Transplantation of Local Nets and Geometric Modular Action on Robertson–Walker Space–Times

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Dedicated to Sergio Doplicher and John E. Roberts

Abstract

A novel method of transplanting algebras of observables from de Sitter space to a large class of Robertson–Walker space–times is exhibited. It allows one to establish the existence of an abundance of local nets on these spaces which comply with a recently proposed condition of geometric modular action. The corresponding modular symmetry groups appearing in these examples also satisfy a condition of modular stability, which has been suggested as a substitute for the requirement of positivity of the energy in Minkowski space. Moreover, they exemplify the conjecture that the modular symmetry groups are generically larger than the isometry and conformal groups of the underlying space–times.

1 Introduction

An important problem in the theory of quantum fields on general curved space–times is how to choose fundamental reference states which could play a role similar to that of the vacuum state for quantum fields on Minkowski space.[1] Interesting suggestions have been made in the recent past for selecting folia of physically relevant states [7, 16, 19, 25, 26, 32]. These approaches have in common that they do not address the question of how to select a fundamental reference state out of these folia.

In [8] a Condition of Geometric Modular Action (henceforth, CGMA) was introduced in order to give a purely algebraic selection criterion for such physically

[1] This problem was apparently first made explicit in [17].

significant states on arbitrary space–times. For a precise statement and brief discussion of the CGMA, see Section 3. It was shown in [10, 13] in the special cases of four–dimensional Minkowski space and three– and four–dimensional de Sitter space that from a state and net of algebras satisfying the CGMA it is possible to derive the isometry group of the respective space–time, along with a strongly continuous unitary representation of this spacetime symmetry group (for further developments, see [9, 11]). Moreover, the initial state is invariant and the initial net of algebras containing the observables of the theory is covariant under the action of this representation. Hence, the spacetime symmetry group and its action upon the observables of the theory were derived from the observables and state and not posited, as is customarily done. Thus, the CGMA is indeed a distinctive feature of the states of interest in these spaces [2, 29].

The primary purpose of this article is to suggest the CGMA's wider range of applicability by providing examples of nets of algebras on a class of Robertson–Walker space–times supplied with states which together satisfy the CGMA. Equally significant, as we shall explain in Section 3, is the fact that in these examples the modular symmetry group is strictly larger than the isometry groups of the space–times. More specifically, the geometric action of the modular symmetry group does not in general implement point transformations.

The essential step in our analysis is of a purely geometric nature. We shall exhibit maps Ξ from a certain specific family of regions in de Sitter space, called wedges, to a corresponding family in the respective Robertson–Walker space–time. These maps are not induced by point transformations, but nevertheless commute with the operation of taking causal complements; moreover they induce an action of the de Sitter group on their images. This fact will enable us to transplant in a local and covariant manner nets of algebras of observables affiliated with wedges in de Sitter space to corresponding nets in the Robertson–Walker spaces. If the underlying de Sitter theory also complies with the CGMA it follows that the resulting Robertson–Walker theory has the same property. This method of transplanting a local net from one space–time to another is akin to the method of "algebraic holography", by which Rehren has proved the Anti-de Sitter – conformal QFT correspondence [27].

Our paper is organized as follows. In Section 2 we outline the geometric background and exhibit the pertinent properties of the maps Ξ. Using these results, we construct in Section 3 nets of algebras and corresponding states on the specified class of Robertson–Walker space–times and establish their desired modular properties. Finally, in Section 4 we discuss the significance of our results.

2 Geometric Considerations

In this section we exhibit a natural correspondence between certain specific families of causally closed regions (wedges) in a large class of Robertson–Walker space–times. Moreover, we establish some basic properties of this correspondence which enter in our construction in the subsequent section.

Robertson–Walker space–times are Lorentzian warped products of a connected open subset of $\mathbb{R}$ with a Riemannian manifold of constant sectional curvature [1, 20, 24]. We restrict our attention here to the case of positive curvature, which may be assumed to be +1. The corresponding Robertson–Walker space–times are homeomorphic to $\mathbb{R} \times S^3$, and one can choose coordinates so that the metric has the form

$$ds^2 = dt^2 - S^2(t)\, d\sigma^2. \tag{1}$$

Here, t denotes time, $S(t)$ is a strictly positive smooth function and $d\sigma^2$ is the time-independent metric on the unit sphere S^3:

$$d\sigma^2 = d\chi^2 + \sin^2(\chi)\; (d\theta^2 + \sin^2(\theta)\, d\phi^2). \tag{2}$$

The isometry group for such space–times contains a subgroup isomorphic to the rotation group $O(4)$. Indeed, for generic Robertson–Walker spaces, the full isometry group is isomorphic to $O(4)$.

Following [20] we define a rescaled time parameter $\tau(t)$ by

$$d\tau/dt = 1/S(t). \tag{3}$$

In terms of this new variable, the metric takes the form

$$ds^2 = S^2(\tau)\big(d\tau^2 - d\sigma^2\big), \tag{4}$$

where $S(\tau)$ is shorthand for $S(t(\tau))$. Since S is everywhere positive, τ is a continuous, strictly increasing function of t; its range is therefore an open interval in $\mathbb{R}$. In this paper we restrict our attention to those Robertson–Walker space–times with functions $S(t)$ such that the range of values of τ is of the form $(-\alpha\frac{\pi}{2}, \alpha\frac{\pi}{2})$, with $\alpha \leq 1$. We henceforth denote by RW any one of this class of Robertson–Walker space–times.

We mention as an aside that, adopting the standard big bang model of a perfect fluid yields the Robertson–Walker space–times as solutions of Einstein's field equations, which determine the range of values of τ. In particular, if the pressure and cosmological constant equal 0, one has the indicated range of values for τ with $\alpha = 1$; if the pressure is strictly positive, one has the case $\alpha < 1$ [20].

As is well known, the four–dimensional de Sitter space–time dS can be embedded into five–dimensional Minkowski space $\mathbb{R}^5$ as follows:

$$dS = \{x \in \mathbb{R}^5 \mid x_0^2 - x_1^2 - \ldots - x_4^2 = -1\}. \tag{5}$$

This space–time is topologically equivalent to $\mathbb{R} \times S^3$, and in the natural coordinates the metric has the form

$$ds^2 = dt^2 - \cosh^2(t)\, d\sigma^2. \tag{6}$$

We recognize dS as a special case of Robertson–Walker space–time with the choice $S(t) = \cosh(t)$. Once again, we change time variables by defining

$$\tau = \arcsin(\tanh(t)) \tag{7}$$

(so that $d\tau/dt = 1/\cosh t$), which takes values $-\frac{\pi}{2} < \tau < \frac{\pi}{2}$. Then the metric becomes

$$ds^2 = \cosh^2(\tau)\left(d\tau^2 - d\sigma^2\right). \tag{8}$$

The infinite past is at $\tau = -\frac{\pi}{2}$ and the infinite future is at $\tau = \frac{\pi}{2}$. The isometry group of dS is the de Sitter group $O(4,1)$, *i.e.* it coincides with the Lorentz group on $\mathbb{R}^5$.

Comparing equations (4) and (8), it is now clear (and well–known [20]) that each of the Robertson–Walker space–times specified above can be conformally embedded into de Sitter space–time, *i.e.* there exists a global conformal diffeomorphism φ (see Definition 9.16 in [1]) from RW onto a subset of dS. How large the embedding in dS is depends on the range of the variable τ in each case examined, which itself depends upon the function $S(t)$. We will consider RW as a submanifold of dS, equipped with both the de Sitter metric g and the Robertson–Walker metric $\underset{\sim}{g}$, which are conformally equivalent:

$$\underset{\sim}{g} = \Omega^{-2}\, g\,, \text{ where } \Omega(p) = \frac{\cosh(t(p))}{S(t(p))}. \tag{9}$$

We shall now use these conformal embeddings to define "wedges" for these space–times in terms of "wedges" in de Sitter space. A wedge in de Sitter space is the causal completion of the worldline of a freely falling observer. The set of these wedges will be denoted by $\mathcal{W}$. Similarly, we define wedges in RW to be the intersections with RW of those de Sitter wedges whose edges[2] are contained in RW. They correspond to the causal completions of the union of worldlines of freely falling observers in RW.

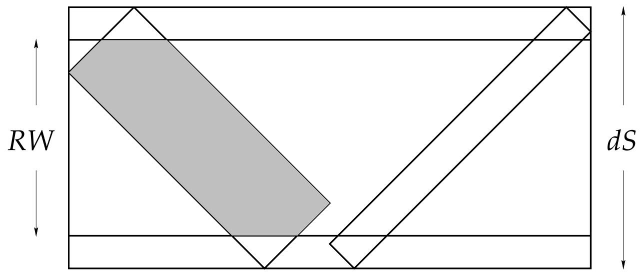

Figure 1: Penrose diagram indicating a wedge in dS (right) and in RW (left).

The set of these Robertson–Walker wedges will be denoted by $\underset{\sim}{\mathcal{W}}$. It inherits useful properties from the family $\mathcal{W}$, which we collect in Lemma 2.1 and which are most easily established with the following alternative characterization of the de Sitter wedges.

Consider the embedding (5) of dS into Minkowski space and let $SO_0(4,1)$ denote the proper orthochronous Lorentz group in five dimensions. Let then $\widetilde{\mathcal{W}}$

[2] See below for a precise definition.

be the family of regions obtained by applying the elements of $SO_0(4,1)$ to a single wedge–shaped region of the form

$$\tilde{W}^{(1)} \doteq \{x \in \mathbb{R}^5 \mid x_1 > |x_0|\}, \tag{10}$$

i.e. this family of regions is $\tilde{\mathcal{W}} \doteq \{\tilde{\gamma}\, \tilde{W}^{(1)} \mid \tilde{\gamma} \in SO_0(4,1)\}$. Then $\mathcal{W}$ is just the collection of intersections $\{\tilde{W} \cap dS \mid \tilde{W} \in \tilde{\mathcal{W}}\}$. There is clearly a one–to–one correspondence between $\mathcal{W}$ and $\tilde{\mathcal{W}}$. For later convenience, we shall therefore denote by $\tilde{W}$ the wedge in $\mathbb{R}^5$ corresponding to a specified $W \in \mathcal{W}$. With this characterization of $\mathcal{W}$, one easily verifies that $\underset{\sim}{\mathcal{W}}$, the set of Robertson–Walker wedges, inherits the following properties from $\mathcal{W}$. These properties have been isolated in [10] as a distinctive feature of wedge regions.

Lemma 2.1 *$\underset{\sim}{\mathcal{W}}$ is stable under the operation of taking causal complements and under the action of the isometry group of rotations $SO(4)$ on RW. Further, the elements of $\underset{\sim}{\mathcal{W}}$ separate spacelike separated points in RW and $\underset{\sim}{\mathcal{W}}$ is a subbase for the topology in RW.*

Note that with the preceding identification of spaces the action on RW of $SO(4)$ is just the restriction to RW of its action on dS.

One obtains a more intrinsic characterization of $\underset{\sim}{\mathcal{W}}$ by noticing that wedges in de Sitter space–time can be characterized by their edges. Let $\tilde{E}^{(1)}$ be the edge of $\tilde{W}^{(1)}$, *i.e.* the three–dimensional subspace $\{x \in \mathbb{R}^5 : x_0 = x_1 = 0\}$. Applying the elements of $SO_0(4,1)$ to $\tilde{E}^{(1)}$, one obtains all three–dimensional spacelike linear subspaces $\tilde{E}$ of $\mathbb{R}^5$. The intersections of these with dS are exactly the two–dimensional, spacelike, totally geodesic, complete, connected submanifolds of dS (in other words, they are just 2-spheres [24, p.105]). These submanifolds will be called *de Sitter edges* and are denoted by E. The causal complement of $\tilde{E}^{(1)}$ has two connected components, one being $\tilde{W}^{(1)}$ and the other one being its causal complement $\tilde{W}^{(1)\prime} \in \tilde{\mathcal{W}}$. Hence also the causal complement of any de Sitter edge has two connected components, each being a wedge, *i.e.* a Lorentz transform of $\tilde{W}^{(1)}$ intersected with dS. So we conclude that the wedges in dS may be characterized as the connected components of the causal complements of de Sitter edges.

Based on this observation, we can give an analogous, intrinsic characterization of wedges in a Robertson–Walker space–time RW after a preparatory lemma. A submanifold F of a semi–Riemannian manifold (M,g) is called *totally umbilic* if there is a vector field Z_F normal to F such that

$$\mathrm{nor}_F \nabla_X Y = g(X,Y)\, Z_F, \tag{11}$$

for all vector fields X, Y tangent to F. In this case, Z_F is called the normal curvature vector field[3] of F. If, in particular, $Z_F = 0$, then F is totally geodesic.

Lemma 2.2 *Let $\underset{\sim}{F}$ be a submanifold of dS. Then $\underset{\sim}{F}$ is a totally geodesic submanifold of (dS, g) contained in RW, if and only if it is a totally umbilic submanifold of $(RW, \underset{\sim}{g})$ with normal curvature vector field*

$$\mathrm{nor}_{\underset{\sim}{F}} \big(\mathrm{grad}(\ln \Omega)\big). \tag{12}$$

[3] Intuitively, F bends away from the normal curvature vector field if F is spacelike.

Here the gradient grad$f$ of a function f denotes the vector field which is metrically equivalent to the differential df.

Proof. Since the metrics $\underset{\sim}{g}$ and g are conformally equivalent as expressed in equation (9), the corresponding connections are related by

$$\underset{\sim}{\nabla}_X Y = \nabla_X Y + \Omega \left((X\Omega^{-1})Y + (Y\Omega^{-1})X - \underset{\sim}{g}(X,Y)\, \mathrm{grad}_{\underset{\sim}{g}}\, \Omega^{-1} \right) \tag{13}$$

(see *e.g.* equation (2.29) in [20]). Now let X, Y be vector fields tangent to $\underset{\sim}{F}$, and denote the normal projections corresponding to $\underset{\sim}{F}$ with respect to $\underset{\sim}{g}$ and g by $\mathrm{nor}_{\underset{\sim}{g}}$ and nor_g, respectively. Then the above equation implies

$$\mathrm{nor}_{\underset{\sim}{g}}\, \underset{\sim}{\nabla}_X Y = \mathrm{nor}_{\underset{\sim}{g}}\, \nabla_X Y + \underset{\sim}{g}(X,Y)\, \mathrm{nor}_{\underset{\sim}{g}}\, \mathrm{grad}_{\underset{\sim}{g}}\, \ln \Omega. \tag{14}$$

Taking into account the fact that $\mathrm{nor}_{\underset{\sim}{g}}\, \nabla_X Y$ vanishes if and only if $\mathrm{nor}_g \nabla_X Y$ does, this proves the claim. $\square$

Obviously, $\underset{\sim}{F}$ is complete and spacelike w.r.t. the metric $\underset{\sim}{g}$ if and only if the same holds for g. Thus the set of de Sitter edges contained in RW coincides with the collection of all two–dimensional, spacelike, totally umbilic, complete, connected submanifolds of $(RW, \underset{\sim}{g})$ with normal curvature vector fields as in (12). These submanifolds will be called *Robertson–Walker edges*, denoted by $\underset{\sim}{E}$. The causal complement of a Robertson–Walker edge $\underset{\sim}{E}$ coincides with the restriction to RW of its causal complement in dS and thus has two connnected components, each being a wedge in RW. Hence, the set of wedges $\underset{\sim}{\mathcal{W}}$ in RW can be intrinsically characterized as the connected components of the causal complements of (the intrinsically defined) Robertson–Walker edges.

We now exhibit a one–to–one map of the set $\mathcal{W}$ of de Sitter wedges onto the set $\underset{\sim}{\mathcal{W}}$ of Robertson–Walker wedges, which will be seen to have properties useful for our "transplantation" of nets performed in the next section. Recall that an element of the class of Robertson–Walker space–times considered here is embedded into dS with a characteristic interval $|\tau| < \alpha\frac{\pi}{2} \leq \frac{\pi}{2}$. If $\alpha = 1$, then the embedded RW coincides with dS. In this case a Robertson–Walker wedge is just a de Sitter wedge and the families $\mathcal{W}$ and $\underset{\sim}{\mathcal{W}}$ are identified by the embedding. To cover the general case $0 < \alpha \leq 1$, we define a diffeomorphism Φ from dS onto RW which bijectively maps the set of de Sitter edges onto the set of Robertson–Walker edges. It is given by

$$\Phi\,(\tau, \chi, \theta, \phi) \doteq (f(\tau), \chi, \theta, \phi), \tag{15}$$

where

$$f(\tau) \doteq \arcsin\left(\sin(\alpha\frac{\pi}{2})\, \sin(\tau)\right). \tag{16}$$

The stated property of Φ as well as its uniqueness are established in the Appendix. The map Φ gives rise to a one–to–one correspondence

$$\Xi : \mathcal{W} \to \underset{\sim}{\mathcal{W}} \tag{17}$$

as follows. Let W be a de Sitter wedge with edge E. The causal complement $\Phi(E)'$ of $\Phi(E)$ in RW has two connected components, exactly one of which has nontrivial intersection with W.

Definition 1 *Let W be a de Sitter wedge with edge E. We define $\Xi(W)$ to be the connected component of $\Phi(E)'$ in RW which has nontrivial intersection with W.*

The map Ξ thus maps the family of de Sitter wedges onto the family of Robertson–Walker wedges. It has the following specific properties which will be a key ingredient in the transplantation of nets in the subsequent section.

Proposition 2.3 *The map $\Xi : \mathcal{W} \to \underset{\sim}{\mathcal{W}}$ is a bijection. It commutes with the operation of taking causal complements in the respective spaces and intertwines the action of the isometry group of rotations $SO(4)$ on $\mathcal{W}$ with its action on $\underset{\sim}{\mathcal{W}}$. If $\alpha < 1$, then Ξ is not induced by a bijective point transformation from dS onto RW,* i.e. *there is no map $p : dS \to RW$ such that for all $W \in \mathcal{W}$*

$$\Xi(W) = \{p(x) \,|\, x \in W\}.$$

The first part of this statement follows from the very construction of Ξ: That Ξ commutes with the operation of taking causal complements is an immediate consequence of the fact that any wedge has the same edge as its causal complement. The intertwining properties of Ξ are due to the fact that the action of $SO(4)$ commutes both with the map Φ and with causal complementation. It is less obvious, however, that Ξ is not induced by a point transformation. This feature originates from the fact that, though the scaling of edges is a diffeomorphism, the subsequent causal complementation is not. A formal proof of the latter statement as well as some further properties of the map Ξ are given below. The reader who wants to skip this quite technical part may proceed at this point directly to the subsequent section.

As a first step in our analysis of Ξ, we give a more explicit formula for its action upon $\mathcal{W}$. Denote by $\tilde{\lambda}_2(t)$ the standard boost in 2-direction of $\mathbb{R}^5$, acting on the 0- and 2- coordinates as

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \tag{18}$$

and by $\lambda_2(\tau)$ the restriction of $\tilde{\lambda}_2(t(\tau))$ to dS.

Lemma 2.4 *Every wedge in dS may be written as*

$$W = \rho\, \lambda_2(\tau)\, W^{(1)}, \tag{19}$$

for some $\rho \in SO(4)$, $\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Similarly, every wedge in RW may be written as the intersection of such a wedge with RW, provided that $|\tau| < \alpha\frac{\pi}{2}$. The bijection $\Xi : \mathcal{W} \to \underset{\sim}{\mathcal{W}}$ acts as

$$\Xi(\rho\, \lambda_2(\tau)\, W^{(1)}) = \rho\, \lambda_2\big(f(\tau)\big)\, W^{(1)} \cap RW. \tag{20}$$

Proof. There is a one–to–one correspondence between wedges W in dS and pairs of lightlike rays in $\mathbb{R}^5$ of the form $\mathbb{R}_+ (\pm 1, \mathbf{e}^{\pm})$, where $\mathbf{e}^{\pm}$ are unit vectors in Euclidean $\mathbb{R}^4$ satisfying $\mathbf{e}^+ \cdot \mathbf{e}^- > -1$. Namely, $\mathbf{e}^+$ and $\mathbf{e}^-$ are the unique unit vectors such that $\tilde{W} + (\pm 1, \mathbf{e}^{\pm}) \subset \tilde{W}$. Conversely, $\tilde{W}$ is the set of all $(x^0, \mathbf{x}) \in \mathbb{R}^5$

satisfying $-\mathbf{x}\cdot\mathbf{e}^- < x^0 < \mathbf{x}\cdot\mathbf{e}^+$. In particular, $\tilde{W}^{(1)}$ corresponds to the rays $\mathbb{R}_+\,(\pm1,1,0,0,0)$. The rays corresponding to $\tilde{\lambda}_2(t)\,\tilde{W}^{(1)}$ are calculated to be

$$\begin{aligned}\tilde{\lambda}_2(t)\,\mathbb{R}_+\,(\pm1,1,0,0,0) &= \mathbb{R}_+\,(\pm1,\mathbf{f}^\pm(t)),\\ \mathbf{f}^\pm(t) &\doteq \big(\cosh(t)^{-1},\pm\tanh(t),0,0\big).\end{aligned} \tag{21}$$

Since $\mathbf{f}^+(t)\cdot\mathbf{f}^-(t) = 2\cosh(t)^{-2}-1$ exhausts all values in $(-1,1]$ for $t\in\mathbb{R}$, one can fix t such that for $\mathbf{e}^\pm$ corresponding to a given wedge W, one has $\mathbf{f}^+(t)\cdot\mathbf{f}^-(t) = \mathbf{e}^+\cdot\mathbf{e}^-$. But then there exists a rotation $\tilde{\rho}\in SO(4)$ satisfying $\tilde{\rho}\,\mathbf{f}^\pm(t)=\mathbf{e}^\pm$. This shows that W is of the form (19). Further, the intersection with RW of a wedge of the form (19) is a Robertson–Walker wedge if and only if its edge is contained in RW. This is the case if and only if $|\tau|<\alpha\frac{\pi}{2}$.

It remains to prove relation (20) for the bijection $\Xi:\mathcal{W}\to\underset{\sim}{\mathcal{W}}$. Denote by $\tilde{f}$ the map which results from f, see equation (16), under the coordinate transformation $\tau\to t$:

$$\tilde{f}(t)\ \doteq \tau^{-1}\big(f(\tau(t))\big)\ =\ \mathrm{arctanh}\big(\sin(\alpha\frac{\pi}{2})\tanh(t)\big). \tag{22}$$

Recall that a point $x\in\mathbb{R}^5$ is in the edge $\tilde{E}^{(1)}$ if and only if it is of the form $x=(0,0,x_2,x_3,x_4)$. For such x, one calculates (see definition (57) and equation (58) in the Appendix)

$$\begin{aligned}\tilde{\Phi}(\tilde{\lambda}_2(t)x) &= \big(\sin(\alpha\tfrac{\pi}{2})\ \sinh(t)\,x_2,0,\cosh(t)\,x_2,x_3,x_4\,\big)\\ &= \tilde{\lambda}_2(\tilde{f}(t))\ \big(0,0,\cosh(t)\ \cosh(\tilde{f}(t))^{-1}\,x_2,x_3,x_4\,\big).\end{aligned} \tag{23}$$

This shows by equation (58) that

$$\Phi(\lambda_2(\tau)\,E^{(1)}) = \lambda_2(f(\tau))\,E^{(1)}, \tag{24}$$

where $E^{(1)}$ denotes the intersection of $\tilde{E}^{(1)}$ with dS. Now, the edge of the wedge W in equation (19) is $E=\rho\,\lambda_2(\tau)\,E^{(1)}$. Since Φ commutes with the rotations, equation (24) implies

$$\Phi(E) = \rho\,\lambda_2(f(\tau))\,E^{(1)}. \tag{25}$$

Obviously, $\rho\,\lambda_2(f(\tau))\,W^{(1)}\cap RW$ is the connected component of the causal complement (in RW) of $\Phi(E)$ which has nontrivial intersection with W. This proves equation (20). $\square$

We now discuss the behavior of intersections of wedges under the map Ξ. Recall that in Minkowski space a double cone is a nonempty intersection of a future lightcone and a past lightcone. This definition makes sense in dS and RW, as well. We consider double cones whose future and past apices differ only in the time coordinate: Let $x=(\tau,\chi,\theta,\phi)\in dS$ and let ε be a positive number such that $|\tau\pm\varepsilon|<\frac{\pi}{2}$. To these data we associate a double cone

$$\mathcal{O}_{x,\varepsilon}\doteq V_+(x_{-\varepsilon})\cap V_-(x_{+\varepsilon})\ ,\ \text{where}\ x_{\pm\varepsilon}\doteq(\tau\pm\varepsilon,\chi,\theta,\phi). \tag{26}$$

Here, $V_\pm(p)$ denotes the future respectively past light cone with apex p. Obviously $\mathcal{O}_{x,\varepsilon}$ is also a double cone in RW if and only if $|\tau\pm\varepsilon|<\alpha\frac{\pi}{2}$, in which case it will be denoted by $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$.

Proposition 2.5 *Let $x \in RW$ be arbitrary, and let ε be a positive number such that $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$ is contained in RW. If $\varepsilon > \frac{\pi}{2}(1-\alpha)$, then the intersection of all wedges W in dS whose corresponding images $\Xi(W)$ in RW contain $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$ is nonempty: it contains the dS double cone $\mathcal{O}_{x,\hat{\varepsilon}}$, where $\hat{\varepsilon} = \frac{\pi}{2} - f^{-1}(\frac{\pi}{2} - \varepsilon)$. If, on the other hand, $\varepsilon < \frac{\pi}{2}(1-\alpha)$, then the above intersection of wedges is empty.*

The last statement implies in particular that, if $\alpha < 1$, there are wedges $\underset{\sim}{W}_i \in \underset{\sim}{\mathcal{W}}$, $i = 1, 2$, with nonempty intersection but with $\Xi^{-1}(\underset{\sim}{W}_i)$ having empty intersection. Hence the map $\Xi : \mathcal{W} \to \underset{\sim}{\mathcal{W}}$ cannot be induced by a bijective point transformation, as claimed in Proposition 2.3.

For the proof of Proposition 2.5 we need some further lemmas.

Lemma 2.6 *Let $\mathcal{O}_{x,\varepsilon}$ be a double cone as in equation (26) with $x = (\tau_0, \frac{\pi}{2}, \frac{\pi}{2}, 0)$. Let further W be a wedge as in equation (19) with $\rho = \rho' \, \rho_{14}(\omega'') \, \rho_{13}(\omega') \, \rho_{12}(\omega)$, where ρ_{1k} denotes a rotation in the 1-k-plane in ambient $\mathbb{R}^5$, $k = 2, 3, 4$, and ρ' is a rotation which leaves the 1-axis fixed. Then W contains $\mathcal{O}_{x,\varepsilon}$ if and only if the following two inequalities hold:*

$$\cos(\omega'') \cos(\omega') \cos(\tau \pm \omega) \geq \cos(\frac{\pi}{2} - \varepsilon \pm \tau_0). \tag{27}$$

Proof. W contains $\mathcal{O}_{x,\varepsilon}$ if and only if the apices of $\lambda_2(-\tau)\rho^{-1}\,\mathcal{O}_{x,\varepsilon}$ are contained in the closure of $W^{(1)}$, *cf.* (19). The 0- and 1-components in ambient $\mathbb{R}^5$ of these two apices are given by

$$\begin{aligned}\left(\lambda_2(-\tau)\rho^{-1}\, x_{\pm\varepsilon}\right)_0 =& \cosh(t)\sinh(t(\tau_0 \pm \varepsilon)) \\ & - b\,\sinh(t)\cosh(t(\tau_0 \pm \varepsilon))\sin(\omega)\,, \\ \left(\lambda_2(-\tau)\rho^{-1}\, x_{\pm\varepsilon}\right)_1 =& b\,\cosh(t(\tau_0 \pm \varepsilon))\cos(\omega),\end{aligned} \tag{28}$$

respectively, where $t \doteq t(\tau)$ and $b \doteq \cos(\omega')\cos(\omega'')$. Hence, taking into account the symmetry $t(-\tau) = -t(\tau)$, the two apices are contained in the closure of $W^{(1)}$ if and only if

$$\tanh(t(\varepsilon \pm \tau_0)) \leq b\left(\cosh(t)^{-1}\cos(\omega) \pm \tanh(t)\sin(\omega)\right). \tag{29}$$

Since relation (7) between t and τ implies $\cosh(t(\tau))^{-1} = \cos(\tau)$, these inequalities are equivalent to

$$\sin(\varepsilon \pm \tau_0) \leq b\,\left(\cos(\tau)\cos(\omega) \pm \sin(\tau)\sin(\omega)\right) = b\,\cos(\tau \mp \omega), \tag{30}$$

which yields the assertions of the lemma. □

Lemma 2.7 *Let W_1 and W_2 be the wedges given by $W_i = \rho_{12}(\omega_i)\,\lambda_2(\tau)\,W^{(1)}$, $i = 1, 2$. If τ and $\omega_- \doteq \frac{1}{2}(\omega_1 - \omega_2)$ satisfy*

$$\cos(\omega_-)\,\cos(\tau + \omega_-) \leq 0, \tag{31}$$

then $W_1 \cap W_2 = \emptyset$.

Proof. As mentioned in the proof of Lemma 2.4, there are unique unit vectors $\mathbf{e}^{\pm} \in \mathbb{R}^4$ corresponding to a wedge $\tilde{W}$ such that $x = (x_0, \mathbf{x}) \in \tilde{W}$ if and only if both inequalities $\pm x_0 < \mathbf{x} \cdot \mathbf{e}^{\pm}$ hold. The unit vectors $\mathbf{e}_i^{\pm}$ corresponding to $\tilde{W}_i$ and hence to W_i are determined by the equation

$$\mathbb{R}_+ \, (\pm 1, \mathbf{e}_i^{\pm}) = \rho_{12}(\omega_i) \, \tilde{\lambda}_2(t(\tau)) \, \mathbb{R}_+ \, (\pm 1, 1, 0, 0, 0) \tag{32}$$

to be

$$\mathbf{e}_i^{\pm} = (\cos(\tau \mp \omega_i) \, , \, \pm \sin(\tau \mp \omega_i) \, , \, 0 \, , \, 0). \tag{33}$$

Let now $x \in W_1 \cap W_2 \subset \tilde{W}_1 \cap \tilde{W}_2$. Then x must satisfy

$$0 < \mathbf{x} \cdot (\mathbf{e}_1^+ + \mathbf{e}_1^- + \mathbf{e}_2^+ + \mathbf{e}_2^-) = 4\cos(\tau)\cos(\omega_-)\big(x_1 \cos(\omega_+) - x_2 \sin(\omega_+)\big) \tag{34}$$

and

$$0 < \mathbf{x} \cdot (\mathbf{e}_1^- + \mathbf{e}_2^+) = 2\cos(\tau + \omega_-)\big(x_1 \cos(\omega_+) - x_2 \sin(\omega_+)\big), \tag{35}$$

where $\omega_+ \doteq \frac{1}{2}(\omega_1 + \omega_2)$. It follows that $\cos(\omega_-)$ and $\cos(\tau + \omega_-)$ are non–zero and have the same sign, since $\cos(\tau) > 0$ by assumption. This contradicts (31). $\square$

We can now prove Proposition 2.5.

Proof of Proposition 2.5. By $SO(4)$ covariance, it suffices to consider the special case $x = (\tau_0, \frac{\pi}{2}, \frac{\pi}{2}, 0)$. First let $\varepsilon > \frac{\pi}{2}(1-\alpha)$ and let $\underset{\sim}{\mathcal{O}}_{x,\varepsilon} \subset \underset{\sim}{W}$, where $\underset{\sim}{W} = \rho\lambda_2(\tau)W^{(1)} \cap RW$, with $|\tau| < \alpha\frac{\pi}{2}$. By Lemma 2.6, this is equivalent to the two conditions for $+$ and $-$, respectively,

$$-\arccos\big(b^{-1}\cos(\varepsilon' \pm \tau_0)\big) \leq \tau \pm \omega \leq \arccos\big(b^{-1}\cos(\varepsilon' \pm \tau_0)\big), \tag{36}$$

where $\varepsilon' \doteq \frac{\pi}{2} - \varepsilon$ and $b \doteq \cos(\omega')\cos(\omega'')$. It must be shown that $W = \Xi^{-1}(\underset{\sim}{W})$ contains $\mathcal{O}_{x,\hat{\varepsilon}}$ with $\hat{\varepsilon} = \frac{\pi}{2} - f^{-1}(\frac{\pi}{2} - \varepsilon)$. According to Lemma 2.6, this is the case if and only if the two above conditions hold with τ replaced by $f^{-1}(\tau)$ and ε' replaced by

$$\hat{\varepsilon}' \doteq \frac{\pi}{2} - \hat{\varepsilon} = f^{-1}(\varepsilon'). \tag{37}$$

First consider the function $h(x) \doteq \arccos(b^{-1}\cos(x))$ for $x \in (0, \pi)$. Then $h(x) - x$ is an increasing function because $h'(x) = \sin(x)\big(b^2 - \cos^2(x)\big)^{-1/2} \geq 1$. Since $\varepsilon' \leq f^{-1}(\varepsilon')$ and $\varepsilon' \leq \frac{\pi}{2}$, this entails the relations

$$h(\varepsilon' \pm \tau_0) - \varepsilon' \leq \; h(f^{-1}(\varepsilon') \pm \tau_0) - f^{-1}(\varepsilon') \tag{38}$$

and

$$h(\varepsilon' + \tau_0) - \varepsilon' \leq \; h(\pi - \varepsilon' + \tau_0) - \pi + \varepsilon'. \tag{39}$$

But $h(x) = \pi - h(\pi - x)$, so the right–hand side of the latter inequality coincides with $-h(\varepsilon' - \tau_0) + \varepsilon'$ and consequently $h(\varepsilon' + \tau_0) + h(\varepsilon' - \tau_0) \leq 2\varepsilon'$. Hence, by adding the inequalities (36) corresponding to "+" to those corresponding to "−", one obtains

$$-\varepsilon' \;\leq\; \tau \;\leq\; \varepsilon'. \tag{40}$$

Recall that $f^{-1}(x) = \arcsin\big(\sin(\alpha\frac{\pi}{2})^{-1}\sin(x)\big)$ with domain $(-\alpha\frac{\pi}{2}, \alpha\frac{\pi}{2})$. This function is odd and has derivative $\cos(x)\big(\sin^2(\alpha\frac{\pi}{2}) - \sin^2(x)\big)^{-1/2} \geq 1$. Hence $f^{-1}(x) - x$ is an increasing function, and (40) implies

$$-f^{-1}(\varepsilon') + \varepsilon' \;\leq\; f^{-1}(\tau) - \tau \;\leq\; f^{-1}(\varepsilon') - \varepsilon'. \tag{41}$$

Combining (41), (38) and the assumption (36) yields

$$-\arccos\big(\frac{1}{a}\cos(f^{-1}(\varepsilon') \pm \tau_0)\big) \;\leq\; f^{-1}(\tau) \pm \omega \;\leq\; \arccos\big(\frac{1}{a}\cos(f^{-1}(\varepsilon') \pm \tau_0)\big), \tag{42}$$

and this shows that W contains $\mathcal{O}_{x,\hat{\varepsilon}}$.

Let now $\varepsilon < \frac{\pi}{2}(1-\alpha)$, *i.e.* $\varepsilon' = \frac{\pi}{2} - \varepsilon > \alpha\frac{\pi}{2}$. The goal is to exhibit wedges W_1, W_2 with empty intersection but satisfying $\underset{\sim}{W}_1 \cap \underset{\sim}{W}_2 \supset \underset{\sim}{\mathcal{O}}_{x,\varepsilon}$. To this end, let for $i = 1, 2$

$$\underset{\sim}{W}_i(\delta) \doteq \rho_{12}(\omega_{i,\delta})\,\lambda_2(\tau_\delta)\;W^{(1)} \cap RW, \tag{43}$$

where $\delta \in (\,0\,,\,\alpha\frac{\pi}{2} - |\tau_0|\,)$ and

$$\tau_\delta = \alpha\frac{\pi}{2} - \delta\;, \quad \omega_{1,\delta} = \tau_0 + \varepsilon' - \tau_\delta\;, \;\text{ and }\; \omega_{2,\delta} = \tau_0 - \varepsilon' + \tau_\delta. \tag{44}$$

For δ in the specified range, Lemma 2.6 asserts that $\underset{\sim}{W}_1(\delta) \cap \underset{\sim}{W}_2(\delta) \supset \underset{\sim}{\mathcal{O}}_{x,\varepsilon}$. On the other hand, the $W_i(\delta)$ are given by $\rho_{12}(\omega_{i,\delta})\,\lambda_2(f^{-1}(\tau_\delta))\;W^{(1)}$, cf. Lemma 2.4. Now for all admissible δ, one has

$$0 \;<\; \frac{1}{2}(\omega_{1,\delta} - \omega_{2,\delta}) = \varepsilon' - \alpha\frac{\pi}{2} + \delta \;<\; \varepsilon' \;<\; \frac{\pi}{2}. \tag{45}$$

Further, the expression $f^{-1}(\tau_\delta) + \frac{1}{2}(\omega_{1,\delta} - \omega_{2,\delta})$ is continuous in δ and approaches the value $\frac{\pi}{2} + \varepsilon' - \alpha\frac{\pi}{2} > \frac{\pi}{2}$ if δ tends to zero. Hence for some $\delta_0 > 0$ this expression is greater $\frac{\pi}{2}$. But then Lemma 2.7 entails $W_1(\delta_0) \cap W_2(\delta_0) = \emptyset$. $\square$

The action of the proper de Sitter group $SO(4,1)$ on dS induces an action on $\mathcal{W}$. Via Ξ, one has then an action of $SO(4,1)$ on the set of Robertson–Walker wedges $\underset{\sim}{\mathcal{W}}$ given by

$$\underset{\sim}{\gamma}\;\underset{\sim}{W} \doteq (\Xi \circ \gamma \circ \Xi^{-1})(\underset{\sim}{W}). \tag{46}$$

We finally discuss the question of whether this action is induced by point transformations on RW, *i.e.* if for $\gamma \in SO(4,1)$ there exists a map $p_\gamma : RW \to RW$ such that

$$\underset{\sim}{\gamma}\;\underset{\sim}{W} = \{p_\gamma(x) \,|\, x \in \underset{\sim}{W}\}. \tag{47}$$

Proposition 2.8 *For the subgroup $SO(4)$ of rotations the action* (46) *on $\underset{\sim}{\mathcal{W}}$ is induced by its action on RW as (isometric) point transformations. However if $\alpha < 1$, then there are elements in $SO(4,1)$ which are not induced by a point transformation in the sense of equation* (47).

Proof. The first statement has been established already in Proposition 2.3, while the latter one is a consequence of the subsequent Lemma 2.9. $\square$

Note that $|f(\tau)| < |\tau|$ if $\alpha < 1$. Hence, in this case it is possible to find $\omega \in (0, \frac{\pi}{2})$ and $\tau > 0$ satisfying

$$f(\tau) + \omega < \frac{\pi}{2}\ ,\ \tau + \omega > \frac{\pi}{2}. \tag{48}$$

For such ω and τ, $\cos(\omega)$ is smaller than $\cos(\frac{\pi}{2} - \tau) = \sin(\tau)$, so that τ_0 is well defined by

$$\sin(\tau_0) = -\cos(\omega)\sin(\tau)^{-1}. \tag{49}$$

Lemma 2.9 *Let $\alpha < 1$ and let $W_\pm \doteq \rho_{12}(\pm\omega)\,\lambda_2(\tau)\,W^{(1)}$, where the parameters $\omega \in (0, \frac{\pi}{2})$, $\tau > 0$ satisfy* (48). *Consider the de Sitter isometry $\gamma_0 \doteq \lambda_2(\tau_0)$, with τ_0 as above. Then $\underset{\sim}{W}_+ \cap \underset{\sim}{W}_- \neq \emptyset$, but $\underset{\sim}{\gamma_0}\,\underset{\sim}{W}_+ \cap \underset{\sim}{\gamma_0}\,\underset{\sim}{W}_- = \emptyset$.*

Proof. Let ω and τ satisfy (48). Then Lemma 2.7 entails $W_+ \cap W_- = \emptyset$. Moreover, the intersection of the wedges $\underset{\sim}{W}_\pm = \rho_{12}(\pm\omega)\,\lambda_2(f(\tau))\,W^{(1)} \cap RW$ is non–empty, because by Lemma 2.6 it contains a double cone $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$, where $x = (0, \frac{\pi}{2}, \frac{\pi}{2}, 0)$ and $\varepsilon > 0$. For the proof of the second part of the statement we proceed to ambient Minkowski space and denote by $\tilde{\gamma}$ the Lorentz transformation corresponding to a given de Sitter transformation γ. Since $\tilde{\lambda}_2(\cdot)$ and $\tilde{\rho}_{12}(\cdot)$ act only on the x_0, x_1 and x_2-coordinates of $\tilde{W}^{(1)}$, it follows from the argument in Lemma 2.4 that there are ω', τ' such that

$$\tilde{\gamma}_0\,\tilde{W}_+ = \tilde{\rho}_{12}(\omega')\,\tilde{\lambda}_2(\tau')\,\tilde{W}^{(1)}. \tag{50}$$

Further, a Lorentz transformation acting only on x_0, x_1 and x_2 leaves $\tilde{W}^{(1)}$ invariant if and only if it leaves the unit vector in the x_2-direction e_2 invariant. Hence, ω' and τ' satisfy the above equation if and only if they satisfy the condition

$$\tilde{\lambda}_2(\tau_0)\,\tilde{\rho}_{12}(\omega)\,\tilde{\lambda}_2(\tau)\,e_2 = \tilde{\rho}_{12}(\omega')\,\tilde{\lambda}_2(\tau')\,e_2, \tag{51}$$

which implies

$$\cot(\omega') = \big(\sin(\tau_0)\sin(\tau) + \cos(\omega)\big)\big(\sin(\omega)\cos(\tau_0)\big)^{-1}. \tag{52}$$

The reflection $\tilde{j}_2$ about the edge $\{x \in \mathbb{R}^5 : x_0 = x_2 = 0\}$ commutes with $\tilde{\lambda}_2(\cdot)$, satisfies $\tilde{j}_2\,\tilde{\rho}_{12}(\omega) = \tilde{\rho}_{12}(-\omega)\,\tilde{j}_2$ and maps $\tilde{W}^{(1)}$ onto itself. Hence, applying $\tilde{j}_2$ to equation (50), it follows that $\tilde{\gamma}_0\,\tilde{W}_- = \tilde{\rho}_{12}(-\omega')\,\tilde{\lambda}_2(\tau')\,\tilde{W}^{(1)}$. Combining the preceding facts one gets $\underset{\sim}{\gamma_0}\,\underset{\sim}{W}_\pm = \rho_{12}(\pm\omega')\,\lambda_2(f(\tau'))\,W^{(1)} \cap RW$. But $\cos(\omega') = 0$ for τ_0 given by equation (49), hence $\underset{\sim}{\gamma_0}\,\underset{\sim}{W}_+ \cap \underset{\sim}{\gamma_0}\,\underset{\sim}{W}_- = \emptyset$ by Lemma 2.7. $\square$

This completes our discussion of the properties of the map Ξ.

3 Transplantation of Nets and States

We now turn to the construction of theories on Robertson–Walker space–times from theories on de Sitter space. Thus we commence with a net of algebras on dS and a corresponding state and define an associated net and state on RW. We shall see that this *transplantation* of net and state preserves the properties of causality and $SO(4)$ covariance. Moreover, the transplanted theory satisfies the CGMA and a Modular Stability Condition if the original theory does. We shall also investigate under which conditions the action of the modular symmetry group upon the transplanted net is induced by point transformations on the corresponding space–time.

For the convenience of the reader we recall the general formulation of the CGMA and the Modular Stability Condition. Let M be a space–time and $\mathcal{W}_M$ be a suitable collection of open subsets of M. Let further $\{\mathcal{A}(W)\}_{W\in\mathcal{W}_M}$ be a net of C^*–algebras indexed by $\mathcal{W}_M$, each of which is a subalgebra of a C^*–algebra $\mathcal{A}$. A state on $\mathcal{A}$ will be denoted by ω and the corresponding GNS representation of $\mathcal{A}$ will be signified by $(\mathcal{H},\pi,\Omega)$; ω will also be said to be a state on the net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}_M}$. For each $W\in\mathcal{W}_M$ the von Neumann algebra $\pi(\mathcal{A}(W))''$ will be denoted by $\mathcal{R}(W)$. The modular involution associated by Tomita–Takesaki theory [4] to the pair $(\mathcal{R}(W),\Omega)$ — whenever Ω is cyclic and separating for $\mathcal{R}(W)$ — will be represented by J_W, while the unitary modular group associated to the same pair will be written as $\{\Delta_W^{it}\}_{t\in\mathbb{R}}$. We emphasize that these modular objects are uniquely determined by the algebraic data.

Definition 2 (Condition of Geometric Modular Action) *Let $\mathcal{W}_M$ be a suitable collection of open subsets of the space–time M, let $\{\mathcal{A}(W)\}_{W\in\mathcal{W}_M}$ be a net of C^*–algebras indexed by $\mathcal{W}_M$, and let ω be a state on $\{\mathcal{A}(W)\}_{W\in\mathcal{W}_M}$. The CGMA is fulfilled if the corresponding net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}_M}$ satisfies*

(i) $W\mapsto\mathcal{R}(W)$ is an order preserving bijection

(ii) Ω is cyclic and separating for $\mathcal{R}(W)$, for each $W\in\mathcal{W}_M$

(iii) the adjoint action of J_{W_0} leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}_M}$ invariant, for each $W_0\in\mathcal{W}_M$.

See [10] for a motivation for and consequences of this condition and also for a discussion of the meaning of "suitable collection". For the class of Robertson–Walker space–times considered here and for de Sitter space–time, the respective wedge regions $\mathcal{W},\underset{\sim}{\mathcal{W}}$ are suitable in this sense, *cf.* Lemma 2.1.

Given a net and state satisfying the CGMA, one is faced with the physically important question of the stability of the state. For Minkowski space theories this stability is usually characterized by the relativistic spectrum condition, *i.e.* the joint spectrum of the generators of the representation of the translation subgroup is contained in the closed forward light cone [18,28]. However, such a subgroup is not to be found in the isometry group of most space–times of physical interest. For this reason, work has been invested in finding a replacement for the relativistic spectrum condition as a stability condition valid for general space–times. We mention, in particular, the interesting microlocal spectrum condition [7,25,26,31]. In [10], an alternative has been suggested which relies on the modular objects.

Definition 3 (Modular Stability Condition) *The modular unitaries are contained in the group $\mathcal{J}$ generated by the modular involutions J_W, $W \in \mathcal{W}_M$,* i.e. $\Delta_W^{it} \in \mathcal{J}$*, for all $t \in \mathbb{R}$ and $W \in \mathcal{W}_M$.*

This condition is formulated without any reference to an underlying space–time; hence, it may be applied in principle to models on any space–time. Though it is certainly not clear *prima facie* that the Modular Stability Condition has anything to do with physical stability, it was shown in [10] that in Minkowski space theories, the CGMA and the Modular Stability Condition *entail* the relativistic spectrum condition.

In the present context it is of interest that there exist many models on de Sitter space satisfying the CGMA and the Modular Stability Condition. We recall that these properties follow, within de Sitter covariant theories, from a characterization of vacuum states via a KMS condition [3], *cf.* also [5]. The latter condition is satisfied, for example, by the (generalized) free field models constructed on de Sitter space in [6] and [23]. Moreover, as proposed in [14], a suitable theory on ambient Minkowski space can be restricted to dS in a specific way such that the resulting theory has, by the Bisognano–Wichmann theorem [2], the above properties.

We now start with such a de Sitter theory: Let $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ and ω be the corresponding net and state, *i.e.* we make the choice $\mathcal{W}_M = \mathcal{W}$, the set of de Sitter wedges. The group $\mathcal{J}$ generated by the modular involutions then provides a continuous (anti)unitary representation of the proper de Sitter group $SO(4,1)$, under which the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is covariant [3]. Moreover, the net satisfies wedge duality and thus is local. We proceed from the given net on dS to a corresponding net $\{\underset{\sim}{\mathcal{A}}(\underset{\sim}{W})\}_{\underset{\sim}{W}\in\underset{\sim}{\mathcal{W}}}$ on RW by *transplantation*, putting

$$\underset{\sim}{\mathcal{A}}(\underset{\sim}{W}) \doteq \mathcal{A}(W), \quad \underset{\sim}{W} = \Xi(W), \tag{53}$$

where $\Xi : \mathcal{W} \to \underset{\sim}{\mathcal{W}}$ is the bijection defined in the preceding section. In addition, we proceed from ω to a corresponding state $\underset{\sim}{\omega}$ on $\{\underset{\sim}{\mathcal{A}}(\underset{\sim}{W})\}_{\underset{\sim}{W}\in\underset{\sim}{\mathcal{W}}}$ by

$$\underset{\sim}{\omega}(A) \doteq \omega(A) \quad \text{for all} \quad A \in \underset{\sim}{\mathcal{A}}(\underset{\sim}{W}),\ \underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}. \tag{54}$$

We thus obtain a net $\{\underset{\sim}{\mathcal{A}}(\underset{\sim}{W})\}_{\underset{\sim}{W}\in\underset{\sim}{\mathcal{W}}}$ and state $\underset{\sim}{\omega}$ on RW, which coincide in the indicated manner with the net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ and ω, but which now are re-interpreted in terms of Robertson–Walker space–time. The physical significance of the operators and state thereby changes.

The modular symmetry group $\mathcal{J}$ induces an action of the de Sitter group $SO(4,1)$ on the Robertson-Walker net, as one easily verifies. More specifically, for any $\gamma \in SO(4,1)$ there is a unique $J \in \mathcal{J}$ such that

$$J\underset{\sim}{\mathcal{R}}(\underset{\sim}{W})J^{-1} = \underset{\sim}{\mathcal{R}}(\underset{\sim}{\gamma}\,\underset{\sim}{W}) \quad \text{for all} \quad \underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}, \tag{55}$$

where $\underset{\sim}{\gamma}\,\underset{\sim}{W} = (\Xi \circ \gamma \circ \Xi^{-1})(\underset{\sim}{W})$ is the action defined in equation (46). It was shown in Proposition 2.8 that this action on $\underset{\sim}{\mathcal{W}}$ is induced by an isometry of RW

if $\gamma \in SO(4)$, but, for arbitrary $\gamma \in SO(4,1)$, need not even arise from a point transformation if $\alpha < 1$.

Theorem 3.1 *With the above definitions, the net* $\{\underset{\sim}{\mathcal{A}}(\underset{\sim}{W})\}_{\underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}}$ *and state* $\underset{\sim}{\omega}$ *satisfy the CGMA and the Modular Stability Condition. The corresponding net of von Neumann algebras* $\{\underset{\sim}{\mathcal{R}}(\underset{\sim}{W})\}_{\underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}}$ *satisfies wedge duality and transforms, in the sense of equation* (55)*, covariantly under* $SO(4,1)$ *under the adjoint action of* $\mathcal{J}$*. In particular, the group* $SO(4)$ *of isometries of* RW *is implemented by a subgroup of* $\mathcal{J}$*. On the other hand, for temporal scaling factor* $\alpha < 1$*, there exist also* $J \in \mathcal{J}$ *which do not implement point transformations of* RW*.*

Proof. The claimed properties follow from the corresponding properties of the underlying de Sitter net, taking into account the specific features of Ξ established in Section 2. □

It is of interest at this point to consider the special case of Robertson–Walker spaces for which the function S is such that the temporal scaling parameter $\alpha = 1$. In this case, the conformal diffeomorphism $\varphi : RW \to dS$ is onto, thus establishing a conformal equivalence of RW and dS. Hence the conformal groups of the two space–times are isomorphic. But the conformal group of dS coincides with its isometry group, the de Sitter group $O(4,1)$. This follows from the characterization in [21] of the de Sitter group as the bijections of dS which preserve separation zero. Hence, the conformal group of RW is isomorphic to the de Sitter group, which acts on RW via the conformal embedding, *i.e.* $\gamma \in 0(4,1)$ acts on RW as $\varphi^{-1} \circ \gamma \circ \varphi$. Taking into account that the map Ξ is induced by φ in these cases, we arrive at the following statement.

Corollary 3.2 *With the above definitions and for Robertson–Walker space–times* RW *with temporal scaling factor* $\alpha = 1$*, the net* $\{\underset{\sim}{\mathcal{R}}(\underset{\sim}{W})\}_{\underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}}$ *satisfies, in addition to the results of Theorem 3.1, conformal covariance. More precisely,* $\mathcal{J}$ *provides a representation of the group of conformal orientation preserving transformations* $SO(4,1)$ *of* RW*, under whose adjoint action the net* $\{\underset{\sim}{\mathcal{R}}(\underset{\sim}{W})\}_{\underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}}$ *is covariant.*

We now turn to the analysis of algebras associated with precompact subsets of RW such as double cones. For double cones $\underset{\sim}{\mathcal{O}} \subset RW$, we define

$$\underset{\sim}{\mathcal{R}}(\underset{\sim}{\mathcal{O}}) \doteq \bigcap_{\underset{\sim}{W} \supset \underset{\sim}{\mathcal{O}}} \underset{\sim}{\mathcal{R}}(\underset{\sim}{W}). \tag{56}$$

These are the maximal algebras one can associate to double cones such that the resulting net including these double cone algebras fulfills the condition of isotony. It is an immediate consequence of wedge duality that the so–defined net is local.

Though the $\underset{\sim}{\mathcal{R}}(\underset{\sim}{\mathcal{O}})$ defined in (56) are the largest algebras one can associate to double cones $\underset{\sim}{\mathcal{O}}$ in a meaningful way, it may be that some or all of them are trivial. This is a matter of some significance. If all double cone algebras are trivial, the net is physically irrelevant, since then no observable can be localized in any bounded spacetime region even though all *real* observations are necessarily so localized. If, however, algebras associated with sufficiently large double cones are non–trivial but those associated with sufficiently small double cones are trivial, then the net describes a system for which there is a length scale below which no observables can be localized. In fact, for the net $\{\underset{\sim}{\mathcal{R}}(\underset{\sim}{\mathcal{O}})\}_{\underset{\sim}{\mathcal{O}} \subset RW}$ this question is settled by the following corollary, which is an immediate consequence of Proposition 2.5 if the underlying net on dS complies with the condition of weak additivity [3] and if intersections of algebras corresponding to disjoint wedges are trivial [29].

Corollary 3.3 *Let $x \in RW$ be arbitrary, and let ε be a positive number such that*[4] *$\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$ is contained in RW. If $\varepsilon > \frac{\pi}{2}(1-\alpha)$, the GNS-vector $\underset{\sim}{\Omega}$ representing $\underset{\sim}{\omega}$ is cyclic for $\underset{\sim}{\mathcal{R}}(\underset{\sim}{\mathcal{O}}_{x,\varepsilon})$, whereas if $\varepsilon < \frac{\pi}{2}(1-\alpha)$, then $\underset{\sim}{\mathcal{R}}(\underset{\sim}{\mathcal{O}}_{x,\varepsilon}) = \mathbb{C} \cdot 1$.*

Note that $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$ can be contained in RW only if $\varepsilon < \alpha\frac{\pi}{2}$. In this case, $\alpha < \frac{1}{2}$ implies $\varepsilon < \frac{\pi}{2}(1-\alpha)$. Hence, only if $\alpha \geq \frac{1}{2}$ are there sufficiently large double cones $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$ such that the associated algebras are non–trivial.

Summarizing, we have shown that for any value of the temporal scaling factor $0 < \alpha \leq 1$, the CGMA and the Modular Stability Condition are satisfied by the net $\{\underset{\sim}{\mathcal{R}}(\underset{\sim}{W})\}_{\underset{\sim}{W} \in \underset{\sim}{\mathcal{W}}}$ and the state $\underset{\sim}{\omega}$. When the scaling factor α equals 1, $\mathcal{J}$ provides a representation of the conformal group $SO(4,1)$ of RW under whose adjoint action the net is covariant. If $\alpha < 1$, $\mathcal{J}$ still induces a geometric action of the group $SO(4,1)$ on the net, but this action can in general not be interpreted in terms of point transformations on RW.

It seems plausible that similar results can be obtained for a broader class of Robertson–Walker space–times. In fact, all Robertson–Walker space–times can be conformally embedded into the Einstein static universe [20], on which one can find well-behaved nets and states [12]. It should be possible to promote these desirable properties to the Robertson–Walker spaces by transplantation. More immediately, a class of Robertson–Walker spaces with negative curvature can be conformally embedded into Minkowski space [20]. In addition, Robertson–Walker spaces with 0 sectional curvature and corresponding to the $\alpha = 1$ case are globally conformally equivalent to Minkowski space. This provides the opportunity to verify results analogous to those worked out above.

4 Concluding Remarks

Making use of the novel method of transplantation of nets, we have been able to make two things clear. First of all, the CGMA is by no means limited in its scope of application to space–times with maximal isometry groups, like the special cases of Minkowski space and de Sitter space worked out in [10, 11, 13]. Second and closely related, the group $\mathcal{J}$ arising from the CGMA can be much larger than the symmetry group of the underlying space–time. This supports our view that the CGMA, coupled with the Modular Stability Condition, could be a useful selection criterion even for space–times with trivial symmetry groups, which, after all, is the generic case.

A further point of interest is that the Modular Stability Condition holds in the examples presented in this paper. We recall that the Modular Stability Condition

[4] Recall the definition (26) of the double cones $\underset{\sim}{\mathcal{O}}_{x,\varepsilon}$.

can be expressed for theories on arbitrary space–times. Moreover, for Minkowski space the CGMA and the Modular Stability Condition together yield the usual condition of physical stability — *i.e.* the relativistic spectrum condition. Recent results concerning theories in de Sitter space [3,5] are consistent with our proposal of the Modular Stability Condition as a criterion for stable vacuum–like systems. Indeed, there are indications [15] of a possible connection between the Modular Stability Condition and the microlocal spectrum condition for non–Minkowskian space–times.

Since the existence of nets and states on certain Robertson–Walker space–times which satisfy the CGMA and the Modular Stability Condition has now been established, it seems rewarding to determine in a next step whether the symmetry group of these space–times along with a corresponding (anti)unitary representation can be recovered from a net and state satisfying these conditions. In this context, it would also be interesting to determine if the net [12] and adiabatic vacuum states [22] (or the adiabatic KMS states constructed in [30]) associated with the free scalar Bose field satisfy the CGMA and the Modular Stability Condition for our choice of wedges $\underset{\sim}{\mathcal{W}}$. We hope to return elswhere to these intriguing aspects of the general program outlined in [10].

A Appendix

Lemma A.1 *Let Φ be the diffeomorphism from dS onto RW given by equations* (15) *and* (16). *Then Φ induces a bijection from the set of de Sitter edges onto the set of Robertson–Walker edges.*

Proof. The first step is to show that Φ arises from the linear map $\tilde{\Phi}$ in $\mathbb{R}^5$ defined by

$$\tilde{\Phi}(x_0, \mathbf{x}) \doteq \big(\sin(\alpha\frac{\pi}{2})\, x_0, \mathbf{x}\big). \tag{57}$$

Since $0 < \sin(\alpha\frac{\pi}{2}) \leq 1$, $\tilde{\Phi}$ leaves the set of spacelike vectors invariant. Denoting $|y| \doteq (-y^2)^{\frac{1}{2}}$ for spacelike y, the claim is that

$$\Phi(x) = \tilde{\Phi}(x)\, |\tilde{\Phi}(x)|^{-1}. \tag{58}$$

To see this, let Φ_1 be the map defined by the right hand side of the above equation. Recall [20] that the t-coordinate of a point $x = (x_0, \mathbf{x}) \in dS$ is given by

$$\tanh(t(x)) = x_0\, \|\mathbf{x}\|^{-1}, \tag{59}$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm of $\mathbf{x}$, and this expression coincides with $\sin(\tau(x))$ by equation (7). Furthermore, the coordinates $(\chi, \theta, \phi)(x)$ are just the natural S^3 coordinates of $\|\mathbf{x}\|^{-1}\mathbf{x}$. Thus one easily verifies that the S^3-coordinates are left invariant by Φ_1, while the τ-coordinate transforms according to

$$\sin(\tau(\Phi_1(x)) = \sin(\alpha\frac{\pi}{2})\sin(\tau(x)). \tag{60}$$

Hence, Φ_1 coincides with Φ, proving equation (58). But this equation implies that for any spacelike linear subspace $\widetilde{L} \subset \mathbb{R}^5$

$$\Phi(\widetilde{L} \cap dS) = \tilde{\Phi}(\widetilde{L}) \cap dS. \tag{61}$$

It follows that Φ leaves invariant the set of intersections of dS with three-dimensional spacelike linear subspaces of $\mathbb{R}^5$, *i.e.* the set of de Sitter edges. Now let $\widetilde{L}$ be a spacelike linear subspace of $\mathbb{R}^5$ whose intersection with dS is contained in RW. Then the τ-coordinates of the intersection are contained in the interval $(-\alpha\frac{\pi}{2}, \alpha\frac{\pi}{2})$. Hence every $x \in \widetilde{L}$ satisfies $|x_0| \, \|\mathbf{x}\|^{-1} < \sin(\alpha\frac{\pi}{2}) \leq 1$, and therefore $\tilde{\Phi}^{-1}(x) = (\sin(\alpha\frac{\pi}{2})^{-1} x_0, \mathbf{x})$ is spacelike. This shows that the preimage under Φ of every edge contained in RW is a de Sitter edge. $\square$

We now turn to the question of uniqueness. Let Φ' be a second diffeomorphism from dS onto RW which bijectively maps the set of de Sitter edges onto the set of Robertson–Walker edges. Then one concludes that $\Phi_0 \doteq \Phi^{-1} \circ \Phi'$ maps the set of spacelike geodesics in dS onto itself. Since any pair of lightlike separated points in dS can be approximated by a pair of spacelike separated ones, one thus verifies that Φ_0 preserves separation zero in both directions. But then Φ_0 is an isometry of dS, as shown in [21]. Since there is no isometry $\Phi_0 \in O(4,1)$ which acts only on the time variable other than the identity and time reversal, we have the following result.

Lemma A.2 *The map Φ is, up to isometries of dS, the only diffeomorphism from dS onto RW which maps de Sitter edges to Robertson–Walker edges. Furthermore, up to time reversal, it is the only such map which acts only on the time variable in the chosen coordinate system.*

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