Riesz-Kolmogorov Compacity Criterion, Ruelle Theorem and Lorentz Convergence on Locally Compact Abelian Groups

by

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Abstract

We prove a version of Ruelle's theorem (concerning the description of the pure point and continuous spectral subspaces of a hamiltonian in terms of bound and scattering states) valid for an arbitrary self-adjoint operator H in $L^2(X)$, the configuration space X being an arbitrary abelian locally compact group. This version follows from the Riesz-Kolmogorov theorem which, in our presentation, gives a description of relatively compact sets of states solely in terms of phase space properties of the system. We also replace in Ruelle's theorem the convergence in the Cesàro mean by convergence in Lorentz sense, which is sharper than any convergence in the mean.

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1 Introduction

Let H be a self-adjoint operator in the Hilbert space $\mathscr{H}$ and E its spectral measure. The pure point spectral subspace $\mathscr{H}_{\mathrm{p}}$ of H is the closed linear span of the set of eigenvectors of H; this is also the set of those $f \in \mathscr{H}$ for which the measure $\mu_f \equiv \|E(\cdot)f\|^2$ is atomic. The continuous spectral subspace $\mathscr{H}_{\mathrm{c}}$ is defined as $\mathscr{H} \ominus \mathscr{H}_{\mathrm{p}}$, so $\mathscr{H}_{\mathrm{c}} = \{f \in \mathscr{H} \mid \mu_f(\{\lambda\}) = 0, \forall \lambda \in \mathbb{R}\}$. A result of Wiener allows one to give a description of these subspaces in terms of the dynamical group e^{itH} (see Lemmas 4.3 and 4.4 for a slightly improved version). For the case $\mathscr{H} = L^2(\mathbb{R}^n)$ and for a certain class of hamiltonians H, D. Ruelle gave in [Ru] a description of $\mathscr{H}_{\mathrm{p}}$ and $\mathscr{H}_{\mathrm{c}}$ which justifies their interpretation as spaces of bound and scattering states respectively. A more general class of operators for which Ruelle's characterization holds has been pointed out in [AG1], where a completely elementary proof (independent of Wiener's theorem) of Ruelle's result can also be found (see also [RS]).

Our aim in this paper is to improve Ruelle's theorem in two directions. First, we consider arbitrary self-adjoint operators in $L^2(X)$ for X an arbitrary locally compact abelian group. Note that the proof in [AG1] is rather abstract and so holds in more general cases than $X = \mathbb{R}^n$ (e.g. one could assume that X is just a metric space); however, the condition of "local compactness" imposed on H (see §5 below) plays an important role in it. The condition of local compactness relatively to the position observable is, of course, very general, but is not satisfied if one keeps as configuration space that of the free system and the hamiltonian contains a very singular potential (this is related to the question of absorption at singularities, see [Pe] and references therein, and also the remarks on page 519 in [AG2]). If X is a locally compact abelian group we obtain a physically meaningful Ruelle type theorem for an arbitrary self-adjoint operator H by combining Wiener's theorem with the Riesz-Kolmogorov compacity criterion in $L^2(X)$. As a preliminary result, we show in §3 that the compactness of a set of states can be physically understood in terms of the position and momentum observables: this is our version of the Riesz-Kolmogorov compacity criterion, see Theorem 3.4.

The second direction in which we improve Ruelle's theorem concerns the behaviour of quantities of the form $\|\varphi(Q)\mathrm{e}^{itH}f\|$ when $|t| \to \infty$, $f \in \mathscr{H}_{\mathrm{c}}$ and $\varphi \in C_0(X)$. Here we assume the usual local compacity condition on H (see §5 for a general H). It is trivial that this quantity tends to zero if f belongs to the subspace of absolute continuity of H. If μ_f is a singularly continuous measure, then $\|\varphi(Q)\mathrm{e}^{itH}f\|$ does not converge to zero in general (however, in [Si] one can find an example of a Schrödinger operator in one dimension with purely singularly continuous spectrum such that all these quantities tend to zero; it is also shown there that the decay is necessarily slow in such a case). One of the important points of Ruelle's theorem is that $\|\varphi(Q)\mathrm{e}^{itH}f\|$ tends to zero in a generalized sense for all $f \in \mathscr{H}_{\mathrm{c}}$. More precisely, it tends to zero in the mean in the sense of Cesàro:

$$\lim_{T\to+\infty} T^{-1}\int_0^T \|\varphi(Q)\,\mathrm{e}^{isH}f\|\,\mathrm{d}s = 0.$$

We describe this type of convergence in terms of filters. Say that a set $A \subset \mathbb{R}$ is *approximately full* if it contains a Borel set A_0 such that $\lim_{T\to+\infty}|[0,T]\cap A_0|/T = 1$ and $\lim_{T\to+\infty}|[-T,0]\cap A_0|/T = 1$ (the Lebesgue measure of the set B is denoted $|\,B\,|$). The family $\mathcal{A}$ of approximately full sets is easily seen to be a filter on $\mathbb{R}$, finer than the filter of neighbourhoods of infinity (this is the Fréchet filter) and convergence to zero of of bounded positive function is equivalent to its convergence to zero along $\mathcal{A}$ (Lemma 4.1). It is natural to call $\mathcal{A}$ the *Cesàro filter*.

Thus a way of improving the convergence in Cesàro mean is to point out a filter $\mathcal{L}$, finer than the Fréchet filter but coarser than the Cesàro filter, such that $\|\varphi(Q)\mathrm{e}^{itH}f\| \to 0$ when $|t| \to \infty$ along $\mathcal{L}$. We point out a filter $\mathcal{L}$, which we call *Lorentz filter*, which has this property and is, in some sense, optimal: it implies convergence in any translation invariant mean sense (see §4). More precisely, $\mathcal{L}$ consists of the sets A which are *full* in the following sense: there is a Borel set $A_0 \subset A$ such that $\lim_{T\to\infty}\inf_{t\in\mathbb{R}}|A_0\cap[t-T,t+T]|/2T = 1$.

To see the difference between a full and an approximately full set, let the complementary set $A^c = \mathbb{R}\backslash A$ be a disjoint union of intervals $[t_i - T_i, t_i + T_i]$. If A is full, then clearly $\sup_i T_i < \infty$; reciprocally, if $\sup_i T_i < \infty$ and the distance between the intervals grows when $t_i \to \infty$, the set A is full. On the other hand, if for instance $A^c = \cup_{n\in\mathbb{N}}[n^2 - \sqrt{n}, n^2 + \sqrt{n}]$, then A is approximately full.

The relation $\|\varphi(Q)\mathrm{e}^{itH}f\| \to 0$ when $|t| \to \infty$ along $\mathcal{L}$ or $\mathcal{A}$ means that for each $\varepsilon > 0$ the set A of t for which $\|\varphi(Q)\mathrm{e}^{itH}f\| \le \varepsilon$ belongs to $\mathcal{L}$ or to $\mathcal{A}$ respectively. The preceding example shows that in the case of Cesàro convergence this set can be much smaller than in the case of Lorentz convergence.

We consider in this paper quantum mechanical systems which have as configuration space an arbitrary locally compact abelian group. Although such a generality could be of no physical interest, this is the setting where the position and momentum observables are defined in the most natural way. Moreover, the algebraic techniques developed in [GI] are very natural in such a general context. Besides the basic examples $X = \mathbb{R}^n$ or $\mathbb{Z}^n$, the case of finite dimensional vector spaces over local fields (e.g. p-adic numbers) is quite interesting and important, see [S-C, T, W]. Another nice example is that of the p-adic torus: the p-adic integers form a compact totally disconnected non-discrete group so its dual (the p-adic torus) is a locally compact non-compact non-discrete abelian group. Many other nontrivial examples can be found in Chapter 2 of [Gu].

2 Framework and auxiliary results

Let X be an abelian locally compact group (with the operation denoted additively) and let $\mathrm{d}x$ be a Haar measure on it (everything we need concerning the harmonic analysis on such groups can be found in [F], but see also [Lo] and [W]). If $\mathscr{H}$ is a Hilbert space then we denote by $B(\mathscr{H})$ and $K(\mathscr{H})$ the algebras of bounded and compact operators on $\mathscr{H}$ respectively and we abbreviate $\mathbb{B}(X) = B(L^2(X))$, $\mathbb{K}(X) = K(L^2(X))$. In order to avoid ambiguities, if φ is a bounded Borel function on X then we denote $\varphi(Q)$ the operator of multiplication by φ in $L^2(X)$. The symbol Q has not a meaning as operator (as in the standard quantum mechanical situation where X is the additive group associated to a real vector space) but the map $\varphi \mapsto \varphi(Q)$ can be interpreted as the position observable. By using this map we identify the algebra $C_{\mathrm{b}}(X)$ of bounded continuous functions on X with a C^*-subalgebra of $\mathbb{B}(X)$, in particular we always embed $C_0(X) \subset \mathbb{B}(X)$, where $C_0(X)$ is the algebra of continuous functions that tend to zero at infinity. We mention two other notations: $C_{\mathrm{c}}(X)$ and $C_{\mathrm{b}}^{\mathrm{u}}(X)$ are the spaces of bounded continuous functions with compact support or uniformly continuous, respectively.

Let X^* be the set of characters of X (continuous homomorphisms $k : X \to \mathbb{C}$ with $|k(x)| = 1$) equipped with the locally compact group structure defined by the operation of multiplication and the topology of uniform convergence on compact sets. We denote the operation in X^* additively and its neutral element by 0 (as in [W], ch.II, §5; note that this convention looks rather strange if $X = \mathbb{Z}^n$ for example). We shall use the notation $\langle\!\langle x, k \rangle\!\rangle = k(x)$. If X is a real finite dimensional vector space then X^* is identified with the vector space dual to X as follows: let $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ be the canonical bilinear map and take $\langle\!\langle x, k \rangle\!\rangle = \mathrm{e}^{i\langle x,k\rangle}$. This is a standard convention also if the field of real numbers is replaced by the field of p-adic numbers (see page 91 in [F]) or by an arbitrary non-discrete locally compact field (see ch.II, §5 in [W] and also [S-C], [T]).

The Fourier transform of $u \in L^1(X)$ is the function $\mathcal{F}u \equiv \widehat{u} : X^* \to \mathbb{C}$ given by $\widehat{u}(k) = \int_X \overline{\langle\!\langle x, k \rangle\!\rangle} u(x)\,\mathrm{d}x$. Then $\mathcal{F}$ is a linear injective contractive map $L^1(X) \to C_0(X^*)$ and we shall equip X^* with the unique Haar measure $\mathrm{d}k$ such that $\mathcal{F}$ induces a unitary map $\mathcal{F} : L^2(X) \to L^2(X^*)$. From $\mathcal{F}^{-1} = \mathcal{F}^*$ we get $(\mathcal{F}^{-1}v)(x) = \int_{X^*} \langle\!\langle x, k\rangle\!\rangle v(k)\,\mathrm{d}k$ for $v \in L^2(X^*)$. We also recall that the dual group $(X^*)^*$ of X^* is identified with X, each $x \in X$ being seen as a character of X^* through the formula $x(k) = k(x)$, or $\langle\!\langle k, x\rangle\!\rangle = \langle\!\langle x, k\rangle\!\rangle$. Then the Fourier transform of $\psi \in L^1(X^*)$ is given by $\widehat{\psi}\,(x) = (\mathcal{F}^*\psi)(-x)$. For each bounded Borel $\psi : X^* \to \mathbb{C}$ we define the operator $\psi(P) \in \mathbb{B}(X)$ by $\psi(P) = \mathcal{F}^* M_\psi \mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$. As before, the symbol P has not an operatorial meaning, but the morphism $\psi \mapsto \psi(P)$ is interpreted as the momentum observable. In particular, the restriction to $C_0(X^*)$ of this map is injective and gives us an embedding $C_0(X^*) \subset \mathbb{B}(X)$

The following result will be needed in our later arguments.

Lemma 2.1 *Let Λ be a neighbourhood of zero in X and let $\varepsilon > 0$. There is $\psi \in C_{\mathrm{c}}(X^*)$ such that $\widehat{\psi} \geq 0$, $\widehat{\psi} \in L^1(X)$, $\psi(0) = \int_X \widehat{\psi}\,(x)\,\mathrm{d}x = 1$, and $\int_{X\setminus\Lambda} \widehat{\psi}\,(x)\,\mathrm{d}x \leq \varepsilon$.*

Proof: We first construct a function $\theta \in C_{\mathrm{c}}(\Lambda)$ such that $\theta(0) = 1$, $0 \leq \theta \leq 1$ and $\widehat{\theta} \geq 0$, $\widehat{\theta} \in L^1(X^*)$. The technique is standard in the analysis on locally compact groups (see the proof of Lemma 4.19 in [F]). Let Ω be a compact neighbourhood of zero such that $\Omega = -\Omega$ and $\Omega + \Omega \subset \Lambda$. Denote by χ the characteristic function of Ω and let $|\Omega| = \int \chi(x)\,\mathrm{d}x$ be the volume of Ω. We shall take $\theta = |\Omega|^{-1}\chi * \chi$. More explicitly,

$$\theta(x) = \frac{1}{|\Omega|}\int_X \chi(x-y)\chi(y)\,\mathrm{d}y = \frac{|\Omega \cap (x+\Omega)|}{|\Omega|}.$$

The only not obvious fact is $\widehat{\theta} \in L^1(X^*)$. But $|\Omega|\widehat{\theta} = \widehat{\chi}^{\,2}$ and $\widehat{\chi}$ is real, so $\widehat{\theta} \geq 0$, thus

$$\int_{X^*} \widehat{\theta}\,(k)\,\mathrm{d}k = |\Omega|^{-1}\|\widehat{\chi}\,\|^2_{L^2(X^*)} = |\Omega|^{-1}\|\chi\|^2_{L^2(X^*)} = 1$$

by the Plancherel formula.

Now let us prove the existence of ψ. Let Γ be a compact neighbourhood of 0 in X^* such that $-\Gamma = \Gamma$ and set $\psi = |\Gamma|^{-1}\chi * \chi$, where χ is the characteristic function of Γ. As above, one has $\psi \in C_{\mathrm{c}}(X^*)$, $\psi(0) = 1$ and $\widehat{\psi} \geq 0$, $\widehat{\psi} \in L^1(X)$. In particular, $\|\widehat{\psi}\,\|_{L^1(X)} = \int_X \widehat{\psi}\,(k)\,\mathrm{d}k = \psi(0) = 1$. If θ is as above, the Plancherel identity gives

$$\int_X (1-\theta)\widehat{\psi}\,\mathrm{d}x = 1 - \int_X \theta\widehat{\psi}\,\mathrm{d}x = 1 - \int_{X^*} \widehat{\theta}\,\psi\,\mathrm{d}k = \int_{X^*} \widehat{\theta}\,(1-\psi)\,\mathrm{d}k.$$

Since $\widehat{\psi} \geq 0$, $1-\theta \geq 0$ and $1-\theta = 1$ on $X \setminus \Lambda$, we get

$$\int_{X\setminus\Lambda} \widehat{\psi}\, \mathrm{d}x \leq \int_{X^*} \widehat{\theta} \left(1 - \frac{|\Gamma \cap (k+\Gamma)|}{|\Gamma|}\right) \mathrm{d}k.$$

Let $K \subset X^*$ be a compact such that $\int_{X^*\setminus K} \widehat{\theta}\, \mathrm{d}k \leq \varepsilon/2$. Then choose Γ large enough such that $1 - |\Gamma \cap (k+\Gamma)| \cdot |\Gamma|^{-1} \leq \varepsilon/2$ for all $k \in K$. This will give us a function ψ with the property $\int_{X\setminus K} \widehat{\psi}\, \mathrm{d}x \leq \varepsilon$. ■

Let $\{U_x\}_{x\in X}$ and $\{V_k\}_{k\in X^*}$ be the strongly continuous unitary representations of X and X^* in $L^2(X)$ defined by $(U_x f)(y) = f(x+y)$ and $(V_k f)(y) = \langle\langle y, k\rangle\rangle f(y)$ respectively. Note that U_x and V_k satisfy the canonical commutation relations $U_x V_k = \langle\langle x, k\rangle\rangle V_k U_x$. The group C^*-algebra of X is the C^*-subalgebra of $\mathbb{B}(X)$ generated by the convolution operators $\int_X u(z) U_z\, \mathrm{d}z$ with $u \in L^1(X)$, and is canonically isomorphic to $C_0(X^*)$. The isomorphism is determined by the formula $\psi(P) = \int_X \widehat{\psi}(z) U_z\, \mathrm{d}z$ for $\psi \in C_0(X^*)$ such that $\widehat{\psi} \in L^1(X)$. One has a similar identification of the group C^*-algebra of X^* with $C_0(X)$.

If $\mathcal{A}$, $\mathcal{B}$ are linear subspaces of $\mathbb{B}(X)$ we denote $[\![\mathcal{A} \cdot \mathcal{B}]\!]$ the closed linear subspace generated by the operators of the form AB with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proposition 2.2 $\mathbb{K}(X) = [\![C_0(X) \cdot C_0(X^*)]\!]$

Proof: If $\varphi \in C_0(X)$, $\psi \in C_0(X^*)$ and $\widehat{\psi} \in L^1(X)$, then $\varphi(Q)\psi(P) = \int_X \varphi(Q)\widehat{\psi}(P) U_y\, \mathrm{d}y$ is an integral operator with kernel $\varphi(x)\widehat{\psi}(y-x) = \varphi(x)(\mathcal{F}^*\psi)(x-y)$. By an approximation argument it follows that if $\varphi \in L^2(X)$, $\psi \in L^2(X^*)$ then $\varphi(Q)\psi(P) \in [\![C_0(X) \cdot C_0(X^*)]\!]$ and it is a Hilbert-Schmidt operator with norm $\|\varphi\|_{L^2(X)} \cdot \|\psi\|_{L^2(X^*)}$. This implies $[\![C_0(X) \cdot C_0(X^*)]\!] \subset \mathbb{K}(X)$.

Reciprocally, it suffices to show that $[\![C_0(X) \cdot C_0(X^*)]\!]$ contains all the Hilbert-Schmidt operators K. Let $K(\cdot,\cdot) \in L^2(X \times X)$ be the kernel of K. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $L^2(X)$. Clearly the functions $f_{ij}(x,y) = e_i(x)e_j(x-y)$ will furnish an orthonormal basis in $L^2(X \times X)$, hence there is a unique family $\{c_{ij}\}_{i,j\in I}$ of complex numbers with $\sum_{ij} |c_{ij}|^2 < \infty$ such that $K(x,y) = \sum_{ij} c_{ij} f_{ij}(x,y)$ in the sense of convergence in $L^2(X \times X)$. But $f_{ij}(x,y) = e_i(x)(\mathcal{F}^*\widehat{e}_{ij})(x-y)$ so we get $K = \sum_{ij} c_{ij} e_i(Q)\widehat{e}_j(P)$ the series being convergent in norm in the space of Hilbert-Schmidt operators, hence in norm in $B(\mathscr{H})$. ■

We shall also consider the *phase space* of X, which is the locally compact abelian group $\Xi = X \oplus X^*$. If $\xi = (x,k)$ and $\eta = (y,p)$ are elements of Ξ, we set $\beta(\xi,\eta) = \langle\!\langle y,k\rangle\!\rangle \cdot \langle\!\langle -x,p\rangle\!\rangle$. Then β is a *bicharacter* of Ξ, i.e. a continuous map of $\Xi \times \Xi$ into $\{z \in \mathbb{C} \mid |z| = 1\}$ which is a character in each variable when the other one is fixed. Let $\alpha : \Xi \to \mathbb{C}$ be a continuous map such that $\alpha(0) = 1$ and $|\alpha(\xi)| = 1$ for all $\xi \in \Xi$ and let us set

$$W(\xi) = \alpha(\xi) U_{-x} V_k \quad \text{and} \quad \omega(\xi,\eta) = \frac{\alpha(\xi+\eta)}{\langle\!\langle y,k\rangle\!\rangle \alpha(\xi)\alpha(\eta)}.$$

Then W is a strongly continuous map of Ξ into the set of unitary operators on $L^2(X)$ satisfying $W(\xi+\eta) = \omega(\xi,\eta)W(\xi)W(\eta)$. In other terms, W is a projective representation of the group Ξ with phase factor ω (note that $W(\xi)W(\eta) = \beta(\xi,\eta)W(\eta)W(\xi)$). By a well known generalization of the Stone-Von Neumann theorem due to G. Mackey (see Theorem C.38 in [RW]), each such representation of Ξ is unitarily equivalent to a multiple of W (which is irreducible).

The representation W allows one to construct a pseudo-differential calculus which depends on α (different choices give different "quantization" rules). The choice of α does not play any role in this paper.

3 Riesz-Kolmogorov Theorem

Let $\{\mathcal{U}_x\}_{x\in X}$ be a strongly continuous unitary representation of X in a Hilbert space $\mathscr{H}$. There is a unique morphism $\mathcal{U} : C_0(X^*) \to B(\mathscr{H})$ such that $\mathcal{U}(\psi) = \int_X \mathcal{U}_x \widehat{\psi}(x)\,\mathrm{d}x$ if $\widehat{\psi} \in L^1(X^*)$ (Theorem 4.44 from [F] gives more than this).

Proposition 3.1 *Let $\mathcal{K} \subset \mathscr{H}$ be a bounded set. The following assertions are equivalent:*
(i) $\lim_{x\to 0} \sup_{f\in\mathcal{K}} \|(\mathcal{U}_x - 1)f\| = 0$;
(ii) $\forall \varepsilon > 0$, $\exists \psi \in C_c(X^*)$ *such that* $\sup_{f\in\mathcal{K}} \|\mathcal{U}(\psi) - 1)f\| < \varepsilon$.

Proof: Assume that (ii) holds. Then for each $f \in \mathcal{K}$

$$\|(\mathcal{U}_x - 1)f\| \le \|(\mathcal{U}_x - 1)(\mathcal{U}(\psi) - 1)f\| + \|(\mathcal{U}_x - 1)\mathcal{U}(\psi)f\| \le 2\varepsilon + M\|(\mathcal{U}_x - 1)\mathcal{U}(\psi)\|$$

where $M = \sup_{f\in\mathcal{K}} \|f\|$. By the properties of the functional calculus $\mathcal{U}$,

$$\|(\mathcal{U}_x - 1)\mathcal{U}(\psi)\| \le \sup_{k\in X^*} |k(x) - 1| \cdot |\psi(k)| \le C \sup\{|k(x) - 1| \mid k \in \operatorname{supp}\psi\}.$$

This quantity tends to zero as $x \to 0$, $\operatorname{supp}\psi$ being compact. Hence we get $\|(\mathcal{U}_x - 1)f\| < \varepsilon$ for x in some neighbourhood of zero independent of $f \in \mathcal{K}$.

Reciprocally, assume that (i) holds. If $\psi \in C_c(X^*)$ has the properties $\psi(0) = 1$ and $\widehat{\psi} \in L^1(X)$ then $\mathcal{U}(\psi) - 1 = \int_X (\mathcal{U}_x - 1)\widehat{\psi}(x)\,\mathrm{d}x$. This gives

$$\|\mathcal{U}(\psi) - 1\| \leq \int_X \|(\mathcal{U}_x - 1)f\| \cdot |\widehat{\psi}(x)|\,\mathrm{d}x.$$

For each $\varepsilon > 0$ there is a neighbourhood Λ of zero in X such that $\|(\mathcal{U}_x - 1)f\| < \varepsilon/2$ for al $x \in \Lambda$ and $f \in \mathcal{K}$. Then

$$\|(\mathcal{U}(\psi) - 1)f\| \leq \frac{\varepsilon}{2} \int_\Lambda |\widehat{\psi}(x)|\,\mathrm{d}x + 2M \int_{X\setminus\Lambda} |\widehat{\psi}(x)|\,\mathrm{d}x.$$

We finally choose ψ such that $\int_\Lambda |\widehat{\psi}(x)|\,\mathrm{d}x = 1$ and $\int_{X\setminus\Lambda} |\widehat{\psi}(x)|\,\mathrm{d}x \leq \varepsilon/(4M)$, which is possible according to Lemma 2.1. ■

Remark: In the case $X = \mathbb{R}$ the following simpler argument gives a slightly better result. Let A be a self-adjoint operator in $\mathscr{H}$, denote E its spectral measure, and let $\chi(|A| \geq r) = E((-\infty, -r] \cup [r, \infty))$. If $\mu(J) = \|E(J)f\|^2$ and $|t| \leq r^{-3/2}$ then

$$\begin{aligned} \|(\mathrm{e}^{itA} - 1)f\|^2 = 2\int_{\mathbb{R}} (1 - \cos tk)\,\mu(\mathrm{d}k) &\leq t^2 r^2 \|f\|^2 + 2\|\chi(|A| > r)f\|^2 \\ &\leq r^{-1}\|f\|^2 + 2\|\chi(|A| > r)f\|^2. \end{aligned}$$

Reciprocally, we show that for all $f \in \mathscr{H}$, $\varepsilon > 0$ one has the estimate

$$2(1 - \sin 1)\,\|\chi(|A| \geq 1/\varepsilon)f\|^2 \leq \frac{1}{\varepsilon}\int_0^\varepsilon \left\|(\mathrm{e}^{itA} - 1)f\right\|^2 \mathrm{d}t. \tag{3.1}$$

This implies

$$\|\chi(|A| \geq r)f\| \leq 1.78 \sup_{0<t<1/r} \left\|(\mathrm{e}^{itA} - 1)f\right\|,$$

but clearly (3.1) gives a sharper result (only a Cesàro limit is involved). It suffices to prove (3.1) for $\varepsilon = 1$. Then a simple computation gives

$$\int_0^1 \left\|(\mathrm{e}^{itA} - 1)f\right\|^2 \mathrm{d}t = 2\langle f, (1 - \sin A/A)f\rangle$$

and we use $\sin x/x < \sin 1$ if $x > 1$.

Corollary 3.2 *If T is a bounded operator in $\mathscr{H}$ then $\lim_{x\to 0} \|(\mathcal{U}_x - 1)T\| = 0$ if and only if for each $\varepsilon > 0$ there is $\psi \in C_c(X^*)$ such that $\|(\mathcal{U}(\psi) - 1)T\| < \varepsilon$.*

The space $\mathscr{H} = L^2(X)$ is equipped with the representation U_x of X and the representation V_k of X^*. We shall adopt the following abbreviation: if S is an operator on $L^2(X)$ then $S^\perp = 1 - S$. Then, as an application of the last corollary, we get:

Corollary 3.3 *If $T \in \mathbb{B}(X)$ then:*
(i) $\lim_{x\to 0} \|(U_x - 1)T\| = 0$ *if and only if* $\forall\varepsilon > 0\ \exists\psi \in C_c(X^*)$ *such that* $\|\psi(P)^\perp T\| < \varepsilon$.
(ii) $\lim_{k\to 0} \|(V_k - 1)T\| = 0$ *if and only if* $\forall\varepsilon > 0\ \exists\varphi \in C_c(X)$ *such that* $\|\varphi(Q)^\perp T\| < \varepsilon$.

The next result is the Riesz-Kolmogorov compacity criterion in $L^2(X)$:

Theorem 3.4 *Let $\mathcal{K}$ be a bounded subset of $L^2(X)$. Then the following assertions are equivalent:*
(i) *$\mathcal{K}$ is a relatively compact subset of $L^2(X)$*
(ii) $\lim_{x\to 0} \sup_{f\in\mathcal{K}} \|(U_x - 1)f\| = 0$ *and* $\lim_{k\to 0} \sup_{f\in\mathcal{K}} \|(V_k - 1)f\| = 0$.
(iii) $\lim_{\xi\to 0} \sup_{f\in\mathcal{K}} \|(W(\xi) - 1)f\| = 0$.
(iv) $\forall\varepsilon > 0\ \exists\varphi \in C_c(X)\ \exists\psi \in C_c(X^*)$ *such that* $\|\varphi(Q)^\perp f\| + \|\psi(P)^\perp f\| \le \varepsilon$ *for all* $f \in \mathcal{K}$.

Proof: If $\mathcal{K}$ is a relatively compact set then it may be approximated by finite subsets of $L^2(X)$ and since U_x and V_k are C_0-groups, (ii) follows from (i). The relation (ii)$\Leftrightarrow$(iii) follows easily from the definition of W. One has (ii)$\Leftrightarrow$(iv) by Proposition 3.1. Finally, a sufficient condition for $\mathcal{K} \subset L^2(X)$ to be a relatively compact set is to find for each $\varepsilon > 0$ a compact operator T such that $\|Tf - f\| < \varepsilon$ for $f \in \mathcal{K}$. If φ and ψ are as in (iv) then the (easy part of) Proposition 2.2 tells us that $\varphi(Q)\psi(P)$ is a compact operator. Moreover

$$\|\varphi(Q)\psi(P)f - f\| = \|\varphi(Q)\,\psi(P)^\perp f + \varphi(Q)^\perp f\| \le c\|\psi(P)^\perp f\| + c\|\varphi(Q)^\perp f\|$$

which is less then $2c\varepsilon$. Hence (iv) implies (i). ■

Corollary 3.5 *An operator $T \in \mathbb{B}(X)$ is compact if and only if it satisfies one of the following equivalent conditions:*
(i) $\lim_{x\to 0} \|(U_x - 1)T\| = 0$ *and* $\lim_{k\to 0} \|(V_k - 1)T\| = 0$.
(ii) $\lim_{\xi\to 0} \|(W(\xi) - 1)T\| = 0$.
(iii) $\forall\varepsilon > 0\ \exists\varphi \in C_c(X)\ \exists\psi \in C_c(X^*)$ *such that* $\|\varphi(Q)^\perp T\| + \|\psi(P)^\perp T\| < \varepsilon$.

Now we prove a noncommutative version of the Ascoli theorem. We make the following convention: if $T^{(*)}$ appears in a relation, then that relation has to be satisfied both by T and by T^*.

Corollary 3.6 *A bounded subset $\mathscr{K} \subset \mathbb{B}(X)$ is a relatively compact set of compact operators if and only if it satisfies the following equivalent conditions:*
(i) $\lim_{x\to 0} \sup_{T\in\mathscr{K}} \|(U_x - 1)T^{(*)}\| = 0$ *and*
$\lim_{k\to 0} \sup_{T\in\mathscr{K}} \|(V_k - 1)T^{(*)}\| = 0$.
(ii) $\lim_{\xi\to 0} \sup_{T\in\mathscr{K}} \|(W(\xi) - 1)T^{(*)}\| = 0$.
(iii) $\forall\varepsilon > 0\ \exists\varphi \in C_c(X)\ \exists\psi \in C_c(X^*)$ *such that* $\|\varphi(Q)^\perp T^{(*)}\| + \|\psi(P)^\perp T^{(*)}\| < \varepsilon$ *for all* $T \in \mathscr{K}$.

Proof: That (i)⇔(iii) is clear from Corollary 3.3 and (i)⇔(ii) is obvious. The implication (i)⇒(ii) is a consequence of Theorem 3.4 and of the fact that the set of adjoint operators is also compact. Now assume (iii) and let B be the unit ball in $L^2(X)$. Then $\Omega := \bigcup_{T\in\mathscr{K}} T(B)$ and $\Omega^* := \bigcup_{T\in\mathscr{K}} T^*(B)$ are relatively compact sets (use Theorem 3.4), so for each $\varepsilon > 0$ there is a finite rank projection F such that $\|F^\perp f\| \le \varepsilon$ for all $f \in \Omega \cup \Omega^*$. Thus $\|F^\perp T^{(*)}\| \le \varepsilon$, hence $\|T - FTF\| \le 3\varepsilon$, for all $T \in \mathscr{K}$. This proves (i). ∎

Let $\mathbf{H}$ be some Hilbert space and consider the space $L^2(X) \otimes \mathbf{H} = L^2(X;\mathbf{H})$. We shall give a "geometric" description of the algebra $\mathbb{K}(X) \otimes B(\mathbf{H})$, in the spirit of the Riesz-Kolmogorov theorem. We begin with the following remark:

Lemma 3.7 *If $\mathscr{E}, \mathscr{F}$ are Hilbert spaces then*

$$[\![K(\mathscr{E}) \otimes 1 \cdot B(\mathscr{E} \otimes \mathscr{F}) \cdot K(\mathscr{E}) \otimes 1]\!] = K(\mathscr{E}) \otimes B(\mathscr{F}).$$

This statement is trivial if $\mathscr{F}$ is finite dimensional, because then $B(\mathscr{E}\otimes\mathscr{F}) = B(\mathscr{E}) \otimes B(\mathscr{F})$, hence $K(\mathscr{E}) \otimes 1 \cdot B(\mathscr{E} \otimes \mathscr{F}) = K(\mathscr{E}) \otimes B(\mathscr{F}) = K(\mathscr{E} \otimes \mathscr{F})$.

Proof: $K(\mathscr{E}) \otimes B(\mathscr{F})$ is the closed linear subspace of $B(\mathscr{E} \otimes \mathscr{F})$ generated by the operators of the form $K \otimes S$ with $K \in K(\mathscr{E})$, $S \in B(\mathscr{F})$. Hence the inclusion $\supset$ is obvious. Since each compact operator is a norm limit of linear combinations of rank one orthogonal projections $P_u = |u\rangle\langle u|$, for the reverse inclusion it suffices to prove that $(P_u\otimes 1)L(P_v\otimes 1) \in K(\mathscr{E})\otimes B(\mathscr{F})$ if u, v are unit vectors in $\mathscr{E}$. Let J_u be the isometry $f \mapsto u\otimes f$ of $\mathscr{F}$ into $\mathscr{E}\otimes\mathscr{F}$. Then ${J_u}^* : \mathscr{E} \otimes \mathscr{F} \to \mathscr{F}$ sends $e \otimes f$ into $\langle u, e\rangle f$, hence $J_u {J_u}^* = P_u \otimes 1$. So if we set $L_{uv} := {J_u}^* L J_v \in B(\mathscr{F})$ then $(P_u \otimes 1)L(P_v \otimes 1) = J_u L_{uv} {J_v}^* = P_{uv} \otimes L_{uv}$ where P_{uv} is the rank one operator in $\mathscr{E}$ defined by $P_{uv}e = u\langle v, e\rangle$. ∎

Theorem 3.8 *An operator $T \in B(L^2(X) \otimes \mathbf{H})$ belongs to $\mathbb{K}(X) \otimes B(\mathbf{H})$ if and only if one has $\|(W(\xi) - 1)\, T^{(*)}\| \to 0$ when $\xi \to 0$ in Ξ.*

Proof: If T belongs to $\mathbb{K}(X)\otimes B(\mathbf{H})$ then it is a norm limit of linear combinations of tensor products $K\otimes S$ with $K \in \mathbb{K}(X)$, hence $\|(W(\xi)-1)\, T^{(*)}\| \to$

0 because $W(\xi) \to 1$ strongly if $\xi \to 0$. Reciprocally, by using Corollary 3.2 we find, for each $\varepsilon > 0$, functions $\varphi_1, \varphi_2 \in C_0(X)$ and $\psi_1, \psi_2 \in C_0(X^*)$ such that $\|T - \varphi_1(Q)\psi_1(P)T\varphi_2(Q)\psi_2(P)\| < \varepsilon$. Since $\varphi_i(Q)\psi_i(P) = K_i \otimes 1$ with $K_i \in \mathbb{K}(X)$, we get $T \in \mathbb{K}(X) \otimes B(\mathbf{H})$ by Lemma 3.7. ■

Remark: The above theorem can also be stated in a form similar to that of Corollary 3.5. We stress that in Corollary 3.5 the conditions are required to hold only for T, while in Theorem 3.8 both T and T^* appear (of course, if $\mathbf{H}$ is finite dimensional there is no such a problem).

4 Convergence in Lorentz sense

Ruelle's characterization of bound and scattering states involves a rather weak notion of limit at infinity, the abelian limit. We shall briefly discuss it here and we shall point out a stronger type of limit, due to G. C. Lorentz, which allows one to state a version of Ruelle's theorem which is, in some sense, optimal.

The description of convergence in terms of filters appears to be quite useful for our purposes. We will need only the simplest facts from the theory of filters, barely more than their definition (see the first few pages from the chapter on filters in [Bo]). Recall that, if $\mathcal{F}$ is a filter on a set X, Y is a topological space, and $\varphi : X \to Y$, one says that φ has a limit y along $\mathcal{F}$, written $\lim_{\mathcal{F}} \varphi = y$, if $\varphi^{-1}(V) \in \mathcal{F}$ for each neighbourhood V of y.

We call *Fréchet filter* on $\mathbb{R}$ the family of subsets V such that $\mathbb{R} \setminus V$ is relatively compact. The limit along the Fréchet filter is the usual notion of limit at infinity (with $+\infty$ and $-\infty$ identified) and is denoted $\lim_{t\to\infty} \varphi(t)$. We are interested in filters $\mathcal{F}$ that are finer than the Fréchet filter (so that a function convergent at infinity in the standard sense converges in the sense of $\mathcal{F}$ too) and which are translation invariant (i.e. if $A \in \mathcal{F}$ and $a \in \mathbb{R}$ then $a + A \in \mathcal{F}$). Such a filter will be called a *TIF*. The second property assures that a function φ and any of its translations $(\tau_a\varphi)(t) = \varphi(t - a)$ have limits at infinity simultaneously and equal. The main object of this section is to point out that there is a quite remarkable *TIF* $\mathcal{L}$ on $\mathbb{R}$, the *Lorentz filter*.

4.1. We first define the filter on $\mathbb{R}$ which corresponds to the limit in the abelian sense at $+\infty$. Our terminology is suggested by geometric measure theory. We say that a real Borel set Ω has density 1 at $+\infty$ if and only if $\lim_{T\to+\infty} T^{-1}|[0,T] \cap \Omega| = 1$, where $|\cdot|$ denotes the Lebesgue measure. This family of sets is stable under finite intersections. We say that a real set is *approximately full at* $+\infty$ if it contains a Borel set of density 1 at $+\infty$. Let

$\mathcal{A}^+$ be the family of approximately full sets at $+\infty$. Clearly $\mathcal{A}^+$ is a filter finer than the filter of neighbourhoods of $+\infty$ (hence finer than the Fréchet filter) and is stable under translations.

Let Y be a topological space and φ a Y-valued map defined on a neighbourhood of $+\infty$ in $\mathbb{R}$. We say that φ *has an approximate limit at* $+\infty$ if φ converges along the filter $\mathcal{A}^+$; in this case we write $(\mathbf{ap})\text{-}\lim_{t\to+\infty}\varphi(t) = \lim_{\mathcal{A}^+}\varphi$.

The notions of "approximately full at $-\infty$" and $(\mathbf{ap})\text{-}\lim_{t\to-\infty}\varphi(t)$ are defined similarly. If the approximate limits at $+\infty$ and $-\infty$ exist and are equal to y we say that φ has an approximate limit at infinity and we set $y = (\mathbf{ap})\text{-}\lim_{t\to\infty}\varphi(t)$.

Lemma 4.1 *Let $\varphi : [0,+\infty) \to \mathbb{C}$ be a bounded Borel function and $c \in \mathbb{C}$. Then the following assertions are equivalent:*
(i) $(\mathbf{ap})\text{-}\lim_{t\to+\infty}\varphi(t) = c$.
(ii) *There is an approximately full at $+\infty$ set $\Omega \subset \mathbb{R}$ such that the restriction of φ to Ω tends to zero at infinity in the usual sense.*
(iii) $\lim_{T\to+\infty} T^{-1}\int_0^T |\varphi(t) - c| = 0$.

Proof: Replacing, if necessary, φ by $|\varphi - c|$, we may assume $\varphi \geq 0$ and $c = 0$. We show (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) (the implication (ii) $\Rightarrow$ (i) is trivial). Assume (i). If $\varepsilon > 0$ there is a Borel set Ω_ε, approximately full at $+\infty$, such that $\varphi(t) < \varepsilon$ if $x \in \Omega_\varepsilon$. For $T > 0$ we shall have $T^{-1}\int_0^T \varphi(t)\,\mathrm{d}t \leq \varepsilon T^{-1}|\Omega_\varepsilon \cap [0,T]| + T^{-1}\sup_{t>0}\varphi(t)\,|[0,T]\setminus\Omega_\varepsilon|$, which furnishes the estimate $\limsup_{t\to+\infty} T^{-1}\int_0^T \varphi(t)\,\mathrm{d}t \leq \varepsilon$. Since ε is arbitrary, we get (iii). Now we assume (iii) and prove (ii) by using an argument from the proof of Lemma 13, §6.1.2 in [BW]. Let $\varepsilon_n = \sup_{k\geq n} k^{-1}\int_0^k \varphi(t)\,\mathrm{d}t$. Then $\varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq \varepsilon_n \to 0$ and if we set $\Omega = \bigcup_{n=1}^\infty \{t \in [n-1,n] \mid \varphi(t) < \sqrt{\varepsilon_n}\}$ then clearly $\varphi(t) \to 0$ if $t \in \Omega$, $t \to +\infty$. Moreover, Ω is approximately full at $+\infty$ because $N^{-1}|[0,N]\setminus\Omega| \leq (N\sqrt{\varepsilon_N})^{-1}\int_0^N \varphi(t)\,\mathrm{d}t \leq \sqrt{\varepsilon_N} \to 0$ ■

4.2. The notion of almost convergence has been introduced by G. C. Lorentz for the case of sequences and later on extended to general semigroups (see [Pa] and references therein). We shall need a more restricted kind of convergence ("absolute" almost convergence) which does not seem to have a name in the literature, although it was studied in several papers (see [C] and also [Pa] for additional references). Since the main idea first appeared in the work of Lorentz, it seems natural to call it convergence in Lorentz sense. The notion of "full set" that we define below and the presentation in terms of filters does not seem to be standard. Note that we shall identify

$+\infty$ and $-\infty$, although one can consider separately the cases $t \to +\infty$ and $t \to -\infty$.

A real Borel set A will be called *full (at infinity)* if

$$\lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{|A \cap [t-T, t+T]|}{2T} = 1.$$

In other terms, A is full if for each $\nu < 1$ there is a number $T(\nu)$ such that $|A \cap J| \geq \nu |J|$ for each interval J of length $\geq T(\nu)$. Observe also that, if we denote $\xi_T = (2T)^{-1}\chi_{[-T,T]}$, then A is full if and only if $\xi_T * \chi_A \to 1$ in $L^\infty(\mathbb{R})$ when $T \to \infty$ (χ_A is the characteristic function of the set A and $*$ is the convolution product).

For example, assume that the complementary set $A = \mathbb{R}\backslash A$ is a disjoint union of intervals $[t_i - T_i, t_i + T_i]$; if A is full, then one must clearly have $\sup_i T_i < \infty$. Reciprocally, if $\sup_i T_i < \infty$ and the distance between the intervals grows when $t_i \to \infty$, the set A is full.

It is clear that the class of full Borel sets is stable under finite intersections, thus the class of sets which contain a full Borel set is a filter $\mathcal{L}$, finer than the Fréchet filter and stable under translations. We shall call $\mathcal{L}$ the *Lorentz filter* on $\mathbb{R}$ and its elements *full sets.* If Y is a separated topological space and $\varphi : \mathbb{R} \to Y$ has a limit along the filter $\mathcal{L}$ then we say that φ *converges in the Lorentz sense* and we write $\mathfrak{Lim}\,\varphi \equiv \mathfrak{Lim}_{t\to\infty}\varphi(t) = \lim_{\mathcal{L}} \varphi$. If $Y = \mathbb{C}$ then the class of Lorentz convergent functions is an algebra and the map $\varphi \mapsto \mathfrak{Lim}\,\varphi$ is an algebraic morphism such that $\mathfrak{Lim}\,\tau_a\varphi = \mathfrak{Lim}\,\varphi$ for all $a \in \mathbb{R}$. Observe that a real set A is full if and only if $\mathfrak{Lim}\,\chi_A = 1$.

Lemma 4.2 *Let φ be a complex bounded Borel function on $\mathbb{R}$ and $c \in \mathbb{C}$. Then $\mathfrak{Lim}\,\varphi = c$ if and only if $\|\xi_T * |\varphi - c|\|_{L^\infty(\mathbb{R})} \to 0$ as $T \to \infty$, i.e.*

$$\lim_{T\to\infty} \sup_{t\in\mathbb{R}} \frac{1}{2T} \int_{t-T}^{t+T} |\varphi(s) - c| \,\mathrm{d}s = 0. \tag{4.1}$$

Proof: It suffices to prove the lemma under the assumptions $\varphi \geq 0,\ c = 0$. Assume first that (4.1) is satisfied, let $\varepsilon > 0$ and N the set of points t such that $\varphi(t) \geq \varepsilon$. Since $\varepsilon\chi_N \leq \varphi$ we get $\varepsilon\xi_T * \chi_N \leq \xi_T * \varphi$, hence $\|\varepsilon\xi_T * \chi_N\|_{L^\infty} \to 0$ as $T \to \infty$, which is equivalent to $A = \mathbb{R} \setminus N \in \mathcal{L}$. But $\varepsilon > 0$ is arbitrary, so $\mathfrak{Lim}\,\varphi = 0$. Reciprocally, assume $\mathfrak{Lim}\,\varphi = 0$ and let ε, N, A be as above. Then A is full hence $\mathfrak{Lim}\,\chi_A = 1,\ \mathfrak{Lim}\,\chi_N = 0$. If M is an upper bound for φ, we have $\varphi \leq \varepsilon\chi_A + M\chi_N$, hence $\xi_T * \varphi \leq \varepsilon\xi_T * \chi_A + M\xi_T * \chi_N$. Taking $T \to \infty$ we get $\limsup_{T\to\infty} \xi_T * \varphi \leq \varepsilon$. In other terms $\|\xi_T * \varphi\|_{L^\infty} \to 0$. ■

In the case of the real line the notion first considered by Lorentz is the following: a bounded Borel function φ is called *almost convergent* to a number c if $\|\xi_T * \varphi - c\|_{L^\infty} \to 0$ as $T \to \infty$. More explicitly, this means

$$(4.2) \qquad \lim_{T\to\infty} \sup_{t\in\mathbb{R}} \left| \frac{1}{2T} \int_{t-T}^{t+T} \varphi(s)\, \mathrm{d}s - c \right| = 0.$$

From the identity

$$\xi_T * |\varphi - c|^2 = \xi_T * |\varphi|^2 - 2\mathfrak{Re}[\xi_T * \varphi\bar{c}] + |c|^2$$

it follows easily that $\mathfrak{Lim}\,\varphi = c$ *if and only if* φ *is almost convergent to* c *and* $|\varphi|^2$ *is almost convergent to* $|c|^2$.

We introduce one more notation. Assume that $\mathscr{H}$ is a Hilbert space and let $\varphi : \mathbb{R} \to \mathscr{H}$ be a bounded function. If φ is Lorentz convergent when $\mathscr{H}$ is equipped with the weak topology, we say that φ is *weakly Lorentz convergent* and denote the limit by $\text{w-}\mathfrak{Lim}\,\varphi$. Clearly, $\text{w-}\mathfrak{Lim}\,\varphi$ *exists if and only if* $\mathfrak{Lim}_{t\to\infty}\langle\varphi(t), f\rangle$ *exists for each* $f \in \mathscr{H}$ and Lemma 4.2 allows us to express this last fact in integral terms. On the other hand, since each compact operator can be approximated by finite rank operators, it is easy to show that this is also equivalent to the existence of a vector $h \in \mathscr{H}$ such that $\mathfrak{Lim}_{t\to\infty}\|K[\varphi(t) - h]\| = 0$ for each compact operator K on $\mathscr{H}$.

4.3. From our presentation it does not seem that the limit in Lorentz sense is more remarkable then other types of limits. We shall now explain why it is (this subsection is not used in the rest of the paper). We are interested in the various meanings that a symbol like $\lim_{t\to\infty}\varphi(t)$ can have, at least for $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$. As we said before, the simplest idea is to replace the Fréchet filter by a finer one which is also translation invariant, i.e. to consider an arbitrary TIF. Such filters can be obtained with the help of the so-called Banach limits, or invariant means. Recall that a state m on the C^*-algebra $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$ such that $m(\tau_a\varphi) = m(\varphi)\ \forall a \in \mathbb{R}$ is called *invariant.* For such a state we shall also have $m(\mu * \varphi) = m(\varphi)$ for each probability measure μ on $\mathbb{R}$, because $\mu * \varphi = \int \tau_a\varphi\, \mathrm{d}a$ (norm convergent integral). Moreover, m has a unique extension to a state m' on $L^\infty(\mathbb{R})$ such that $m'(\mu * \varphi) = m'(\varphi)$ if the probability μ is absolutely continuous and $\varphi \in L^\infty(\mathbb{R})$ (just define the right hand side by the left hand side for a fixed μ). A state m' on $L^\infty(\mathbb{R})$ which has the preceding property is called a *topological invariant mean* on $\mathbb{R}$; obviously it also has the property $m'(\tau_a\varphi) = m'(\varphi)$ for all $a \in \mathbb{R}$ and all $\varphi \in L^\infty(\mathbb{R})$, hence its restriction to $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$ is an invariant state. Thus we get a canonical bijective map between invariant states on $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$ and topological invariant means on $\mathbb{R}$. From now on we shall not distinguish between m and m': we shall call any one of them a TIM. As usual, if φ is a bounded Borel complex valued function on $\mathbb{R}$ and if $\widetilde{\varphi}$ is the class of φ in $L^\infty(\mathbb{R})$, we set $m(\varphi) = m(\widetilde{\varphi})$. It is easy to see that m is uniquely determined by its restriction to characteristic functions. We set $m(A) = m(\chi_A)$ if A is a measurable subset of $\mathbb{R}$; thus m may also be thought as a translation invariant absolutely continuous *finitely* additive probability on the family of (Lebesgue) measurable subsets of $\mathbb{R}$.

If m is a TIM and φ is a bounded Borel complex valued function on $\mathbb{R}$ then $m(\varphi)$ should be interpreted as a *limit at infinity* of φ. Indeed, denote $\mathcal{K}$ the set of compact subsets K of $\mathbb{R}$ such that $|K| \neq 0$. For each such K we set $\xi_K = |K|^{-1}\chi_{-K}$, so that $\xi_K * \varphi(t) = |K|^{-1}\int_{t+K}\varphi(s)\,\mathrm{d}s$. Let us define $Q_+(\varphi) = \inf_{K\in\mathcal{K}}\limsup_{t\to\infty}\xi_K * \varphi(t)$ and $Q_-(\varphi) = -Q(-\varphi) = \sup_{K\in\mathcal{K}}\liminf_{t\to\infty}\xi_K * \varphi(t)$ for real φ. Then it is easy to see that

$$\liminf_{t\to\infty}\varphi(t) \le Q_-(\varphi) \le m(\varphi) \le Q_+(\varphi) \le \limsup_{t\to\infty}\varphi(t).$$

One important property of the usual notion of limit is not shared by the operation m: it can be shown that a TIM is never a character, hence m is not multiplicative. In order to avoid this problem, we consider the following more restrictive notion (the terminology is not standard). We say that a bounded Borel function φ on $\mathbb{R}$ is *m-convergent* if $m(|\varphi - m(\varphi)|) = 0$ (it is easy to show that this is equivalent to $m(|\varphi|^2) = |m(\varphi)|^2$). If this is the case, then we define $m\text{-}\lim\varphi \equiv m\text{-}\lim_{t\to\infty}\varphi(t) = m(\varphi)$.

We describe now the m-convergence in terms of filters. A subset of $\mathbb{R}$ will be called *m-full* if it contains a measurable set A such that $m(A) = 1$. It is clear that the family $\mathcal{F}_m$ of m-full sets is a filter on $\mathbb{R}$ and that this filter is translation invariant. Moreover, $\mathcal{F}_m$ is finer than the Fréchet filter (if $K \subset \mathbb{R}$ is compact, for each natural n there are points $x_1, \ldots, x_n$ such that the sets $x_i + K$ are pairwise disjoint and of equal m measure, thus $m(K) = 0$). Thus, if m is a TIM, then $\mathcal{F}_m$ is a TIF. It is not difficult to prove that each TIF is included in $\mathcal{F}_m$ for some m. Moreover, it is easy to see that a complex bounded Borel function φ on $\mathbb{R}$ is m-convergent if and only if φ is convergent along the filter $\mathcal{F}_m$. And in this case one has $m\text{-}\lim\varphi = \lim_{\mathcal{F}_m}\varphi$.

The following is the original version of Lorentz's definition: one says that φ is almost convergent if $m(\varphi)$ is independent of the TIM m. The fact that this is equivalent to (4.2) has been proved by Lorentz. This class of functions has the disadvantage of not being an algebra, for reasons which have been explained before. In order to avoid this problem, one considers the next class of functions: φ is called *absolutely almost convergent* if it is m-convergent for each TIM m and $m(\varphi)$ is independent of m. Or, in other terms, if there is $c \in \mathbb{C}$ such that $m(|\varphi - c|) = 0$ for each m. Now it is clear that this is equivalent to Lorentz convergence and that $\mathfrak{Lim}\, \varphi = c$.

4.4. We are interested in the behaviour at infinity of functions of the form $\varphi(t) = \int_{\mathbb{R}} \mathrm{e}^{itk}\mu(\mathrm{d}k)$ with μ an integrable measure on $\mathbb{R}$. We denote $\mu = \widetilde{\varphi}$ and $\mathrm{e}_k(t) = \mathrm{e}^{itk}$. Note that the operator $f \mapsto \xi_T * f$ is just $\sin(TP)/(TP)$ (with the convention $\sin x/x = 1$ if $x = 0$), in particular $\xi_T * \mathrm{e}_k = (\sin Tk/Tk)\mathrm{e}_k$. Since $\varphi = \int \mathrm{e}_k \widetilde{\varphi}\,(\mathrm{d}k)$, we obtain

$$\|\xi_T * \varphi - \widetilde{\varphi}\,(\{0\})\|_{L^\infty} \leq \int_{\mathbb{R}\setminus\{0\}} \|\xi_T * \mathrm{e}_k\|_{L^\infty} |\widetilde{\varphi}\,|(\mathrm{d}k) = \int_{\mathbb{R}\setminus\{0\}} |\sin Tk/Tk| \cdot |\widetilde{\varphi}\,|(\mathrm{d}k)$$

which tends to zero as $T \to \infty$ by the dominated convergence theorem. Thus we proved that φ *is almost convergent to* $\widetilde{\varphi}\,(\{0\})$.

Let us denote $\mathcal{N}$ the closure in $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$ of the set of Fourier transforms of integrable measures on $\mathbb{R}$. This is a unital C^*-algebra stable under translations. Since the space of almost convergent bounded functions is closed in L^∞, we see that the functions from $\mathcal{N}$ are almost convergent; but most of them are not Lorentz convergent, as we will show below. It is obvious that for each $\varphi \in \mathcal{N}$ one has

$$\widetilde{\varphi}\,(\{0\}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(t)\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} \varphi(t)\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} \varphi(t)\,\mathrm{d}t.$$

In particular, the functions from $\mathcal{N}$ have, in the mean, the same behaviour at $+\infty$ and $-\infty$. For example, $\mathcal{N}$ does not contain a function which has different limits at $\pm\infty$.

As we explained before, a function φ which almost converges to c is Lorentz convergent if and only if $|\varphi|^2$ is almost convergent to $|c|^2$. If φ is the Fourier transform (normalized as above) of an integrable measure $\widetilde{\varphi}$, then $|\varphi|^2$ is the Fourier tansform of the measure $\widetilde{\varphi}^{\,\dagger} * \widetilde{\varphi}$, where $\widetilde{\varphi}^{\,\dagger}(A) = \overline{\widetilde{\varphi}\,(-A)}$. Thus $|\varphi|^2$ almost converges to:

$$\widetilde{|\varphi|^2}\,(\{0\}) = \widetilde{\varphi}^{\,\dagger} * \widetilde{\varphi}\,(\{0\}) = \sum_{k\in\mathbb{R}} |\widetilde{\varphi}\,(\{k\})|^2.$$

Hence *the Fourier transform of an integrable measure* $\widetilde{\varphi}$ *is Lorentz convergent if and only if* $\widetilde{\varphi}\,(\{k\}) = 0$ *if* $k \neq 0$. And this is also equivalent to the relation $\|\xi_T * |\varphi - \widetilde{\varphi}\,(\{0\})|\|_{L^\infty} \to 0$ as $T \to \infty$ or, more explicitly:

$$\lim_{T\to\infty} \sup_{t\in\mathbb{R}} \frac{1}{2T}\int_{t-T}^{t+T} |\varphi(s) - \widetilde{\varphi}\,(\{0\})|\,\mathrm{d}s = 0. \tag{4.3}$$

4.5. Let H be a self-adjoint operator in the Hilbert space $\mathscr{H}$ and E its spectral measure. The pure point spectral subspace $\mathscr{H}_{\mathrm{p}}$ of H is the closed linear span of the set of eigenvectors of H; this is also the set of those $f \in \mathscr{H}$ for which the measure $\mu_f \equiv \|E(\cdot)f\|^2$ is atomic. The continuous spectral subspace $\mathscr{H}_{\mathrm{c}}$ is defined as $\mathscr{H} \ominus \mathscr{H}_{\mathrm{p}}$, so $\mathscr{H}_{\mathrm{c}} = \{f \in \mathscr{H} \mid \mu_f(\{\lambda\}) = 0, \forall \lambda \in \mathbb{R}\}$. Let $f \in \mathscr{H}$. For each $g \in \mathscr{H}$ we have:

$$\langle g, \mathrm{e}^{itH} f\rangle = \int_{\mathbb{R}} \mathrm{e}^{itk} \langle g, E(\,\mathrm{d}k) f\rangle,$$

so the remarks from §4.4 imply

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle g, \mathrm{e}^{itH} f\rangle \,\mathrm{d}t \quad = \quad \langle g, E(\{0\}) f\rangle, \tag{4.4}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\langle g, \mathrm{e}^{itH} f\rangle|^2 \,\mathrm{d}t \quad = \quad \sum_{\lambda\in\mathbb{R}} |\langle g, E(\{\lambda\}) f\rangle|^2. \tag{4.5}$$

The second relation is an observation of Wiener. In §4.4 we proved more, namely that the functions $\varphi(t) = \langle g, \mathrm{e}^{itH} f\rangle$ and $|\varphi|^2$ are almost convergent to the r.h.s. of (4.4) and (4.5) respectively. Also, we saw that we have $\mathfrak{Lim}_{t\to\infty} \langle g, \mathrm{e}^{itH} f\rangle = \langle g, E(\{0\}) f\rangle$ if and only if $\langle g, E(\{\lambda\}) f\rangle = 0$ for $\lambda \neq 0$. This furnishes the following dynamic characterization of $\mathscr{H}_{\mathrm{c}}$:

Lemma 4.3 *$f \in \mathscr{H}_{\mathrm{c}}$ if and only if one of the following equivalent relations is satisfied:*
(i) $(\mathbf{ap})$-$\lim_{t\to+\infty} \langle f, \mathrm{e}^{itH} f\rangle = 0$;
(ii) $\exists\, \{t_n\}_{n\in\mathbb{N}}$, $t_n \to +\infty$, *such that* $\mathrm{w}\text{-}\lim_{n\to\infty} \mathrm{e}^{it_n H} f = 0$;
(iii) $\mathrm{w}\text{-}\mathfrak{Lim}_{t\to\infty} \mathrm{e}^{itH} f = 0$;
(iv) *if* $K \in K(\mathscr{H})$ *then* $\mathfrak{Lim}_{t\to\infty} \|K\, \mathrm{e}^{itH} f\| = 0$

Proof: Notice first that the last comment we made in §4.2 furnishes the equivalence between (iii) and (iv). Also, (iii) implies (i) trivially. Now assume that (ii) holds and write $f = f_{\mathrm{p}} + f_{\mathrm{c}}$ with $f_{\mathrm{p}} \in \mathscr{H}_{\mathrm{p}}$ and $f_{\mathrm{c}} \in \mathscr{H}_{\mathrm{c}}$. If g is an eigenvector of H corresponding to the eigenvalue λ, then $\mathrm{e}^{-it\lambda} \langle f_{\mathrm{p}}, g\rangle = \langle \mathrm{e}^{itH} f, g\rangle$ tends to zero when $t = t_n \to \infty$. Thus $\langle f_{\mathrm{p}}, g\rangle = 0$; since g is an arbitrary eigenvector, we get $f_{\mathrm{p}} = 0$, i.e. $f \in \mathscr{H}_{\mathrm{c}}$. Further, if this is the case, then the r.h.s. of (4.5) is zero, which means, by Lemma 4.1, that $(\mathbf{ap})$-w-$\lim_{t\to+\infty} \mathrm{e}^{itH} f = 0$ so (i) is true. Assume now (i). By (4.5) this implies that $\|E(\{\lambda\} f\| = 0$ for all $\lambda \in \mathbb{R}$, which, by the comments we made after (4.5) furnishes (iii). Finally, suppose (ii) false. Then for any $g \in \mathscr{H}$ there is a $c > 0$ such that $|\langle \mathrm{e}^{it_n H} f, g\rangle| > c$, so (iii) cannot hold. ■

We close this subsection with a short proof of a well known result which, together with Theorem 3.4, will give an alternative characterization of the spectral subspaces $\mathscr{H}_{\mathrm{p}}$ and $\mathscr{H}_{\mathrm{c}}$. If $f \in \mathscr{H}$ we shall denote by $[f]^H_{\pm}$, $[f]^H$ the closures of the sets $\{\mathrm{e}^{itH}f \mid t \in \mathbb{R}_{\pm}\}$ and $\{\mathrm{e}^{itH}f \mid t \in \mathbb{R}\}$ respectively.

Lemma 4.4 *If $f \in \mathscr{H}_{\mathrm{p}}$ then $[f]^H$ is a compact subset of $\mathscr{H}$. Reciprocally, if $[f]^H_+$ (or $[f]^H_-$) is compact in $\mathscr{H}$, then $f \in \mathscr{H}_{\mathrm{p}}$.*

Proof: For any $f \in \mathscr{H}_{\mathrm{p}}$ and $\varepsilon > 0$ there are eigenvectors $f_1, \dots, f_n$ corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$ of H such that $\|f - \sum_{j=1}^n c_j f_j\| \le \varepsilon$ for some complex constants $c_1, \dots, c_n$. Hence $\|\mathrm{e}^{itH}f - \sum_{j=1}^n c_j \mathrm{e}^{-it\lambda_j} f_j\| \le \varepsilon$. But the set $\{\sum_{j=1}^n c_j \mathrm{e}^{-it\lambda_j} f_j \mid t \in \mathbb{R}\}$ is bounded in a finite dimensional subspace of $\mathscr{H}$, thus is relatively compact. Hence $[f]^H$ is compact. We show now that if $[f]^H_+$ is compact then $f \in \mathscr{H}_{\mathrm{p}}$. Let $f = f_{\mathrm{p}} + f_{\mathrm{c}}$ be the decomposition of f determined by $\mathscr{H} = \mathscr{H}_{\mathrm{p}} \oplus \mathscr{H}_{\mathrm{c}}$. By Lemma 4.3 there is a sequence $t_n \to \infty$ such that $\text{w-lim}_{n\to\infty} \mathrm{e}^{-it_nH} f_{\mathrm{c}} = 0$. Since $[f]^H_+$ and $[f_{\mathrm{p}}]^H_+$ are compact subsets of $\mathscr{H}$, the set $[f_{\mathrm{c}}]^H_+$ will have the same property, so there is a subsequence $s_m \equiv t_{n_m}$ of $\{t_n\}$ and a vector $f_0 \in \mathscr{H}$ such that $\text{s-lim}_{m\to\infty} \mathrm{e}^{-is_mH} f_{\mathrm{c}} = f_0$. Hence $f_0 = 0$. Then $0 = \|f_0\| = \lim_{m\to\infty} \|\mathrm{e}^{-is_mH} f_{\mathrm{c}}\| = \|f_{\mathrm{c}}\|$, so $f_{\mathrm{c}} = 0$, i.e. $f = f_{\mathrm{p}} \in \mathscr{H}_{\mathrm{p}}$. ■

5 Ruelle's Theorem

In this paragraph we show how the Riesz-Kolmogorov compacity result (Theorem 3.4) allows a complete characterization, in terms of phase-space properties, of the pure point and continuous spectral subspaces $\mathscr{H}_{\mathrm{p}}$ and $\mathscr{H}_{\mathrm{c}}$ corresponding to a self-adjoint operator H in the Hilbert space $\mathscr{H} = L^2(X)$; in particular, this extends Ruelle's characterization in $L^2(\mathbb{R}^n)$.

Theorem 5.1 **(a)** *$f \in \mathscr{H}_{\mathrm{p}}$ if and only if for each $\varepsilon > 0$ there are $\varphi \in C_{\mathrm{c}}(X)$ and $\psi \in C_{\mathrm{c}}(X^*)$ such that*

$$\|\varphi(Q)^{\perp}\mathrm{e}^{itH}f\| + \|\psi(P)^{\perp}\mathrm{e}^{itH}f\| \le \varepsilon \quad \text{for all} \quad t \ge 0. \tag{5.6}$$

(b) *$f \in \mathscr{H}_{\mathrm{c}}$ if and only if for each $\varphi \in C_{\mathrm{c}}(X)$ and $\psi \in C_{\mathrm{c}}(X^*)$ one of the following equivalent conditions is valid:*

$$\textbf{(ap)}\text{-}\lim_{t\to+\infty} \|\psi(P)\varphi(Q)\mathrm{e}^{itH}f\| = 0 \tag{5.7}$$

$$\mathfrak{Lim}_{\,t\to\infty}\|\psi(P)\varphi(Q)\mathrm{e}^{itH}f\| = 0 \tag{5.8}$$

Remark that by (a) and Corollary 3.3 the relation $f \in \mathscr{H}_{\rm p}$ is also equivalent to the condition $\lim_{\xi\to 0}\sup_{t\geq 0}\|(W(\xi)-1)\mathrm{e}^{itH}f\| = 0$.

Proof: Assertion (a) follows immediately from Theorem 3.4 and Lemma 4.4. Now we prove (b). $f \in \mathscr{H}_{\rm c}$ is equivalent to (5.8) because of Lemma 4.3(iv), Proposition 2.2 and the last comment from §4.2. The equivalence with (5.7) follows similarly if one takes into account that (i) from Lemma 4.3 is equivalent to the assertion $(\mathbf{ap})\text{-w-}\lim_{t\to+\infty}\mathrm{e}^{itH}f = 0$. ■

Let us say that Q *is locally H-compact*[(a)] if $\varphi(Q)\phi(H) \in \mathbb{K}(X)$ for all $\varphi \in C_{\rm c}(X)$ and $\phi \in C_{\rm c}(\mathbb{R})$ (hence for all $\varphi \in C_0(X)$ and $\phi \in C_0(\mathbb{R})$). Then $\varphi(Q)[f]^H$ is a relatively compact set for each $\varphi \in C_0(X)$ (indeed, there are $\phi \in C_0(\mathbb{R})$ and $f_0 \in \mathscr{H}$ such that $f = \phi(H)f_0$). We stress that only this fact will play a role in the proof of Corollary 5.2.

The assumption of local H-compactness on Q is a quite general local (in $X-$space) condition on the domain $D(H)$ of H (see [RS]). For example, in the case $X = \mathbb{R}^n$ it suffices that $D(H)$ be included in the local Sobolev space $\mathscr{H}^s_{\rm loc}$ for some $s > 0$. However, the generality of our context makes that in rather simple situations such an assumption cannot be satisfied (e.g. if X is compact and H has not a purely discrete spectrum). But note that in the next result, which is an improvement of Ruelle's theorem, one can replace Q by P (the local H-compactness of P being defined in an obvious way).

Corollary 5.2 *If Q is locally H-compact, then:*
(a) $f \in \mathscr{H}_{\rm p} \iff \forall\varepsilon > 0\ \exists\varphi \in C_{\rm c}(X)$ such that $\|\varphi(Q)^\perp\mathrm{e}^{itH}f\| \leq \varepsilon\ \forall t \geq 0$.
(b) $f \in \mathscr{H}_{\rm c}$ *if and only if* $(\mathbf{ap})\text{-}\lim_{t\to+\infty}\|\varphi(Q)\mathrm{e}^{itH}f\| = 0$ *for all* $\varphi \in C_{\rm c}(X)$ *and also if and only if* $\mathfrak{Lim}_{\,t\to\infty}\|\varphi(Q)\mathrm{e}^{itH}f\| = 0$ *for all* $\varphi \in C_{\rm c}(X)$.

Remark: We saw in §4.1 that the relation $(\mathbf{ap})\text{-}\lim_{t\to+\infty}\|\varphi(Q)\mathrm{e}^{itH}f\| = 0$ is equivalent to $\lim_{T\to+\infty}T^{-1}\int_0^T\|\varphi(Q)\,\mathrm{e}^{isH}f\|^2\,\mathrm{d}s = 0$. On the other hand, $\mathfrak{Lim}_{\,t\to\infty}\|\varphi(Q)\mathrm{e}^{itH}f\| = 0$ is equivalent to the stronger condition

$$\lim_{T\to+\infty}\sup_{t\in\mathbb{R}}\frac{1}{T}\int_t^{t+T}\|\varphi(Q)\mathrm{e}^{isH}f\|^2\,\mathrm{d}s = 0.$$

Proof: To see that if Q is locally H-compact then the relation involving P can be eliminated, observe first that the relatively compact set $\varphi(Q)[f]^H$ can be approximated by a finite set, hence for each $\varepsilon > 0$ there is $\psi \in C_{\rm c}(X^*)$,

[(a)]The next condition is equivalent to the compactness of the operator $\varphi(Q)(H+i)^{-1}$ for all $\varphi \in C_{\rm c}(X)$. For this reason the standard terminology "H is locally compact" seems to us misleading: $\varphi(Q)$ has to be relatively compact with respect to H and not inversely.

$0 \le \psi \le 1$ such that $\|\psi(P)^\perp g\| < \varepsilon$ for all $g \in \varphi(Q)[f]^H$. Thus the first term in the right hand side of the inequality

$$\|\psi(P)^\perp \mathrm{e}^{itH} f\| \le \|\psi(P)^\perp \varphi(Q) \mathrm{e}^{itH} f\| + \|\psi(P)^\perp \varphi(Q)^\perp \mathrm{e}^{itH} f\|$$

is $< \varepsilon$. The second term is made $< \varepsilon$ by making small the first term in (5.6). This gives (a). As for the assertion (b), if Q is locally H-compact and if for $\varepsilon > 0$ we choose ψ as before, we have

$$\|\varphi(Q)\mathrm{e}^{itH} f\| \le \|\psi(P)\varphi(Q)\mathrm{e}^{itH} f\| + \|\psi(P)^\perp \varphi(Q)\mathrm{e}^{itH} f\|.$$

The first term on the right hand side is controlled by (5.7) and (5.8).

The implication $f \in \mathscr{H}_\mathrm{c} \Rightarrow \mathfrak{Lim}_{t\to\infty} \|\varphi(Q)\mathrm{e}^{itH} f\| = 0$ for $\varphi \in C_0(X)$ is also a consequence of Lemma 4.3. Indeed, one can write $f = \phi(H) f_0$ with $\phi \in C_0(\mathbb{R})$ and $\varphi(Q)\phi(H)$ is compact. ■

References

[AG1] W. Amrein and V. Georgescu, *On the characterization of bound states and scattering states in quantum mechanics*, Helv. Phys. Acta **46** (1973), 635-658.

[AG2] W. Amrein and V. Georgescu, *Strong asymptotic completeness of wave operators for highly singular potentials*, Helv. Phys. Acta **47** (1974), 517–533.

[BW] H. Baumgärtel and M. Wollenberg, *Mathematical scattering theory*, Akademie-Verlag, Berlin, 1983.

[Bo] N. Bourbaki, *Eléments de mathématique : topologie générale, Ch. 1-4*, Paris, Hermann, 1971.

[C] C. Chou, *Weakly almost periodic functions and almost convergent functions on a group*, Trans. Amer. Math. Soc. **206** (1975), 175–200.

[F] G. B. Folland, *A Course in abstract harmonic analysis*, CRC Press, 1995.

[GI] V. Georgescu and A. Iftimovici, *C^*-algebras of energy observables: I. General Theory and Bumps Algebras* and *II. Graded symplectic algebras*, available at `http://www.ma.utexas.edu/mp_arc/`.

[Gu] V. P. Gurarii, *Group methods in commutative harmonic analysis* , in Commutative Harmonic Analysis II, eds. V. P. Havin, N. K. Nikolski, Springer, Encyclopedia of Mathematical Sciences, **25**, 1998.

[Lo] L. H. Loomis, *An Introduction to abstract harmonic analysis*, Van Nostrand, 1953.

[Lr] G. C. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[Pa] A. T. Paterson, *Amenability*, Mathematical Surveys and Monographs **29**, American mathematical Society, Providence, Rhode Island 1988.

[Pe] D. B. Pearson, *Quantum scattering and spectral theory*, Techniques of physics **9**, Academic Pres: London, San Diego, Sydney 1988.

[RW] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, American Mathematical Society, 1998.

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics. III: scattering theory*, Academic Press, 1979.

[Ro] D.W. Robinson, *Propagation properties in scattering theory*, J. Austral. Math. Soc. Ser. B **21** (1979/80), no. 4, 474–485.

[Ru] D. Ruelle, *A remark on bound states in potential-scattering theory*, Nuovo Cimento A (10) **61** (1969), 655–662.

[S-C] L. Saloff-Coste, *Opérateurs pseudo-différentiels sur un corps local*, thèse de troisième cycle, Université Pierre et Marie Curie, 1983.

[Si] K. Sinha, *On the absolutely and singularly continuous subspaces in scattering theory*, Ann. Inst. H. Poincaré Sect. A (N.S.) **26** (1977), no. 3, 263–277.

[T] M. H. Taibleson, *Fourier analysis on local fields*, Princeton University Press, 1975.

[W] A. Weil, *Basic number theory*, Springer-Verlag, 1973.