const>0 if $i\in J$, and there is $\varkappa\in\widetilde{I}$ such that $J\in\varkappa$. Then clearly $\lim_{i,\varkappa}S_i\neq 0$. Thus (2.8) induces a canonical embedding $\mathfrak{A}^{\langle I\rangle}\hookrightarrow l^\infty(\widetilde{I};\mathfrak{A})$ which, however, is *not* surjective: for each $S\in\mathfrak{A}^{[I]}$ the limits $S_\varkappa$ belong to the closure of the range of S, hence one has $\mathfrak{A}^{\langle I\rangle}\hookrightarrow\mathfrak{A}^{[\widetilde{I}]}\subsetneq l^\infty(\widetilde{I};\mathfrak{A})$. Remark, moreover, that neither the embedding $\mathfrak{A}^{\langle I\rangle}\hookrightarrow\mathfrak{A}^{[\widetilde{I}]}$ is surjective: in fact, there is a natural topology on $\widetilde{I}$ such that $\widetilde{I}$ becomes a compact space (see ch.2 in [HiSt] or Théorème T.2,XXII,6;5 in [Sch1]) and then

$$\mathfrak{A}^{\langle I\rangle}\cong C(\widetilde{I};\mathfrak{A}). \tag{2.9}$$

This isomorphism may be obtained as follows. There is a canonical identification of $\widetilde{I}$ with the spectrum of the quotient C^*-algebra $l^\infty(I)/c_0(I)$, so that this quotient equals $C(\widetilde{I})$. In other terms, we have a surjective morphism $l^\infty(I)\to C(\widetilde{I})$ whose kernel is $c_0(I)$. Then, by taking its tensor product with the identity map, we get a surjective morphism sending $l^\infty(I)\otimes\mathfrak{A}\cong\mathfrak{A}^{[I]}$ onto $C(\widetilde{I})\otimes\mathfrak{A}\cong C(\widetilde{I};\mathfrak{A})$ whose kernel is $c_0(I)\otimes\mathfrak{A}\cong\mathfrak{A}^{(I)}$. This gives (2.9).

We now describe a certain class of C^*-subalgebras of $\mathfrak{A}^{[I]}$ containing the ideal $\mathfrak{A}^{(I)}$. Let $\mathcal{I}$ be a partition of I consisting of infinite sets. Then

$$\mathfrak{A}^{\mathcal{I}} := \{(S_i)_{i\in I}\in\mathfrak{A}^{[I]}\mid \lim_{i\to\infty, i\in J} S_i\equiv S_J \text{ exists in } \mathfrak{A},\ \forall J\in\mathcal{I}\}. \tag{2.10}$$

Clearly, this is the set of $S\in\mathfrak{A}^{[I]}$ such that, for each $J\in\mathcal{I}$ and for each ultrafilter $\varkappa\in\widetilde{I}$ with $J\in\varkappa$, the limit $S_\varkappa=\lim_{i,\varkappa}S_i$ depends only on J. Note that for each $\varkappa\in\widetilde{I}$ there is a unique $J\in\mathcal{I}$ such that $J\in\varkappa$ (see page TG I.39 in [Bou2]) so $\mathcal{I}$ defines also a partition of $\widetilde{I}$ (consisting of subsets which are open and closed, as it can be easily shown). The following fact is a straightforward consequence of the definition (2.10): if $\mathcal{I}$ is *finite*, then

$$\mathfrak{A}^{\mathcal{I}}\big/\mathfrak{A}^{(I)}\cong\textstyle\bigoplus_{J\in\mathcal{I}}\mathfrak{A}. \tag{2.11}$$

For example, in the simplest case when the partition $\mathcal{I}$ of I is trivial (i.e. $\mathcal{I} = \{I\}$) we get $\mathfrak{A}^{\{I\}}/\mathfrak{A}^{(I)} \cong \mathfrak{A}$ by the following argument. Let $c_\infty(I)$ be the set of maps $I \to \mathbb{C}$ which have a limit at infinity, so $c_\infty(I) = \mathbb{C} + c_0(I)$ and $\mathfrak{A}^{\{I\}} = c_\infty(I) \otimes \mathfrak{A}$. Then

$$\mathfrak{A}^{\{I\}}\Big/\mathfrak{A}^{(I)} \cong c_\infty(I) \otimes \mathfrak{A}\Big/c_0(I) \otimes \mathfrak{A} \cong \left[c_\infty(I)\Big/c_0(I)\right] \otimes \mathfrak{A} \cong \mathbb{C} \otimes \mathfrak{A} \cong \mathfrak{A}.$$

3 Crossed Products

3.1. In this section we first recall the definition of crossed products in the particular case of abelian groups and then we discuss several results which we have not been able to locate in the literature in a form convenient to us. We fix a locally compact abelian group X (with the operation denoted additively) and let $\mathrm{d}x$ be a Haar measure on it. Note that the crossed products $\mathscr{A} \rtimes X$ defined below will not depend on the choice of this measure.

We shall say that a C^*-algebra $\mathscr{A}$ is a X-*algebra* if a homomorphism $\alpha : x \mapsto \alpha_x$ of X into the group of automorphisms of $\mathscr{A}$ is given, such that for each $a \in \mathscr{A}$ the map $x \mapsto \alpha_x(a)$ is continuous. A subalgebra of $\mathscr{A}$ is called *stable* if it is left invariant by all the automorphisms α_x. If $(\mathscr{A}, \alpha)$ and $(\mathscr{B}, \beta)$ are two X-algebras, a morphism $\phi : \mathscr{A} \to \mathscr{B}$ is called X-*morphism* (or *covariant* morphism) if $\phi[\alpha_x(a)] = \beta_x[\phi(a)]$ for all $x \in X$ and $a \in \mathscr{A}$.

Let $\mathscr{A}$ be a X-algebra and let $L^1(X; \mathscr{A})$ the Banach $*$-algebra constructed as follows. As a Banach space it is just the space of (Bochner) integrable (equivalence classes of) functions $S : X \to \mathscr{A}$. The product and the involution are defined by:

$$(S \cdot T)(x) = \int_X S(y)\, \alpha_y[T(x-y)]\, \mathrm{d}y, \tag{3.12}$$

$$S^*(x) = \alpha_x[S(-x)^*]. \tag{3.13}$$

Note that $C_\mathrm{c}(X; \mathscr{A})$, the space of continuous functions $X \to \mathscr{A}$ with compact support, is a dense $*$-subalgebra of $L^1(X; \mathscr{A})$. Moreover, the algebraic tensor product $\mathscr{A} \otimes_{\mathrm{alg}} C_\mathrm{c}(X)$ is a dense subspace (identified with the set of elements from $C_\mathrm{c}(X; \mathscr{A})$ whose ranges are contained in finite dimensional subspaces of $\mathscr{A}$).

Assume, furthermore, that $\mathscr{A}$ is realized on a Hilbert space $\mathscr{H}$ and let $\mathscr{H}_X = L^2(X; \mathscr{H})$. Then one has a faithful representation of $L^1(X; \mathscr{A})$ on $\mathscr{H}_X$, the so-called *left regular representation*: one defines the action of

$S \in L^1(X;\mathscr{A})$ onto $\xi \in \mathscr{H}_X$ by

$$(S \bullet \xi)(x) = \int_X \alpha_{-x}[S(x-y)]\,\xi(y)\,\mathrm{d}y. \tag{3.14}$$

In particular, $L^1(X;\mathscr{A})$ *is an* A^**-algebra* (see §6.1 in the Appendix).

Definition 3.1 *If $\mathscr{A}$ is a X-algebra, then the* ***crossed product*** *of $\mathscr{A}$ by the action α of X, is the enveloping C^*-algebra of the A^*-algebra $L^1(X;\mathscr{A})$. This C^*-algebra is denoted by $\mathscr{A} \rtimes X$.*

Thus $\mathscr{A} \rtimes X$ is the completion of $L^1(X;\mathscr{A})$ under the largest C^*-norm on it, and each representation of $L^1(X;\mathscr{A})$ extends to a representation of $\mathscr{A} \rtimes X$ (see the §6.1 in the Appendix). Due to the fact that X is abelian (hence amenable) the crossed product defined above coincides with the so-called "reduced crossed product" (cf. Theorems 7.7.5 and 7.7.7 in [Ped]):

Theorem 3.2 *The left regular representation of $L^1(X;\mathscr{A})$ extends to a faithful representation of $\mathscr{A} \rtimes X$. In particular, $\mathscr{A} \rtimes X$ is canonically isomorphic to the closure in $B(\mathscr{H}_X)$ of the $*$-algebra of operators of the form (3.14).*

Heuristically, one should think of $\mathscr{A} \rtimes X$ as a kind of twisted tensor product of algebras $\mathscr{A}$ and $C_0(X^*)$, where X^* is the group dual to X. In fact, if the action of X on $\mathscr{A}$ is trivial, then $\mathscr{A} \rtimes X = \mathscr{A} \otimes C_0(X^*)$.

3.2. The correspondence $\mathscr{A} \mapsto \mathscr{A} \rtimes X$ extends to a covariant functor from the category of X-algebras (with X-morphisms as morphisms) into the category of C^*-algebras. Indeed, if $\phi : \mathscr{A} \to \mathscr{B}$ is a X-morphism, then it clearly induces a morphism $\phi_0 : L^1(X;\mathscr{A}) \to L^1(X;\mathscr{B})$ by the formula $(\phi_0 S)(x) := \phi[S(x)]$. Hence we may define the morphism $\phi_* : \mathscr{A} \rtimes X \to \mathscr{B} \rtimes X$ as the canonical extension of ϕ_0 to the enveloping algebras, i.e. $\phi_* = (\phi_0)_*$.

Theorem 3.3 *Let $\mathscr{J}$, $\mathscr{A}$, $\mathscr{B}$ be X-algebras and let*

$$0 \longrightarrow \mathscr{J} \xrightarrow{\ \phi\ } \mathscr{A} \xrightarrow{\ \psi\ } \mathscr{B} \longrightarrow 0$$

be an exact sequence of X-morphisms. Then

$$0 \longrightarrow \mathscr{J} \rtimes X \xrightarrow{\ \phi_*\ } \mathscr{A} \rtimes X \xrightarrow{\ \psi_*\ } \mathscr{B} \rtimes X \longrightarrow 0$$

is an exact sequence.

Proof: It suffices to prove that

$$0 \longrightarrow L^1(X;\mathscr{J}) \xrightarrow{\ \phi_0\ } L^1(X;\mathscr{A}) \xrightarrow{\ \psi_0\ } L^1(X;\mathscr{B}) \longrightarrow 0$$

is an exact sequence of Banach $*$-algebras; then we use Theorem 6.1. The injectivity of ϕ_0 and the relation $\psi_0 \circ \phi_0 = 0$ are obvious. If $S \in L^1(X;\mathscr{A})$ and $\psi_0(S) = 0$ then $\psi[S(x)] = 0$ for a.e. $x \in X$, i.e. $S(X) \in \phi(\mathscr{J})$ for a.e. x. But ϕ is an isometry, so there is $T \in L^1(X;\mathscr{J})$ such that $S(X) = \phi[T(x)]$ for a.e. x. This proves that $\ker \psi_0 = \phi_0(L^1(X;\mathscr{J}))$.

The surjectivity of ϕ_0 is a consequence of the following general property (see §3.5 in [DeFl]). Let $\mathbf{E}$ be a Banach space and $F \in L^1(X;\mathbf{E})$. Then for each number $\varepsilon > 0$ there are sequences $f_n \in L^1(X)$ and $e_n \in \mathbf{E}$ such that $F = \sum_{n=1}^{\infty} e_n \otimes f_n$ and

$$\sum_{n=1}^{\infty} \|e_n\| \int |f_n(x)|\,\mathrm{d}x \leq (1+\varepsilon) \int \|F(x)\|\,\mathrm{d}x.$$

Note also that since the map $\mathscr{A}/\ker\psi \to \mathscr{B}$ induced by ψ is an isometric bijection, for each $b \in \mathscr{B}$ there is $a \in \mathscr{A}$ such that $\psi(a) = b$ and $\|a\| = (1+\varepsilon)\|b\|$. ■

Let $\mathscr{J}$ be a stable ideal of a X-algebra $\mathscr{A}$. According to Theorem 3.3, if $j : \mathscr{J} \to \mathscr{A}$ is the inclusion map, then $j_* : \mathscr{J} \rtimes X \to \mathscr{A} \rtimes X$ is an isometric morphism of $\mathscr{J} \rtimes X$ onto an ideal of $\mathscr{A} \rtimes X$. From now on *we shall identify* $\mathscr{J} \rtimes X$ *with its image under* j_*. More explicitly, $\mathscr{J} \rtimes X$ is just the closure in $\mathscr{A} \rtimes X$ of the ideal $L^1(X;\mathscr{J})$ of $L^1(X;\mathscr{A})$.

Now the quotient C^*-algebra $\mathscr{B} = \mathscr{A}/\mathscr{J}$ has a natural structure of X-algebra such that the canonical morphism $\mathscr{A} \to \mathscr{A}/\mathscr{J}$ is a X-morphism. The Theorem 3.3 says also that the morphism $\mathscr{A} \rtimes X \to [\mathscr{A}/\mathscr{J}] \rtimes X$ associated to it has $\mathscr{J} \rtimes X$ as kernel. We thus get the following reformulation of Theorem 3.3:

Theorem 3.4 *If $\mathscr{J}$ is a stable ideal of a X-algebra $\mathscr{A}$ then*

$$\mathscr{A} \rtimes X \big/ \mathscr{J} \rtimes X \;\cong\; [\mathscr{A}/\mathscr{J}] \rtimes X. \tag{3.15}$$

The simplest case of the preceding situation is that when the exact sequence splits, so that $\mathscr{A}/\mathscr{J}$ can be realized as a stable C^*-subalgebra of $\mathscr{A}$. Then we have:

Corollary 3.5 *Let $\mathscr{A}$ be a X-algebra, $\mathscr{J}$ a stable ideal, and $\mathscr{B}$ a stable C^*-subalgebra such that $\mathscr{A} = \mathscr{B} + \mathscr{J}$ direct linear sum. Then $\mathscr{J} \rtimes X$ is an ideal in $\mathscr{A} \rtimes X$, $\mathscr{B} \rtimes X$ is a C^*-subalgebra of $\mathscr{A} \rtimes X$, and $\mathscr{A} \rtimes X = \mathscr{B} \rtimes X + \mathscr{J} \rtimes X$ direct linear sum.*

Corollary 3.6 *Let $\mathscr{A}$, $\mathscr{B}$ be X-algebras and let $\mathscr{A} \oplus \mathscr{B}$ be equipped with the natural X-algebra structure. Then*

$$(\mathscr{A} \oplus \mathscr{B}) \rtimes X \cong (\mathscr{A} \rtimes X) \oplus (\mathscr{B} \rtimes X). \tag{3.16}$$

We mention one more fact:

Proposition 3.7 *If $\phi : \mathscr{A} \to \mathscr{B}$ is an injective or surjective X-morphism then $\phi_* : \mathscr{A} \rtimes X \to \mathscr{B} \rtimes X$ is injective or surjective respectively. In particular, if $\mathscr{A}$ is a stable C^*-subalgebra of the X-algebra $\mathscr{B}$, then $\mathscr{A} \rtimes X$ can be identified with a C^*-subalgebra of $\mathscr{B} \rtimes X$.*

The assertion is obvious in the surjective case. For the injective case, see Proposition 7.7.9 in [Ped]. So what we proved above for ideals is valid for subalgebras too.

Proposition 3.8 *Let $\mathscr{A}$ be a X-algebra and let $\mathscr{B}$ be a nuclear (e.g. abelian) C^*-algebra. Equip $\mathscr{A} \otimes \mathscr{B}$ with the X-algebra structure defined by $\alpha_x(a \otimes b) = \alpha_x(a) \otimes b$. Then*

$$(\mathscr{A} \otimes \mathscr{B}) \rtimes X \cong (\mathscr{A} \rtimes X) \otimes \mathscr{B}. \tag{3.17}$$

Proposition 2.4 in [Tak] asserts more than this. Since Takai gives no proof, we shall sketch a simple one in §6.2 of the Appendix.

3.3. We discuss now the behavior of the crossed product under infinite direct products and sums. Let $\{\mathscr{A}_i\}_{i \in I}$ be an arbitrary family of C^*-algebras. Assume that each $\mathscr{A}_i$ is a X-algebra, the corresponding group of automorphisms being α^i. Then one may define $\alpha : X \to \mathrm{Aut}(\prod_{i \in I} \mathscr{A}_i)$ by $\alpha_x[(a_i)_{i \in I}] = (\alpha^i_x[a_i])_{i \in I}$. In this way we do not (in general) get a X-algebra structure on $\prod_{i \in I} \mathscr{A}_i$ because the continuity condition is not satisfied. However, we may define an "equicontinuous product" algebra as the largest subalgebra on which α acts continuously:

$$\textstyle\prod^X_{i \in I} \mathscr{A}_i = \{(a_i)_{i \in I} \in \prod_{i \in I} \mathscr{A}_i \mid \lim_{x \to 0} \sup_{i \in I} \|\alpha^i_x[a_i] - a_i\| = 0\}.$$

This is naturally a X-algebra which contains $\bigoplus_{i \in I} \mathscr{A}_i$ as a stable subalgebra, so $\bigoplus_{i \in I} \mathscr{A}_i$ becomes a X-algebra too.

Proposition 3.9 $\left(\bigoplus_{i \in I} \mathscr{A}_i\right) \rtimes X \cong \bigoplus_{i \in I} (\mathscr{A}_i \rtimes X)$.

Proof: Denote $\mathscr{A} = \bigoplus_{i \in I} \mathscr{A}_i$. Since each $\mathscr{A}_i$ is an ideal in $\mathscr{A}$, we have canonical embeddings of $\mathfrak{A}_i \equiv \mathscr{A}_i \rtimes X$ as ideals in $\mathfrak{A} \equiv \mathscr{A} \rtimes X$. Now it suffices to show two things: 1) $\mathfrak{A}_i \cdot \mathfrak{A}_j = 0$ if $i \neq j$ and 2) the linear subspace $\sum_{i \in I} \mathfrak{A}_i$ generated by $\bigcup_{i \in I} \mathfrak{A}_i$ is dense in $\mathfrak{A}$. Both assertions follow easily from the fact that $\mathfrak{A}_i$ is the closure of $L^1(X; \mathscr{A}_i)$ in $\mathfrak{A}$. ■

We shall go beyond direct sums only in the particular case we need. Assume that $\mathscr{A}$ is a X-algebra and I is a set. The algebras $\mathscr{A}^{(I)}$ and $\mathscr{A}^{[I]}$ have been introduced in §2.5. Then the C^*-algebra $\mathscr{A}^{[I]}$ is a X-algebra, and $\mathscr{A}^{(I)}$ is a stable ideal in it, if we set $\alpha_x[(a_i)_{i \in I}] = (\alpha_x[a_i])_{i \in I}$. Indeed, for each $\varepsilon > 0$ there is a finite set $K \subset \mathscr{A}$ such that $\operatorname{dist}(a_i, K) < \varepsilon$ for all $i \in I$. Then

$$\|\alpha_x[(a_i)_{i \in I}] - (a_i)_{i \in I}\| = \sup_{i \in I} \|\alpha_x[a_i] - a_i\| \leq 2\varepsilon + \sup_{b \in K} \|\alpha_x[b] - b\|$$

and the last term is $< \varepsilon$ if x is in a suitable neighborhood of zero in X. Note that we could also consider the X-algebra $\mathscr{A}^{[I,X]} = \prod_{i \in I}^{X} \mathscr{A}$ which depends on the action of X on $\mathscr{A}$ and which contains $\mathscr{A}^{[I]}$ as a stable ideal.

Proposition 3.10 *If $\mathscr{A}$ is a X-algebra and I a set, then*

$$\mathscr{A}^{(I)} \rtimes X \cong (\mathscr{A} \rtimes X)^{(I)} \quad \text{and} \quad \mathscr{A}^{[I]} \rtimes X \cong (\mathscr{A} \rtimes X)^{[I]}. \tag{3.18}$$

Moreover, the I-asymptotic algebra $\mathscr{A}^{\langle I \rangle}$ has a canonical structure of X-algebra and one has

$$(\mathscr{A} \rtimes X)^{[I]} \Big/ (\mathscr{A} \rtimes X)^{(I)} \;\cong\; (\mathscr{A} \rtimes X)^{\langle I \rangle} \cong C(\widetilde{I}\,; \mathscr{A} \rtimes X). \tag{3.19}$$

The first identification of (3.18) is a particular case of Proposition 3.9. The second one is a consequence of Proposition 3.8 and of the Lemma 2.3. The last part of the proposition follows from the Theorem 3.4 and the representation (2.9).

3.4. We introduce now several new notations and recall facts concerning the harmonic analysis on X that we will need later on (see [Fol2, Loo, Wei] for details). Note that the Hilbert space $L^2(X) = L^2(X, \mathrm{d}x)$ depends on the choice of the Haar measure $\mathrm{d}x$, but the C^*-algebras $\mathbb{B}(X) = B(L^2(X))$, $\mathbb{K}(X) = K(L^2(X))$ do not. We shall embed the C^*-algebras $C_0(X)$, $C_{\mathrm{b}}^{\mathrm{u}}(X)$, $C_{\mathrm{b}}(X)$ in $\mathbb{B}(X)$ by associating to $\varphi \in C_{\mathrm{b}}(X)$ the operator of multiplication by the function φ. In order to avoid ambiguities we often denote this operator by $\varphi(Q)$ (by analogy with the standard quantum mechanical situation where X is the additive group associated to a real vector space and Q is the X valued position observable).

Let X^* be the set of characters of X (continuous homomorphisms $k : X \to \mathbb{C}$ with $|k(x)| = 1$) equipped with the locally compact group structure defined by the operation of multiplication and the topology of uniform convergence on compact sets. We denote the operation in X^* additively and its neutral element by 0 (as in [Wei], ch.II, §5). This convention looks rather strange if $X = \mathbb{Z}^n$ for example, but stresses the fact that we are especially interested in the case when X is the additive group of a vector space. We shall use the notation $\langle\!\langle x, k \rangle\!\rangle = k(x)$. If X is a real finite dimensional vector space then X^* is identified with the vector space dual to X by the following convention: let $\langle \cdot , \cdot \rangle : X \times X^* \to \mathbb{R}$ be the canonical bilinear map and take $\langle\!\langle x, k \rangle\!\rangle = \mathrm{e}^{i\langle x,k \rangle}$. This convention is natural also if the field of real numbers is replaced by the field of p-adic numbers, see page 91 in [Fol2]. For the case of an arbitrary non-discrete locally compact field, see ch.II, §5 in [Wei] (and also [Sa-C, Tai]).

The Fourier transform of $u \in L^1(X) = L^1(X, \mathrm{d}x)$ is the function $\mathcal{F}u \equiv \widehat{u} : X^* \to \mathbb{C}$ given by $\widehat{u}\,(k) = \int_X \overline{\langle\!\langle x, k \rangle\!\rangle} u(x)\,\mathrm{d}x$. Then $\mathcal{F}$ is a linear injective contractive map $L^1(X) \to C_0(X^*)$ and we shall equip X^* with the unique Haar measure $\mathrm{d}k$ such that $\mathcal{F}$ induces a unitary map $\mathcal{F} : L^2(X, \mathrm{d}x) \to L^2(X^*, \mathrm{d}k)$. From $\mathcal{F}^{-1} = \mathcal{F}^*$ we get $(\mathcal{F}^{-1}v)(x) = \int_{X^*} \langle\!\langle x, k \rangle\!\rangle v(k)\,\mathrm{d}k$ for $v \in L^2(X^*)$. We also recall that the dual group $(X^*)^*$ of X^* is identified with X, each $x \in X$ being seen as a character of X^* through the formula $x(k) = k(x)$, or $\langle\!\langle k, x \rangle\!\rangle = \langle\!\langle x, k \rangle\!\rangle$. Then the Fourier transform of $\psi \in L^1(X^*)$ is given by $\widehat{\psi}\,(x) = (\mathcal{F}^*\psi)(-x)$. For each $\psi \in C_{\mathrm{b}}(X^*)$ we define the operator $\psi(P) \in \mathbb{B}(X)$ by $\psi(P) = \mathcal{F}^* M_\psi \mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$ (the symbol P has no signification here but, as we already said, it will be convenient for us to keep notations close to those standard in the quantum mechanical setting where P is the X^* valued momentum observable). The injective morphism $\psi \mapsto \psi(P)$ gives us an embedding $C_{\mathrm{b}}(X^*) \subset \mathbb{B}(X)$.

We have two strongly continuous unitary representations $\{U_x\}_{x\in X}$ and $\{V_k\}_{k\in X^*}$ of X and X^* in $L^2(X)$ defined by $(U_x f)(y) = f(x + y)$ and $(V_k f)(y) = \langle\!\langle y, k \rangle\!\rangle f(y)$ respectively. Note that U_x and V_k satisfy the so called canonical commutation relations $U_x V_k = \langle\!\langle x, k \rangle\!\rangle V_k U_x$. The group C^*-algebra of X is the C^*-subalgebra of $\mathbb{B}(X)$ generated by the convolution operators $\int_X u(z) U_z\,\mathrm{d}z$ with $u \in L^1(X)$, and is canonically isomorphic to $C_0(X^*)$. The isomorphism is determined by the formula $\psi(P) = \int_X \widehat{\psi}\,(z) U_z\,\mathrm{d}z$ for $\psi \in C_0(X^*)$ such that $\widehat{\psi} \in L^1(X)$. One has a similar identification of the group C^*-algebra of X^* with $C_0(X)$. By using the preceding representation of $\psi(P)$ one can prove the following result (the proof given in [DaG1] for X

a real vector space extends with minor modifications to the case of abelian groups).

Proposition 3.11 *Let $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ and let $\psi \in C_0(X^*)$ be the Fourier transform of a function from $L^1(X)$ (the set of such ψ is a dense $*$-subalgebra of $C_0(X^*)$). Then for each number $\varepsilon > 0$ there are points $x_1, \dots, x_n \in X$ and functions $\psi_1, \dots, \psi_n \in C_0(X^*)$ such that*

$$\|\psi(P)\varphi(Q) - \textstyle\sum_{1\le i\le n}\varphi(Q+x_i)\psi_i(P)\| \le \varepsilon.$$

In particular, if $\mathscr{A}$ is a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ stable under translations then $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ is a C^*-algebra of operators on $L^2(X)$.

3.5. The group X acts in a natural way on $C_{\mathrm{b}}^{\mathrm{u}}(X)$: if $x \in X$ and if we denote by $\tau_x\varphi$ the function $y \mapsto \varphi(y-x)$ then for $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ we have $\tau_x\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $x \mapsto \tau_x\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ is norm continuous. We consider a C^*-subalgebra $\mathscr{A}$ of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ stable under translations: $\tau_x\varphi \in \mathscr{A}$ if $x \in X$ and $\varphi \in \mathscr{A}$. Then $\mathscr{A}$ is a X-algebra and we are interested in the crossed product $\mathfrak{C} = \mathscr{A} \rtimes X$ of $\mathscr{A}$ by the action $\alpha := \tau$ of X. In such a situation the crossed product $\mathscr{A} \rtimes X$ has an especially useful faithful representation that we shall describe below.

Recall that we have embedded $C_{\mathrm{b}}^{\mathrm{u}}(X) \subset \mathbb{B}(X)$; then we have $(\tau_x\varphi)(Q) \equiv \tau_x[\varphi(Q)] = U_x^*\varphi(Q)U_x$. In particular $\mathscr{A} \subset \mathbb{B}(X)$ and our purpose is to show that $\mathscr{A} \rtimes X$ can also be realized as a C^*-algebra of operators on the Hilbert space $L^2(X)$.

Theorem 3.12 *Let $\mathscr{A}$ be a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ stable under translations. Then the linear subspace $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ is a C^*-algebra on the Hilbert space $L^2(X)$ and*

$$[\![\mathscr{A} \cdot C_0(X^*)]\!] \cong \mathscr{A} \rtimes X \tag{3.1}$$

in the sense that there is a unique isomorphism $\Phi : [\![\mathscr{A} \cdot C_0(X^*)]\!] \to \mathscr{A} \rtimes X$ such that $\Phi\,[\varphi(Q)\psi(P)] = S_{\varphi,\psi}$ for all $\varphi \in \mathscr{A}$ and $\psi \in C_0(X^*)$ with $\widehat{\psi} \in L^1(X)$. Here $S_{\varphi,\psi}$ is the element $y \mapsto S_{\varphi,\psi}(\cdot, y) \in \mathscr{A}$ of $L^1(X;\mathscr{A})$ defined by the function $S_{\varphi,\psi}(x,y) = \varphi(x)\widehat{\psi}\,(y)$.

Proof: The fact that $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ is a C^*-algebra has already been noticed above, but it is also a consequence of the next arguments. Due to Theorem 3.2 we have the following description of $\mathscr{A} \rtimes X$. Let

$$\mathscr{H}_X = L^2(X;\mathscr{H}) \cong \mathscr{H} \otimes L^2(X) \cong L^2(X \times X).$$

To each integrable function $S : X \to \mathscr{A}$ we associate an operator $S\bullet \in B(\mathscr{H}_X)$ in the following manner: if $\xi : X \to \mathscr{H}$ is L^2, then

$$(S \bullet \xi)(y) = \int_X \tau_y[S(y-z)]\,\xi(z)\,\mathrm{d}z = \int_X U_y^* S(z) U_y\,\xi(y-z)\,\mathrm{d}z. \tag{3.2}$$

The map $S \mapsto S\bullet$ of $L^1(X;\mathscr{A})$ into $B(\mathscr{H}_X)$ is linear and injective. One equips $L^1(X;\mathscr{A})$ with a structure of $*$-algebra by asking that $S \mapsto S\bullet$ be a $*$-morphism; then one sets $\|S\| := \|S\bullet\|_{B(\mathscr{H}_X)}$. The completion of $(L^1(X;\mathscr{A}), \|\cdot\|)$ will then be a C^*-subalgebra of $B(\mathscr{H}_X)$ and this C^*-algebra is (canonically isomorphic to) the crossed product $\mathscr{A} \rtimes X$.

This representation, however, is not convenient for our purposes. We thus construct a new one with the help of the unitary operator $W : \mathscr{H}_X \to \mathscr{H}_X$ defined as $(W\xi)(x,y) := \xi(x-y,x)$. Note that its adjoint is given by $(W^*\xi)(x,y) = \xi(y,y-x)$.

If $S \in L^1(X;\mathscr{A})$ then S may also be viewed as a function $S : X \times X \to \mathbb{C}$ with the convention $S(y) = S(\cdot,y) \in \mathscr{A}$. Similarly, an element $\xi : X \to \mathscr{H}$ of $\mathscr{H}_X$ is interpreted as a function $\xi : X \times X \to \mathbb{C}$ by setting $\xi(y) = \xi(\cdot,y)$. Then (3.2) may be written as:

$$(S \bullet \xi)(x,y) = \int_X S(x-y,z)\,\xi(x,y-z)\,\mathrm{d}z,$$

which allows us to compute:

$$\begin{aligned}
&(W^* S \bullet W\xi)(x,y) = (S \bullet W\xi)(y,y-x) \\
&= \int_X S(x,z)\,(W\xi)(y,y-x-z)\,\mathrm{d}z = \int_X S(x,z)\,\xi(x+z,y)\,\mathrm{d}z \\
&= \int_X S(x,z)\,[(U_z \otimes 1)\xi]\,(x,y)\,\mathrm{d}z = \int_X \{[S(Q,z)\,U_z \otimes 1]\,\xi\}\,(x,y)\,\mathrm{d}z.
\end{aligned}$$

In other terms,

$$W^*\,[S\bullet]\,W = \left[\int_X S(Q,z)\,U_z\,\mathrm{d}z\right] \otimes 1.$$

Consider the particular case when $S(x,y) = S_{\varphi,\psi}(x,y) = \varphi(x)\widehat{\psi}\,(y)$ as in the statement of the theorem. Then the above integral is equal to $\varphi(Q)\psi(P)$, cf. the comments made before the statement of Proposition 3.11. So we have

$$W^*\,[S_{\varphi,\psi}\bullet]\,W = [\varphi(Q)\psi(P)] \otimes 1.$$

Since the subspace generated by the elements of the form $S_{\varphi,\psi}$ is dense in $L^1(X;\mathscr{A})$, the assertions of the theorem follow easily. ∎

3.6. The *phase space* of X is the locally compact abelian group $\Xi = X \oplus X^*$. If $\xi = (x,k)$ and $\eta = (y,p)$ are elements of Ξ, we set $\beta(\xi,\eta) = \langle\!\langle y,k\rangle\!\rangle \cdot \langle\!\langle -x,p\rangle\!\rangle$. Then β is a *bicharacter* of Ξ, i.e. a continuous map of $\Xi \times \Xi$ into $\{z \in \mathbb{C} \mid |z| = 1\}$ which is a character in each variable when the other one is fixed. Let $\alpha : \Xi \to \mathbb{C}$ be a continuous map such that $\alpha(0) = 1$ and $|\alpha(\xi)| = 1$ for all $\xi \in \Xi$ and let us set

$$W(\xi) = \alpha(\xi) U_{-x} V_k \quad \text{and} \quad \omega(\xi,\eta) = \frac{\alpha(\xi+\eta)}{\langle\!\langle y,k\rangle\!\rangle \alpha(\xi)\alpha(\eta)}. \tag{3.3}$$

Then W is a strongly continuous map of Ξ into the set of unitary operators on $L^2(X)$ satisfying $W(\xi+\eta) = \omega(\xi,\eta)W(\xi)W(\eta)$. In other terms, W is a projective representation of the group Ξ with phase factor ω (note that $W(\xi)W(\eta) = \beta(\xi,\eta)W(\eta)W(\xi)$). By a well known generalization of the Stone-Von Neumann theorem due to G. Mackey (see Theorem C.38 in [RaWi]), each such representation of Ξ is unitarily equivalent to a multiple of W (which is irreducible).

The representation W allows one to construct a pseudo-differential calculus which depends on α (different choices give different "quantization" rules). In this first part of our paper the choice of α does not play any role, so we shall not specify it. However, in the second part [GI2] of the paper, when X is a vector space, we shall find convenient to work with the Weyl calculus (although the algebras we construct do not depend on this choice), so we introduce here the notations specific to this situation.

Assume that X is a (real or p-adic) finite dimensional vector space. Then we equip Ξ with the symplectic form (i.e. bilinear antisymmetric nondegenerate form) σ defined by $\sigma(\xi,\eta) = \langle y,k\rangle - \langle x,p\rangle$. Then $\beta = \mathrm{e}^{i\sigma}$ and we take $\alpha(\xi) = \mathrm{e}^{\frac{i}{2}\langle x,k\rangle}$, which gives $\omega = \mathrm{e}^{-\frac{i}{2}\sigma}$. It is convenient in this case to use the quantum mechanical conventions and to write

$$U_x = \mathrm{e}^{i\langle x,P\rangle}, \qquad V_k = \mathrm{e}^{i\langle Q,k\rangle}. \tag{3.4}$$

Thus we will have

$$W(\xi) = \mathrm{e}^{\frac{i}{2}\langle x,k\rangle}\mathrm{e}^{-i\langle x,P\rangle}\mathrm{e}^{i\langle Q,k\rangle} = \mathrm{e}^{-\frac{i}{2}\langle x,k\rangle}\mathrm{e}^{i\langle Q,k\rangle}\mathrm{e}^{-i\langle x,P\rangle} = \mathrm{e}^{i(\langle Q,k\rangle - \langle x,P\rangle)}.$$

More explicitly, for $f \in L^2(X)$ we have

$$(W(\xi)f)(y) = \mathrm{e}^{i\langle y-\frac{x}{2},k\rangle} f(y-x). \tag{3.5}$$

We state now a version of the Riesz-Kolmogorov compacity criterion in $L^2(X)$ (see [GI1]) which may be considered as a geometric (or phase space) description of $\mathbb{K}(X)$. We shall adopt the following abbreviation: if S is an operator on $L^2(X)$ then $S^\perp = 1 - S$.

Proposition 3.13 *An operator $T \in \mathbb{B}(X)$ is compact if and only if it satisfies one of the following equivalent conditions:*
(i) $\lim_{x\to 0} \|(U_x - 1)T\| = 0$ *and* $\lim_{k\to 0} \|(V_k - 1)T\| = 0$.
(ii) $\lim_{\xi\to 0} \|(W(\xi) - 1)T\| = 0$.
(iii) $\forall \varepsilon > 0 \ \exists \varphi \in C_{\mathrm{c}}(X) \ \exists \psi \in C_{\mathrm{c}}(X^*)$ *such that* $\|\varphi(Q)^\perp T\| + \|\psi(P)^\perp T\| < \varepsilon$.

Two other descriptions of the algebra $\mathbb{K}(X)$ are furnished by the result:

Proposition 3.14 $\mathbb{K}(X) = [\![C_0(X) \cdot C_0(X^*)]\!] \cong C_0(X) \rtimes X$.

For the proof of the first equality, see [GI1] for example. Then the canonical isomorphism with $C_0(X) \rtimes X$ follows from Theorem 3.12 (for another proof of the isomorphism of $\mathbb{K}(X)$ with $C_0(X) \rtimes X$ see Proposition 3.3 in [Tak]).

3.7. Later on we will need the following noncommutative version of the Ascoli theorem (which follows from Proposition 3.13, see [GI1] for details).

Proposition 3.15 *A bounded subset $\mathscr{K} \subset \mathbb{B}(X)$ is a relatively compact set of compact operators if and only if it satisfies the following equivalent conditions:*
(i) $\lim\limits_{x\to 0} \sup\limits_{T\in\mathscr{K}} \|(U_x - 1)T^{(*)}\| = 0$ *and* $\lim\limits_{k\to 0} \sup\limits_{T\in\mathscr{K}} \|(V_k - 1)T^{(*)}\| = 0$.
(ii) $\lim_{\xi\to 0} \sup_{T\in\mathscr{K}} \|(W(\xi) - 1)T^{(*)}\| = 0$.
(iii) *For each $\varepsilon > 0$ there are $\varphi \in C_{\mathrm{c}}(X)$ and $\psi \in C_{\mathrm{c}}(X^*)$ such that*

$$\|\varphi(Q)^\perp T^{(*)}\| + \|\psi(P)^\perp T^{(*)}\| < \varepsilon \quad \textit{for all } T \in \mathscr{K}.$$

Finally, we prove a result mentioned in the Introduction which is the main justification for considering crossed products as C^*-algebras of energy observables.

Proposition 3.16 *Let $\mathscr{A}$ be a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ which contains the constants and is stable under translations. Let $h : X^* \to \mathbb{R}$ be a continuous non-constant function such that $\lim_{k\to\infty} |h(k)| = \infty$. Then $\mathscr{A} \rtimes X$ is the C^*-algebra generated by the self-adjoint operators of the form $h(P + k) + V(Q)$, with $k \in X^*$ and $V \in \mathscr{A}$ real.*

Proof: Let $\mathfrak{C}$ be the C^*-algebra generated by the operators $H = h(P + k) + V(Q) \equiv H_0 + V(Q)$, with $k \in X^*$ and $V \in \mathscr{A}$ real. By making a norm convergent series expansion for large z

$$(z - H)^{-1} = \sum_{n\geq 0} (z - H_0)^{-1} [V(Q)(z - H_0)^{-1}]^n$$

we get $\mathfrak{C} \subset \mathscr{A} \rtimes X$. It remains to prove the opposite inclusion. For each $\mu \in \mathbb{R}$ the operator $H_\mu = h(P+k)+\mu V(Q)$ is affiliated to $\mathfrak{C}$ and the function $\mu \mapsto (H_\mu - i)^{-1}$ is norm derivable at $\mu = 0$ with derivative $-(H_0 - i)^{-1}V(Q)(H_0 - i)^{-1}$. Hence $(H_0 - i)^{-1}V(Q)(H_0 - i)^{-1} \in \mathfrak{C}$. Let $\theta \in C_c(\mathbb{R})$ with $\theta(0) = 1$ and $\varepsilon > 0$. Since H_0 is affiliated to $\mathfrak{C}$, we get $\theta(\varepsilon H_0)(H_0 - i) \in \mathfrak{C}$, and so $\theta(\varepsilon H_0)V(Q)(H_0 - i)^{-1} \in \mathfrak{C}$. From the uniform continuity of V, and since $(h(p+k)-i)^{-1} \to 0$ when $p \to \infty$ in X^*, we get $\|(U_x - 1)V(Q)(H_0 - i)^{-1}\| \to 0$ if $x \to 0$ in X. This implies $\lim_{\varepsilon \to 0} \theta(\varepsilon H_0)V(Q)(H_0 - i)^{-1} = V(Q)(H_0 - i)^{-1}$ in norm in $B(L^2(X))$ (indeed, for $T \in \mathbb{B}(X)$ we have $\lim_{x \to 0} \|(U_x - 1)T\| = 0$ if and only if for each $\delta > 0$ there is $\eta \in C_c(X^*)$ such that $\|\eta(P)^\perp T\| < \delta$). Hence $V(Q)(h(P + k) - i)^{-1} \in \mathfrak{C}$ for each $k \in X^*$ and each $V \in \mathscr{A}$ real. But H_0 is affiliated to $\mathfrak{C}$, so this implies $\varphi(Q)\psi(P) \in \mathfrak{C}$ for all $\varphi \in \mathscr{A}$ (not necessarily real) and all ψ in the $*$-subalgebra $\mathscr{B} \subset C_0(X^*)$ generated by functions of the form $p \mapsto \xi(h(p + k))$ with $\xi \in C_c(\mathbb{R})$ and $k \in X^*$. By the Stone-Weierstrass theorem, $\mathscr{B}$ is dense in $C_0(X^*)$. Hence, since the set of $\psi \in C_0(X^*)$ such that $\varphi(Q)\psi(P) \in \mathfrak{C}$ is norm closed and contains $\mathscr{B}$, we finally obtain $\varphi(Q)\psi(P) \in \mathfrak{C}$ for all $\varphi \in \mathscr{A}, \psi \in C_0(X^*)$. ■

The proposition can be restated as follows: *$\mathscr{A} \rtimes X$ is the smallest C^*-algebra of operators on $L^2(X)$ which contains all the resolvents $(h(P) + V(Q) + i)^{-1}$ with $V \in \mathscr{A}$ real and which is stable under the automorphisms* $S \mapsto V_k S V_k^*$ (see (3.4)).

4 Algebras of hamiltonians: examples

4.1. In this section we shall describe several algebras of hamiltonian operators of physical interest. Most of them are treated in a rather succint manner either because they present no difficulties, or because they will be considered in detail in other publications. Two examples are studied in depth in Section 5 and in the second part [GI2] of this paper. Roughly speaking, there will be four types of such algebras:

(i) Crossed products of translation invariant unital C^*-subalgebras of $C_b^u(X)$ by the natural action of X. These correspond to Q-anisotropic systems (X being the configuration space).

(ii) The simplest type of phase space (or Q-P) anisotropy, where the algebra is (essentially) of the form $[\![\mathscr{A} \cdot \mathscr{B}]\!]$, where $\mathscr{A}$ and $\mathscr{B}$ are C^*-subalgebras of $C_b^u(X)$ and $C_b^u(X^*)$ respectively.

(iii) Full phase space anisotropy: here we consider C^*-algebras naturally associated to the Heisenberg group, which is the simplest type of nilpotent group. The algebra we construct will be called *graded symplectic algebra* and will be studied in detail in the second part of this paper. This also suggests the consideration of algebras associated to unitary representations of more general groups, which we shall, however, not do here. More specifically, we have in mind groups related to non-euclidean (hyperbolic) geometries.

(iv) Algebras of creation-anihilation operators on a Fock space. These are generated by hamiltonians of quantum fields with a particle number cut-off. Their quotient with respect to the ideal of compact operators has an especially nice form: we shall describe the result here but the proof will be given elsewhere ([GM]; see also [G]).

The main emphasis of this paper is on the technique of crossed products, which allows one to do the computations at an abelian level. Assuming that X *is non compact*, the strategy is as follows. We first choose a C^*-algebra $\mathscr{A}$, interpreted as the algebra of classical interactions (or potentials), such that:

$$C_\infty(X) \subset \mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X) \text{ and } \mathscr{A} \text{ is stable under translations.} \tag{4.1}$$

We recall that $C_\infty(X) = \mathbb{C} + C_0(X)$, so $\mathscr{A}$ is unital (i.e. it contains the constant functions) and $C_0(X)$ is an ideal in it. Now, if we can give a convenient description of $\mathscr{A}/C_0(X)$, then by using Proposition 3.14 and Theorem 3.4 we get

$$[\mathscr{A} \rtimes X]/\mathbb{K}(X) \;\cong\; [\mathscr{A}/C_0(X)] \rtimes X. \tag{4.2}$$

The simplest situation is that when $\mathscr{A} = C_\infty(X)$. Then we get algebra $\mathbb{T}(X)$ of energy observables corresponding to a two-body quantum system (or a particle in an external field):

$$\mathbb{T}(X) = C_\infty(X) \rtimes X = (\mathbb{C} + C_0(X)) \rtimes X = C_0(X^*) + \mathbb{K}(X). \tag{4.3}$$

Here we used the Corollary 3.5. We trivially have $\mathbb{T}(X)/\mathbb{K}(X) = C_0(X^*)$.

Notice that $C_\infty(X)$ is related to the simplest compactification of X, the one-point (Alexandroff) compactification: $X_\infty \equiv X \cup \{\infty\}$. Indeed, we have a natural identification $C_\infty(X) = C(X_\infty)$. In fact the algebras $\mathscr{A}$ with the properties (4.1) are in bjective correspondence with a certain class of compactifications of X: the spectrum $X(\mathscr{A})$ of $\mathscr{A}$ is a compact space and X is homeomorphically and densely embedded in $X(\mathscr{A})$; then $\mathscr{A}$ is just the set of functions in $C_{\mathrm{b}}^{\mathrm{u}}(X)$ which have continuous extensions to $X(\mathscr{A})$, i.e. $\mathscr{A} = C(X(\mathscr{A}))$. These compactifications can be described in terms of ultrafilters on X: they are quotients of the Stone-Čech compactification βX, see [GiJe, HiSt, Wal] (see also [Sim1] for concrete constructions of nontrivial compactifications in elementary situations). We shall, however, not insist on this point of view, because in non trivial cases the spaces $X(\mathscr{A})$ are rather complicated.

4.2. Before going on with the construction of algebras satisfying (4.1) we shall state and discuss a formula describing the essential spectrum of an observable affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ in terms of its "localizations at infinity". Detailed proofs of the results presented below will be found in [GI3].

Theorem 4.1 *Let H be an observable affiliated to the algebra $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$. Then the strong limit $\text{s-lim}_{x,\varkappa} U_x H U_x^* = H_\varkappa$ exists for each ultrafilter $\varkappa$ on X finer than the Fréchet filter and one has*

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \textstyle\bigcup_\varkappa \boldsymbol{\sigma}(H_\varkappa). \tag{4.4}$$

The statement of the theorem is independent of the representation of the crossed product $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ on $L^2(X)$ (the action of translations on this algebra being defined in an obvious way). Then by strong limit we mean norm convergence as $x \to \infty$ along the filter $\varkappa$ of all quantities of the form $U_x \varphi(H) U_x^* K$ with $K \in \mathbb{K}(X) = C_0(X) \rtimes X$ and $\varphi \in C_0(\mathbb{R})$ (it suffices in fact to have this for $\varphi(t) = (t \pm i)^{-1}$). We stress that even if we work in the standard representation on $L^2(X)$ and H is a densely defined self-adjoint operator, the limits $H_\varkappa$ are observables but are not densely defined self-adjoint operators on $L^2(X)$ in general. For example, H could be an operator with purely discrete spectrum, so $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \emptyset$; then $H_\varkappa = \infty$ and $\boldsymbol{\sigma}(H_\varkappa) = \emptyset$ for all $\varkappa$.

The following corollary is a consequence of the characterization of the self-adjoint operators affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ obtained in [DaG2] in the case of $X = \mathbb{R}^n$. This result is probably true for all locally compact abelian groups, but one step of the proof from [DaG2] does not extend easily to the general case.

Corollary 4.2 *Let $X = \mathbb{R}^n$ and H a self-adjoint operator in $L^2(X)$ such that:*

$$\lim_{\substack{x \to 0 \\ x \in X}} \|(U_x - 1)(H+i)^{-1}\| = 0 \quad \textit{and} \quad \lim_{\substack{k \to 0 \\ k \in X^*}} \|[V_k, (H+i)^{-1}]\| = 0.$$

Then, for each ultrafilter $\varkappa$ on X finer than the Fréchet filter the limit $H_\varkappa = \text{s-lim}_{x,\varkappa} U_x H U_x^*$ exists in strong resolvent sense, and $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_\varkappa \boldsymbol{\sigma}(H_\varkappa)$.

The operators $H_\varkappa$ will be called *localizations at infinity* of H. To be precise we should say Q-infinity. Indeed, the "infinity" of other observables (e.g. the momentum P) could play a role too and in fact such a pure configuration space description of the essential spectrum is not valid in general (see below the case of the algebra $[\![C(\overline{\mathbb{R}}) \cdot C(\overline{\mathbb{R}^*})]\!]$). More complicated situations appear naturally in connection with algebras of hamiltonians associated to symplectic spaces, where the boundary at infinity of the phase space plays a role.

The conditions imposed on H in this corollary are easy to check in many situations. On the other hand, the affiliation criteria from §2.3 can also be used to check the affiliation to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ of operators of the form $H = h(P) + V(Q)$.

Some particular situations covered by the preceding corollary have been treated by different methods in [HeM]. We mention only two examples, the first one being borrowed from [HeM]. Let $H = \Delta + V(x)$ in $L^2(\mathbb{R})$. If $V(x) = \langle x \rangle \cos^2 \sqrt{\langle x \rangle}$, then $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \mathbb{N} + 1/2$ (here $\langle x \rangle = (1 + |x|^2)^{1/2}$). If $V(x) = \sin \sqrt{|x|}$, then $\boldsymbol{\sigma}_{\mathrm{ess}}(H) = [-1, \infty)$. To get this it suffices to determine the operators $H_\varkappa$. It is easy to show (the computations are clarified by the notion of localization at infinity for the case of functions introduced below) that in the first case these are exactly the operators $\Delta + (\lambda \pm Q/2)^2$ with $\lambda \in \mathbb{R}$, while for the second example they are $\Delta + \lambda$ with $\lambda \in [-1, 1]$.

We shall now give the main ideea of the proof of Theorem 4.1 in order to make the connection with the algebraic framework of the rest of this paper.

An operator $S \in \mathbb{B}(X)$ will be called *regular* if $\{U_x S^{(*)} U_x^* \mid x \in X\}$ is a strongly relatively compact part of $\mathbb{B}(X)$. Clearly the set of regular operators is a unital C^*-subalgebra of $\mathbb{B}(X)$ which contains $\mathbb{K}(X)$. From the Riesz-Kolmogorov compacity criterion it follows easily that it also contains $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ (see Proposition 3.15). A self-adjoint operator H on $L^2(X)$ will be called regular if it is affiliated to the algebra of regular operators.

We recall now a result of Stone concerning the Stone-Čech compactification of X: if Φ is a continuous map from X to a Hausdorff topological space Y and if the range of Φ is a relatively compact set, then Φ has a unique continuous extension to βX (this property is known as the universal property of βX). This is related to the fact that the limit of Φ along any ultrafilter exists (under the same conditions). Recall also that X is an open dense subset of βX (because X is locally compact) and denote $\widetilde{X} = \beta X \setminus X$ the boundary of X in βX.

If S is a regular operator then the map $x \mapsto U_x S U_x^*$ is continuous and has relatively compact range if $\mathbb{B}(X)$ is equipped with the strong operator topology, hence it extends to a strongly continuous map $\varkappa \mapsto S_\varkappa$ from βX to $\mathbb{B}(X)$. The operators $S_\varkappa$ with $\varkappa \in \widetilde{X}$ will be called *localizations at infinity* of S. It is clear now how to define the localizations at infinity of a regular (possibly unbounded) self-adjoint operator (of course, these localizations will not be densely defined operators in general, so it is more convenient to work from the begining with observables, as we have done in Theorem 4.1). The inclusion $\boldsymbol{\sigma}_{\text{ess}}(H) \supset \bigcup_{\varkappa\in\widetilde{X}} \boldsymbol{\sigma}(H_\varkappa)$ and the fact that the right hand side is closed are not difficult to show. It is also clear that in this generality one cannot have equality[(a)]. But if H is affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ then one has

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{\varkappa\in\widetilde{X}} \boldsymbol{\sigma}(H_\varkappa). \tag{4.5}$$

The description of $\widetilde{X}$ in terms of z-ultrafilters (which can be found in [GiJe] or [Wal]) shows that this is a strengthened version of the formula (4.4).

Our proof of the equality (4.5) is based on a certain description of the quotient $C_{\mathrm{b}}^{\mathrm{u}}(X)/C_0(X)$, which we present below. By the universal property of βX each $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ extends to an element of $C(\beta X)$ and this gives an identification of the algebras $C_{\mathrm{b}}^{\mathrm{u}}(X)$ and $C(\beta X)$. The restriction map $\varphi \mapsto \varphi|_{\widetilde{X}}$ induces an isomorphism between $C_{\mathrm{b}}^{\mathrm{u}}(X)/C_0(X)$ and $C(\widetilde{X})$. However, this fact does not seem us directly relevant for the proof of (4.5). Instead, we consider the "localizations at infinity" of $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ which can be defined (as we did in the case of operators) by using the next result: *A function $\varphi \in C_{\mathrm{b}}(X)$ belongs to $C_{\mathrm{b}}^{\mathrm{u}}(X)$ if and only if for some (and hence for all) $\theta \in C_0(X), \theta \neq 0$, the set of functions of the form $\theta\, \tau_x\varphi$, $x \in X$, is relatively compact in $C_0(X)$* (this is equivalent to the relative compactness of the set $\{\tau_x\varphi | x \in X\}$ in the strict topology of $C_{\mathrm{b}}(X)$). Thus if $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ we can define $\varphi_\varkappa = \lim_{x,\varkappa} \tau_{-x}\varphi$ for each ultrafilter $\varkappa$ on X finer then the Fréchet filter (the limit exists locally uniformly on X). Observe that if $\varkappa$ is a free z-ultrafilter then we have $\varphi_\varkappa(x) = \varphi(x + \varkappa)$ (in the right hand side appears the extension of φ to $\widetilde{X}$ and the canonical action of the group X on βX which leaves $\widetilde{X}$ invariant); this allows one to relate the two descriptions of the quotient $C_{\mathrm{b}}^{\mathrm{u}}(X)/C_0(X)$ we work with.

It can be shown that the map $\varphi \mapsto \{\varphi_\varkappa\}_{\varkappa\in\widetilde{X}}$ is a morphism from $C_{\mathrm{b}}^{\mathrm{u}}(X)$ into $C(\widetilde{X}; C_{\mathrm{b}}^{\mathrm{u}}(X)_{\text{st}})$ with $C_0(X)$ as kernel (here the index "st" means "strict topology"). Thus we get a canonical embedding $C_{\mathrm{b}}^{\mathrm{u}}(X)/C_0(X) \subset$

[(a)]Note that the definition of regularity makes sense if X is an arbitrary group and $L^2(X)$ is replaced by a Hilbert space equipped with a projective representation of X. For example, one may consider the phase space Ξ and the representation W_ξ in $L^2(X)$. Then the formula for the essential spectrum of the regular operators will involve the boundary of the phase space in its Stone-Čech compactification.

$C(\widetilde{X}\,; C^{\rm u}_{\rm b}(X)_{\rm st})$ and this induces the embedding $\mathfrak{C}/\mathbb{K}(X) \subset C(\widetilde{X}\,; \mathfrak{C}_{\rm st})$ where $\mathfrak{C} = C^{\rm u}_{\rm b}(X) \rtimes X$ which, in turn, allows one to prove (4.5).

4.3. Instead of considering explicitly the Stone-Čech compactification of X one can construct algebras $\mathscr{A}$ with the properties (4.1) by using limits at infinity along certain filters. Although a filter can be identified with the set of ultrafilters finer than it, this point of view is more fruitful in applications, allowing one to get less abstract results.

Let us say that a filter $\mathcal{F}$ is a TIF if it is translation invariant ($A \in \mathcal{F}, x \in X \Rightarrow x + A \in \mathcal{F}$) and finer than the Fréchet filter. To each such filter we associate the space of $\varphi \in C^{\rm u}_{\rm b}(X)$ which have a limit at infinity along the filter:

$$C_{\mathcal{F}}(X) = \{\varphi \in C^{\rm u}_{\rm b}(X) \mid \lim_{\mathcal{F}} \varphi \text{ exists}\}. \tag{4.6}$$

It is clear that $C_{\mathcal{F}}(X)$ is a unital C^*-subalgebra of $C^{\rm u}_{\rm b}(X)$ such that (4.1) is satisfied. If $\mathcal{F}$ is the Fréchet filter then $C_{\mathcal{F}}(X) = C_\infty(X)$. There is a largest non-trivial ideal in $C_{\mathcal{F}}(X)$, namely $C_{\mathcal{F},0}(X) = \{\varphi \in C^{\rm u}_{\rm b}(X) \mid \lim_{\mathcal{F}} \varphi = 0\}$, and $C_{\mathcal{F}}(X) = \mathbb{C} + C_{\mathcal{F},0}(X)$. Since $C_0(X) \subset C_{\mathcal{F},0}(X)$, we are reduced to the computation of the quotient $C_{\mathcal{F},0}(X)/C_0(X)$. This can be done with the help of βX, but in general the resulting object is too abstract to be useful in practical computations of essential spectra, for example. We consider now a particular class of algebras of the preceding form.

If $L \subset X$ is closed and Λ is compact in X then we set $L_\Lambda \equiv L + \Lambda = \{x + y \mid x \in L,\ y \in \Lambda\}$ and $L^c_\Lambda = X \setminus L_\Lambda$. If X is equipped with an invariant metric and Λ is the closed ball of radius r, then L^c_Λ is the set of points at distance $> r$ from L. Note that $\Lambda \subset \Lambda' \Rightarrow L_\Lambda \subset L_{\Lambda'}$, in particular $L^c_{\Lambda_1 \cup \Lambda_2} \subset L^c_{\Lambda_1} \cap L^c_{\Lambda_2}$. Moreover, for each $x \in X$ we have $x + L^c_\Lambda = L^c_{x+\Lambda}$.

Now let L be a closed set such that $L_\Lambda \neq X$ if Λ is compact (this means that there are points as far as we wish from L). Then the family of open sets $\{L^c_\Lambda \mid \Lambda \subset X \text{compact}\}$ is the base of a filter $\mathcal{F}_L$ which is a TIF by the preceding remarks. We denote $C_L(X) = C_{\mathcal{F}_L}(X)$ the corresponding C^*-algebra. It will also be convenient to denote by L-$\lim \varphi$ the limit along the filter $\mathcal{F}_L$. Thus, if $\varphi : X \to \mathbb{C}$ then L-$\lim \varphi$ exists if and only if there is a complex number c with the property: for each $\varepsilon > 0$ one can find a compact set $\Lambda \subset X$ such that $|\varphi(x) - c| < \varepsilon$ if $x \notin L_\Lambda$. Then $\lim_{\mathcal{F}_L} \varphi \equiv L$-$\lim \varphi = c$. Now $C_L(X)$ is the C^*-algebra of functions $\varphi \in C^{\rm u}_{\rm b}(X)$ such that L-$\lim \varphi$ exists.

The set L will be called *sparse* if it is locally finite and for each compact Λ of X there is a finite set $F \subset L$ such that if $l \in M = L \setminus F$ and $l' \in L \setminus \{l\}$ then

$(l+\Lambda)\cap(l'+\Lambda)=\emptyset$. For this special category of sets the algebra $C_L(X)$ will be studied in detail in Section 5, in particular we shall compute its quotient by $C_0(X)$ and we shall see that it has a quite interesting structure. We shall prove that the operators affilated to $C_L(X)\rtimes X$ contain those introduced by Klaus in the particular case $X=\mathbb{R}$, cf. [Kla]. In fact, the class of hamiltonians affiliated to $C_L(X)\rtimes X$ is much larger and we can describe the essential spectrum for each of them. It would be interesting to find other classes of sets L or, more generally, TIF's, for which such "explicit" computations can be done. The difficulty of the problem appears already in the seemingly elementary case when L is a straight line in $X=\mathbb{R}^2$.

We have considered a natural generalization of the preceding situation. We do not give the details because it does not involve essentially new ideas; we shall, however, describe it here succinctly. Let L be the union of a family $\mathcal{B}$ of pairwise disjoint compact sets such that for each compact Λ of X there is a finite set $\mathcal{B}_\Lambda\subset\mathcal{B}$ such that $(B+\Lambda)\cap(B'+\Lambda)=\emptyset$ if $B\notin\mathcal{B}_\Lambda$ and $B'\neq B$. Then we say that L is a *dispersed* set. If there is a compact set K such that each $B\in\mathcal{B}$ is a subset of a translate of K (i.e. L is "uniformly" dispersed), then L is equivalent to a sparse set, in the sense that $\mathcal{F}_L$ coincides to $\mathcal{F}_{L^0}$ for some sparse set L^0. Indeed, it suffices to replace each compact set $B\in\mathcal{B}$ by a point sitting inside it. As we shall see, in the case of a sparse set the main role in the computation of the quotient is played by the algebra $C_0(X)^{[L]}$ consisting of relatively compact families $\{\varphi_l\}_{l\in L}$ of elements of $C_0(X)$. For a general dispersed set this has to be replaced with the algebra $C_0(X)^{[L,X]}$ (see §3.3 for the notations) consisting of equicontinous such families (compare with the statement of the Ascoli theorem in §5.1).

4.4. Among the TIF's on X there is a rather remarkable one, the Lorentz filter. We shall define it only for metric groups, although it can be defined in the general case. So we assume that a map $x\mapsto|x|$ is given such that $|x|$ is a positive real number, $|x|=0\Leftrightarrow x=0$, $|x+y|\leq|x|+|y|$, $|-x|=|x|$, and the topology of X is defined by the metric $d(x,y)=|x-y|$. Denote $B(x;r)$ the open ball consisting of the points y such that $|y-x|<r$, and let $B(r)=B(0;r)$, so that $B(x;r)=x+B(r)$. Denote $\lambda(A)$ the Haar measure of a Borel set A and let $v(r)=\lambda(B(r))$. We shall explicitly make the hypothesis $\lim_{r\to\infty}\lambda(B(x;r)\setminus B(r))v(r)^{-1}=0$ locally uniformly in x, although we think it is always satisfied. We say that a Borel set $A\subset X$ is *full (at infinity)* if

$$\lim_{r\to\infty}\inf_{x\in X}\frac{\lambda(A\cap B(x;r))}{v(r)}=1. \tag{4.7}$$

In other terms, A is full if and only if for each $\nu < 1$ there is a positive number $r(\nu)$ such that $\lambda(A \cap B) \geq \nu\lambda(B)$ for each ball B of radius $r \geq r(\nu)$. An arbitrary set containing a Borel full set will also be called full. Clearly, the family of full sets is a TIF; we denote it $\mathcal{L}(X)$ and call it the *Lorentz filter* on X. The limit along the filter $\mathcal{L}(X)$ will be called *Lorentz limit* and will be denoted $\mathfrak{Lim}\,\varphi$ or $\mathfrak{Lim}_{x\to\infty}\varphi(x)$.

For example, assume that $A^c = X \setminus A$ is a disjoint union of balls $B_l = B(l; R_l)$; if A is full, then one must clearly have $\sup_l R_l < \infty$. Reciprocally, if $\sup_l R_l < \infty$ and the distance between the balls growth when $l \to \infty$, the set A is full. In particular, if L is a sparse set, then L^c_Λ is full, hence $\mathcal{F}_L \subset \mathcal{L}(X)$ (if L is only dispersed this inclusion does not hold: in this case, one has to consider abelian limits instead of limits in Lorentz sense).

The C^*-algebra $C_{\mathcal{L}}(X) = C_{\mathcal{L}(X)}(X)$ associated to $\mathcal{L}(X)$ is called *Lorentz algebra* of X. There is an interesting description of the elements of $C_{\mathcal{L}}(X)$. By definition, this algebra consists of all functions $\varphi \in C^{\mathrm{u}}_{\mathrm{b}}(X)$ such that $\mathfrak{Lim}_{x\to\infty}\varphi(x)$ exists. But a bounded Borel function φ converges in Lorentz sense to some number c if and only if

$$\lim_{r\to\infty}\sup_{x\in X}\frac{1}{v(r)}\int_{B(x;r)}|\varphi(y) - c|\,\lambda(\mathrm{d}y) = 0. \tag{4.8}$$

This makes the connection with the notion of almost convergence introduced by G. C. Lorentz for the case of sequences and extended to general semigroups by several other people (see [Pat] and references therein). The "absolute" almost convergence that we denoted $\mathfrak{Lim}$ above was studied especially by C. Chou, see [Cho] and also [Pat] for additional references. The notion of "full set" and the presentation in terms of filters does not seem to be standard, but is quite natural and it is easy to deduce the characterization (4.8) from the results of Chou as presented in [Pat] (see [GI1] for the case $X = \mathbb{R}$).

By an observation made above, we have $C_L(X) \subset C_{\mathcal{L}}(X)$ for each sparse set L. Thus an "explicit" description of the quotient $C_{\mathcal{L}}(X)/C_0(X)$ would provide an L-independent expression for the essential spectra of the hamiltonians affiliated to $C_L(X) \rtimes X$. We shall see below that $C_{\mathcal{L}}(X) \rtimes X$ also contains the N-body type algebras, so this algebra allows a unified treatment of sparse and N-body type interactions. Unfortunately, we do not have any reasonably simple description of $C_{\mathcal{L}}(X)/C_0(X)$.

We construct another class of algebras with the properties (4.1) defined in terms of invariant means on the group X. A state m on the C^*-algebra $C^{\mathrm{u}}_{\mathrm{b}}(X)$ such that $m(\tau_x\varphi) = m(\varphi)\ \forall x \in X$ is called *invariant* (their existence is a nontrivial consequence of the Hahn-Banach theorem). The number

$m(\varphi)$ is, in some sense, a *limit at infinity* of φ. Indeed, if φ is real then $\liminf_{x\to\infty}\varphi(x) \leq m(\varphi) \leq \limsup_{x\to\infty}\varphi(x)$. One important property of the usual notion of limit is not shared by the operation m: it is not multiplicative. But one can easily point out the maximal subspace on which m is multiplicative (see [Cho]). Say that φ is $m-$*convergent* if $m(|\varphi - m(\varphi)|) = 0$ (this is equivalent to $m(|\varphi|^2) = |m(\varphi)|^2$). Now it is clear that

$$C_m(X) = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid \varphi \text{ is } m-\textit{convergent}\} \tag{4.9}$$

is a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$, containing $C_\infty(X)$ and stable under translations. It is easy to see that $C_{\mathcal{L}}(X) \subset \bigcap_m C_m(X)$ where m runs over the set of all TIM on X. This already shows the special role played by the Lorentz algebra, but this fact can be better understood by the following remarks.

Our purpose is to express the $m-$convergence in terms of filters. The main point is that m has a unique extension to a state m' on $L^\infty(X)$ such that $m'(\mu * \varphi) = m'(\varphi)$ if μ is an absolutely continuous probability and $\varphi \in L^\infty(X)$ (this implies $m'(\tau_x\varphi) = m'(\varphi)$ for all $x \in X$). A state m' on $L^\infty(X)$ which has the preceding property is called a *topological invariant mean* on X. The map $m \mapsto m'$ is a bijection from invariant states on $C_{\mathrm{b}}^{\mathrm{u}}(X)$ onto topological invariant means on X, so we keep for m' the notation m. Clearly m is uniquely determined by its restriction to characteristic functions. We set $m(A) = m(\chi_A)$ if A is a measurable subset of X; thus m may also be thought as a translation invariant absolutely continuous *finitely* additive probability on the family of (Haar) measurable subsets of X.

A subset of X will be called $m-$*full* if it contains a measurable set A such that $m(A) = 1$. It is easy to show that the family $\mathcal{F}_m$ of $m-$full sets is a TIF on X. We define the $m-$convergence of a complex bounded Borel function on X in the same way as for functions from $C_{\mathrm{b}}^{\mathrm{u}}(X)$. Then it is possible to prove that such a function is $m-$convergent if and only if it is convergent along the filter $\mathcal{F}_m$ and then $m(\varphi) = \lim_{\mathcal{F}_m}\varphi$. In particular we get $C_m(X) = C_{\mathcal{F}_m}(X)$. Finally, one can prove that $\mathcal{L}(X) = \bigcap \mathcal{F}_m$, where m runs over the set of all topological invariant means on X. This explains the remarkable character of the Lorentz filter.

4.5. The intersection of a family of C^*-algebras with the properties (4.1) has the properties (4.1). Thus if $\mathcal{M}$ is a family of TIF's then $C_{\mathcal{M}}(X) := \bigcap_{\mathcal{F}\in\mathcal{M}} C_{\mathcal{F}}(X)$ is a C^*-algebra satisfying (4.1). Note that $\mathcal{G} := \bigcap_{\mathcal{F}\in\mathcal{M}}\mathcal{F}$ is also a TIF and $C_{\mathcal{G}}(X) \subset C_{\mathcal{M}}(X)$ strictly in general. For example, if $L_1, \ldots, L_n$ are closed sets such that $L_i + \Lambda \neq X$ if Λ is compact, one may consider the algebra $\bigcap_{i=1}^n C_{L_i}(X)$ with a simple geometrical interpretation.

Here we consider the simplest anisotropic behavior: one dimensional physical systems with different asymptotics at plus and minus infinity in configuration space. More precisely, let $X = \mathbb{R}$, $\mathbb{R}_\pm = \{x | \pm x \geq 0\}$. Then $C_{\mathbb{R}_+}(\mathbb{R}) \cap C_{\mathbb{R}_-}(\mathbb{R})$ is the algebra of continuous bounded complex functions on $\mathbb{R}$ which have limits at $\pm\infty$. If we define $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ as the two-point compactification of $\mathbb{R}$ with the natural topology, then this intersection is just $C(\overline{\mathbb{R}})$, the set of functions in $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$ which extend continuously to $\overline{\mathbb{R}}$. Note that $\mathcal{F}_{\mathbb{R}_+} \cap \mathcal{F}_{\mathbb{R}_-}$ is the Fréchet filter so the algebra associated to it is $C_\infty(\mathbb{R})$, strictly included in $C(\overline{\mathbb{R}})$.

It is very easy to describe the quotient $C(\overline{\mathbb{R}})/C_0(\mathbb{R})$. One has two morphisms $\varphi \mapsto \lim_{x\to\pm\infty} \varphi(x)$ from $C(\overline{\mathbb{R}})$ onto $\mathbb{C}$ and the intersection of their kernels is $C_0(\mathbb{R})$, so we get an identification $C(\overline{\mathbb{R}})/C_0(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C}$. Taking the cross product by the action of $\mathbb{R}$ is also easy. If $\mathfrak{C} = C(\overline{\mathbb{R}}) \rtimes \mathbb{R}$ then we consider the morphisms $\mathcal{P}_\pm : \mathfrak{C} \to C_0(\mathbb{R}^*)$ given by $\mathcal{P}_\pm[T] = \text{s-}\lim_{x\to\pm\infty} U_x T U_x^*$. The map $T \mapsto (\mathcal{P}_-[T], \mathcal{P}_+[T])$ is a surjective morphism of $\mathfrak{C}$ onto $C_0(\mathbb{R}^*) \oplus C_0(\mathbb{R}^*)$ and its kernel equals $\mathbb{K}(\mathbb{R})$. Thus we get

$$\mathfrak{C}/\mathbb{K}(\mathbb{R}) \cong C_0(\mathbb{R}^*) \oplus C_0(\mathbb{R}^*).$$

Physically interesting self-adjoint operators affiliated to $\mathfrak{C}$ include Schrödinger operators with locally singular potentials and different asymptotics at $\pm\infty$.

An algebraically trivial modification of the algebra $\mathfrak{C}$, obtained by taking the tensor product with an algebra of compact operators, allows one to greately improve the applications of the theory.

Let $\mathbf{E}$ be a complex Hilbert space and $\mathscr{H} = L^2(\mathbb{R}; \mathbf{E}) \equiv L^2(\mathbb{R}) \otimes \mathbf{E}$ the Hilbert space of $\mathbf{E}$-valued square integrable functions on $\mathbb{R}$. We shall use the notation $\varphi(Q)$ in the more general case when $\varphi : \mathbb{R} \to B(\mathbf{E})$ is a weakly Borel function ($\varphi(Q)$ is the operator of multiplication in $\mathscr{H}$ by the operator-valued function φ). The operator $\varphi(P)$ is similarly defined. Let $C_0^{\mathbf{E}}(\mathbb{R})$ be the C^*-algebra of functions $\varphi : \mathbb{R} \to K(\mathbf{E})$ which are norm-continuous and norm-convergent to zero at infinity; in other terms $C_0^{\mathbf{E}}(\mathbb{R}) = C_0(\mathbb{R}) \otimes K(\mathbf{E})$ (tensor product of C^*-algebras). The algebra $C^{\mathbf{E}}(\overline{\mathbb{R}})$ is similarly defined and we have $C^{\mathbf{E}}(\overline{\mathbb{R}}) = C(\overline{\mathbb{R}}) \otimes K(\mathbf{E})$. Then $C_0^{\mathbf{E}}(\mathbb{R}^*), C^{\mathbf{E}}(\overline{\mathbb{R}^*})$ are obtained with the help of a Fourier transformation (i.e. by replacing Q by P).

Now we consider the spaces

$$\begin{aligned}
\mathfrak{C}^{\mathbf{E}} &= [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C_0^{\mathbf{E}}(\mathbb{R}^*)]\!] = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C_0(\mathbb{R}^*)]\!], \\
\mathfrak{B}^{\mathbf{E}} &= [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C^{\mathbf{E}}(\overline{\mathbb{R}^*})]\!] = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C(\overline{\mathbb{R}^*})]\!].
\end{aligned}$$

These are C^*-algebras, $\mathfrak{C}^{\mathbf{E}} \subset \mathfrak{B}^{\mathbf{E}}$, and it is clear that

$$\begin{aligned} \mathfrak{C}^{\mathbf{E}} &= \mathfrak{C} \otimes K(\mathbf{E}) = C^{\mathbf{E}}(\overline{\mathbb{R}}) \rtimes \mathbb{R}, \\ \mathfrak{B}^{\mathbf{E}} &= \mathfrak{B} \otimes K(\mathbf{E}), \end{aligned}$$

where $\mathfrak{C}$ and $\mathfrak{B}$ are the algebras corresponding to the choice $\mathbf{E} = \mathbb{C}$. We stress the fact that $\mathfrak{B}$ (and more generally $\mathfrak{B}^{\mathbf{E}}$) is *not* a crossed product. This suggests the study of general objects of the form $[\![\mathscr{A} \cdot \mathscr{B}]\!]$ for some C^*-subalgebras $\mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X), \mathscr{B} \subset C_{\mathrm{b}}^{\mathrm{u}}(X^*)$.

It is possible to give a geometric description of these spaces (of the same nature as that of Theorem 5.11). If $T \in \mathfrak{B}^{\mathbf{E}}$ then $T_{\pm} = \mathrm{s}\text{-}\lim_{x \to \pm\infty} U_x T U_x^*$ and $T^{\pm} = \mathrm{s}\text{-}\lim_{k \to \pm\infty} V_k^* T V_k$ exist and the map $T \mapsto (T_-, T_+, T^-, T^+)$ is a morphism of $\mathfrak{B}^{\mathbf{E}}$ into

$$C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}})$$

with kernel equal to $K(\mathscr{H})$ and range given by the compatibility relations $(T_-)^{\pm} = (T^{\pm})_-$ and $(T_+)^{\pm} = (T^{\pm})_+$. The restriction of $T \mapsto (T_-, T_+)$ to the subalgebra $\mathfrak{C}^{\mathbf{E}}$ is a surjective morphism onto $C_0^{\mathbf{E}}(\mathbb{R}^*) \oplus C_0^{\mathbf{E}}(\mathbb{R}^*)$. Thus we get natural identifications:

$$\begin{aligned} \mathfrak{B}^{\mathbf{E}}/K(\mathscr{H}) &\hookrightarrow C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}), \\ \mathfrak{C}^{\mathbf{E}}/K(\mathscr{H}) &\cong C_0^{\mathbf{E}}(\mathbb{R}^*) \oplus C_0^{\mathbf{E}}(\mathbb{R}^*). \end{aligned}$$

Hence if H is an observable affiliated to $\mathfrak{B}^{\mathbf{E}}$ then one can associate to it four asymptotic observables $H_{\pm}, H^{\pm}$ (the first two $H_{\pm}$ correspond to $Q \to \pm\infty$ while the other ones $H^{\pm}$ correspond to $P \to \pm\infty$) and spectral properties like essential spectrum, thresholds, Mourre estimate, of H can be expressed in terms of these observables. For example

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \boldsymbol{\sigma}(H_-) \cup \boldsymbol{\sigma}(H_+) \cup \boldsymbol{\sigma}(H^-) \cup \boldsymbol{\sigma}(H^-).$$

This is an elementary but physically interesting situation when the essential spectrum of H is not determined by its localizations at infinity if the infinity is interpreted only in the Q sense.

Detailed proofs of the preceding statements and of the Mourre estimate for operators affiliated to $\mathfrak{C}^{\mathbf{E}}$ or $\mathfrak{B}^{\mathbf{E}}$ will be presented in [GI4]. The applications cover the spectral theory of elliptic operators (with operator valued coefficients) on asymptotically cylindric (star shaped) domains in $\mathbb{R}^n$ (with different asymptotics at various ends) and on some manifolds with cylindrical ends (our work on these questions has been especially motivated by

[Be, CrIf, DDI1, DDI2] and references therein). By taking $\mathbf{E} = \mathbb{C}^2$ we cover one dimensional Dirac operators with different asymptotics at $\pm\infty$ (in fact, by taking an arbitrary $\mathbf{E}$ in this context one gets a generalized Dirac operator which is also covered by the formalism).

Finally, note that $\mathbb{R}$ could be replaced by an arbitrary finite dimensional vector space X and $\overline{\mathbb{R}}$ by the spherical compactification $\overline{X}$ defined by the algebra $C(\overline{X})$ of functions $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ such that $\lim_{\lambda\to+\infty}\varphi(\lambda x)$ exists locally uniformly in $x \in X$. It is easy to compute the quotient $C(\overline{X})/C_0(X)$ in this case but the applications are less interesting at the physical level.

4.6. We consider here a class of C^*-algebras which appear in the study of N-body systems and their generalizations. A detailed presentation of this subject including references to previous works can be found in [ABG, BoGe] for the case of finite semilattices and [DaG1, DaG2] for the case of arbitrary semilattices. Besides the non-relativistic N-body hamiltonians, this framework covers the dispersive case (see [Dam, DaG2] and references therein) and the class of pluristratified media first considered by Dermenjian and Iftimie in [DeIf], see [DaG2]. Moreover, N-body hamiltonians with constant magnetic fields (cf. [GeLa]) also belong to this framework, but this question will be discussed in [GI2].

A family $\{\mathfrak{C}_i\}_{i\in I}$ of subalgebras of an algebra $\mathfrak{C}$ is *linearly independent* if for each family $\{S_i\}_{i\in I}$ such that $S_i \in \mathfrak{C}_i\ \forall i, S_i \neq 0$ for at most a finite number of i and $\sum_{i\in I} S_i = 0$, one has $S_i = 0$ for all $i \in I$. The sums of algebras which appear below are understood in the sense of linear spaces.

Let $\mathcal{L}$ be a *semilattice*, i.e. a partially ordered set in which each pair of elements a, b has a lower bound $a \wedge b$. For $a \in \mathcal{L}$ we set $\mathcal{L}_a = \{b \in \mathcal{L} | b \geq a\}$; this is also a semilattice.

We say that a C^*-algebra $\mathfrak{C}$ is *$\mathcal{L}$-graded* if a linearly independent family $\{\mathfrak{C}(a)\}_{a\in\mathcal{L}}$ of C^*-subalgebras of $\mathfrak{C}$ has been given such that:

(i) $\mathfrak{C}(a)\cdot\mathfrak{C}(b) \subset \mathfrak{C}(a \wedge b)$ for all $a, b \in \mathcal{L}$;

(ii) if $\mathcal{E} \subset \mathcal{L}$ is finite then $\sum_{a\in\mathcal{E}} \mathfrak{C}(a)$ is a closed subspace of $\mathfrak{C}$;

(iii) $\sum_{a\in\mathcal{L}} \mathfrak{C}(a)$ is dense in $\mathfrak{C}$.

For each $\mathcal{E} \subset \mathcal{L}$ we define $\mathfrak{C}(\mathcal{E})$ as the closure of $\mathfrak{C}(\mathcal{E})^\circ = \sum_{a\in\mathcal{E}} \mathfrak{C}(a)$. If $a \in \mathcal{L}$ we set $\mathfrak{C}_a = \mathfrak{C}(\mathcal{L}_a)$. It is clear that $\mathfrak{C}_a$ is a $\mathcal{L}_a$-graded C^*-subalgebra of $\mathfrak{C}$. There is a natural map $\mathcal{P}_a^\circ : \mathfrak{C}(\mathcal{L})^\circ \to \mathfrak{C}(\mathcal{L}_a)^\circ$ defined by $\mathcal{P}_a^\circ \sum_{b\in\mathcal{L}} T(b) = \sum_{b\in\mathcal{L}_a} T(b)$ if $T(b) \in \mathfrak{C}(b)$ and $T(b) \neq 0$ only for a finite numbers of b. This map is clearly a surjective morphism of $*$-algebras. It can be shown that this map is continuous, so it extends to a surjective morphism $\mathcal{P}_a : \mathfrak{C} \to \mathfrak{C}_a$. Moreover, $\mathcal{P}_a$ is a projection (in the sense of linear spaces) and its kernel is

$\mathfrak{C}(\mathcal{L}'_a)$, where $\mathcal{L}'_a = \{b \in \mathcal{L} | b \not\geq a\}$.

Assume that $\mathcal{L}$ has a least element $\min \mathcal{L}$, denote $\mathcal{M}$ the set of atoms of $\mathcal{L}$ (i.e. minimal elements of $\mathcal{L} \setminus \{\min \mathcal{L}\}$) and assume that $\mathcal{L}$ is atomic (i.e. each $a \neq \min \mathcal{L}$ is minorated by an atom). Observe that $\mathfrak{C}(\min \mathcal{L})$ is a closed self-adjoint ideal in $\mathfrak{C}$, so the C^*-algebra $\mathfrak{C}/\mathfrak{C}(\min \mathcal{L})$ is well defined. The important fact is that one can explicitly realize this algebra as follows. There is a natural morphism

$$\mathfrak{C} \ni T \longmapsto (\mathcal{P}_a[T])_{a \in \mathcal{M}} \in \textstyle\prod_{a \in \mathcal{M}} \mathfrak{C}_a,$$

where the direct product is in the C^*-algebra sense, and the kernel of this morphism is equal to $\mathfrak{C}(\min \mathcal{L})$. It is easy to see that for each $T \in \mathfrak{C}$ the set $\{\mathcal{P}_a[T] \mid a \in \mathcal{M}\}$ is relatively compact in $\mathfrak{C}$. Thus, with a notation introduced at the end of §2.5, we get a canonical embedding

$$\mathfrak{C}/\mathfrak{C}(\min \mathcal{L}) \hookrightarrow \textstyle\prod^{\mathrm{rc}}_{a \in \mathcal{M}} \mathfrak{C}_a.$$

As usual, such a result allows one to compute the essential spectrum and to prove the Mourre estimate for the observables affilated to $\mathfrak{C}$. Assume that $\mathfrak{C}$ is realized as a C^*-algebra of operators on a Hilbert space $\mathscr{H}$ such that $\mathfrak{C}(\min \mathcal{L}) = K(\mathscr{H})$. If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathfrak{C}$ and $a \in \mathcal{L}$, then $H_a = \mathcal{P}_a[H]$ is an observable affiliated to $\mathfrak{C}_a$, hence to $\mathfrak{C}$, and $\boldsymbol{\sigma}_{\mathrm{ess}}(H)$ is equal to the closure of $\bigcup_{a \in \mathcal{M}} \boldsymbol{\sigma}(H_a)$. Moreover, if for each $T \in \mathfrak{C}$ the set $\{\mathcal{P}_a[T] \mid a \in \mathcal{M}\}$ is compact in $\mathfrak{C}$ (and this property holds in many cases, e.g. in the two explicit examples considered below) then one has

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_{a \in \mathcal{M}} \boldsymbol{\sigma}(H_a).$$

We now give the first example of graded C^*-algebra which appears in the N-body problem. Let X be a finite dimensional real vector space and Y a linear subspace. We denote π_Y the canonical surjection of X onto the quotient vector space X/Y. We shall embed $C_0(X/Y) \subset C^{\mathrm{u}}_{\mathrm{b}}(X)$ with the help of the map $\varphi \mapsto \varphi \circ \pi_Y$ (for $Y = X$ we have $C_0(X/X) = \mathbb{C}$). Clearly this is a translation invariant C^*-subalgebra of $C^{\mathrm{u}}_{\mathrm{b}}(X)$ and so we may construct the crossed product $\mathfrak{C}(Y) = C_0(X/Y) \rtimes X$. The norm closure of $\sum_Y C_0(X/Y)$ is a translation invariant C^*-subalgebra of $C^{\mathrm{u}}_{\mathrm{b}}(X)$ containing $C_\infty(X)$, so we may consider its crossed product by X; denote it $\mathfrak{C}$. If we take as semilattice $\mathcal{L}$ the set of all linear subspaces of X equipped with the natural order relation (so $\mathcal{L} \equiv \mathbb{G}(X)$ the grassmannian of X), then the family $\{\mathfrak{C}(Y)\}_{Y \in \mathcal{L}}$ is a $\mathcal{L}$-grading on $\mathfrak{C}$. The operators affiliated to $\mathfrak{C}$ are studied in [DaG2].

The second example will be studied in detail in the second part of this paper [GI2], so again we do not enter into details. We shall present it in a way which suggests generalizations to nilpotent groups more general than the Heisenberg group (which is implicit below).

We consider a system $(\Xi, \sigma; \mathscr{H}, W)$ consisting of a symplectic space (Ξ, σ) and a representation W of it on a Hilbert space $\mathscr{H}$. So Ξ is a real finite dimensional vector space equipped with a real antisymmetric nondegenerate bilinear form σ and W is a strongly continuous map from Ξ to the set of unitary operators on $\mathscr{H}$ satisfying, for all $\xi, \eta \in \Xi$:

$$W(\xi+\eta) = \mathrm{e}^{-\frac{i}{2}\sigma(\xi,\eta)} W(\xi) W(\eta).$$

We recall that to each abelian group one can associate a structure of this type, Ξ being only an abelian group in general (see §3.6). If E is a vector subspace of Ξ we set

$$E^{\sigma} := \{\xi \in \Xi \mid \sigma(\xi, \eta) = 0, \text{ for all } \eta \in E\}.$$

Then $E^{\sigma\sigma} = E$ and $(E \cap F)^{\sigma} = E^{\sigma} + F^{\sigma}$. Note also that E^{σ} can also be defined as the set of those $\xi \in \Xi$ such that $W(\xi)W(\eta) = W(\eta)W(\xi)$ for all $\eta \in E$. We shall take $\mathcal{L} = \mathbb{G}(\Xi)$ the set of all vector subspaces of Ξ. If $E \in \mathbb{G}(\Xi)$, then we define $\mathfrak{C}(E)$ as the set of operators $T \in B(\mathscr{H})$ such that:
(i) $\| [W(\xi), T] \| \to 0$ if $\xi \to 0$ in Ξ,
(ii) $[W(\xi), T] = 0$ if $\xi \in E$,
(iii) $\| (W(\xi) - 1) T^{(*)} \| \to 0$ if $\xi \in E^{\sigma}$ and $\xi \to 0$.
Let $\mathfrak{C}$ be the closure of $\sum_E \mathfrak{C}(E)$. Then the family of C^*-subalgebras $\mathfrak{C}(E)$ provides $\mathfrak{C}$ with a $\mathbb{G}(\Xi)$-graded C^*-algebra structure.

We mention only one important theorem: *If W is an irreducible representation, then $\mathfrak{C}(E)$ coincides with the closure in $B(\mathscr{H})$ of the set of operators $W(\mu)$ with μ an E^{σ}-a.c. integrable measure.* Here, an integrable measure μ on Ξ is F-a.c. for some vector subspace F of Ξ if there are a Haar measure λ_F on F and $\rho \in L^1(F) = L^1(F, \lambda_F)$ such that $\mu = \rho\lambda_F$. Of course, $W(\mu) = \int_{\Xi} W(\xi)\mu(\mathrm{d}\xi)$.

4.7. We shall give a last example of C^*-algebra of energy observables relevant in the study of boson quantum field models. This topic is developed in [G, GM]. Here we treat only situations when the number of bosons does not exceed a finite number N.

Let $\mathcal{H}$ be the (complex, infinite dimensional) Hilbert space of one boson states. We denote $\mathcal{H}^{\vee n}$ the symmetric tensor power of order n of $\mathcal{H}$

and $\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n}$ the Fock space associated to $\mathcal{H}$. Then $\Gamma_N(\mathcal{H}) = \bigoplus_{n=0}^{N} \mathcal{H}^{\vee n}$ is the state space of the system consisting of at most N bosons. The interaction between bosons is a (generalized) polynomial in $a(u)$, $a^*(u)$, the anihilation and creation operators of a boson in the state $u \in \mathcal{H}$ (the normalizations and other conventions are as in [DeG2, DeG3]). We introduce the truncated anihilation and creation operators $a_N(u) = 1_N a(u) 1_N$, $a_N^*(u) = 1_N a^*(u) 1_N$, where 1_N the orthogonal projection $\Gamma(\mathcal{H}) \to \Gamma_N(\mathcal{H})$. The set $K(\Gamma_N(\mathcal{H}))$ of compact operators on $\Gamma_N(\mathcal{H})$ will be denoted $\mathcal{K}_N$.

If $S \in B(\mathcal{H})$ then we define $S^{\vee n} = S^{\otimes n} | \mathcal{H}^{\vee n}$ and

$$\Gamma_N(S) = \oplus_{n=0}^{N} S^{\vee n} \in B(\Gamma_N(\mathcal{H})).$$

Now if $\mathcal{C} \subset B(\mathcal{H})$ is a unital C^*-subalgebra, then $\mathcal{C}_N$ is the C^*-algebra of operators on $\Gamma_N(\mathcal{H})$ generated by the operators of the form $\Gamma_N(S)$ and $a_N(u)$ with $S \in \mathcal{C}$ and $u \in \mathcal{H}$. So $\mathcal{C}_N$ is the smallest C^*-algebra which contains $\Gamma_N(\mathcal{C}) \bigcup \mathcal{A}_N$, where $\mathcal{A}_N$ is the C^*-algebra generated by the operators $a_N(u)$. A rather natural notation for $\mathcal{C}_N$ is $\mathcal{A}_N \rtimes \mathcal{C}$ (to suggest crossed-product).

Theorem 4.3 *Assume that $\mathcal{C} \subset B(\mathcal{H})$ is a unital C^*-subalgebra with no nonzero finite rank projections. Then there is a unique morphism $\mathcal{P}_N : \mathcal{C}_N \to \mathcal{C} \otimes \mathcal{C}_{N-1}$ such that for all $u \in \mathcal{H}$ and $S \in \mathcal{C}$*

$$\begin{aligned} \mathcal{P}_N[a_N(u)] &= 1 \otimes a_{N-1}(u) \\ \mathcal{P}_N[\Gamma_N(S)] &= S \otimes \Gamma_{N-1}(S). \end{aligned}$$

One has $\ker \mathcal{P}_N = \mathcal{K}_N$.

Thus we have a canonical embedding

$$\mathcal{C}_N / \mathcal{K}_N \equiv \widehat{\mathcal{C}_N} \hookrightarrow \mathcal{C} \otimes \mathcal{C}_{N-1}.$$

Note that the inclusion $\widehat{\mathcal{C}_N} \subset \mathcal{C} \otimes \mathcal{C}_{N-1}$ is an equality if and only if $\mathcal{C} = \mathbb{C}$.

5 Bumps Algebras

5.1. The class of hamiltonians considered in this section generalizes that of the Schrödinger operators with "widely separated bumps" introduced by M.Klaus in [Kla] (see also [CFKS, HuK]). As we shall see, this example fits very nicely in the framework of Section 2, the quotient algebra having an especially interesting structure.

Let X be an abelian locally compact group and L a sparse subset of X, more precisely (see §4.3) L is a locally finite set and for each compact Λ of X there is a finite set $F \subset L$ such that if $l \in M = L \setminus F$ and $l' \in L \setminus \{l\}$ then $(l+\Lambda) \cap (l'+\Lambda) = \emptyset$. We shall use the notations L_Λ and L_Λ^{c} and the notion L-lim introduced in §4.3. We recall that $C_L(X)$ is the translation invariant C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ defined by

$$C_L(X) = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid L\text{-}\lim \varphi \text{ exists }\}.$$

We want to apply the general theory from Section 3 to the algebra $\mathscr{A} = C_L(X)$, hence we have to give an explicit description of the quotient algebra $C_L(X)/C_0(X)$. Let us introduce the following two subsets of $\mathscr{C}_L$:

$$\begin{aligned} C_{L,0}(X) &= \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid L\text{-}\lim \varphi = 0\} \\ C_{L,\mathrm{c}}(X) &= \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X) \mid \exists \Lambda \subset X \text{ compact such that } \operatorname{supp} \varphi \subset L_\Lambda\}. \end{aligned}$$

Clearly $C_{L,0}(X)$ is an ideal of $C_L(X)$ (of $C_{\mathrm{b}}^{\mathrm{u}}(X)$, in fact) and one has $C_L(X) = \mathbb{C} + C_{L,0}(X)$, so $C_L(X)$ is the unital algebra associated to $C_{L,0}(X)$. Since $C_0(X) \subset C_{L,0}(X)$, we have

$$C_L(X)\Big/C_0(X) \;\cong\; \mathbb{C} + C_{L,0}(X)\Big/C_0(X)\,,$$

i.e. $C_L(X) \,/\, C_0(X)$ is the unital algebra associated to $C_{L,0}(X) \,/\, C_0(X)$. So we are reduced to the problem of computing this last quotient.

We begin by describing some properties of the space $C_{L,\mathrm{c}}(X)$. For the convenience of the reader we recall first the Ascoli theorem for the case of a locally compact space X: *a bounded subset $\mathcal{K}$ of $C_0(X)$ is relatively compact in $C_0(X)$ if and only if $\mathcal{K}$ is an equicontinuous family of functions and for each $\varepsilon > 0$ there is a compact set $\Lambda \subset X$ such that $|\varphi(x)| < \varepsilon$ for all $\varphi \in \mathcal{K}$ and $x \in X \setminus \Lambda$.*

If Λ is a compact subset of X we shall denote by $C_{\mathrm{c}}(\Lambda)$ the set of continuous functions on X with support included in Λ.

Lemma 5.1 (i) *$C_{L,\mathrm{c}}(X)$ is a dense self-adjoint ideal in $C_{L,0}(X)$.*
(ii) *A function φ belongs to $C_{L,\mathrm{c}}(X)$ if and only if there is a compact set $\Lambda \subset X$ and an equicontinuous bounded family $\{\varphi_l\}_{l \in L}$ of elements of $C_{\mathrm{c}}(\Lambda)$ such that $\varphi = \sum_{l \in L} \tau_l \varphi_l$.*
(iii) *The linear subspace generated by functions of the form $\sum_{l \in M} \tau_l \varphi$, where M is a subset of L and $\varphi \in C_{\mathrm{c}}(X)$, is dense in $C_{L,0}(X)$.*

Proof: (i) Clearly $C_{L,\mathrm{c}}(X)$ is a self-adjoint (non-closed) ideal in $C_L(X)$. We shall prove its density in $C_{L,0}(X)$. Let Λ be a compact neighbourhood of 0 in X and let $\theta \in C_c(\Lambda)$ such that $0 \le \theta \le 1$ and $\theta = 1$ on a neighbourhood Λ_0 of zero. Denote n the maximal number of sets $l + \Lambda$, with $l \in L$, which have a non-empty intersection. Since L is a sparse subset of X the number n is finite. Then $\Theta(x) = \sum_{l \in L} \theta(x - l)$ is well defined, $0 \le \Theta(x) \le n$ (for each x there are at most n nonzero terms in the sum) and

$$|\Theta(x) - \Theta(y)| \le n \sup_{l \in L} |\theta(x - l) - \theta(y - l)| \le n \|\tau_{y-x}\theta - \theta\|,$$

where $\|\cdot\|$ is the sup norm. So Θ is uniformly continuous and $\Theta \in C_{L,c}(X)$.

Let $\varphi \in C_{L,0}(X)$ and $\varepsilon > 0$. Then there is a compact neighbourhood Λ of zero in X such that $|\varphi(x)| < \varepsilon$ if $x \notin L + \Lambda$. Choose $F \subset L$ finite such that $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$ if $l \in M \equiv L \setminus F$ and $l' \in L$, $l' \neq l$, and denote K the compact set $\bigcup_{l \in F}[l + \Lambda]$. Observe that if $x \notin K$ and θ is as above then $\Theta(x) = \sum_{l \in M} \theta(x - l)$ and the supports of the functions $\tau_l\theta$ are disjoint if $l \in M$; in particular $0 \le \Theta(x) \le 1$. Let η be a continuous function such that $0 \le \eta \le 1$, $\eta = 0$ on a neighborhood of K and $\eta = 1$ on a neighborhood of infinity. If we denote $\eta\varphi = \psi$ we have

$$\varphi - (1 - \eta)\varphi - \Theta\psi = \psi - \Theta\psi.$$

If $x \in K$ then the r.h.s. above takes the value zero in x. If $x \notin K$ and $x \in L + \Lambda_0$ then there is a unique $l \in M$ such that $x \in l + \Lambda_0$, hence $\Theta(x) = \theta(x - l) = 1$ and $\psi(x) - \Theta(x)\psi(x) = 0$. If $x \notin K$ and $x \notin L + \Lambda_0$ then $|\psi(x)| < \varepsilon$ and $0 \le \Theta(x) \le 1$, so $|\psi(x) - \Theta(x)\psi(x)| < \varepsilon$. Thus we have

$$\|\varphi - (1 - \eta)\varphi - \Theta\psi\| \le \varepsilon.$$

Since $(1 - \eta)\varphi \in C_c(X)$ and $\Theta\psi \in C_{L,c}(X)$, we proved that for each $\varepsilon > 0$ there is $\varphi_\varepsilon \in C_{L,c}(X)$ such that $\|\varphi - \varphi_\varepsilon\| < \varepsilon$. Hence $C_{L,c}(X)$ is dense in $C_{L,0}(X)$.

(ii) If $\varphi \in C_{L,c}(X)$ then there is a compact $\Lambda \subset X$ such that $\operatorname{supp}\varphi \subset L + \Lambda$. Since L is sparse one can write $L + \Lambda = K \cup \bigcup_{l \in M}(l + \Lambda)$, where the compact set $K \subset X$ and M are chosen such that the sets which appear in the preceding union be pairwise disjoint. For $l \in M$ we define φ_l by $\varphi_l(x) := \varphi(x + l)$ if $x \in \Lambda$ and $\varphi_l = 0$ otherwise. If $l \in L \setminus M$ the definition of φ_l is to a large extent arbitrary, e.g. we may take $\varphi_l = 0$ for all but one $l \equiv l_0$ and choose conveniently φ_{l_0}; this is possible if K is large enough. Reciprocally, it suffices to notice that the equicontinuity of the family $\{\varphi_l\}_{l \in L}$ implies the uniform continuity of φ.

(iii) Because of the first part of the lemma it suffices to prove that for each φ as in (ii) and for each $\varepsilon > 0$ there is a partition $\{L_1, \ldots, L_k\}$ of L and there are functions $\phi_1, \dots, \phi_k \in C_c(\Lambda)$ such that $\|\varphi - \sum_{i=1}^k \sum_{l \in L_i} \tau_l \phi_i\| < \varepsilon$. By the Ascoli theorem, $\{\varphi_l \mid l \in L\}$ is a relatively compact subset of $C_c(\Lambda)$, hence there is a finite number of functions $\phi_1 := \varphi_{l_1}, \dots, \phi_k := \varphi_{l_k}$ and there is a partition $\{L_1, \dots, L_k\}$ of L such that $\|\varphi_l - \phi_i\| < \varepsilon/n$ for $l \in L_i$, where n is the maximal number of sets of the form $l + \Lambda$ which have non-empty intersection. Then, for each x,

$$\begin{aligned} |\varphi(x) - \sum_{i=1}^{k} \sum_{l \in L_i} \tau_l \phi_i(x)| &= \left| \sum_{i=1}^{k} \sum_{l \in L_i} \tau_l (\varphi_l - \phi_i)(x) \right| \\ &\leq n \sup_{i=1,\dots,k} \sup_{l \in L_i} \|\tau_l (\varphi_l - \phi_i)\| \leq \varepsilon. \end{aligned}$$ ■

We are now ready to compute the quotient $C_{L,0}(X)/C_0(X)$. We recall the notation $C_0(X)^{\langle L \rangle} \equiv C_0(X)^{[L]}/C_0(X)^{(L)}$ (see §2.5) and denote by π the canonical morphism $C_0(X)^{[L]} \to C_0(X)^{\langle L \rangle}$.

Theorem 5.2 *There is a unique morphism $\mathcal{J} : C_{L,0}(X) \to C_0(X)^{\langle L \rangle}$ such that $\mathcal{J}(\varphi) = \pi[(\varphi_l)_{l \in L}]$ if $\varphi = \sum_{l \in L} \tau_l \varphi_l$, with $\{\varphi_l\}_{l \in L}$ an equicontinuous bounded family in $C_0(\Lambda)$ for some compact $\Lambda \subset X$. The morphism $\mathcal{J}$ is surjective and* $\ker \mathcal{J} = C_0(X)$. *In particular, $\mathcal{J}$ induces a canonical isomorphism:*

$$C_{L,0}(X) \Big/ C_0(X) \;\cong\; C_0(X)^{[L]} \Big/ C_0(X)^{(L)} \;\equiv\; C_0(X)^{\langle L \rangle} \tag{5.1}$$

Remark: As a consequence of Lemma 5.1 and of the identifications (2.6), $\mathcal{J}$ is the unique morphism $C_{L,0}(X) \to C_0(X)^{\langle L \rangle}$ such that for each subset M of L and each function $\varphi \in C_c(X)$ one has $\mathcal{J}(\sum_{l \in M} \tau_l \varphi) = \pi(\chi_M \otimes \varphi)$, where χ_M is the characteristic function of the set M. There is another description of $\mathcal{J}$ based on the identification (2.9), but we shall make it explicit only in the case of the algebra $C_{L,0}(X) \rtimes X$.

Proof: The uniqueness of $\mathcal{J}$ is a consequence of (i) from Lemma 5.1. The surjectivity of $\mathcal{J}$ is also easy to prove: since the range of a morphism is closed, it suffices to show that the set of elements of the form $(\varphi_l)_{l \in L}$, where the family $\{\varphi_l\}_{l \in L}$ is as in the statement of the theorem, is dense in $C_0(X)^{[L]}$. But this is a straightforward consequence of the Ascoli theorem, because $C_0(X)^{[L]}$ consists of relatively compact families of elements of $C_0(X)$ (see the definition (2.5)).

We prove the existence of $\mathcal{J}$. Note first that for a given $\varphi \in C_{L,c}(X)$ a family $\{\varphi_l\}_{l \in L}$ which verifies $\varphi = \sum_{l \in L} \tau_l \varphi_l$ is not unique. However, L being a sparse set, the functions φ_l which correspond to large enough l

are uniquely defined. In particular, the image of $\{\varphi_l\}_{l\in L}$ in the quotient $C_0(X)^{[L]}/C_0(X)^{(L)}$ depends only on φ. Thus $\mathscr{J}$ is well defined on $C_{L,\mathrm{c}}(X)$ and it is clearly a morphism with $\|\mathscr{J}\| \le 1$. This allows us to extend it by continuity to all $C_{L,0}(X)$.

It remains to prove that the kernel of $\mathscr{J}$ is $C_0(X)$. We first make a preliminary general remark. *Let $\mathscr{A}$ be a C^*-algebra and let $\boldsymbol{S} = (S_l)_{l\in L} \in \mathscr{A}^{[L]}$ such that $\|\widehat{\boldsymbol{S}}\| < \varepsilon$ for some $\varepsilon > 0$, where $\widehat{\boldsymbol{S}}$ is the image of $\boldsymbol{S}$ in $\mathscr{A}^{\langle L\rangle} \equiv \mathscr{A}^{[L]}/\mathscr{A}^{(L)}$. Then there is a finite set $F \subset L$ such that $\|S_l\| < 2\varepsilon$ if $l \notin F$.* Indeed,

$$\|\widehat{\boldsymbol{S}}\| = \inf\{\|(S_l - T_l)_{l\in L}\| \mid \boldsymbol{T} \equiv (T_l)_{l\in L} \in \mathscr{A}^{(L)}\}$$

so there is $\boldsymbol{T} \in \mathscr{A}^{(L)}$ such that $\|(S_l - T_l)_{l\in L}\| = \sup_{l\in L}\|S_l - T_l\| < 3\varepsilon/2$. Then $\|S_l\| < \|T_l\| + 3\varepsilon/2$ and $\|T_l\| \to 0$ as $l \to \infty$. So there is a finite set $F \subset L$ such that $\|T_l\| < \varepsilon/2$ if $l \notin F$, which proves the remark.

Let $\varphi \in C_{L,0}(X)$ such that $\mathscr{J}(\varphi) = 0$ and let $\varepsilon > 0$. Then there is $\psi \in C_{L,\mathrm{c}}(X)$ such that $\|\varphi - \psi\| < \varepsilon$ and $\|\mathscr{J}(\psi)\| < \varepsilon$. Choose a compact $\Lambda \subset X$ and a bounded equicontinuous family $\{\psi_l\}_{l\in L}$ in $C_0(\Lambda)$ such that $\psi = \sum_{l\in L}\tau_l\psi_l$. Then $(\psi_l)_{l\in L} \in C_0(\Lambda)^{[L]}$ and $\|\pi[(\psi_l)_{l\in L}]\| = \|\mathscr{J}(\psi)\| < \varepsilon$ hence, by the preceding remark, there is a finite set $F \subset L$ such that $\|\psi_l\| < 2\varepsilon$ if $l \notin F$. But $\psi(x) = \sum_{l\in L}\psi_l(x-l)$ and if x is outside some compact then at most one term in the sum is non-zero, so $|\psi(x)| < \varepsilon$ for x in some neighbourhood of infinity. Then $|\varphi(x)| \le |\varphi(x) - \psi(x)| + |\psi(x)| < 3\varepsilon$ for such x. Since ε is arbitrary, this shows $\varphi \in C_0(X)$. ■

Corollary 5.3 *The quotient algebra $C_L(X)/C_0(X)$ is canonically isomorphic to the unital C^*-algebra associated to $C_0(X)^{\langle L\rangle}$. In particular, there is a natural embedding:*

$$C_L(X)\Big/C_0(X) \hookrightarrow C_\infty(X)^{[L]}\Big/C_0(X)^{(L)} \tag{5.2}$$

Proof: We have

$$\begin{aligned} C_L(X)/C_0(X) &\cong (\mathbb{C} + C_{L,0}(X))/C_0(X) \cong \mathbb{C} + C_{L,0}(X)/C_0(X) \\ &\cong \mathbb{C} + C_0(X)^{[L]}/C_0(X)^{(L)} \cong (\mathbb{C} + C_0(X)^{[L]})/C_0(X)^{(L)} . \end{aligned}$$

Moreover, by Lemma 2.3 one has

$$\begin{aligned} \mathbb{C} + C_0(X)^{[L]} &\cong \mathbb{C} + l^\infty(L) \otimes C_0(X) \\ &\hookrightarrow l^\infty(L) \otimes [\mathbb{C} + C_0(X)] = l^\infty(L) \otimes C_\infty(X) \cong C_\infty(X)^{[L]}. \end{aligned}$$

In other terms, the canonical embedding $\mathbb{C} + C_0(X)^{[L]} \hookrightarrow C_\infty(X)^{[L]}$ associates to $\lambda + (\varphi_l)_{l\in L}$ the element $(\lambda + \varphi_l)_{l\in L}$. ∎

Note that we have a simple description of the range of the embedding (5.2): this is the quotient of the space of the elements of the form $(\lambda+\varphi_l)_{l\in L}$ with $\lambda \in \mathbb{C}$ and $(\varphi_l)_{l\in L} \in C_0(X)^{[L]}$.

5.2. We are now ready to introduce the C^*-algebra of energy observables corresponding to quantum systems with interactions having sparse supports. This will be done by "quantifying" the algebra of potentials, i.e. by taking the crossed product of $C_L(X)$ by the action of translations on the locally compact group X. The second equality below is a consequence of the general Theorem 3.12.

Definition 5.4 $\mathfrak{C}_L := C_L(X) \rtimes X = [\![C_L(X) \cdot C_0(X^*)]\!]$

In the same manner we may define the smaller C^*-algebra (without unit)

$$\mathfrak{C}_{L,0} := C_{L,0}(X) \rtimes X = [\![C_{L,0}(X) \cdot C_0(X^*)]\!]. \tag{5.3}$$

Then, by (3.16) we can write $\mathfrak{C}_L$ as a linear direct sum

$$\mathfrak{C}_L = C_0(X^*) + \mathfrak{C}_{L,0}. \tag{5.4}$$

The algebra $\mathfrak{C}_{L,0}$ is an ideal of $\mathfrak{C}_L$ and $\mathfrak{C}_L \to C_0(X^*)$ is a surjective morphism which gives the pure kinetic energy part, and $\mathfrak{C}_L/\mathfrak{C}_{L,0} \cong C_0(X^*)$. On the other hand, $C_0(X)$ is a stable ideal of $C_L(X)$ thus the crossed product subalgebra $C_0(X) \rtimes X$ is also an ideal of $\mathfrak{C}_L$. We recall that $C_0(X) \rtimes X = \mathbb{K}(X)$ is the algebra of compact operators (Proposition 3.14).

We mention now some elements of the multiplier algebra of $\mathfrak{C}_L$:
(i) if $\varphi : X \to \mathbb{C}$ is bounded, Borel and tends to zero at infinity, then $\varphi(Q)T$ and $T\varphi(Q)$ are compact operators, for all $T \in \mathfrak{C}_L$;
(ii) if $\varphi \in C_\mathrm{b}^\mathrm{u}(X)$ and $T \in \mathfrak{C}_{L,0}$, then the operators $\varphi(Q)T$ and $T\varphi(Q)$ belong to $\mathfrak{C}_{L,0}$;
(iii) if $\psi \in C_\mathrm{b}(X^*)$ and $T \in \mathfrak{C}_L$ (or $\mathfrak{C}_{L,0}$), then the operators $\psi(P)T$ and $T\psi(P)$ belong to $\mathfrak{C}_L$ ($\mathfrak{C}_{L,0}$ respectively).

The general theory exposed in section 3 allows us to give a complete characterization both of the quotient algebras $\mathfrak{C}_{L,0}/\mathbb{K}(X)$ and $\mathfrak{C}_L/\mathbb{K}(X)$ in terms of much simpler objects involving only the compact operator algebra $\mathbb{K}(X)$ and the two-body algebra (see (4.3)):

Theorem 5.5 *The quotient algebra $\mathfrak{C}_{L,0}/\mathbb{K}(X)$ is canonically isomorphic to the L-asymptotic algebra $\mathbb{K}(X)^{\langle L\rangle}$ of the algebra of compact operators $\mathbb{K}(X)$. One has a natural embedding:*

$$\mathfrak{C}_L \big/ \mathbb{K}(X) \hookrightarrow \mathbb{T}(X)^{[L]} \big/ \mathbb{K}(X)^{(L)} \tag{5.5}$$

Proof: The first assertion follows from Proposition 3.10 because of (5.1). In order to prove the second assertion we start with the imbedding (5.2) and use (3.15), Proposition 3.7 and (3.18) to get

$$\begin{aligned}
\mathfrak{C}_L \big/ \mathbb{K}(X) &\equiv (C_L(X) \rtimes X) \big/ (C_0(X) \rtimes X) = C_L(X) \big/ C_0(X) \rtimes X \\
&\hookrightarrow C_\infty(X)^{[L]} \big/ C_0(X)^{(L)} \rtimes X = (C_\infty(X)^{[L]} \rtimes X) \big/ (C_0(X)^{(L)} \rtimes X) \\
&= (C_\infty(X) \rtimes X)^{[L]} \big/ (C_0(X) \rtimes X)^{(L)} = \mathbb{T}(X)^{[L]} \big/ \mathbb{K}(X)^{(L)} . \quad \blacksquare
\end{aligned}$$

Theorem 5.5 is the main result of this Section: it allows, via Theorem 2.2, to compute the essential spectrum of a hamiltonian affiliated to the algebra of energy observables $\mathfrak{C}_L$ in terms of spectra of hamiltonians affiliated to the two-body algebra.

Notice that, as in the abelian case, we have a precise description of the range of the embedding (5.5): it is the quotient with respect to $\mathbb{K}(X)^{(L)}$ of the subspace of $\mathbb{T}(X)^{[L]}$ consisting of sequences of the form $(\psi(P) + K_l)_{l\in L}$ for some $\psi \in C_0(X^*)$ and $(K_l)_{l\in L} \in \mathbb{K}(X)^{[L]}$.

If H is an observable affiliated to $\mathfrak{C}_L$ and if $\widehat{H}$ is its image through the canonical morphism $\mathfrak{C}_L \to \mathfrak{C}_L/\mathbb{K}(X)$, then there is a family $(H_l)_{l\in L}$ of observables affiliated to the two-body algebra $\mathbb{T}(X)$ such that the quotient of $\prod_{l\in L} H_l$ with respect to the ideal $\mathbb{K}(X)^{(L)}$ be equal to the image of $\widehat{H}$ through the embedding (5.5). Such a family $(H_l)_{l\in L}$ will be called a *representative* of H. By the discussion above we have $\prod_{l\in L}(H_l - z)^{-1} \in \mathbb{T}(X)^{[L]}$ and the component of $(H_l - z)^{-1}$ in $C_0(X^*)$ is independent of $l \in L$, so $\boldsymbol{\sigma}_{\rm ess}(H_l)$ is independent of l. Thus the next result is a consequence of the Theorem 2.2.

Theorem 5.6 *If H is an observable affiliated to $\mathfrak{C}_L$ and $\{H_l\}_{l\in L}$ is a representative of H, then*

$$\boldsymbol{\sigma}_{\rm ess}(H) = \bigcap_{\substack{F\subset L \\ F \text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}.$$

Let us give now a second description of the quotient algebra $\mathfrak{C}_L/\mathbb{K}(X)$, based on the formalism we developped in §2.5. According to the notations introduced there, we shall denote by $\widetilde{L}$ the set of ultrafilters on L finer than the Fréchet filter; $\widetilde{L}$ is a compact topological space.

Theorem 5.7 *For each $S \in \mathfrak{C}_L$ and $\varkappa \in \widetilde{L}$ the limit $\text{s-lim}_{l,\varkappa} U_l S U_l^* = S_\varkappa$ exists in the strong operator topology and belongs to $\mathbb{T}(X) = C_0(X^*)+\mathbb{K}(X)$. The component of $S_\varkappa$ in $C_0(X^*)$ is equal to that of S in $C_0(X^*)$. The map $S \mapsto (S_\varkappa)_{\varkappa\in\widetilde{L}}$ is a morphism of $\mathfrak{C}_L$ into $C(\widetilde{L};\mathbb{T}(X))$ with kernel $\mathbb{K}(X)$. Its range is equal to the set of $(S_\varkappa)_{\varkappa\in\widetilde{L}}$ such that the component of $S_\varkappa$ in $C_0(X^*)$ is independent of $\varkappa$.*

Proof: One has a unique decomposition of S into a sum $T + S'$ with $T \in C_0(X^*)$ and $S' \in \mathfrak{C}_{L,0}$. Since $U_l T U_l^* = T$, it suffices to consider $T = 0$. Then by (iii) of Lemma 5.1 and (5.3) it suffices to take $S = \varphi(Q)\psi(P)$ with $\varphi = \sum_{m\in M} \tau_m\varphi_0$ for some subset $M \subset L$ and some $\varphi_0 \in C_0(\Lambda)$, $\Lambda \subset X$ compact, and with $\psi \in C_0(X^*)$. Since $U_l S U_l^* = (\tau_{-l}\varphi)(Q)\psi(P)$, it suffices to show that $\lim_{l,\varkappa} \tau_{-l}\varphi$ exists uniformly on compacts on X. There are only two possibilities: either $M \in \varkappa$, or $L \setminus M \in \varkappa$; in the first case we shall prove that $\lim_{l,\varkappa} \tau_{-l}\varphi = \varphi_0$ and in the second one that $\lim_{l,\varkappa} \tau_{-l}\varphi = 0$. Indeed, let $K \subset X$ be a compact set and let $x \in K$. Then $\tau_{-l}\varphi(x) = \sum_{m\in M} \varphi_0(x + l - m)$. If $\varphi_0(x + l - m) \neq 0$ then $l \in m + (\Lambda - K)$ and $\Lambda - K$ is a compact set. So if l is large enough then $\varphi_0(x+l-m) \neq 0$ only if $l = m$ (L being sparse). So for large l one has either $\tau_{-l}\varphi(x) = \varphi_0(x)$ or $\tau_{-l}\varphi(x) = 0$ (independently of $x \in K$).

We have thus shown that the limit $\text{s-lim}_{l,\varkappa} U_l S U_l^* := S_\varkappa$ exists for each $\varkappa \in \widetilde{L}$. The argument also gives the explicit form of the limit for a class of operators S which is dense in $\mathfrak{C}_L$. Namely, assume that S is of the form $\psi_0(P) + \sum_{i=1}^n \varphi_i(Q)\psi_i(P) \equiv T + S'$, with $\psi_1, \dots, \psi_n \in C_0(X^*)$ and $\varphi_i = \sum_{j=1}^{k_i} \sum_{l\in L_{ij}} \tau_l\varphi_{ij}$, where, for each i, $\{L_{i1}, \dots, L_{ik_i}\}$ is a partition of L and $\varphi_{ij} \in C_c(X)$. For each i there is a unique $j(i) \in \{1, \dots, k_i\}$ such that $L_{ij(i)} \in \varkappa$. Then

$$S_\varkappa = \psi_0(P) + \sum_{i=1}^n \varphi_{ij(i)}(Q)\psi_i(P). \tag{5.6}$$

Thus $S_\varkappa \in \mathbb{T}(X)$ and its projection on $C_0(X^*)$ is $\psi_0(P)$, which is the component of S in $C_0(X^*)$. This remains valid for all S by continuity and density.

Finally, consider the image $\widehat{S'}$ of S' in $\mathbb{K}(X)^{\langle L\rangle}$ given by Theorem 5.5 and identify $\mathbb{K}(X)^{\langle L\rangle} \cong C(\widetilde{L};\mathbb{K}(X))$, cf. (2.9). Then $\widehat{S'}$ will be the family of operators $S_\varkappa$ defined by (5.6), so the theorem is proved. ■

The following result is a straightforward consequence of Theorem 5.7.

Theorem 5.8 *Let H be an observable affiliated to $\mathfrak{C}_L$. Then the limit* $\text{s-lim}_{l,\varkappa} U_l H U_l^* := H_\varkappa$ *exists in the strong resolvent sense for each $\varkappa \in \widetilde{L}$ and*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{\varkappa \in \widetilde{L}} \boldsymbol{\sigma}(H_\varkappa).$$

If $\{H_l\}_{l\in L}$ is a representative of H, then for $\varkappa \in \widetilde{L}$ one also has $H_\varkappa =$ $\text{u-lim}_{l,\varkappa} H_l$ *(limit in the norm resolvent sense).*

It is quite easy to give examples of self-adjoint operator affiliated to $\mathfrak{C}_L$ with a nontrivial essential spectrum. Let $h : X^* \to \mathbb{R}$ be a continuous divergent function (by "divergent" we mean $\lim_{\kappa\to\infty} h(\kappa) = \infty$). Then $H_0 = h(P)$ is a self-adjoint operator strictly affiliated to $\mathfrak{C}_L$, hence if V is a self-adjoint operator in the multiplier algebra of $\mathfrak{C}_L$ then $H = H_0 + V$ is also strictly affiliated to $\mathfrak{C}_L$ and we may apply to it the Theorems 5.6 and 5.8. More explicitly, we may take $V = \sum_{l\in L} \tau_l \varphi_l(Q)$, where $\{\varphi_l\}$ is as in (ii) of Lemma 5.1, in which case the operators H_l of Theorem 5.6 are given by $H_l = h(P) + \varphi_l(Q)$. Much more singular perturbations are, however, allowed, as we shall show later on.

We close this paragraph by pointing out the interesting particular case when there is only a finite number of *types of bumps*. Let $\mathcal{L}$ be a finite partition of L consisting of infinite sets M and let $\mathfrak{C}_L^{\mathcal{L}}$ be the space of $S \in \mathfrak{C}_L$ such that $\text{s-lim}_{l\in M, l\to\infty} U_l S U_l^* := S_M$ exists for each $M \in \mathcal{L}$. This is clearly a C^*-subalgebra of $\mathfrak{C}_L$ which contains $\mathbb{K}(X)$. $\mathfrak{C}_L^{\mathcal{L}}$ is the set of $S \in \mathfrak{C}_L$ such that for each $M \in \mathcal{L}$ and each ultrafilter $\varkappa \in \widetilde{L}$ with $M \in \varkappa$ the limit $S_\varkappa$ is independent of $\varkappa$. By using the remarks made in the last part of §2.5 one can prove that

$$\mathfrak{C}_L^{\mathcal{L}} \big/ \mathbb{K}(X) \hookrightarrow \bigoplus_{M\in\mathcal{L}} \mathbb{T}(X). \tag{5.7}$$

If H is an observable affiliated to $\mathfrak{C}_L^{\mathcal{L}}$ then $\text{s-lim}_{l\in M, l\to\infty} U_l H U_l^* := H_M$ exists in the strong resolvent sense for each $M \in \mathcal{L}$ and

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{M\in\mathcal{L}} \boldsymbol{\sigma}(H_M). \tag{5.8}$$

5.3. We shall describe here a class of elements of the algebra $\mathfrak{C}_{L,0}$. This will give us a version of Theorem 5.5 independent of the constructions from the abelian case.

Proposition 5.9 *Let $\{K_l\}_{l\in L}$ be a relatively compact family of compact operators on $\mathscr{H}$. Assume that there is a compact set $\Lambda \subset X$ such that $K_l = \chi_\Lambda(Q)\, K_l\, \chi_\Lambda(Q)$ for all $l \in L$. Then the series $\sum_{l\in L} U_l^*\, K_l\, U_l$ converges in the strong operator topology and its sum belongs to $\mathfrak{C}_{L,0}$. The set of operators of the preceding form is dense in $\mathfrak{C}_{L,0}$. More precisely, the linear subspace generated by the operators $\sum_{l\in M} U_l^* K U_l$, where M is a subset of L and $K \in \mathbb{K}(X)$ has the property $K = \chi_\Lambda(Q)\, K\, \chi_\Lambda(Q)$ for some compact set $\Lambda \subset X$, is dense in $\mathfrak{C}_{L,0}$.*

Remark: According to Proposition 3.15 the relative compacity of a family $\{K_l\}_{l\in L}$ satisfying $K_l = \chi_\Lambda(Q)\, K_l\, \chi_\Lambda(Q)$ is equivalent to

$$\lim_{x\to 0} \sup_{l\in L} \|(U_x - 1)K_l^{(*)}\| = 0. \tag{5.9}$$

Proof: Notice first that finite sums $\sum_{l\in F} U_l^*\, K_l\, U_l$ are compact operators, so belong to $\mathfrak{C}_{L,0}$. Hence we just have to prove the first part of the theorem under the assumption $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$ if $K_l \neq 0$, $K_{l'} \neq 0$. The series $T = \sum_{l\in L} U_l^*\, K_l\, U_l$ converges strongly because the operators $U_l^*\, K_l\, U_l$ are paiwise orthogonal for large l. The family $\{K_l\}_{l\in L}$ being relatively compact, for each $\varepsilon > 0$ there is a finite subset I of L such that L decomposes into a disjoint union $\bigsqcup_{i\in I} L_i$ and for each $i \in I$ we have $\|K_l - K_i\| < \varepsilon$ for all $l \in L_i$. Let then $T_\varepsilon \equiv \sum_{i\in I} \sum_{l\in L_i} U_l^*\, K_i\, U_l$ and estimate for each $f \in \mathscr{H}$:

$$\begin{aligned}
\|(T - T_\varepsilon)f\|^2 &= \|\textstyle\sum_{i\in I} \sum_{l\in L_i} \chi_\Lambda(Q-l) U_l^*\,(K_l - K_i)\, U_l \chi_\Lambda(Q-l) f\,\|^2 \\
&= \textstyle\sum_{i\in I} \sum_{l\in L_i} \|\chi_\Lambda(Q-l)\, U_l^*\,(K_l - K_i)\, U_l \chi_\Lambda(Q-l) f\|^2 \\
&\leq \varepsilon^2 \textstyle\sum_{l\in L} \|\chi_\Lambda(Q-l)\, f\|^2 = \varepsilon^2 \|\sum_{l\in L} \chi_{\Lambda+l}(Q)\, f\,\|^2 \leq \varepsilon^2 \|f\|^2.
\end{aligned}$$

Thus it suffices to show that $T_\varepsilon \in \mathfrak{C}_{L,0}$ which actually means that it suffices to prove the proposition for the case when $K_l \equiv K$ is independent of l. Note that in the arguments below one can substitute to L any subset of it.

So let K be a compact operator and Λ a compact subset of X such that $K = \chi_\Lambda(Q) K \chi_\Lambda(Q)$. We shall prove that $\widetilde{K} \equiv \sum_{l\in L} U_l^*\, K\, U_l \in \mathfrak{C}_{L,0}$ (the series being strongly convergent by the same argument as above). The set of $\psi \in C_0(X^*)$ such that $\psi = \widehat{\eta}$ for some $\eta \in C_c(X)$ is dense in $C_0(X^*)$ (see (4.13) in [Fol2]). By using also Proposition 3.14, we see that for each $\varepsilon > 0$ there are functions $\varphi_1, \ldots, \varphi_n, \eta_1, \ldots, \eta_n$ in $C_c(X)$ such that $\|K - S\| < \varepsilon$, where $S = \sum_{i=1}^{n} \varphi_i(Q)\widehat{\eta}_i(P)$. Since $(\widehat{\eta}(P)f)(x) = \int_X \eta(x-y)f(y)\,\mathrm{d}y$ we have $(Sf)(x) = \int_X \left(\sum_i \varphi_i(x)\eta_i(x-y)\right) f(y)\,\mathrm{d}y$. Let Γ be a compact set such that the supports of the functions φ_i, η_i be included in Γ and let Ω be the compact set $\Lambda \cup \Gamma \cup (\Gamma - \Gamma)$. If $\chi_\Omega \equiv \chi_\Omega(Q)$ then $K = \chi_\Omega K \chi_\Omega$, $S = \chi_\Omega S \chi_\Omega$. This shows in particular that the series $\sum_{l\in L} U_l^* S U_l$ is strongly convergent (L being sparse) and its sum $\widetilde{S}$ can be computed:

$$\widetilde{S} = \textstyle\sum_i \sum_l U_l^* \varphi_i(Q)\widehat{\eta}_i(P) = \sum_i [\sum_l \varphi_i(Q-l)]\widehat{\eta}_i(P) \equiv \sum_i \phi_i(Q)\widehat{\eta}_i(P).$$

ϕ_i belongs to $C_{L,0}(X)$ by Lemma 5.1, so $\widetilde{S} \in \mathfrak{C}_{L,0}$. We have $\widetilde{K} - \widetilde{S} = \sum_l \chi_{l+\Omega} U_l^*(K-S)U_l \chi_{l+\Omega}$ and there is a finite set $F \subset L$ such that, if $l, l' \in M \equiv L \setminus F$, $l \neq l'$, then $(l+\Omega) \cap (l'+\Omega) = \emptyset$. Then

$$\begin{aligned} \left\| \widetilde{K} f - \widetilde{S} f - \textstyle\sum_{l\in F} U_l^*(K-S)U_l f \right\|^2 &= \left\| \textstyle\sum_{l\in M} \chi_{l+\Omega} U_l^*(K-S)U_l \chi_{l+\Omega} f \right\|^2 \\ = \quad \textstyle\sum_{l\in M} \|\chi_{l+\Omega} U_l^*(K-S)U_l \chi_{l+\Omega} f\|^2 &\leq \varepsilon^2 \textstyle\sum_{l\in M} \|\chi_{l+\Omega} f\|^2 \leq \varepsilon^2 \|f\|^2. \end{aligned}$$

Thus for each $\varepsilon > 0$ there is an operator $T = \widetilde{S} - \sum_{l\in F} U_l^*(K-S)U_l \in \mathfrak{C}_{L,0}$ such that $\|\widetilde{k} - T\| \leq \varepsilon$. Hence $\widetilde{K} \in \mathfrak{C}_{L,0}$.

The fact that the linear subspace generated by the operators $\sum_{l\in M} U_l^* \varphi(Q)\widehat{\eta}(P)U_l$, with $M \subset L$ and $\varphi, \eta \in C_{\mathrm{c}}(X)$, is dense in $\mathfrak{C}_{L,0}$ follows from Theorem 3.12, Lemma 5.1 and the preceding arguments (where L can be replaced by M). ∎

The next result, a more explicit version of Theorem 5.5, is a straightforward consequence of Proposition 5.9 and Theorem 5.5.

Theorem 5.10 *There is a unique morphism $\mathfrak{C}_L \to \mathbb{T}(X)^{[L]}/\mathbb{K}(X)^{(L)}$ such that the image of an element of the form $\psi(P) + \sum_{l\in M} U_l^* K U_l$, where $\psi \in C_0(X^*)$, $M \subset L$, and $K \in \mathbb{K}(X)$ is such that $K = \chi_\Lambda(Q)\, K\, \chi_\Lambda(Q)$ for some compact set $\Lambda \subset X$, be the quotient of the element $\chi_M \otimes (\psi(P) + K) \in \mathbb{T}(X)^{[L]}$ with respect to the ideal $\mathbb{K}(X)^{(L)}$. The kernel of this morphism is $\mathbb{K}(X)$ and its restriction to $\mathfrak{C}_{L,0}$ induces the canonical isomorphism of $\mathfrak{C}_{L,0}/\mathbb{K}(X)$ with the L-asymptotic algebra of compact operators $\mathbb{K}(X)^{\langle L\rangle}$.*

5.4. The aim of this paragraph is to give a description of the algebras $\mathfrak{C}_{L,0}$ and $\mathfrak{C}_L$ in terms of geometric properties of its elements in the phase-space $\Xi = X \oplus X^*$. We use the notations introduced in §3.4; let us also recall the notation $L_\Lambda = L + \Lambda$ and set L_Λ^c for $X \setminus L_\Lambda$. As usual, by $\chi_{L_\Lambda^c}(Q)$ we denote the operator of multiplication by the characteristic function of the set L_Λ^c.

Theorem 5.11 *A bounded operator T on $L^2(X)$ belongs to $\mathfrak{C}_{L,0}$ iff*
(i) $\lim_{x\to 0}\|(U_x-1)T^{(*)}\|=0$,
(ii) $\lim_{k\to 0}\|V_kTV_k^*-T\|=0$,
(iii) *For each $\varepsilon>0$ there is a compact set $\Lambda\subset X$ such that $\|\chi_{L^c_\Lambda}(Q)T^{(*)}\|<\varepsilon$.*

Corollary 5.12 *A bounded operator T on $L^2(X)$ belongs to $\mathfrak{C}_L$ if and only if it satisfies conditions* (i), (ii) *of Theorem 5.11 and there exists $\widetilde{T}\in C_0(X^*)$ such that for each $\varepsilon>0$ there is a compact set $\Lambda\subset X$ with $\|\chi_{L^c_\Lambda}(Q)\,(T-\widetilde{T}\,)^{(*)}\|<\varepsilon$.*

Notice that an equivalent formulation of (ii) is $\lim_{k\to 0}\|[V_k,T]\|=0$. We denote by $\mathfrak{A}$ the set of operators verifying the conditions (i)-(iii) of the theorem. Clearly $\mathfrak{A}$ is a C^*-algebra. If one replaces U_x by V_k in (i) one obtains a condition stronger than (ii), hence Proposition 3.13 already shows that $\mathbb{K}(X)\subset\mathfrak{A}$. Moreover, one has $\mathfrak{C}_{L,0}\subset\mathfrak{A}$. Indeed, by (5.3) it suffices to show that operators of the form $\varphi(Q)\psi(P)$ with $\varphi\in C_{L,\mathrm{c}}(X)(X)$ and $\psi\in C_0(X^*)$ belong to $\mathfrak{A}$. Since $U_x\varphi(Q)U_x^*=\varphi(Q+x)$, we have

$$\|(U_x-1)\varphi(Q)\psi(P)\|\le\|\varphi(Q+x)-\varphi(Q)\|\,\|\psi(P)\|+\|\varphi(Q)\|\,\|(U_x-1)\psi(P)\|.$$

The function φ is uniformly continuous so the first term in the r.h.s. above tends to zero as $x\to 0$. Since

$$\|(U_x-1)\psi(P)\|=\sup_{k\in X^*}|k(x)-1|\,|\psi(k)|$$

and $\psi(k)\to 0$ as $k\to\infty$ in X^* we see that this norm also tends to zero as $x\to 0$. Thus (i) is satisfied. Further, (ii) is an immediate consequence of the uniform continuity of ψ because

$$V_k\varphi(Q)\psi(P)V_k^*-\varphi(Q)\psi(P)=\varphi(Q)\,[\psi(P+k)-\psi(P)].$$

Also, $\chi_{L^c_\Lambda}(Q)T=\chi_{L^c_\Lambda}(Q)\varphi(Q)\psi(P)$ is zero if Λ is large enough, by the properties of $\varphi\in C_{L,\mathrm{c}}(X)(X)$. In order to treat the term $T\chi_{L^c_\Lambda}(Q)$ of (iii) we recall that $C_0(X^*)\cdot C_{L,\mathrm{c}}(X)(X)$ is dense in $\mathfrak{C}_{L,0}$, so for each $\varepsilon>0$ one may find $\psi_1,\dots,\psi_n\in C_0(X^*)$ and $\varphi_1,\dots,\varphi_n\in C_{L,\mathrm{c}}(X)(X)$ such that $\|T-\sum_1^n\psi_i(P)\varphi_i(Q)\|<\varepsilon$. For Λ large enough we have $\varphi_i(Q)\chi_{L^c_\Lambda}(Q)=0$ for all i, hence $\|T\chi_{L^c_\Lambda}(Q)\|<\varepsilon$. This finishes the proof of $\mathfrak{C}_{L,0}\subset\mathfrak{A}$. The reciprocal assertion is less elementary and we devote the rest of the subsection to its proof.

In what follows we shall need three groups of automorphisms of $B(\mathscr{H})$, namely $\{\mathcal{U}_x\}_{x\in X}$, $\{\mathcal{V}_k\}_{k\in X^*}$ and $\{\mathcal{W}_\xi\}_{\xi\in X\times X^*}$, defined on every $T\in B(\mathscr{H})$

by $\mathcal{U}_x[T] := U_x T U_x^*$, $\mathcal{V}_k[T] := V_k T V_k^*$ and $\mathcal{W}_{(x,k)} := \mathcal{U}_x \mathcal{V}_k$ respectively. Notice that $[\mathcal{U}_x, \mathcal{V}_k] = 0$ for each couple $(x,k) \in X \times X^*$. Hence $\mathcal{W}_\xi$ is a representation on $B(\mathscr{H})$ of the locally compact group $\Xi := X \times X^*$ equipped with the Haar measure $\mathrm{d}\xi = \mathrm{d}x \otimes \mathrm{d}k$. This representation is continuous if we equip $B(\mathscr{H})$ with the strong operator topology but it is not norm continuous. It is clear that

$$C^0_{\mathrm{u}}(\Xi) := \{T \in B(\mathscr{H}) \,|\, \Xi \ni \xi \mapsto \mathcal{W}_\xi[T] \in B(\mathscr{H}) \text{ is norm continuous}\} \tag{5.10}$$

is a C^*-subalgebra of $B(\mathscr{H})$.

If $T \in B(\mathscr{H})$ and $u \in L^1(X)$, $v \in L^1(X^*)$, we denote

$$T_{u,v} := \int_\Xi \mathcal{W}_\xi[T]\,(u \otimes v)(\xi)\; \mathrm{d}\xi. \tag{5.11}$$

In the next three lemmas we give properties of this object which show that $T_{u,v}$ is a "regularization" of T in a similar manner in which the convolution of a function is a smoothing of this function. We mention that the regularization in x, realized by u, is not needed for the proof of Theorem 5.11, but will be useful later on.

Lemma 5.13 *For each $T \in B(\mathscr{H})$ the following statements are equivalent:*
(i) *For each $\varepsilon > 0$ there are $u \in L^1(X)$ and $v \in L^1(X^*)$ such that $\|T_{u,v} - T\| < \varepsilon$*
(ii) $T \in C^0_{\mathrm{u}}(\Xi)$
(iii) $\lim_{x\to 0} \|(\mathcal{U}_x - 1)T\| = 0$ *and* $\lim_{k \to 0} \|(\mathcal{V}_k - 1)T\| = 0$.
Moreover, if one of these conditions is satisfied, the functions u and v from (i) *may be chosen such that their Fourier transforms $\widehat{u}$, $\widehat{v}$ have compact support.*

Proof: First, (ii) is equivalent to (iii), as a consequence of

$$\|\mathcal{W}_{(x,k)}[T] - T\| = \|\mathcal{U}_x(\mathcal{V}_k[T]-T) + (\mathcal{U}_x[T]-T)\| \le \|\mathcal{V}_k[T]-T\| + \|\mathcal{U}_x[T]-T)\|.$$

We prove now the equivalence between (i) and (ii). For each couple $(y,p) \in X \times X^*$ we have

$$\mathcal{W}_{(y,p)}[T_{u,v}] \equiv (\mathcal{U}_y \mathcal{V}_p)[T_{u,v}] = \int_X \int_{X^*} (\mathcal{U}_x \mathcal{V}_k)[T]\, u(x-y)\, v(k-p)\, \mathrm{d}x\, \mathrm{d}k.$$

Since the translations act continuously on L^1 we deduce that the map $(y,p) \mapsto \mathcal{W}_{(y,p)}[T_{u,v}]$ is norm continuous. So $T_{u,v} \in C^0_{\mathrm{u}}(\Xi)$ for each $T \in B(\mathscr{H})$. Then if (i) holds we get (ii) because $C^0_{\mathrm{u}}(\Xi)$ is a norm closed subspace of $B(\mathscr{H})$. Reciprocally, assume that (iii) holds. It can be shown that for every open set $\Lambda \neq \emptyset$ of X and for every $\varepsilon > 0$ there is $u \in L^1(X)$ such that $u \geq 0$, $\int_X u = 1$, $\int_{X\setminus\Lambda} u \leq \varepsilon$ and $\widehat{u} \in C_{\mathrm{c}}(X^*)$ (put $u = \widehat{\psi}$ in Lemma 2.1 from [GI1]). Similarly, for each neighbourhood Γ of 0 in X^* there is $v \in L^1(X^*)$ with $v \geq 0$, $\int_{X^*} v = 1$, $\int_{X^*\setminus\Gamma} v \leq \varepsilon$ and $\widehat{v} \in C_{\mathrm{c}}(X)$. Then we have

$$\begin{aligned}
\|T_{u,v} - T\| &= \left\| \int_{X\times X^*} (\mathcal{W}_{(x,k)}[T] - T)\, u(x)\, v(k)\, \mathrm{d}x\, \mathrm{d}k \right\| \\
&\leq \int_X \|(\mathcal{U}_x - 1)T\|\, u(x)\, \mathrm{d}x + \int_{X^*} \|(\mathcal{V}_k - 1)T\|\, v(k)\, \mathrm{d}k \\
&\leq \sup_{x\in\Lambda} \|(\mathcal{U}_x - 1)T\| + \sup_{k\in\Gamma} \|(\mathcal{V}_k - 1)T\| + 4\varepsilon\|T\|.
\end{aligned}$$

By choosing Λ, Γ such that the first two terms in the last member above be small, we see that (i) holds. Moreover, we also proved the last assertion of the lemma. ■

Lemma 5.14 $\mathfrak{A} \subset C^0_{\mathrm{u}}(\Xi)$ *and if* $T \in \mathfrak{A}$ *and* $u \in L^1(X)$, $v \in L^1(X^*)$ *then* $T_{u,v} \in \mathfrak{A}$.

Proof: The condition (i) of Theorem 5.11 is stronger than $\lim_{x\to 0} \|(\mathcal{U}_x - 1)T\| = 0$ from (iii) of Lemma 5.13, hence each $T \in \mathfrak{A}$ verifies the statements (i)-(iii) of this lemma. It remains to prove that $T_{u,v} \in \mathfrak{A}$, which obviously follows by dominated convergence if $\mathcal{W}_\xi[T] \in \mathfrak{A}$ for any $\xi \in X \times X^*$.

Let, more generally, ψ be the Fourier transform of an integrable measure on X and $T \in \mathfrak{A}$. We shall prove that $\psi(P)T \in \mathfrak{A}$. Since $U_x = x(P)$ the operator $\psi(P)T$ clearly satisfies condition (i) of Theorem 5.11. Then

$$[V_k, \psi(P)T] = \{V_k\psi(P)V_k^* - \psi(P)\}V_kT + \psi(P)[V_k, T].$$

Since $V_k\psi(P)V_k^* = \psi(P + k)$ and $\psi \in C^{\mathrm{u}}_{\mathrm{b}}(X^*)$, condition (ii) of Theorem 5.11 is also satisfied. Finally, to check (iii) we use the representation $\psi(P) = \int_X U_x\widehat{\psi}\,(\,\mathrm{d}x)$. If Λ is a a compact subset of X then

$$\|\chi_{L^{\mathrm{c}}_\Lambda}(Q)\psi(P)T\| \leq \int_X \|\chi_{L^{\mathrm{c}}_\Lambda}(Q)U_xT\|\; |\widehat{\psi}\,|(dx).$$

Let $\varepsilon > 0$. Since $\widehat{\psi}$ is an integrable measure, there is a compact subset K of X such that $\int_{X\setminus K} |\widehat{\psi}\,|(dx) < \varepsilon/(2\|T\|)$, hence

$$\|\chi_{L^{\mathrm{c}}_\Lambda}(Q)\psi(P)T\| \leq \frac{\varepsilon}{2} + \int_K \|\chi_{L^{\mathrm{c}}_\Lambda}(Q - x)T\|\; |\widehat{\psi}\,|(dx).$$

Since $T \in \mathfrak{A}$, there is a compact set M in X such that $\|\chi_{L^c_M}(Q)T\| \leq \varepsilon[2|\widehat{\psi}|(K)]^{-1}$. If we take $\Lambda = M - K$ then Λ is also a compact in X and for each $x \in K$ we have $M \subset x + \Lambda$, so $L_M \subset L_\Lambda + x$. But $\chi_{L^c_\Lambda}(Q - x) = \chi_{(L_\Lambda + x)^c}(Q)$ so $\|\chi_{L^c_\Lambda}(Q - x)T\| \leq \|\chi_{L^c_M}(Q)T\|$. With this choice of Λ we shall thus have $\|\chi_{L^c_\Lambda}(Q)\psi(P)T\| \leq \varepsilon$. ■

Lemma 5.15 *Let $T_{u,v}$ be given by (5.11) with $\widehat{v}$ of compact support. Then there is a compact set Λ in X such that for $\theta_1,\ \theta_2 \in C_c(X)$ with*

$$\operatorname{supp}\theta_1 \bigcap (\operatorname{supp}\theta_2 + \Lambda) = \emptyset$$

one has $\theta_1(Q)\, T_{u,v}\, \theta_2(Q) = 0$.

Proof: We have to prove

$$\theta_1(Q) \int_X \int_{X^*} V_k U_x T U_x^* V_k^*\, u(x)\, v(k)\ \mathrm{d}x\, \mathrm{d}k\ \theta_2(Q) = 0$$

for $\operatorname{supp}\theta_1$ and $\operatorname{supp}\theta_2$ sufficiently far away one from another and for all $T \in B(\mathscr{H})$. By the weak density of the finite rank operators, it suffices to assume T of rank one and, by the polarization identity, we may take T of the form $|g\rangle\langle g|$ for some $g \in \mathscr{H}$. Then for any $f_1,\ f_2 \in \mathscr{H}$ one has

$$\begin{aligned}
&\langle f_1, \theta_1(Q) T_{u,v} \theta_2(Q) f_2\rangle \\
&\quad = \iint_{X\times X^*} \langle f_1, \theta_1(Q) V_k U_x g\rangle\, \langle V_k U_x g, \theta_2(Q) f_2\rangle\, u(x)\, v(k)\ \mathrm{d}x\, \mathrm{d}k \\
&= \iint_{X\times X^*} \mathrm{d}x\, \mathrm{d}k\, u(x)\, v(k)\ \times \\
&\quad \times \iint_{X\times X} \overline{f_1}(x_1)\, \theta_1(x_1)\, g(x_1 + x)\, k(x_1)\overline{k}(x_2)\, \overline{g}(x_2 + x)\, \theta_2(x_2)\, f_2(x_2)\ \mathrm{d}x_1\, \mathrm{d}x_2 \\
&= \iiint_{X\times X\times X} \mathrm{d}x\, \mathrm{d}x_1\, \mathrm{d}x_2\, u(x)\, \overline{f_1}(x_1)\, \theta_1(x_1)\, g(x_1 + x)\, \overline{g}(x_2 + x)\, \theta_2(x_2)\, f_2(x_2) \times \\
&\quad \times \int_{X^*} k(x_1 - x_2)\, v(k)\, \mathrm{d}k.
\end{aligned}$$

The last integral over X^* equals $(\mathcal{F}^{-1}v)(x_1 - x_2)$, thus the triple integral above will be non zero only if $x_1 - x_2$ belongs to the compact set $\operatorname{supp}\mathcal{F}^{-1}v$. Then $x_1 = x_2 + (x_1 - x_2) \in (\operatorname{supp}\theta_2 + \operatorname{supp}\mathcal{F}^{-1}v) \bigcap \operatorname{supp}\theta_1$ shows that it suffices to choose $\Lambda = \operatorname{supp}\mathcal{F}^{-1}v$. ■

As a consequence of the previous results, for each $T \in \mathfrak{A}$ and each $\varepsilon > 0$ there are $u \in L^1(X)$ and $v \in L^1(X^*)$ with $\widehat{v}$ compactly supported such that $T_{u,v} \in \mathfrak{A}$, $\|T_{u,v} - T\| < \varepsilon$ and such that the conclusion of Lemma 5.15 be satisfied. Hence there is (and we may fix it) a compact set $\Lambda \subset X$ such that Lemma 5.15 be valid and for which we have

$$\|\chi_{L^c_\Lambda}(Q)T_{u,v}\| + \|T_{u,v}\chi_{L^c_\Lambda}(Q)\| < \varepsilon. \tag{5.12}$$

Let now $\theta \in C_c(X)$ with $0 \le \theta \le 1$ and $\theta = 1$ on Λ. For each $l \in L$ we denote simply by θ^l both the map $x \mapsto \theta(x - l)$ and the operator of multiplication by this function (in what follows we shall freely use the same condensed notation for other functions on X too). Since the set L is sparse, we may find a subset $M \subset L$ with a finite complementary such that

$$\operatorname{supp}\theta^l \bigcap (\operatorname{supp}\theta^{l'} + \Lambda) = \emptyset \quad \text{if} \;\; l,\, l' \in M \text{ and } l \neq l'.$$

Lemma 5.15 gives then $\theta^l T_{u,v}\theta^{l'} = 0$ for all the pairs l, l' as above. Hence, setting $\phi = \sum_{l\in M}\theta^l$ one obtains

$$\phi\, T_{u,v}\, \phi = \sum_{l,l'\in M} \theta^l T_{u,v}\theta^{l'} = \sum_{l\in M}\theta^l T_{u,v}\theta^l.$$

On the other hand, since $0 \le \phi \le 1$ there is a bounded, compactly supported function φ such that $1 - \phi = \varphi + (1-\phi)\chi_{L^c_\Lambda}$. This gives the following decomposition of $T_{u,v}$:

$$\begin{aligned} T_{u,v} &= \phi\, T_{u,v}\, \phi + (1-\phi)T_{u,v}\, \phi + T_{u,v}(1-\phi) \\ &= \sum_{l\in M}\theta^l T_{u,v}\theta^l + (\varphi\, T_{u,v}\, \phi + T_{u,v}\, \varphi) + ((1-\phi)\chi_{L^c_\Lambda}T_{u,v}\, \phi + T_{u,v}\chi_{L^c_\Lambda}(1-\phi)) \end{aligned}$$

Let us observe that for each $S \in \mathfrak{A}$ and each bounded function with compact support φ on X the operators $\varphi(Q)S$ and $S\varphi(Q)$ are compact. Indeed, choose $\phi \in C_c(X)$ such that $\varphi\phi = \varphi$. It suffices thus to show that $\phi(Q)S$ is compact. The second member of the estimate

$$\|(U_x - 1)\phi(Q)S\| \le \|\phi(Q)\|\, \|(U_x - 1)S\| + \|U_x\phi(Q)U_x^* - \phi(Q)\|\, \|S\|,$$

tends to zero because $S \in \mathfrak{A}$ and ϕ is uniformly continuous. This shows that $\phi(Q)S$ satisfies the hypothesis of the compacity criterion Proposition 3.13.

So $\varphi\, T_{u,v}\, \phi + T_{u,v}\, \varphi$ is a compact operator K. Thus we may use (5.12) to get

$$\|T_{u,v} - \sum_{l\in M}\theta^l T_{u,v}\theta^l - K\| < \varepsilon.$$

In this manner, we are reduced to the proof of the assertion $\sum_{l\in M}\theta^l T_{u,v}\theta^l \in \mathfrak{C}_{L,0}$. This may be reformulated as $\sum_{l\in M} U_l^* K_l U_l \in \mathfrak{C}_{L,0}$ if we take into account that $\theta^l \equiv \theta(Q-l) = U_l^*\theta(Q)U_l$ and if we denote by K_l the compact operator $\theta(Q)U_l T_{u,v}U_l^*\theta(Q)$. It is straightforward to check that the family of these compacts verifies the hypotheses of the Proposition 5.9, cf. the Remark after that proposition. This finishes the proof of the inclusion $\mathfrak{A} \subset \mathfrak{C}_{L,0}$, hence that of Theorem 5.11.

5.5. In this paragraph we shall point out a large class of hamiltonians affiliated to the algebra $\mathfrak{C}_L$. Although we consider explicitly only the case $X = \mathbb{R}^n$, our arguments easily extend to other groups (the case $X = \mathbb{Z}^n$ being much easier).

We begin with a remark concerning the definition of the hamiltonians. Let $\mathscr{H}^s = \mathscr{H}^s(\mathbb{R}^n)$ be the scale of Sobolev spaces; here $s \in \mathbb{R}$, $\mathscr{H}^0 = L^2(\mathbb{R}^n) \equiv \mathscr{H}$. Let s, t be real numbers such that $0 \le t < s$. Let H_0 be a self-adjoint operator in $\mathscr{H}$ with $D(|H_0|^{1/2}) \subset \mathscr{H}^s$ and let $W : \mathscr{H}^t \to \mathscr{H}^{-t}$ be a symmetric operator. Then $\langle P\rangle^{2s} \le C(|H_0|+1)$ for some constant C and for each $\varepsilon > 0$ there is a constant $c < \infty$ such that $\pm W \le \varepsilon\langle P\rangle^{2s} + c$. Thus the form sum $H_0 + W$ defines a self-adjoint operator H in $\mathscr{H}$ with form domain equal to that of H_0. The self-adjoint operators from the next theorem should be interpreted in this sense.

Theorem 5.16 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that*

$$C^{-1}|x|^{2s} \le |h(x)| \le C|x|^{2s} \text{ for } |x| > R, \tag{5.13}$$

for some constants $s > 0$, $C > 0$ and $R < \infty$; denote $H_0 = h(P)$. Let $t \in [0, s)$ real, let L be a sparse subset of $\mathbb{R}^n$, and let $\{V_l\}_{l\in L}$ be a family of symmetric operators in $B(\mathscr{H}^t, \mathscr{H}^{-t})$ with the following property: there is a number $a > 2n$ such that

$$\sup_{l\in L} \|\langle Q\rangle^a V_l\|_{B(\mathscr{H}^t,\mathscr{H}^{-t})} < \infty. \tag{5.14}$$

Then the series $\sum_{l\in L} \mathrm{e}^{-ilP} V_l \mathrm{e}^{ilP}$ converges in the strong topology of $B(\mathscr{H}^t, \mathscr{H}^{-t})$ and its sum is a symmetric operator $V : \mathscr{H}^t \to \mathscr{H}^{-t}$. Let $H = H_0 + V$, $H_l = H_0 + V_l$ be the self-adjoint operators in $\mathscr{H}$ defined as form sums. Then H is strictly affiliated to $\mathfrak{C}_L$, H_l is strictly affiliated to $\mathbb{T}(X)$, and the family $\{H_l\}_{l\in L}$ is a representative of H. In particular:

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset L\\ F \text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}.$$

If $\varkappa$ is an ultrafilter on L finer than the Fréchet filter, then $\text{u-lim}_{l\in\varkappa} H_l := H_\varkappa$ *exists in the norm resolvent sense, one has* $H_\varkappa = \text{s-lim}_{l,\varkappa} \mathrm{e}^{ilP} H \mathrm{e}^{-ilP}$ *in the strong resolvent sense, and*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{\varkappa \in \widetilde{L}} \boldsymbol{\sigma}(H_\varkappa).$$

Remarks: (i) If $s \leq n/2$ and $V_l : \mathbb{R}^n \to \mathbb{R}$ are Borel functions satisfying the condition $\int_{|y-x|<1} |V_l(y)| \cdot |y-x|^{-n+2s-\lambda} \, \mathrm{d}y \leq c\langle x\rangle^{-a} \ \forall x \in \mathbb{R}^n$ for some constans $c, \lambda > 0$, then the operators V_l of multiplication by the functions V_l satisfy (5.14) for some $t < s$. If $s > n/2$ then the simpler condition $\int_{|y-x|<1} |V_l(y)| \, \mathrm{d}y \leq c\langle x\rangle^{-a}$ suffices. If s is an integer and V_l is a differential operator of order less than $2s$, similar explicit conditions on its coefficients can be stated, see the end of Section 1.3 in [ABG].
(ii) The condition $a > 2n$ is not natural and should be improved to $a > n$, but the version of the Cotlar-Stein lemma that we use in the proof does not allow us to get such a result. However, the assumption on a may be relaxed in terms of the "degree of rarefaction" of L: the greatest lower bound for a is actually inversely proportional to it, see Lemma 5.18.
(iii) There is no essential change in our formalism if we replace the Hilbert space $L^2(X)$ of physical states by $L^2(X; \mathbf{E})$ where $\mathbf{E}$ is a finite dimensional Hilbert space. This allows us to treat, for example, Dirac hamiltonians H_0 perturbed by the same class of potentials V. The condition (5.13) is satisfied with $s = 1/2$, $|h(x)|$ being interpreted as $[h(x)^2]^{1/2}$ ($h(x)$ is a self-adjoint operator in $\mathbf{E}$). Note that H_0 is not semibounded in this case. Theorems 5.6 and 5.8 remain valid without any change in this context.
(iv) The assumption $t < s$ is not essential and can be improved to $t = s$, which allows one to treat perturbations of the same order as H_0. But then one must add other conditions in order to give a sense to the sums $H_0 + V_l$ and $H_0 + V$ as self-adjoint operators. This question is of some importance if one wants to treat Dirac operators with Coulomb potentials or second order perturbations of the Laplace operator, but is outside the main scope of this paper.

We need the Cotlar-Stein lemma that we recall below (see [Tor] or [Fol1]):

Lemma 5.17 *Let $\{B_l\}_{l\in L}$ be a family of operators in $B(\mathscr{H}_1, \mathscr{H}_2)$ for some Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$. Assume that*

$$\sup_{l\in L} \sum_{m\in L} \max\left\{ \|B_l B_m^*\|_{B(\mathscr{H}_2)}^{1/2} \ , \ \|B_l^* B_m\|_{B(\mathscr{H}_1)}^{1/2} \right\} = b < \infty.$$

Then the sum $\sum_{l\in L} B_l \equiv B$ *exists in the strong operator topology and* $\|B\|_{B(\mathscr{H}_1,\mathscr{H}_2)} \leq b$.

Theorem 5.16 will be a consequence of the next lemma. We denote by $\|\cdot\|_{u,v}$ the norm in $B(\mathscr{H}^u, \mathscr{H}^v)$.

Lemma 5.18 *Let* $L \subset \mathbb{R}^n$ *such that* $|l-m| \geq \text{const.} > 0$ *if* l, m *are distinct points of* L, *and let* $a > 2n$. *Then there is* $C > 0$ *such that, for each family of operators* $V_l \in B(\mathscr{H}^u, \mathscr{H}^v)$ *with* $V_l = 0$ *for all but a finite number of* l, *the following estimate holds:*

$$\left\| \textstyle\sum_{l\in L} \mathrm{e}^{-ilP} V_l \mathrm{e}^{ilP} \right\|_{u,v} \leq C \sup_{l\in L} \max\left\{ \|\langle Q\rangle^a V_l\|_{u,v} \ , \ \|V_l \langle Q\rangle^a\|_{u,v} \right\}. \tag{5.15}$$

Proof: Let us denote $B_l \equiv \mathrm{e}^{-ilP}\langle P\rangle^v V_l \langle P\rangle^{-u} \mathrm{e}^{ilP}$ and check that the hypotheses of the Cotlar-Stein lemma are satisfied. For each couple l, m of points of L we estimate:

$$\begin{aligned}
\|B_l B_m^*\| &= \|\langle P\rangle^v V_l \langle P\rangle^{-2u} \mathrm{e}^{i(l-m)P} V_m^* \langle P\rangle^v\| \\
&\leq \ \|V_l\langle Q\rangle^a\|_{u,v} \cdot \|\langle P\rangle^u \langle Q\rangle^{-a} \langle P\rangle^{-2u} \mathrm{e}^{i(l-m)P} \langle Q\rangle^{-a} \langle P\rangle^u\| \ \cdot \|\langle Q\rangle^a V_m^*\|_{-v,-u}.
\end{aligned}$$

We have $\|V_m\langle Q\rangle^a\|_{u,v} = \|\langle Q\rangle^a V_m^*\|_{-v,-u}$. By standard commutator estimates (see Proposition 4.1.2 in [ABG]) there is a bounded operator S such that $\langle P\rangle^u\langle Q\rangle^{-a}\langle P\rangle^{-2u} = S\langle P\rangle^{-u}\langle Q\rangle^{-a}$, thus the middle norm in the last term of the above inequality may be majorated by $\|S\|$ times the quantity:

$$\|\langle P\rangle^{-u}\langle Q\rangle^{-a}\mathrm{e}^{i(l-m)P}\langle Q\rangle^{-a}\mathrm{e}^{-i(l-m)P}\langle P\rangle^u\| = \|\langle Q\rangle^{-a}\langle Q-(m-l)\rangle^{-a}\|_{u,u}.$$

Let us denote by C a generic positive finite constant. By interpolation between 0 and an integer $N > |u|$ the above quantity is dominated by

$$C\,\|\langle Q\rangle^{-a}\langle Q-(m-l)\rangle^{-a}\|_{N,N} \leq C \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\leq N} \left|\varphi_{lm}^{(\alpha)}(x)\right|,$$

where $\varphi_{lm}(x) \equiv \langle x\rangle^{-a}\langle x-(m-l)\rangle^{-a}$. Furthermore, for each α there is a constant c_α such that

$$\left|\varphi_{lm}^{(\alpha)}(x)\right| \leq c_\alpha\ |\varphi_{lm}(x)| \leq C\,\langle l-m\rangle^{-a}.$$

Hence $\|B_l B_m^*\| \leq C\,\langle l-m\rangle^{-a}\sup_{l\in L}\|V_l\langle Q\rangle^a\|_{u,v}$. Similarly we obtain the estimate $\|B_l^* B_m\| \leq C\,\langle l-m\rangle^{-a}\sup_{l\in L}\|\langle Q\rangle^a V_l\|_{u,v}$ which finally yields

$$\max\left\{\|B_l B_m^*\| \ , \ \|B_l^* B_m\|\right\} \leq C\,\langle l-m\rangle^{-a}\max\left\{\|\langle Q\rangle^a V_l\|_{u,v} \ , \ \|V_l\langle Q\rangle^a\|_{u,v}\right\}.$$

The hypothesis $a > 2n$ is then sufficient to ensure that $\sum_{m\in L}\langle l-m\rangle^{-a/2} \leq$ const. $< \infty$ independently of $l \in L$, which means that the hyptheses of Lemma 5.17 are verified. ■

If H_0 is as in the statement of Theorem 5.16 then it is obviously affiliated to $C_0(X^*) \subset \mathfrak{C}_L$. According to the abstract affiliation criterion Theorem 2.1, the decomposition (for some $\lambda \geq 0$)

$$\varphi(H_0)V(|H_0|+1)^{-1/2} = \varphi(H_0)\langle P\rangle^{s+\lambda} \cdot \langle P\rangle^{-s-\lambda}V\langle P\rangle^{-s} \cdot \langle P\rangle^{s}(|H_0|+1)^{-1/2}$$

and the fact that the symmetric operator $\langle P\rangle^{s}(|H_0|+1)^{-1/2}$ belongs to $\mathcal{M}(\mathfrak{C}_L)$ (because H_0 is strictly affiliated to $\mathfrak{C}_L$, see Proposition 3.16 and §2.2) reduce the problem to the proof of the statement

$$\langle P\rangle^{-s-\lambda}V\langle P\rangle^{-s} \in \mathfrak{C}_L \text{ for some } \lambda \geq 0. \tag{5.16}$$

These general remarks allow one to consider the case when $t = s$ in the statement of Theorem 5.16, but here we shall consider only the situation $t < s$, when it suffices to take $\lambda = 0$ above. We shall prove (5.16) by constructing a family of symmetric operators $\{V_\varepsilon\}$ of $\mathfrak{C}_L$ which approximates V in the norm of $B(\mathscr{H}^s, \mathscr{H}^{-s})$.

Choose $\theta \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \theta \leq 1$ and $\theta(0) = 1$, and set $\Theta_\varepsilon = \theta(\varepsilon Q)\theta(\varepsilon P)$. Let $V_\varepsilon = \sum_{l\in L} \mathrm{e}^{-ilP}V_{l,\varepsilon}\mathrm{e}^{ilP}$, where $V_{l,\varepsilon} = \Theta_\varepsilon V_l \Theta_\varepsilon^*$. Observe first that, for each $\varepsilon \in (0,1]$, $\{V_{l,\varepsilon}\}_{l\in L}$ is a relatively compact family of compact symmetric operators on $\mathscr{H}$. Hence, by Proposition 5.9, V_ε belongs to $\mathfrak{C}_{L,0}$ for each $\varepsilon > 0$. It remains thus only to show the convergence $\|V_\varepsilon - V\|_{s,-s} \to 0$ as $\varepsilon \to 0$. Let $2n < \alpha < a$, where a is as in Theorem 5.16. We shall use Lemma 5.18 with V replaced by $V_\varepsilon - V$, a by α, and $u = s$, $v = -s$. Then the first norm in the r.h.s. of the corresponding inequality of type (5.15) is estimated as follows:

$$\begin{aligned}
\|\langle Q\rangle^\alpha(V_{l,\varepsilon} - V_l)\|_{s,-s} &\leq \|\langle Q\rangle^\alpha(\Theta_\varepsilon - 1)V_l\Theta_\varepsilon^*\|_{s,-s} + \|\langle Q\rangle^\alpha V_l(\Theta_\varepsilon^* - 1)\|_{s,-s} \\
&\leq \|\langle Q\rangle^\alpha(\Theta_\varepsilon - 1)\langle Q\rangle^{-a}\|_{-t,-s} \cdot \|\langle Q\rangle^a V_l\|_{t,-t} \cdot \|\Theta_\varepsilon^*\|_{s,t} \\
&\leq \|\langle Q\rangle^\alpha V_l\langle Q\rangle^{a-\alpha}\|_{t,-s} \cdot \|\langle Q\rangle^{\alpha-a}(\Theta_\varepsilon^* - 1)\|_{s,t}
\end{aligned}$$

We shall use the scale of spaces $\mathscr{H}_v^u$ defined by the norms $\|\langle P\rangle^u\langle Q\rangle^v \cdot\|$; their interpolation properties are described in [ABG] and will be used below. By hypothesis, the family of operators V_l is bounded in $B(\mathscr{H}^t, \mathscr{H}_a^{-t})$. By interpolation, and since $t < s$, we get that it is also bounded in $B(\mathscr{H}_{\alpha-a}^t, \mathscr{H}_\alpha^{-s})$. So in order to show that $\|\langle Q\rangle^\alpha(V_{l,\varepsilon} - V_l)\|_{s,-s} \to 0$ if $\varepsilon \to 0$ it suffices to prove the next two relations:

$$\begin{aligned}
&\|\langle P\rangle^{-s}\langle Q\rangle^{\alpha}(\Theta_\varepsilon-1)\langle Q\rangle^{-a}\langle P\rangle^{t}\| \\
&\quad = \|\langle P\rangle^{-s}\langle Q\rangle^{\alpha}(\Theta_\varepsilon-1)\langle Q\rangle^{-\alpha}\langle P\rangle^{s}\cdot\langle P\rangle^{-s}\langle Q\rangle^{\alpha-a}\langle P\rangle^{t}\| \to 0 \\
&\|\langle P\rangle^{-s}(\Theta_\varepsilon-1)\langle Q\rangle^{\alpha-a}\langle P\rangle^{t}\| \\
&\quad = \|\langle P\rangle^{-s}(\Theta_\varepsilon-1)\langle P\rangle^{s}\cdot\langle P\rangle^{-s}\langle Q\rangle^{\alpha-a}\langle P\rangle^{t}\| \to 0
\end{aligned}$$

The operator $\langle P\rangle^{-s}\langle Q\rangle^{\alpha-a}\langle P\rangle^{t}$ is compact, so it sufices to show that $\Theta_\varepsilon \to 1$ strongly in $B(\mathscr{H}_v^u)$ when $\varepsilon \to 0$, for each $u, v \in \mathbb{R}$. But this is an easy consequence of the next more precise lemma and its analog with the roles of Q and P interchanged. Theorem 5.16 is proved. ■

Lemma 5.19 *Let θ be in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and let $u, v \in \mathbb{R}$. Then*

$$\lim_{\varepsilon\to 0}\|[\langle Q\rangle^{v}, \theta(\varepsilon P)]\langle Q\rangle^{-v}\|_{B(\mathscr{H}^u)} = 0.$$

Proof: We use the Fourier representation $\theta(\varepsilon P) = \int e^{i\varepsilon x P}\widehat{\theta}(x)\,dx$ in order to compute

$$\begin{aligned}
[\langle Q\rangle^{v}, \theta(\varepsilon P)]\langle Q\rangle^{-v} &= \int e^{i\varepsilon x P}(e^{-i\varepsilon x P}\langle Q\rangle^{v}e^{i\varepsilon P x}\langle Q\rangle^{-v}-1)\,\widehat{\theta}(x)\,dx \\
&= \int e^{i\varepsilon x P}(\langle Q-\varepsilon x\rangle^{v}\langle Q\rangle^{-v}-1)\,\widehat{\theta}(x)\,dx.
\end{aligned}$$

It is clear that $\|\langle Q-x\rangle^{v}\langle Q\rangle^{-v}\|_{B(\mathscr{H}^u)} \le C\langle x\rangle^{r}$ for some positive numbers C and r, so we shall have $\|[\langle Q\rangle^{v}, \theta(\varepsilon P)]\langle Q\rangle^{-v}\|_{B(\mathscr{H}^u)} \le$ const. Then, by an easy interpolation argument, it suffices to prove the lemma in the case $u = 0$. Now the dominated convergence theorem shows that it is enough to prove $\|\langle Q-x\rangle^{v}\langle Q\rangle^{-v}-1\| \to 0$ when $x \to 0$. But this is a consequence of $(1+|x|)^{-1} \le \langle y-x\rangle\langle y\rangle^{-1} \le (1+|x|)$. ■

5.6. We finish with a remark concerning the existence of wave operators for perturbations of $H_0 = \Delta$ in $L^2(\mathbb{R}^n)$ by Klaus type potentials $V(x) = \sum_{l\in L} V_l(x-l)$. It has been observed by Hundertmark and Kirsch in [HuK] that the wave operators for the pair (H, H_0) exist if $H = \Delta + V(x)$ and $\int_{|x|<r}|V(x)|^2\,dx \le Cr^{n-2-\varepsilon}$ for some $\varepsilon > 0$ (a better result, cf. §6.3, has been obtained in 1978 by M. Demuth by the same method) and that this criterion can be used when V is of Klaus form. Note that this condition implies $V = 0$ if $n = 1, 2$. In §6.3 we give an elementary proof of a theorem of Hörmander [Hor] which gives non-trivial results in all dimensions. For example, it suffices that V be a real function such that $\int_{\mathbb{R}^n} |V(x)|^2 \langle x\rangle^{-N}\,\mathrm{d}x < \infty$ for some N and that there is an open subset Ω of the unit sphere $S^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$, with $S^{n-1} \setminus \Omega$ of spherical measure zero, and for each $\omega \in \Omega$ there is $\vartheta > 0$ such that:

$$\int_1^\infty \left(\frac{1}{r^n}\int_{C(\omega;\vartheta,r)} |V(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}r < \infty.$$

Here $C(\omega;\vartheta,r)$ is the truncated cone consisting of the points x such that $\langle x,\omega\rangle > |x|\cos\vartheta$ and $r < |x| < 2r$. Unfortunately, this criterion is useless for Klaus potentials if $n = 1$ (in this case one can in fact use [DeWo] in order to show that the wave operators *do not* exist in certain cases). However, if $n \geq 2$, one can easily get nontrivial results:

Proposition 5.20 *Let W be a real square integrable function with compact support on $\mathbb{R}^n$, $n \geq 2$, and $L \subset \mathbb{R}^n$ be a sparse set. Let H be a self-adjoint operator on $L^2(\mathbb{R}^n)$ equal to $\Delta + \sum_{l\in L} W(x-l)$ on $\mathscr{S}(\mathbb{R}^n)$ and let us denote by H_0 the usual self-adjoint realization of Δ on $L^2(\mathbb{R}^n)$. For $\omega \in S^{n-1}, \vartheta > 0, r > 0$, denote $N_{\omega,\vartheta}(r)$ the number of points in $L \cap C(\omega;\vartheta,r)$. Assume that there is an open set $\Omega \subset S^{n-1}$ with $S^{n-1}\setminus\Omega$ of measure zero, and for each $\omega \in \Omega$ there is $\vartheta > 0$ such that $\int_1^\infty \sqrt{N_{\omega,\vartheta}(r)}\, r^{-n/2}\,\mathrm{d}r < \infty$. Then the wave operators for the pair (H, H_0) exist.*

Proof: If $K = \operatorname{supp} W$ and r is large enough, then $C(\omega;\vartheta,r)$ contains at most $c(n)N_{\omega,\vartheta}(r)$ sets of the form $l + K$ $(l \in L)$ and these sets are pairwise disjoint. Thus, if $V(x) = \sum_{l\in L} W(x-l)$, then $\int_{C(\omega;\vartheta,r)} |V(x)|^2\,\mathrm{d}x \leq c(n)N_{\omega,\vartheta}(r)\|W\|_{L^2}^2$, so we may use the preceding remarks. ∎

We point out the following simple consequence of the theorem: *if the accumulation points of the set $\{l/|l| \mid l \in L, l \neq 0\}$ form a set of spherical measure zero, then the wave operators for the pair (H, H_0) exist.* A second consequence is:

Corollary 5.21 *Let $\delta_{\omega,\vartheta}(r)$ be the minimum of the distances $|l_1 - l_2|$ for $l_1, l_2 \in L \cap C(\omega;\vartheta,r)$, $l_1 \neq l_2$. Assume that there is Ω as above and for each $\omega \in \Omega$ there is $\vartheta > 0$ such that $\int_1^\infty \delta_{\omega,\vartheta}(r)^{-n/2}\,\mathrm{d}r < \infty$. Then the wave opearators for the pair (H, H_0) exist.*

For the proof it suffices to note that $N_{\omega,\vartheta}(r)\delta_{\omega,\vartheta}(r) \leq c_n r^n$ for large r. Observe that the condition of the corollary is satisfied if for each $\omega \in \Omega$ there are $\vartheta, c, \varepsilon > 0$ such that $\delta_{\omega,\vartheta}(r) \geq c r^{2/n+\varepsilon}$ for all r.

6 Appendix

6.1. Enveloping C^*-Algebras

The purpose of this paragraph is to summarize several elementary facts concerning enveloping C^*-algebras which we need in the main text. We refer to §2.7 in [Dix] for details in the case when an approximate unit exists (which is true in the situation of crossed products).

Let $(\mathcal{A}, \|\cdot\|)$ be a Banach $*$-algebra. A C^*-(semi)norm on $\mathcal{A}$ is a (semi)norm $|\cdot|$ satisfying $|ST| \leq |S| \cdot |T|$ and $|S|^2 = |S^*S|$ for all $S, T \in \mathcal{A}$. We recall that each morphism (i.e. $*$-morphism) from $\mathcal{A}$ into a C^*-algebra is a contraction. Hence we have $|\cdot| \leq \|\cdot\|$ for each C^*-(semi)norm on $\mathcal{A}$. Since the upper bound of any family of C^*-(semi)norms is a C^*-(semi)norm, there is a largest C^*-(semi)norm on $\mathcal{A}$: we call it *the C^*-(semi)norm of $\mathcal{A}$* and we denote it by $\|\cdot\|_*$ (note that $\|\cdot\|_* \leq \|\cdot\|$). The C^*-algebra obtained by separation and completion of $(\mathcal{A}, \|\cdot\|_*)$ is denoted $\mathcal{A}_*$ (with norm denoted again by $\|\cdot\|_*$), and is called *the enveloping C^*-algebra of $\mathcal{A}$*.

Let $\theta_\mathcal{A} : \mathcal{A} \to \mathcal{A}_*$ be the canonical morphism. Then $\theta_\mathcal{A}$ is continuous (contractive) with dense range and it is injective if and only if $\|\cdot\|_*$ is a norm on $\mathcal{A}$, so if and only if there is a C^*-norm on $\mathcal{A}$. In this case we say that $\mathcal{A}$ is an *A^*-algebra* and we identify $\mathcal{A} \subset \mathcal{A}_*$, thus $\mathcal{A}$ becomes a dense $*$-subalgebra of $\mathcal{A}_*$ and $\|A\|_* \leq \|A\|$ if $A \in \mathcal{A}$.

The algebra $\mathcal{A}_*$ obviously has the following *universal property*: if $\mathcal{C}$ is a C^*-algebra and $\phi : \mathcal{A} \to \mathcal{C}$ is a morphism then there is a unique morphism $\phi_* : \mathcal{A}_* \to \mathcal{C}$ such that $\phi = \phi_* \circ \theta_\mathcal{A}$. As a consequence, we get a (covariant) functor from the category of Banach $*$-algebras to that of C^*-algebras. Indeed, if $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism between two Banach $*$-algebras, then clearly there is a morphism $\phi_* : \mathcal{A}_* \to \mathcal{B}_*$ such that $\phi_* \circ \theta_\mathcal{A} = \theta_\mathcal{B} \circ \phi_*$.

It is easy to see that *if ϕ has dense range then ϕ_* is surjective*. But we stress that *ϕ_* could be non-injective even if ϕ is injective.*

For example, if there are several distinct C^*-norms on $\mathcal{A}$ (which is the case if $\mathcal{A} = L^1(G)$ and G is a non-amenable locally compact group), then there is a C^*-norm $|\cdot|$ on $\mathcal{A}$ distinct from $\|\cdot\|_*$. So $|\cdot| \leq \|\cdot\|_*$ and if $\mathcal{A}_\bullet$ is the algebra obtained by completing $(\mathcal{A}, |\cdot|)$ then there is a canonical morphism $\mathcal{A}_\bullet \to \mathcal{A}_*$ which is surjective but not injective.

For similar reasons it may happen that the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ of a closed $*$-subalgebra $\mathcal{A}$ of the Banach $*$-algebra $\mathcal{B}$ induces a morphism $\mathcal{A}_* \to \mathcal{B}_*$ which is not injective. So if $\mathcal{B}$ is an A^*-algebra then $\mathcal{A}_*$ cannot (in general) be identified with the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathcal{B}_*$ (but there is a canonical surjection $\mathcal{A}_* \to \overline{\mathcal{A}}$, so $\overline{\mathcal{A}}$ is a quotient of $\mathcal{A}_*$).

Such unpleasant features do not occur, however, in the case of ideals (by *ideal* in a Banach $*$-algebra we mean "closed $*$-ideal"), as a consequence of the following result.

Theorem 6.1 *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\ \phi\ } \mathcal{A} \xrightarrow{\ \psi\ } \mathcal{B} \longrightarrow 0$$

be an exact sequence of Banach $*$-algebras. Then

$$0 \longrightarrow \mathcal{J}_* \xrightarrow{\ \phi_*\ } \mathcal{A}_* \xrightarrow{\ \psi_*\ } \mathcal{B}_* \longrightarrow 0$$

is an exact sequence of C^*-algebras.

Proof: a) We need only one non-trivial result, namely the Theorem VI.19.11 from [FeD], which says that if $\mathcal{I}$ is an ideal in a Banach $*$-algebra $\mathcal{A}$ and $\boldsymbol{\pi}$ is a non-degenerate representation of $\mathcal{I}$ on a Hilbert space $\mathscr{H}$, the there is a unique representation $\tilde{\boldsymbol{\pi}}$ of $\mathcal{A}$ on $\mathscr{H}$ which extends $\boldsymbol{\pi}$. This implies that the C^*-seminorm $\|\cdot\|_*^{\mathcal{I}}$ of $\mathcal{I}$ is equal to the restriction of the C^*-seminorm $\|\cdot\|_*^{\mathcal{A}}$ of $\mathcal{A}$ to $\mathcal{I}$. Indeed, note that e.g. the C^*-norm of $\mathcal{A}$, is also given by:

$$\begin{aligned}\|S\|_*^{\mathcal{A}} &= \sup\{\|\boldsymbol{\pi}(S)\| \mid \boldsymbol{\pi} \text{ is a representation of } \mathcal{A}\}\\ &= \sup\{\|\boldsymbol{\pi}(S)\| \mid \boldsymbol{\pi} \text{ is a non-degenerate representation of } \mathcal{A}\}\end{aligned}$$

b) We apply these remarks with the choice $\mathcal{I} = \phi(\mathcal{J}) = \ker\psi$. Since ϕ is a bijective morphism of $\mathcal{J}$ onto $\mathcal{I}$, we have $\|\phi(S)\|_*^{\mathcal{I}} = \|S\|_*^{\mathcal{J}}$, hence also $\|S\|_*^{\mathcal{J}} = \|\phi(S)\|_*^{\mathcal{A}}$. So ϕ is an isometry of $(\mathcal{J}, \|\cdot\|_*^{\mathcal{J}})$ into $(\mathcal{A}, \|\cdot\|_*^{\mathcal{A}})$ with range equal to $\mathcal{I}$. This implies that ϕ_* is an isometry of $\mathcal{J}_*$ into $\mathcal{A}_*$ with range equal to $\overline{\mathcal{I}}$, the closure in $\mathcal{A}_*$ of $\theta_{\mathcal{A}}(\mathcal{I})$.
c) Now we compute for $S \in \mathcal{A}$ the norm of $\theta_{\mathcal{A}} \circ \psi(S)$ by noticing that $\boldsymbol{\pi} \mapsto \boldsymbol{\pi} \circ \psi$ realizes a bijective correspondence between the representations of $\mathcal{B}$ and the representations of $\mathcal{A}$ which are zero on $\mathcal{I}$ (denote by $\mathrm{Rep}\,(\mathcal{A},\mathcal{I})$ the set of unitary equivalence classes of such representations):

$$\begin{aligned}\|\psi(S)\|_*^{\mathcal{B}} &= \sup\{\|\boldsymbol{\pi} \circ \psi(S)\| \mid \boldsymbol{\pi} \text{ is a representation of } \mathcal{B}\}\\ &= \sup\{\|\boldsymbol{\rho}(S)\| \mid \boldsymbol{\rho} \in \mathrm{Rep}\,(\mathcal{A},\mathcal{I})\}\end{aligned}$$

The map $\boldsymbol{\rho} \mapsto \boldsymbol{\rho}_*$ is a bijective correspondence between the representations of $\mathcal{A}$ and the representations of $\mathcal{A}_*$ (with inverse $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \circ \theta_{\mathcal{A}}$) which sends $\mathrm{Rep}\,(\mathcal{A},\mathcal{I})$ onto the space of representations of $\mathcal{A}_*$ which are zero on $\overline{\mathcal{I}}$. Hence

$$\|\psi(S)\|_*^{\mathcal{B}} = \sup\{\|\boldsymbol{\lambda} \circ \theta_{\mathcal{A}}(S)\| \mid \boldsymbol{\lambda} \in \mathrm{Rep}\,(\mathcal{A}_*, \overline{\mathcal{I}})\}.$$

The r.h.s here is equal to the norm of the image of $\theta_{\mathcal{A}}(S)$ into the quotient C^*-algebra $\mathcal{A}_*/\overline{\mathcal{I}}$. So we see that $\psi_* : \mathcal{A}_* \to \mathcal{B}_*$ factorize to a surjective isometry $\mathcal{A}_*/\overline{\mathcal{I}} \to \mathcal{B}_*$. Hence $\ker \psi_* = \overline{\mathcal{I}} = \phi_*(\mathcal{J}_*)$. ■

We shall reformulate a particular case of the preceding theorem in a form which is particularly useful for us. Let $\mathcal{A}$ be an A^*-algebra and $\mathcal{J}$ an ideal in $\mathcal{A}$ such that the quotient Banach $*$-algebra $\mathcal{B} = \mathcal{A}/\mathcal{J}$ is an A^*-algebra (which is not always the case). Then one has a dense embedding $\mathcal{A} \subset \mathcal{A}_*$ and $\mathcal{J}_*$ is canonically identified with the closure $\overline{\mathcal{J}}$ of $\mathcal{J}$ in $\mathcal{A}_*$ (the C^*-norm of $\mathcal{J}$ being equal to the restriction to $\mathcal{J}$ of the C^*-norm of $\mathcal{A}$). Moreover, the canonical map $\mathcal{A} \to \mathcal{A}/\mathcal{J}$ induces a surjective morphism $\mathcal{A}_* \to (\mathcal{A}/\mathcal{J})_*$ whouse kernel is equal to $\mathcal{J}_* \equiv \overline{\mathcal{J}}$, so we have a canonical identification $(\mathcal{A}/\mathcal{J})_* = \mathcal{A}_*/\mathcal{J}_*$. In other terms, the natural map $\mathcal{A}_* \to \mathcal{A}_*/\mathcal{J}_*$ induces an isomorphism of $(\mathcal{A}/\mathcal{J})_*$ onto $\mathcal{A}_*/\mathcal{J}_*$.

6.2. Proof of Proposition 3.8

We assume that $\mathscr{A}$ is realized on a Hilbert space $\mathscr{H}$ and $\mathscr{B}$ on another Hilbert space $\mathscr{G}$. Then $\mathscr{A} \otimes \mathscr{B}$ can be identified with the norm closure in $B(\mathscr{H} \otimes \mathscr{G})$ of the algebraic tensor product $\mathscr{A} \otimes_{\mathrm{alg}} \mathscr{B}$ (realized on $\mathscr{H} \otimes \mathscr{G}$). The C^*-norm on $L^1(X; \mathscr{A} \otimes \mathscr{B})$ can be obtained by using the left regular representation on

$$\mathscr{E} \equiv L^2(X; \mathscr{H} \otimes \mathscr{G}) = L^2(X; \mathscr{H}) \otimes \mathscr{G} = \mathscr{H}_X \otimes \mathscr{G}$$

If $S \in L^1(X; \mathscr{A} \otimes \mathscr{B})$ is of the form $S(x) = T(x) \otimes b$ with $T \in L^1(X; \mathscr{A})$ and $b \in \mathscr{B}$, and if $\xi(x) = \eta(x) \otimes k$ with $\xi \in \mathscr{H}_X$ and $k \in \mathscr{G}$, we have

$$(S \bullet \xi)(x) = \int_X \alpha_{-x}[T(x-y) \otimes b] \cdot \eta(y) \otimes k \,\mathrm{d}y = \left(\int_X \alpha_{-x}[T(x-y)]\, \eta(y)\, \mathrm{d}y \right) \otimes bk,$$

hence $S \bullet \xi = (T \bullet \eta) \otimes (bk)$. If we denote by $\pi_{\mathscr{A} \otimes \mathscr{B}}$ and $\pi_{\mathscr{A}}$ the left regular representations of $L^1(X; \mathscr{A} \otimes \mathscr{B})$ and $L^1(X; \mathscr{A})$ respectively, we see that $\pi_{\mathscr{A} \otimes \mathscr{B}} = \pi_{\mathscr{A}} \otimes 1$ in $\mathscr{E} = \mathscr{H}_X \otimes \mathscr{G}$ which proves the Proposition 3.8. ■

6.3. Wave operators

Here we present a general criterion for the existence of wave operators in the case when the laplacian Δ on $X = \mathbb{R}^n$ is perturbed by a (possibly non local) operator V. When V is a differential operator the results of L. Hörmander [Hor] are very precise even when Δ is replaced by an arbitrary differential operator with constant coefficients. Our purpose is to show that for operators $\Delta + V$ one can get very easily results which cover those of

Hörmander: the only aspect that we want to stress is the simplicity of our arguments. Interesting *necessary* conditions on V for the existence of the wave operators have been found by Demuth and Wollenberg in [Dem] and [DeWo].

Theorem 6.2 *Let V be a symmetric operator on $L^2(\mathbb{R}^n)$ whose domain contains $\mathscr{S}(\mathbb{R}^n)$ and such that the operator $\Delta + V$, with domain $\mathscr{S}(\mathbb{R}^n)$, has a self-adjoint extension H. Assume that there is $N < \infty$ such that:*
(i) *The operator $V\langle P\rangle^{-N}\langle Q\rangle^{-N}$ extends to a continuous map $C_0(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*
(ii) *There is a closed cone of measure zero $\Sigma \subset \mathbb{R}^n$ such that for each $x_0 \in \mathbb{R}^n \setminus \Sigma$ there is $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, with $\varphi = 1$ on a neighbourhood of x_0, such that:*

$$\int_1^\infty \|V\langle P\rangle^{-N}\varphi(Q/t)\|_{B(C_0,L^2)}\, t^{-n/2}\,\mathrm{d}t < \infty. \tag{6.1}$$

Then $\operatorname{s-lim}_{t\to\infty} \mathrm{e}^{itH}\mathrm{e}^{-it\Delta}$ *exists on* $L^2(\mathbb{R}^n)$.

Remarks: Our proof shows that one can relax the assumption (i): it suffices to assume that for each $\theta \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ there is $N < \infty$ such that $V\theta(P)\langle Q\rangle^{-N} \in B(C_0(\mathbb{R}^n), L^2(\mathbb{R}^n))$. Also, note that there is an equivalent version of (i) in which the space C_0 is replaced by L^2. Indeed, one can use the estimate

$$\|V\langle P\rangle^{-N}\langle Q\rangle^{-N}\|_{B(C_0,L^2)} \le \|V\langle P\rangle^{-N}\langle Q\rangle^{-N+\alpha}\|_{B(L^2)}\;\|\langle Q\rangle^{-\alpha}\|_{B(C_0,L^2)}$$

with $\alpha > n/2$, while the reciprocal assertion follows from the simplest Sobolev inequality. Finally, (6.1) can be replaced with

$$\int_1^\infty \|V\langle P\rangle^{-N}\varphi(Q/t)\|_{B(L^2)}\,\mathrm{d}t < \infty \tag{6.2}$$

because, if $K = \operatorname{supp}\varphi$ and χ_K is its characteristic function, then

$$\|V\langle P\rangle^{-N}\varphi(Q/t)\|_{B(C_0,L^2)} \le \|V\langle P\rangle^{-N}\varphi(Q/t)\|_{B(L^2)}\;\|\chi_K(Q/t)\|_{B(C_0,L^2)}$$

and the last factor is $Ct^{n/2}$. But (6.2) is strictly stronger than (6.1): if, for example, the support of φ is the region $1 \le |x| \le 2$, then (6.2) is a standard short-range type condition, while (6.1) allows a much more general behavior (as we have seen above in the case $N = 0$).

The simplest application of Theorem 6.2 is to the case when V is a differential operator; note that the order of V is arbitrary. The following is a particular case of the results of Hörmander.

Corollary 6.3 *Let* $V = \sum_{|\alpha|\le m} V_\alpha(x)\partial^\alpha$ *be a differential operator whose coefficients* V_α *are complex Borel functions such that* $\int_{\mathbb{R}^n} |V_\alpha(x)|^2 \langle x\rangle^{-N}\,\mathrm{d}x < \infty$ *for some* N. *Assume that the operator* $\Delta + V$ *with domain* $\mathscr{S}(\mathbb{R}^n)$ *in* $L^2(\mathbb{R}^n)$ *has a self-adjoint extension* H *and that there is an open subset* Ω *of the unit sphere* $S^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$, *with* $S^{n-1}\setminus\Omega$ *of spherical measure zero, and for each* $\omega\in\Omega$ *there is* $\vartheta > 0$ *such that:*

$$\int_1^\infty \Big(\frac{1}{r^n}\int_{C(\omega;\vartheta,r)} |V_\alpha(x)|^2\,\mathrm{d}x\Big)^{\frac12}\,\mathrm{d}r < \infty.$$

Here $C(\omega;\vartheta,r)$ *is the truncated cone consisting of the points* x *such that* $\langle x,\omega\rangle > |x|\cos\vartheta$ *and* $r < |x| < 2r$. *Then* $\operatorname{s-lim}_{t\to\infty} \mathrm{e}^{itH}\mathrm{e}^{-it\Delta}$ *exists on* $L^2(\mathbb{R}^n)$.

In particular, a suficient decay condition on V_α is

$$\int_1^\infty \Big[\frac{1}{r^n}\int_{r<|x|<2r} |V_\alpha(x)|^2\,\mathrm{d}x\Big]^{\frac12}\,\mathrm{d}r < \infty.$$

In the case $m = 0$ the existence of wave operators has been proved by another method by M. Demuth (in 1978) under the condition $\int_1^\infty [r^{-n}\int_{|x|<r}|V_0(x)|^2\,\mathrm{d}x]^{1/2}\,\mathrm{d}r < \infty$. Observe that this implies $V_0 = 0$ if $n = 1, 2$.

For the proof of the Theorem 6.2 we need the next decay estimate (the restrictions we put on a, b are needed only to make the proof completely elementary). By $\widehat{f}$ we denote the Fourier transform of f.

Lemma 6.4 *Let* $a, b \in \mathbb{N}$. *Then there is a constant* C *such that for all* $t \ge 1$ *and* $f \in \mathscr{S}(\mathbb{R}^n)$*:*

$$\|\langle Q\rangle^{2b}\chi_\Omega(Q/(2t))\mathrm{e}^{-it\Delta}f\|_{C_0} \le Ct^{2b-a-n/2}\|\langle Q\rangle^{2(a+b)}\langle P\rangle^{2b}f\|_{L^1}\,, \tag{6.3}$$

where $\Omega = \mathbb{R}^n\setminus\operatorname{supp}\widehat{f}$.

Proof: Note that

$$\|\langle Q\rangle^{2b}\chi_\Omega(Q/(2t))\mathrm{e}^{-it\Delta}f\|_{C_0} = \|\langle 2tQ\rangle^{2b}\chi_\Omega(Q)(\mathrm{e}^{-it\Delta}f)(2t\,\cdot)\|_{C_0}$$

and recall the identity

$$(\mathrm{e}^{-it\Delta}f)(x) = (4\pi it)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \mathrm{e}^{i|x-y|^2/4t}f(y)\,\mathrm{d}y.$$

In particular, we have $(\mathrm{e}^{-it\Delta}f)(2t\,\cdot) = (2it)^{-n/2}\mathrm{e}^{itQ^2}\mathrm{e}^{i\Delta/(4t)}\widehat{f}$. Since $\langle 2tx\rangle \leq 2t\langle x\rangle$ if $2t \geq 1$, we have:

$$\|\langle Q\rangle^{2b}\chi_\Omega(Q/(2t))\mathrm{e}^{-it\Delta}f\|_{C_0} \leq (2t)^{2b-n/2}\|\langle Q\rangle^{2b}\chi_\Omega(Q)\mathrm{e}^{i\Delta/(4t)}\widehat{f}\,\|_{C_0}$$

On the other hand, the Taylor expansion $\mathrm{e}^z = \sum_{k<a}\frac{z^k}{k!} - \frac{z^a}{a!}\int_0^1 \mathrm{e}^{\tau z}\,\mathrm{d}(1-\tau)^a$, valid for each $z \in \mathbb{C}$ and $a \in \mathbb{N}$, furnishes, for each $\lambda \in \mathbb{R}$, the expansion

$$\mathrm{e}^{i\lambda\Delta} = \sum_{k<a}\frac{(i\lambda)^k}{k!}\Delta^k + \frac{(i\lambda)^a}{a!}\Delta^a\int_0^1 \mathrm{e}^{i\tau\lambda\Delta}\,\mathrm{d}(1-\tau)^a.$$

We have $\chi_\Omega(Q)\Delta^k\widehat{f} = 0$ because Δ is a local operator. Hence

$$|\langle Q\rangle^{2b}\chi_\Omega(Q)\mathrm{e}^{i\Delta/(4t)}\widehat{f}\,| = \frac{1}{a!(4t)^a}\left|\int_0^1 \chi_\Omega(Q)\langle Q\rangle^{2b}\mathrm{e}^{i\tau\Delta/(4t)}\Delta^a\widehat{f}\,\mathrm{d}(1-\tau)^a\right|.$$

Thus we obtain

$$\|\langle Q\rangle^{2b}\chi_\Omega(Q/(2t))\mathrm{e}^{-it\Delta}f\|_{C_0} \leq \frac{(2t)^{2b-n/2}}{a!(4t)^a}\sup_{0<\tau<1}\|\langle Q\rangle^{2b}\mathrm{e}^{i\tau\Delta/(4t)}\Delta^a\widehat{f}\,\|_{C_0}.$$

We use now the relations $\mathrm{e}^{-i\lambda\Delta}Q\mathrm{e}^{i\lambda\Delta} = Q - 2\lambda P$ and

$$\|\mathrm{e}^{i\lambda\Delta}g\|_{C_0} = \|\mathcal{F}\mathrm{e}^{i\lambda Q^2}\mathcal{F}^*g\|_{C_0} \leq \|\mathcal{F}^*g\|_{L^1}$$

where $\mathcal{F}$ denotes the Fourier transformation, in order to get

$$\|\langle Q\rangle^{2b}\chi_\Omega(Q/(2t))\mathrm{e}^{-it\Delta}f\|_{C_0} \leq \frac{(2t)^{2b-a-n/2}}{a!2^a}\sup_{0<\tau<1}\|\mathcal{F}^*\langle Q - \frac{\tau}{2t}P\rangle^{2b}P^{2a}\mathcal{F}f\|_{L^1}.$$

This proves the estimate (6.3). ■

Proof of Theorem 6.2: By the Cook criterion, it suffices to prove that $\int_1^\infty \|V\mathrm{e}^{-it\Delta}f\|\,\mathrm{d}t < \infty$ for functions f with a Fourier transform $\widehat{f} \in C_\mathrm{c}^\infty(\mathbb{R}^n\setminus\Sigma)$ (these functions form a dense linear subspace in L^2). By using a convenient partition of unity on $\operatorname{supp}\widehat{f}$ one finds $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^n\setminus\Sigma)$ such that $\varphi(x) = 1$ on $\operatorname{supp}\widehat{f}$ and $\int_1^\infty \|W\varphi(Q/t)\|_{B(C_0,L^2)}\,t^{-n/2}\,\mathrm{d}t < \infty$, where $W = V\langle P\rangle^{-N}$ and N is as in the statement of the theorem. If $g = \langle P\rangle^N f$ then g has properties similar to f and $\operatorname{supp}\widehat{g} = \operatorname{supp}\widehat{f}$, $f = \langle P\rangle^{-N}g$. Set $\phi_t = \varphi(Q/(2t))$, $\phi_t^\perp = 1-\phi$, then

$$\begin{aligned}\|V\mathrm{e}^{-it\Delta}f\| &\leq \|W\phi_t\,\mathrm{e}^{-it\Delta}g\| + \|W\phi_t^\perp\mathrm{e}^{-it\Delta}g\|\\ &\leq \|W\phi_t\|_{B(C_0,L^2)}\|\mathrm{e}^{-it\Delta}g\|_{C_0} + \|W\langle Q\rangle^{-N}\|_{B(C_0,L^2)}\|\langle Q\rangle^N\phi_t^\perp\mathrm{e}^{-it\Delta}g\|_{C_0}.\end{aligned}$$

The first term in the last member is integrable with respect to t on $(1,\infty)$ because of $\|\mathrm{e}^{-it\Delta}g\|_{C_0} \leq (4\pi t)^{-n/2}\|g\|_{L^1}$. The second one is rapidly decaying as $t \to \infty$ by Lemma 6.4, so it is integrable too. This finishes the proof of the theorem. ■

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