GEOMETRIC QUANTIZATION AND THE GENERALIZED SEGAL-BARGMANN TRANSFORM FOR LIE GROUPS OF COMPACT TYPE

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Abstract. Let K be a connected Lie group of compact type and let $T^*(K)$ be its cotangent bundle. This paper considers geometric quantization of $T^*(K)$, first using the vertical polarization and then using a natural Kähler polarization obtained by identifying $T^*(K)$ with the complexified group $K_{\mathbb{C}}$. The first main result is that the Hilbert space obtained by using the Kähler polarization is naturally identifiable with the generalized Segal-Bargmann space introduced by the author from a different point of view, namely that of heat kernels. The second main result is that the pairing map of geometric quantization coincides with the generalized Segal-Bargmann transform introduced by the author. This means that in this case the pairing map is a constant multiple of a unitary map. For both results it is essential that the half-form correction be included when using the Kähler polarization. Together with results of the author with B. Driver, these results may be seen as an instance of "quantization commuting with reduction."

Contents

1. Introduction

For quantum mechanics of a particle moving in $\mathbb{R}^n$, there are several different ways of expressing the quantum Hilbert space, including the position Hilbert space (or Schrödinger representation) and the Segal-Bargmann (or Bargmann, or Bargmann-Fock) space. The position Hilbert space is $L^2(\mathbb{R}^n)$, with $\mathbb{R}^n$ thought of as the position variables. The Segal-Bargmann space is the space of holomorphic functions on $\mathbb{C}^n$ that are square-integrable with respect to a Gaussian measure, where $\mathbb{C}^n = \mathbb{R}^{2n}$ is the phase space. (There are also the momentum Hilbert space and the Fock symmetric tensor space.) There is a natural unitary map that relates the position Hilbert space to the Segal-Bargmann space, namely the Segal-Bargmann transform.

One way to understand these constructions is in terms of geometric quantization. (See Section 5.) In geometric quantization one first constructs a pre-quantum Hilbert space over the phase space $\mathbb{R}^{2n}$. The prequantum Hilbert space is essentially just $L^2\left(\mathbb{R}^{2n}\right)$. It is generally accepted that this Hilbert space is "too big"; for example, the space of position and momentum operators does not act irreducibly. To get an appropriate Hilbert space one chooses a "polarization," that is (roughly) a choice of n out of the $2n$ variables on $\mathbb{R}^{2n}$. The quantum Hilbert space is then the space of elements of the prequantum Hilbert space that are independent of the chosen n variables. So in the "vertical polarization" one considers functions that are independent of the momentum variables, hence functions of the position only. In this case the quantum Hilbert space is just the position Hilbert space $L^2\left(\mathbb{R}^n\right)$. Alternatively, one may identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and consider complex variables $z_1, \cdots, z_n$, and $\bar{z}_1, \cdots, \bar{z}_n$. The Hilbert space is then the space of functions that are "independent of the $\bar{z}_k$'s," that is, holomorphic. In this case the quantum Hilbert space is the Segal-Bargmann space.

More precisely, the prequantum Hilbert space for a symplectic manifold (M, ω) is the space of sections of a line-bundle-with-connection L over M, where the curvature of L is given by the symplectic form ω. A real polarization for M is a foliation of M into Lagrangian submanifolds. A Kähler polarization is a choice of a complex structure on M that is compatible with the symplectic structure, in such a way that M becomes a Kähler manifold. The quantum Hilbert space is then the space of sections that are covariantly constant along the leaves of the foliation (for a real polarization) or covariantly constant in the $\bar{z}$-directions (for a complex polarization). Since the leaves of a real polarization are required to be Lagrangian, the curvature of L (given by ω) vanishes along the leaves and so there exist, at least locally, polarized sections. Similarly, the compatibility condition between the complex structure and the symplectic structure in a complex polarization guarantees the existence, at least locally, of polarized sections.

A further ingredient is the introduction of "half-forms," which is a technical necessity in the case of the vertical polarization and which can be useful even for a Kähler polarization. The inclusion of half-forms in the Kähler-polarized Hilbert space is essential to the results of this paper.

If one has two different polarizations on the same manifold then one gets two different quantum Hilbert spaces. Geometric quantization gives a canonical way of constructing a map between these two spaces, called the pairing map. The pairing map is not unitary in general, but in the case of the vertical and Kähler polarizations on $\mathbb{R}^{2n}$ it is unitary. In the $\mathbb{R}^{2n}$ case, this unitarity can be explained by the Stone-von Neumann theorem. I do of the calculations for the $\mathbb{R}^{2n}$ case in Section 5; the reader may wish to begin with that section.

Besides the $\mathbb{R}^{2n}$ case, there have not been many examples where pairing maps have been studied in detail. In particular, the only works I know of that address unitarity of the pairing map outside of $\mathbb{R}^{2n}$ are those of J. Rawnsley [R] and K. Furutani and S. Yoshizawa [FY]. Rawnsley considers the cotangent bundle of spheres, with the vertical polarization and also a certain Kähler polarization. Furutani and Yoshizawa consider a similar construction on the cotangent bundle of complex and quaternionic projective spaces. In these cases the pairing map is not unitary (nor a constant multiple of a unitary map).

An interesting class of symplectic manifolds having two different natural polarizations is the following. Let X be a real-analytic Riemannian manifold and let $M = T^*(X)$. Then M has a natural symplectic structure and a natural vertical polarization, in which the leaves of the Lagrangian foliation are the fibers of $T^*(X)$. By a construction of Guillemin and Stenzel [GStenz1, GStenz2] and Lempert and Szőke [LS], $T^*(X)$ also has a canonical "adapted" complex structure, defined in a neighborhood of the zero section. This complex structure is compatible with the symplectic structure and so defines a Kähler polarization on an open set in $T^*(X)$.

This paper considers the special case in which X is a Lie group K with a bi-invariant Riemannian metric. Lie groups that admit a bi-invariant metric are said to be of "compact type"; these are precisely the groups of the form (compact) $\times \mathbb{R}^n$. In this special case, the adapted complex structure is defined on all of $T^*(K)$, so $T^*(K)$ has two polarizations, the vertical polarization and the Kähler polarization coming from the adapted complex structure. If $K = \mathbb{R}^n$ then the complex structure is just the usual one on $T^*(\mathbb{R}^n) = \mathbb{R}^{2n} = \mathbb{C}^n$.

There are two main results, generalizing what is known in the $\mathbb{R}^n$ case. First, the Kähler-polarized Hilbert space constructed over $T^*(K)$ is naturally identifiable with the generalized Segal-Bargmann space defined in [H1] in terms of heat kernels. Second, the pairing map between the vertically polarized and the Kähler-polarized Hilbert space over $T^*(K)$ coincides (up to a constant) with the generalized Segal-Bargmann transform of [H1]. Thus by [H1, Thm. 2] a constant multiple of the pairing map is unitary in this case. Both of these results hold only if one includes the "half-form correction" in the construction of the Kähler-polarized Hilbert space. In the case $K = \mathbb{R}^n$ everything reduces to the Segal-Bargmann space and the Segal-Bargmann transform (Section 5).

The results are surprising for two reasons. First, the constructions in [H1] involve heat kernels, whereas geometric quantization seems to have nothing to do with heat kernels or the heat equation. Second, in the absence of something like the Stone-von Neumann theorem there does not seem to be any reason for pairing maps to be unitary. The discussion in Section 4 gives some partial explanation for the occurrence of the heat kernel. (See also [JL].)

If one considers Yang-Mills theory over a space-time cylinder, in the temporal gauge, the "unreduced phase space" is a certain infinite-dimensional linear space of connections. The reduced phase space, obtained by "reducing" by a suitable gauge group, is the finite-dimensional symplectic manifold $T^*(K)$, where K is the structure group for the Yang-Mills theory. Thus the symplectic manifold $T^*(K)$ considered here can also be viewed as the "symplectic quotient" of an infinite-dimensional linear space by an infinite-dimensional group. It is reasonable to ask whether "quantization commutes with reduction," that is, whether one gets the same results by first quantizing and then reducing as by first reducing and then quantizing. Surprisingly (to me), the answer in this case is yes, as described in Section 3.

I conclude this introduction by discussing two additional points. First, it is reasonable to consider the more general situation where the group K is allowed to be a symmetric space of compact type. In that case the geometric quantization constructions make perfect sense, but the main results apparently do not hold. Specifically, the Kähler-polarized Hilbert space does not coincide with the heat kernel Hilbert space of M. Stenzel [St], and I do not know whether the pairing map of geometric quantization is unitary. This discrepancy reflects special properties that compact Lie groups have among all compact symmetric spaces. See the discussion at the end of Section 2.3.

Second, one could attempt to construct for a momentum Hilbert space for $T^*(K)$. In the case $K=\mathbb{R}^n$ this may be done by considering the natural horizontal polarization. The pairing map between the vertically polarized and horizontally polarized Hilbert spaces is in this case just the Fourier transform. By contrast, if K is non-commutative, then there is no natural horizontal polarization. (For example, the foliation of $T^*(K)$ into the left orbits of K is not Lagrangian.) Thus, even though there is a sort of momentum representation given by the Peter-Weyl theorem, it does not seem possible to obtain a momentum representation by means of geometric quantization.

2. The main results

2.1. **Preliminaries.** Let K be a connected Lie group of **compact type**. A Lie group is said to be of compact type if it is locally isomorphic to some compact Lie group. Equivalently, a Lie group K is of compact type if there exists an inner product on the Lie algebra of K that is invariant under the adjoint action of K. So $\mathbb{R}^n$ is of compact type, being locally isomorphic to a d-torus, and every compact Lie group is of compact type. It can be shown that every connected Lie group of compact type is isomorphic to a product of $\mathbb{R}^n$ and a connected compact Lie group. So all of the constructions described here for Lie groups of compact type include as a special case the constructions for $\mathbb{R}^n$. On the other hand, all the new information (beyond the $\mathbb{R}^n$ case) is contained in the compact case. See [He, Chap. II, Sect. 6] (including Proposition 6.8) for information on Lie groups of compact type.

Let $\mathfrak{k}$ denote the Lie algebra of K. We fix once and for all an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}$ that is invariant under the adjoint action of K. For example we may take $K=\mathrm{SU}(n)$, in which case $\mathfrak{k}=\mathrm{su}(n)$ is the space of skew matrices with trace zero. An invariant inner product on $\mathfrak{k}$ is $\langle X,Y\rangle=\mathrm{Re}\left[\mathrm{trace}\left(X^*Y\right)\right]$.

Now let $K_{\mathbb{C}}$ be the *complexification* of K. If K is simply connected then the complexification of K is the unique simply connected Lie group whose Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is $\mathfrak{k}+i\mathfrak{k}$. In general, $K_{\mathbb{C}}$ is defined by the following three properties. First, $K_{\mathbb{C}}$ should be a connected complex Lie group whose Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is equal to $\mathfrak{k}+i\mathfrak{k}$. Second, $K_{\mathbb{C}}$ should contain K as a closed subgroup (whose Lie algebra is $\mathfrak{k}\subset\mathfrak{k}_{\mathbb{C}}$). Third, every homomorphism of K into a complex Lie group H should extend to a holomorphic homomorphism of $K_{\mathbb{C}}$ into H. The complexification of a connected Lie group of compact type always exists and is unique. (See [H1, Sect. 3].)

Example 2.1. If $K=\mathbb{R}^n$ then $K_{\mathbb{C}}=\mathbb{C}^n$. If $K=\mathrm{SU}(n)$ then $K_{\mathbb{C}}=\mathrm{SL}(n;\mathbb{C})$. If $K=\mathrm{SO}(n)$ then $K_{\mathbb{C}}=\mathrm{SO}(n;\mathbb{C})$. In the first two examples, K and $K_{\mathbb{C}}$ are simply connected. In the last example, neither K nor $K_{\mathbb{C}}$ is simply connected.

We have the following structure result for Lie groups of compact type. This result is a modest strengthening of Corollary 2.2 of [Dr] and allows all the relevant results for Lie groups of compact type to be reduced to two cases, the compact case and the $\mathbb{R}^n$ case.

Proposition 2.2. *Suppose that K is a connected Lie group of compact type, with a fixed Ad-invariant inner product on its Lie algebra $\mathfrak{k}$. Then there exists a isomorphism $K \cong H \times \mathbb{R}^n$, where H is compact and where the associated Lie algebra isomorphism $\mathfrak{k} = \mathfrak{h} + \mathbb{R}^n$ is orthogonal.*

The proof of this result is given in an appendix.

2.2. **Prequantization.** We let θ be the canonical 1-form on $T^*(K)$, normalized so that in the usual sort of coordinates we have

$$\theta = \sum p_k \, dq_k.$$

We then let ω be the canonical 2-form on $T^*(K)$, which I normalize as $\omega = -d\theta$, so that in coordinates $\omega = \Sigma dq_k \wedge dp_k$. We then consider a trivial complex line bundle L on $T^*(K)$

$$L = T^*(K) \times \mathbb{C}$$

with trivial Hermitian structure. Sections of this bundle are thus just functions on $T^*(K)$. We define a connection (or covariant derivative) on L by

$$\nabla_X = X - \frac{1}{i\hbar}\theta(X).$$

The curvature of this connection is given by

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \frac{1}{i\hbar}\omega(X,Y).$$

We let ε denote the Liouville volume form on $T^*(K)$, given by

$$\varepsilon = \frac{1}{n!}\omega^n,$$

where $n = \dim K = (1/2)\dim T^*(K)$. Integrating this form gives the associated Liouville volume measure. Concretely we have the identification

$$T^*(K) \cong K \times \mathfrak{k} \tag{2.1}$$

by means of left-translation and the inner product on $\mathfrak{k}$. Under this identification we have [H3, Lem. 4]

$$\int_{T^*(K)} f\,\varepsilon = \int_{\mathfrak{k}}\int_K f(x,Y)\; dx\, dY \tag{2.2}$$

where dx is Haar measure on K, normalized to coincide with the Riemannian volume measure, and dY is Lebesgue measure on $\mathfrak{k}$, normalized by means of the inner product. The prequantum Hilbert space is then the space of sections of L that are square integrable with respect to ε. This space may be identified with $L^2(T^*(K), \varepsilon)$.

One motivation for this construction is the existence of a natural mapping $\mathcal{Q}$ from functions on $T^*(K)$ into the space of symmetric operators on the prequantum Hilbert space satisfying $[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\mathcal{Q}(\{f,g\})$, where $\{f,g\}$ is the Poisson bracket. Explicitly, $\mathcal{Q}(f) = i\hbar\nabla_{X_f} + f$, where X_f is the Hamiltonian vector field associated to f. This "prequantization map" will not play an important role in this paper. See [Wo, Chap. 8] for more information.

2.3. **The Kähler-polarized subspace.** Let me summarize what the results of this subsection will be. The cotangent bundle $T^*(K)$ has a natural complex structure that comes by identifying it with the "complexification" of K. This complex structure allows us to define a notion of Kähler-polarized sections of the bundle L. There exists a natural trivializing polarized section s_0 such that every other polarized section is a holomorphic function times s_0. The Kähler-polarized Hilbert space is then identifiable with an L^2 space of holomorphic functions on $T^*(K)$, where the measure is the Liouville measure times $|s_0|^2$. We then consider the "half-form" bundle δ_1. The half-form corrected Kähler Hilbert space is the space of polarized sections of $L \otimes \delta_1$. This may be identified with an L^2 space of holomorphic functions on $T^*(K)$, where now the measure is the Liouville measure ε times $|s_0|^2 |\beta_0|^2$, where β_0 is a trivializing polarized section of δ_1. The main result is that this last measure coincides up to a constant with the heat kernel measure on $T^*(K)$ introduced in [H1]. Thus the half-form-corrected Kähler-polarized Hilbert space of geometric quantization coincides (up to a constant) with the generalized Segal-Bargmann space of [H1, Thm. 2].

We let $K_{\mathbb{C}}$ denote the complexification of K, as described in Section 2.1, and we let $T^*(K)$ denote the cotangent bundle of K. There is a diffeomorphism of $T^*(K)$ with $K_{\mathbb{C}}$ as follows. We identify $T^*(K)$ with $K \times \mathfrak{k}^*$ by means of left-translation and then with $K \times \mathfrak{k}$ by means of the inner product on $\mathfrak{k}$. We consider the map $\Phi : K \times \mathfrak{k} \to K_{\mathbb{C}}$ given by

$$\Phi(x, Y) = xe^{iY}, \quad x \in K, Y \in \mathfrak{k}. \tag{2.3}$$

The map Φ is a diffeomorphism. If we use Φ to transport the complex structure of $K_{\mathbb{C}}$ to $T^*(K)$, then the resulting complex structure on $T^*(K)$ is compatible with the symplectic structure on $T^*(K)$, so that $T^*(K)$ becomes a Kähler manifold. (See [H3, Sect. 3].)

Consider the function $\kappa : T^*(K) \to \mathbb{R}$ given by

$$\kappa(x, Y) = |Y|^2. \tag{2.4}$$

This function is a *Kähler potential* for the complex structure on $T^*(K)$ described in the previous paragraph. Specifically we have

$$\operatorname{Im}\left(\bar{\partial}\kappa\right) = \theta \tag{2.5}$$

and

$$\partial\bar{\partial}\kappa = i\omega. \tag{2.6}$$

An important feature of this situation is the natural explicit form of the Kähler potential. This formula for κ comes as a special case of the general construction Guillemin-Stenzel and Lempert-Szőke ([GStenz1, Sect. 5] and [LS, Cor. 5.5]). It this case one can compute directly that κ satisfies (2.5) and (2.6) (see the first appendix).

We define a section s of L to be *Kähler-polarized* if

$$\nabla_X s = 0$$

for all vectors of type $(0,1)$. Equivalently s is polarized if $\nabla_{\partial/\partial\bar{z}_k} s = 0$ for all k, in holomorphic local coordinates. The *Kähler-polarized Hilbert space* is then the space of square-integrable Kähler-polarized sections of L. (See [Wo, Sect. 9.2].)

Proposition 2.3. *If we think of sections s of L as functions on $T^*(K)$ then the Kähler-polarized sections are precisely the functions s of the form*

$$s = e^{-|Y|/2\hbar} F,$$

with F holomorphic and $|Y|^2 = \kappa(x, Y)$ the Kähler potential (2.4). The notion of holomorphic is via the identification (2.3) of $T^*(K)$ with $K_{\mathbb{C}}$.

Proof. If we work in holomorphic local coordinates $z_1, \cdots, z_n$ then we want sections s such that $\nabla_{\partial/\partial \bar{z}_k} s = 0$ for all k. The condition (2.5) on κ says that in these coordinates

$$\theta = \frac{1}{2i} \sum_k \left(\frac{\partial \kappa}{\partial \bar{z}_k} d\bar{z}_k - \frac{\partial \kappa}{\partial z_k} dz_k \right).$$

So

$$\theta\left(\frac{\partial}{\partial \bar{z}_k}\right) = \frac{1}{2i} \frac{\partial \kappa}{\partial \bar{z}_k}.$$

Note first that

$$\begin{aligned}
\nabla_{\partial/\partial \bar{z}_k} e^{-\kappa/2\hbar} &= \frac{\partial}{\partial \bar{z}_k} e^{-\kappa/2\hbar} - \frac{1}{i\hbar}\theta\left(\frac{\partial}{\partial \bar{z}_k}\right) e^{-\kappa/2\hbar} \\
&= \left(-\frac{1}{2}\frac{\partial \kappa}{\partial \bar{z}_k} - \frac{1}{i\hbar}\frac{1}{2i}\frac{\partial \kappa}{\partial \bar{z}_k} \right) e^{-\kappa/2\hbar} = 0.
\end{aligned}$$

Now any section s can be written uniquely as $s = \exp(-\kappa/2\hbar) F$, where F is a complex valued function. Such a section is polarized precisely if

$$\begin{aligned}
0 &= \nabla_{\partial/\partial \bar{z}_k} \left(e^{-\kappa/2\hbar} F \right) \\
&= \left(\nabla_{\partial/\partial \bar{z}_k} e^{-\kappa/2\hbar} \right) F + e^{-\kappa/2\hbar} \frac{\partial F}{\partial \bar{z}_k} \\
&= e^{-\kappa/2\hbar} \frac{\partial F}{\partial \bar{z}_k}
\end{aligned}$$

for all k, that is, precisely if F is holomorphic. □

The norm of a polarized section s is computed as

$$\begin{aligned}
\|s\|^2 &= \int_{T^*(K)} |F|^2 e^{-\kappa/\hbar} \varepsilon \\
&= \int_{\mathfrak{k}} \int_K \left| F\left(xe^{iY}\right) \right|^2 e^{-|Y|^2/\hbar} \, dx \, dY.
\end{aligned}$$

Here F is a holomorphic function on $K_{\mathbb{C}}$ which we are "transporting" to $T^*(K)$ by means of the map $\Phi(x, Y) = xe^{iY}$. (Recall (2.1) and (2.2).)

If we identify the section s with the holomorphic function F, then the Kähler-polarized Hilbert space may be identified with

$$\mathcal{H}L^2(T^*(K), e^{-|Y|^2/\hbar}\varepsilon)$$

Here ε is the Liouville volume measure and $\mathcal{H}L^2$ denotes the space of holomorphic functions that are square-integrable with respect to the indicated measure. In Section 7 of [H4] I compared the measure $e^{-|Y|^2/\hbar}\varepsilon$ to the "K-invariant heat kernel measure" $\nu_\hbar$ on $K_{\mathbb{C}} \cong T^*(K)$. In the commutative case the two measures agree up to a constant. However, in the non-commutative case the two measures differ by a non-constant function of Y, and it is easily seen that this discrepancy cannot be eliminated by choosing a different trivializing polarized section of L. In the remainder of this section we will see that this discrepancy between the heat kernel measure and the geometric quantization measure can be eliminated by the "half-form correction." I am grateful to Dan Freed for suggesting to me that this could be the case.

We now consider the canonical bundle for $T^*(K)$ relative to the complex structure obtained from $K_{\mathbb{C}}$. The canonical bundle is the complex line bundle whose sections are complex-valued n-forms of type $(n,0)$. The forms of type $(n,0)$ may be described as those n-forms α for which

$$X \lrcorner \alpha = 0$$

for all vectors of type $(0,1)$. We then define the polarized sections of the canonical bundle to be the $(n,0)$-forms such that

$$X \lrcorner d\alpha = 0$$

for all vector fields of type $(0,1)$. (Compare [Wo, Eq. (9.3.1)].) These are nothing but the holomorphic n-forms. We define a Hermitian structure on the canonical bundle by defining for two $(n,0)$-forms α,β

$$(\alpha,\beta) = \frac{\bar{\alpha}\wedge\beta}{(2i)^n\,\varepsilon}.$$

Here the ratio means the only thing that is reasonable: (α,β) is the unique function such that $(\alpha,\beta)\,(2i)^n\,\varepsilon = \bar{\alpha}\wedge\beta$.

In this situation the canonical bundle may be trivialized as follows. We think of $T^*(K)$ as $K_{\mathbb{C}}$, since at the moment the symplectic structure is not relevant. If $Z_1,\cdots,Z_n$ are linearly independent left-invariant holomorphic 1-forms on $K_{\mathbb{C}}$ then their wedge product is a nowhere vanishing holomorphic n-form.

We now choose a square root δ_1 of the canonical bundle in such a way that there exists a exists a smooth section of δ_1 whose square is $Z_1\wedge\cdots\wedge Z_n$. This section of δ_1 will be denoted by the mnemonic $\sqrt{Z_1\wedge\cdots\wedge Z_n}$. There then exists a unique notion of polarized sections of δ_1 such that 1) a locally defined, smooth, nowhere-zero section ν of δ_1 is polarized if and only if ν^2 is a polarized section of the canonical bundle, and 2) if ν is a locally defined, nowhere-zero, polarized section of δ_1 and F is a smooth function, then $F\nu$ is polarized if and only if F is holomorphic. (See [Wo, p. 186].) Concretely the polarized sections of δ_1 are of the form

$$s = F(g)\sqrt{Z_1\wedge\cdots\wedge Z_n}$$

with F a holomorphic function on $K_{\mathbb{C}}$. The absolute value of such a section is defined as

$$|s|^2 := \sqrt{(s^2,s^2)} = |F|^2\sqrt{\frac{\bar{Z}_1\wedge\cdots\wedge\bar{Z}_n\wedge Z_1\wedge\cdots\wedge Z_n}{(2i)^n\,\varepsilon}}.$$

Now the *half-form corrected Kähler-polarized Hilbert space* is the space of square-integrable polarized sections of $L\otimes\delta_1$. (The polarized sections of $L\otimes\delta_1$ are precisely those that can be written locally as the product of a polarized section of L and a polarized section of δ_1.) Such sections are precisely those that can be expressed as

$$s = e^{-|Y|^2/2\hbar}F\otimes\sqrt{Z_1\wedge\cdots\wedge Z_n} \tag{2.7}$$

with F holomorphic. The norm of such a section is computed as

$$\|s\|^2 = \int_{T^*(K)} |F|^2 e^{-|Y|^2/\hbar} \eta\, \varepsilon,$$

where η is the function given by

$$\eta = \sqrt{\frac{\bar{Z}_1 \wedge \cdots \wedge \bar{Z}_n \wedge Z_1 \wedge \cdots \wedge Z_n}{(2i)^n \varepsilon}}. \tag{2.8}$$

If we identify the section s with the holomorphic function F in (2.7) then we have the following result.

Theorem 2.4. *If we write elements of the half-form corrected Kähler Hilbert space in the form (2.7) then this Hilbert space may be identified with*

$$\mathcal{H}L^2(T^*(K), \gamma_\hbar)$$

where $\gamma_\hbar$ is the measure given by

$$\gamma_\hbar = e^{-|Y|^2/\hbar} \eta\, \varepsilon.$$

Here ε is the canonical volume form on $T^*(K)$, $|Y|^2$ is the Kähler potential (2.4), and η is the "half-form correction" given in (2.8). Here as elsewhere $\mathcal{H}L^2$ denotes the space of square-integrable holomorphic functions.

Note that $\bar{Z}_1 \wedge \cdots \wedge \bar{Z}_n \wedge Z_1 \wedge \cdots \wedge Z_n$ is a left-invariant $2n$-form on $K_{\mathbb{C}}$, so that the associated measure is simply a multiple of Haar measure on $K_{\mathbb{C}}$. Meanwhile ε is just the Liouville volume form on $T^*(K)$. Thus η is simply the square root of the density of Haar measure with respect to Liouville measure, under our identification of $K_{\mathbb{C}}$ with $T^*(K)$. Both measures are K-invariant, so in our (x, Y) coordinates on $T^*(K)$, η will be a function of Y only. By [H3, Lem. 5] we have that $\eta(Y)$ is the unique Ad-K-invariant function on $\mathfrak{k}$ such that in a maximal abelian subalgebra

$$\eta(Y) = \prod_{\alpha \in R^+} \frac{\sinh \alpha(Y)}{\alpha(Y)}, \tag{2.9}$$

where R^+ is a set of positive roots.

Meanwhile there is the "K-invariant heat kernel measure" $\nu_\hbar$ on $K_{\mathbb{C}} \cong T^*(K)$, as described in [H1]. When written in terms of the polar decomposition $g = xe^{iY}$, $\nu_\hbar$ is given explicitly as

$$d\nu_\hbar = (\pi\hbar)^{-n/2} e^{-|\rho|^2\hbar} e^{-|Y|^2/\hbar} \eta(Y)\; dx\, dY.$$

(See [H3, Eq. (13)].) Here ρ is half the sum of the positive roots for the group K. Thus apart from an overall constant, the measure $T^*(K)$ coming from geometric quantization coincides exactly with the heat kernel measure of [H1]. So we have proved the following result.

Theorem 2.5. *For each $\hbar > 0$ there exists a constant $c_\hbar$ such that the measure $\gamma_\hbar$ coming from geometric quantization and the heat kernel measure $\nu_\hbar$ are related by*

$$\nu_\hbar = c_\hbar \gamma_\hbar$$

where

$$c_\hbar = (\pi\hbar)^{-n/2} e^{-|\rho|^2\hbar}$$

and where ρ is half the sum of the positive roots for the group K.

Let us try to understand, at least in part the seemingly miraculous agreement between these two measures. The cotangent bundle $T^*(K)$ has a complex structure obtained by identification with $K_{\mathbb{C}}$. The metric tensor on K then has an analytic continuation to a holomorphic n-tensor on $T^*(K)$. The restriction of the analytically continued metric tensor to the fibers of $T^*(K)$ is the negative of a Riemannian metric g. Each fiber, with this metric, is isometric to the non-compact symmetric space $K_{\mathbb{C}}/K$. This reflects the well-known duality between compact and non-compact symmetric spaces. Each fiber is also identified with $\mathfrak{k}$, and under this identification the Riemannian volume measure with respect to g is given by

$$\sqrt{g}dY = \eta(Y)^2\ dY.$$

That is, the "half-form factor" η is simply the square root of the Jacobian of the exponential mapping for $K_{\mathbb{C}}/K$.

Now on any Riemannian manifold the heat kernel measure (at a fixed base point, written in exponential coordinates) has an asymptotic expansion of the form

$$d\mu_\hbar(Y) \sim (\pi\hbar)^{-n/2} e^{-|Y|^2/\hbar} \left(j^{1/2}(Y) + ta_1(Y) + t^2 a_2(Y) + \cdots\right) dY. \tag{2.10}$$

Here $j(Y)$ is the Jacobian of the exponential mapping, also known as the Van Vleck–Morette determinant. (I have written $\hbar$ for the time variable and normalized the heat equation to be $du/dt = (1/4)\Delta u$.) Note that this is the expansion for the heat kernel *measure*; in the expansion of the heat kernel *function* one has $j^{-1/2}$ instead of $j^{1/2}$. In the case of the manifold $K_{\mathbb{C}}/K$ we have a great simplification. All the higher terms in the series are just constant multiples of $j^{1/2}$ and we get an exact convergent expression of the form

$$d\mu_\hbar(Y) = (\pi\hbar)^{-n/2} e^{-|Y|^2/\hbar} j^{1/2}(Y) f(t)\ dY. \tag{2.11}$$

Here explicitly $f(t) = \exp(-|\rho|^2 t)$, where ρ is half the sum of the positive roots. The measure $\nu_\hbar$ in [H1] is then simply this measure times the Haar measure dx in the K-directions. So we have

$$d\nu_\hbar = e^{-|\rho|^2 t} (\pi\hbar)^{-n/2} e^{-|Y|^2/\hbar} j^{1/2}(Y)\ dx\, dY.$$

So how does geometric quantization produce a multiple of $\nu_\hbar$? The Gaussian factor comes from the simple explicit form of the Kähler potential, and the factor of $j^{1/2}$ is the half-form correction.

If we begin with a compact symmetric space X then much of the analysis goes through: $T^*(X)$ has a natural complex structure, $|Y|^2$ is a Kähler potential, and the fibers are identifiable with non-compact symmetric spaces. (See [St, p. 48].) Furthermore, the half-form correction is still the square root of the Jacobian of the exponential mapping. What goes wrong is that the heat kernel expansion (2.10) does not simplify to an expression of the form (2.11). So the heat kernel measure of [St] and the measure coming from geometric quantization will not agree up to a constant. Nevertheless the two measures do agree "to leading order in $\hbar$." I do not know whether the geometric quantization pairing map is unitary in the case of general compact symmetric spaces X. Nevertheless, there is a unitary Segal-Bargmann-type transform, given in terms of heat kernels and described in [St].

2.4. **The vertically polarized Hilbert space.** After much sound and fury, the vertically polarized Hilbert space will be identified simply with $L^2(K, dx)$, where dx is Haar measure on K. Nevertheless, the fancy constructions described below are important for two reasons. First, the vertically polarized Hilbert space does not depend on a choice of measure on K. The Hilbert space is really a space of "half-forms." If one chooses a smooth measure μ on K then this choice gives an identification of the vertically polarized Hilbert space with $L^2(K,\mu)$ (and of course Haar measure is the obvious choice). But the choice of measure is needed only to give a concrete realization of the space as an L^2 space; the vertically polarized Hilbert space exists independent of this choice. Second, the description of the vertically polarized Hilbert space as space of half-forms will be essential to the construction of the pairing map in Section 2.5.

The following description follows Section 9.3 of [Wo]. Roughly speaking our Hilbert space will consist of objects whose squares are n-forms on $T^*(K)$ that are constant along the fibers and thus descend to n-forms on K. The norm of such an object is computed by squaring and then integrating the resulting n-form over K.

We consider sections of L that are covariantly constant in the directions parallel to the fibers of $T^*(K)$. Note that each fiber of $T^*(K)$ is a Lagrangian submanifold of $T^*(K)$, so that $T^*(K)$ is naturally foliated into Lagrangian submanifolds. Suppose that X is a tangent vector to $T^*(K)$ that is parallel to one of the fibers. Then it is easily seen that $\theta(X) = 0$, where θ is the canonical 1-form on $T^*(K)$. Thus thinking of the sections of L as functions on $T^*(K)$, the vertically polarized sections are the functions that are constant along the fibers. Such a section cannot be square-integrable with respect to the Liouville measure unless it is zero almost everywhere. This means that we cannot construct the vertically polarized Hilbert space as a subspace of the prequantum Hilbert space.

So we consider the canonical bundle of $T^*(K)$ relative to the vertical polarization. This is the *real* line bundle whose sections are n-forms α such that

$$X \lrcorner \alpha = 0 \tag{2.12}$$

for all vectors parallel to the fibers of $T^*(K)$. We call such a section polarized if in addition we have

$$X \lrcorner d\alpha = 0 \tag{2.13}$$

for all vectors X parallel to the fibers. (See [Wo, Eq. (9.3.1)].)

Now let Q be the space of fibers (or the space of leaves of our Lagrangian foliation). Clearly Q may be identified with K itself, the "configuration space" corresponding to the "phase space" $T^*(K)$. Let $pr : T^*(K) \to K$ be the projection map. It is not hard to verify that if α is a n-form on $T^*(K)$ satisfying (2.12) and (2.13) then there exists a unique n-form β on K such that

$$\alpha = pr^*(\beta).$$

We may think of such an n-form α as being constant along the fibers, so that it descends unambiguously to an n-form β on K. In this way the polarized sections of the canonical bundle may be identified with n-forms on K.

Since K is a Lie group it is orientable. So let us pick an orientation on K, which we think of as an equivalence class of nowhere-vanishing n-forms on K. Then if β is a nowhere-vanishing oriented n-form on K, we define the "positive" part of each fiber of the canonical bundle to be the half-line in which $pr^*(\beta)$ lies. We may then construct a unique trivial line bundle δ_2 such that 1) the square of δ_2 is the canonical bundle and 2) if γ is a nowhere-vanishing section of δ_2 then γ^2 lies in the positive part of the canonical bundle. We have a natural notion of polarized sections of δ_2, such that 1) a locally defined, smooth, nowhere-zero section ν of δ_2 is polarized if and only if ν^2 is a polarized section of the canonical bundle and 2) if ν is a locally defined, nowhere-zero, polarized section of δ_2 and f is a smooth function, then $f\nu$ is polarized if and only if f is constant along the fibers.

Now let β be any nowhere vanishing oriented n-form on K. Then there exists a polarized section of δ_2 (unique up to an overall sign) whose square is $pr^*(\beta)$. This section is denoted $\sqrt{pr^*(\beta)}$. Any other polarized section of δ_2 is then of the form

$$f(x)\sqrt{pr^*(\beta)},$$

where $f(x)$ denotes a real-valued function on $T^*(K)$ that is constant along the fibers.

Finally we consider polarized sections of $L\otimes\delta_2$, i.e. those that are locally the product of a vertically polarized section of L and a polarized section of δ_2. These are precisely the sections that can be expressed in the form

$$s=f(x)\otimes\sqrt{pr^*(\beta)},$$

where f is a complex-valued function on $T^*(K)$ that is constant along the fibers. The norm of such a section is computed as

$$\|s\|^2=\int_K |f(x)|^2\,\beta.$$

It is easily seen that this expression for $\|s\|$ is independent of the choice of β. Note that the integration is over the quotient space K, not over $T^*(K)$.

In particular we may choose linearly independent left-invariant 1-forms $\eta_1,\cdots,\eta_n$ on K so that $\eta_1\wedge\cdots\wedge\eta_n$ is oriented. Then every polarized section of $L\otimes\delta_2$ is of the form

$$s=f(x)\otimes\sqrt{pr^*(\eta_1\wedge\cdots\wedge\eta_n)}$$

and the norm of a section is computable as

$$\begin{aligned}\|s\|^2&=\int_K |f(x)|^2\,\eta_1\wedge\cdots\wedge\eta_n\\ &=\int_K |f(x)|^2\,dx,\end{aligned}\tag{2.14}$$

where dx is Haar measure on K. Thus we may identify the vertically polarized Hilbert space with $L^2(K,dx)$. More precisely, if we assume up to now that all sections are smooth, then we have the subspace of $L^2(K,dx)$ consisting of smooth functions. The vertically polarized Hilbert space is then the completion of this space, which is just $L^2(K,dx)$.

2.5. **Pairing.** Geometric quantization gives a way to define a *pairing* between the Kähler-polarized and vertically polarized Hilbert spaces, that is, a sesquilinear map from $H_{Kähler}\times H_{Vertical}$ into $\mathbb{C}$. This pairing then induces a linear map between the two spaces, called the *pairing map*. The main results are: 1) the pairing map coincides up to a constant with the generalized Segal-Bargmann transform of [H1], and 2) a constant multiple of the pairing map is unitary from the vertically polarized Hilbert space onto the Kähler-polarized Hilbert space.

Now the elements of the Kähler-polarized Hilbert space are polarized sections of $L\otimes\delta_1$ and the elements of the vertically polarized Hilbert space are polarized sections of $L\otimes\delta_2$. Here δ_1 and δ_2 are square roots of the canonical bundle for the

Kähler polarization and the vertical polarization, respectively. The pairing of the Hilbert spaces will be achieved by appropriately pairing the sections at each point and then integrating over $T^*(K)$ with respect to the canonical volume form ε. (See [Wo, p. 234].)

A polarized section s_1 of $L \otimes \delta_1$ can be expressed as $s_1 = f_1 \otimes \beta_1$, where f_1 is a Kähler-polarized section of L and β_1 is a polarized section of δ_1. Similarly, a polarized section of $L \otimes \delta_2$ is expressible as $s_2 = f_2 \otimes \beta_2$ with f_2 a vertically polarized section of L and β_2 a polarized section of δ_2. We define a pairing between β_1 and β_2 by

$$(\beta_1, \beta_2) = \sqrt{\frac{\overline{\beta_1^2} \wedge \beta_2^2}{i^n \varepsilon}}.$$

Note that β_1^2 and β_2^2 are n-forms on $T^*(K)$, so that $\overline{\beta_1^2} \wedge \beta_2^2$ is a $2n$-form on $T^*(K)$. Note that (β_1, β_2) is a complex-valued function on $T^*(K)$. There are at most two continuous ways of choosing the sign of the square root, which differ just by a single overall sign. That there is at least one such choice will be evident below.

We then define the pairing of two sections s_1 and s_2 (as in the previous paragraph) by

$$\langle s_1, s_2 \rangle_{pair} = \int_{T^*(K)} (f_1, f_2)(\beta_1, \beta_2)\, \varepsilon \tag{2.15}$$

whenever the integral is well-defined. Here as usual ε is the Liouville volume form on $T^*(K)$. It is easily seen that this expression is independent of the decomposition of s_i as $f_i \otimes \beta_i$. The quantity (f_1, f_2) is computed using the (trivial) Hermitian structure on the line bundle L. Although the integral in (2.15) may not be absolutely convergent in general, there are dense subspaces of the two Hilbert spaces for which it is. Furthermore, Theorem 2.6 shows that the pairing can be extended by continuity to all s_1, s_2 in their respective Hilbert spaces.

Now, we have expressed concretely the polarized sections of $L \otimes \delta_1$ in the form

$$e^{-|Y|^2/2\hbar} F \otimes \sqrt{Z_1 \wedge \cdots \wedge Z_n}$$

where F is a holomorphic function on $K_{\mathbb{C}}$ and $Z_1, \cdots, Z_n$ are left-invariant holomorphic 1-forms on $K_{\mathbb{C}}$. As always we identify $K_{\mathbb{C}}$ with $T^*(K)$ as in (2.3). The function $|Y|^2$ is the Kähler potential (2.4). We have expressed the polarized section of $L \otimes \delta_2$ in the form

$$f(x) \otimes \sqrt{pr^*(\eta_1 \wedge \cdots \wedge \eta_n)}.$$

Here $f(x)$ is a function on $T^*(K)$ that is constant along the fibers, $\eta_1, \cdots, \eta_n$ are left-invariant 1-forms on K, and $pr : T^*(K) \to K$ is the projection map.

Thus we have the following expression for the pairing

$$\langle F, f \rangle_{pair} = \int_K \int_{\mathfrak{k}} \overline{F(xe^{iY})} f(x)\, e^{-|Y|^2/2\hbar} \zeta(Y)\, dx\, dY, \tag{2.16}$$

where ζ is the function on $T^*(K)$ given by

$$\zeta = \sqrt{\frac{\bar{Z}_1 \wedge \cdots \wedge \bar{Z}_n \wedge pr^*(\eta_1 \wedge \cdots \wedge \eta_n)}{i^n \varepsilon}}. \tag{2.17}$$

I have expressed things in terms of the functions F and f, and I have used the identification (2.1) of $T^*(K)$ with $K \times \mathfrak{k}$. It is easily seen that $\zeta(x, Y)$ is independent of x, and so I have written $\zeta(Y)$.

Theorem 2.6. *Let us identify the vertically polarized Hilbert space with $L^2(K)$ as in (2.14) and the Kähler-polarized Hilbert space with $\mathcal{H}L^2(T^*(K),\gamma_\hbar)$ as in Theorem 2.4. Then there exists a unique bounded linear operator $\Pi_\hbar : L^2(K) \to \mathcal{H}L^2(T^*(K),\gamma_\hbar)$ such that*

$$\langle F, f\rangle_{pair} = \langle F, \Pi_\hbar f\rangle_{L^2(K)} = \langle \Pi_\hbar^* F, f\rangle_{\mathcal{H}L^2(T^*(K),\gamma_\hbar)}$$

for all $f \in L^2(K)$ and all $F \in \mathcal{H}L^2(T^*(K),\gamma_\hbar)$. We call $\Pi_\hbar$ the ***pairing map****.*

The pairing map has the following properties.

1) There exists a constant $a_\hbar$ such that for any $f \in L^2(K)$, $\Pi_\hbar f$ is the unique holomorphic function on $T^*(K)$ whose restriction to K is given by

$$(\Pi_\hbar f)|_K = a_\hbar e^{\hbar\Delta_K/2} f.$$

Equivalently,

$$\Pi_\hbar f(g) = a_\hbar \int_K \rho_\hbar(gx^{-1}) f(x)\, dx, \quad g \in K_{\mathbb{C}},$$

where $\rho_\hbar$ is the heat kernel on K, analytically continued to $K_{\mathbb{C}}$.

2) The map $\Pi_\hbar^*$ may be computed as

$$(\Pi_\hbar^* F)(x) = \int_{\mathfrak{k}} F\left(xe^{iY}\right) e^{-|Y|^2/2\hbar} \zeta(Y)\, dY,$$

where ζ is defined by (2.17) and computed in Proposition 2.7.

3) There exists a constant $b_\hbar$ such that $b_\hbar \Pi_\hbar$ is a unitary map of $L^2(K)$ onto $\mathcal{H}L^2(T^*(K),\gamma_\hbar)$. Thus $\Pi_\hbar^* = b_\hbar^{-2}\Pi_\hbar^{-1}$.

The constants $a_\hbar$ and $b_\hbar$ are given explicitly as $a_\hbar = (2\pi\hbar)^{n/2} e^{-|\rho|^2\hbar/2}$ and $b_\hbar = (4\pi\hbar)^{-n/4}$, where ρ is half the sum of the positive roots for K.

Remarks. 1) In the first expression for $\Pi_\hbar f$, the analytic continuation is in the space variable, from K to $T^*(K) \cong K_{\mathbb{C}}$, with $\hbar$ fixed. The map $\Pi_\hbar$ coincides (up to the constant $a_\hbar$) with the generalized Segal-Bargmann transform for K, as described in [H1, Thm. 2].

2) The formula for $\Pi_\hbar^* F$ may be taken literally on a dense subspace of $\mathcal{H}L^2(T^*(K),\gamma_\hbar)$. For general F, however, one should integrate over a ball of radius R in $\mathfrak{k}$ and then take a limit in $L^2(K)$, as in [H2, Thm. 1].

3) The formula for $\Pi_\hbar^*$ is an immediate consequence of the formula (2.16) for the pairing. By computing $\zeta(Y)$ explicitly we may recognize $\Pi_\hbar^*$ as simply a constant times the *inverse Segal-Bargmann transform* for K, as described in [H2].

4) In [H2] I deduce the unitarity of the generalized Segal-Bargmann transform from the inversion formula. However, I do not know how to prove the unitarity of the pairing map without recognizing that the measure in the formula for $\Pi_\hbar^*$ is related to the heat kernel measure for $K_{\mathbb{C}}/K$.

5) Since F is holomorphic, there can be many different formulas for $\Pi_\hbar^*$ (or $\Pi_\hbar^{-1}$). In particular, if one takes the second expression for $\Pi_\hbar$ and computes the adjoint in the obvious way, one will *not* get the given expression for $\Pi_\hbar^*$. Nevertheless, the two expressions for $\Pi_\hbar^*$ do agree on holomorphic functions.

Proof. We begin by writing the explicit formula for ζ.

Proposition 2.7. *The function ζ is an Ad-K-invariant function on $\mathfrak{k}$ which is given on a maximal abelian subalgebra by*

$$\zeta(Y) = \prod_{\alpha \in R^+} \frac{\sinh \alpha(Y/2)}{\alpha(Y/2)},$$

where R^+ is a system of positive roots.

The proof of this proposition is a straightforward but tedious calculation, which I defer to an appendix.

Directly from the formula (2.15) for the pairing map we see that

$$\langle F, f \rangle_{pair} = \langle \Pi_\hbar^* F, f \rangle_{L^2(K)}, \tag{2.18}$$

where $\Pi_\hbar^*$ is defined by

$$(\Pi_\hbar^* F)(x) = \int_{\mathfrak{k}} F\left(xe^{iY}\right) e^{-|Y|^2/2\hbar} \zeta(Y)\, dY.$$

At the moment it is not at all clear that $\Pi_\hbar$ is a bounded operator, but there is a dense subspace of $\mathcal{H}L^2(T^*(K), \gamma_\hbar)$ on which $\Pi_\hbar$ makes sense and for which (2.18) holds. We will see below that $\Pi_\hbar$ extends to a bounded operator on all of $\mathcal{H}L^2(T^*(K), \gamma_\hbar)$, for which (2.18) continues to hold. Then by taking the adjoint of $\Pi_\hbar^*$ we see that $\langle F, f \rangle_{pair} = \langle F, \Pi_\hbar f \rangle_{\mathcal{H}L^2(T^*(K), \gamma_\hbar)}$ as well.

Using the explicit formula for ζ and making the change of variable $Y' = \frac{1}{2}Y$ we have

$$(\Pi_\hbar^* F)(x) = 2^n \int_{\mathfrak{k}} F\left(xe^{2iY'}\right) \left[e^{-2|Y'|^2/\hbar} \prod_{\alpha \in R^+} \frac{\sinh \alpha(Y')}{\alpha(Y')}\, dY' \right]$$

We recognize from [H3] the expression in square brackets as a constant times the heat kernel measure on $K_{\mathbb{C}}/K$, written in exponential coordinates and evaluated at time $t = \hbar/2$. It follows from the inversion formula of [H2] that

$$\Pi_\hbar^* = c_\hbar C_\hbar^{-1},$$

for some constant $c_\hbar$ and where $C_\hbar$ is the generalized Segal-Bargmann transform of [H1, Thm. 2].

Now, $C_\hbar$ is unitary if we use on $K_{\mathbb{C}} \cong T^*(K)$ the heat kernel measure $\nu_\hbar$. But in Theorem 1 we established that this measure coincides up to a constant with the measure $\gamma_\hbar$. Thus $\Pi_\hbar$ is a constant multiple of a unitary and coincides with $C_\hbar$ up to a constant. This gives us what we want except for computing the constants, which I leave as an exercise for the reader. □

3. Quantization versus reduction

Let me summarize the results of this section before explaining them in detail. It is possible to realize a compact Lie group K as the quotient $K = \mathcal{A}/\mathcal{L}(K)$ where $\mathcal{A}$ is a certain infinite-dimensional Hilbert space and $\mathcal{L}(K)$ is the based loop group over K, which acts freely and isometrically on $\mathcal{A}$. (Here $\mathcal{A}$ is to be interpreted as a space of connections over S^1 and $\mathcal{L}(K)$ as a gauge group.) The cotangent bundle of $\mathcal{A}$ may be identified with the associated complex Hilbert space $\mathcal{A}_{\mathbb{C}}$ and the symplectic quotient $\mathcal{A}_{\mathbb{C}}//\mathcal{L}(K)$ is identifiable with $T^*(K)$. The results of [DH, Wr] (see also the exposition in [H8]) together with the results of this paper may be interpreted as saying that in this case *quantization commutes with reduction.* This means two things. First, if we perform geometric quantization on $\mathcal{A}_{\mathbb{C}}$ and then reduce by

$\mathcal{L}(K)$ the resulting Hilbert space is naturally *unitarily* equivalent to the result of first reducing by $\mathcal{L}(K)$ and then quantizing the reduced manifold $\mathcal{A}_{\mathbb{C}}//\mathcal{L}(K) = T^*(K)$. This result holds using either the vertical or the Kähler polarization; in the Kähler case it is necessary to include the half-form correction. Second, the pairing map between the vertically polarized and Kähler-polarized Hilbert spaces over $\mathcal{A}_{\mathbb{C}}$ descends to the reduced Hilbert spaces and then coincides (up to a constant) with the pairing map for $T^*(K)$. Additional discussion of these ideas is found in [H7, H8]. The first result contrasts with those of Guillemin and Sternberg in [GStern]. That paper considers the geometric quantization of compact Kähler manifolds, without half-forms, and exhibits (under suitable regularity assumptions) a one-to-one onto linear map between the "first quantize then reduce" space and the "first reduce and then quantize" space. However, they do not show that this map is unitary, and it seems very unlikely that it is unitary in general. In the case considered in this paper and [DH], quantization commutes *unitarily* with reduction.

Consider then a Lie group K of compact type, with a fixed Ad-K-invariant inner product on its Lie algebra $\mathfrak{k}$. Then consider the real Hilbert space

$$\mathcal{A} := L^2([0,1];\mathfrak{k}).$$

Let $\mathcal{L}(K)$ denote the *based loop group* for K, namely the group of maps $l:[0,1]\to K$ such that $l_0 = l_1 = e$. (For technical reasons I also assume that l has one derivative in L^2, i.e. that l has "finite energy.") There is a natural action of $\mathcal{L}(K)$ on $\mathcal{A}$ given by

$$(l\cdot A)_\tau = l_\tau A_\tau l_\tau^{-1} - \frac{dl}{d\tau} l_\tau^{-1}. \tag{3.1}$$

Here l is in $\mathcal{L}(K)$, A is in $\mathcal{A}$, and τ is in $[0,1]$. Then we have the following result: the based loop group $\mathcal{L}(K)$ acts freely and isometrically on $\mathcal{A}$, and the quotient $\mathcal{A}/\mathcal{L}(K)$ is a finite-dimensional manifold that is isometric to K. Thus K, which is finite-dimensional but with non-trivial geometry, can be realized as a quotient of $\mathcal{A}$, which is infinite-dimensional but flat.

Explicitly the quotient map is given in terms of the *holonomy*. For $A\in\mathcal{A}$ we define the holonomy $h(A)\in K$ by the "path-ordered integral"

$$\begin{aligned} h(A) &= \mathcal{P}\left(e^{\int_0^1 A_\tau\,d\tau}\right) \\ &= \lim_{N\to\infty} e^{\int_0^{1/N} A_\tau d\tau} e^{\int_{1/N}^{2/N} A_\tau d\tau}\cdots e^{\int_{(N-1)/N}^{1} A_\tau d\tau}. \end{aligned} \tag{3.2}$$

Then it may be shown that A and B are in the same orbit of $\mathcal{L}(K)$ if and only if $h(A) = h(B)$. Furthermore, every $x\in K$ is the holonomy of some $A\in\mathcal{A}$, and so the $\mathcal{L}(K)$-orbits are in one-to-one correspondence with points in K. The motivation for these constructions comes from gauge theory. The space $\mathcal{A}$ is to be thought of as the space of connections for a trivial principal K-bundle over S^1, in which case $\mathcal{L}(K)$ is the based gauge group and (3.1) is a gauge transformation. For connections A over S^1 the only quantity invariant under (based) gauge transformations is the holonomy $h(A)$ around the circle. See [DH] or [H8] for further details.

Meanwhile, we may consider the cotangent bundle of $\mathcal{A}$, $T^*(\mathcal{A})$, which may be identified with

$$\mathcal{A}_{\mathbb{C}} := L^2([0,1];\mathfrak{k}_{\mathbb{C}}).$$

Then $\mathcal{A}_{\mathbb{C}}$ is an infinite-dimensional flat Kähler manifold. The action of the based loop group $\mathcal{L}(K)$ on $\mathcal{A}$ extends in a natural way to an action on $\mathcal{A}_{\mathbb{C}}$ (given by the same formula). Starting with $\mathcal{A}_{\mathbb{C}}$ we may construct the symplectic (or Marsden-Weinstein) quotient $\mathcal{A}_{\mathbb{C}}//\mathcal{L}(K)$. This quotient is naturally identifiable with $T^*(\mathcal{A}/\mathcal{L}(K)) = T^*(K)$. One may also realize the symplectic quotient as $\mathcal{A}_{\mathbb{C}}/\mathcal{L}(K_{\mathbb{C}})$ where $\mathcal{L}(K_{\mathbb{C}})$ is the based loop group over $K_{\mathbb{C}}$. The quotient $\mathcal{A}_{\mathbb{C}}/\mathcal{L}(K_{\mathbb{C}})$ is naturally identifiable with $K_{\mathbb{C}}$. So we have ultimately

$$T^*(K) \cong T^*(\mathcal{A}/\mathcal{L}(K)) \cong \mathcal{A}_{\mathbb{C}}/\mathcal{L}(K_{\mathbb{C}}) \cong K_{\mathbb{C}}.$$

The resulting identification of $T^*(K)$ with $K_{\mathbb{C}}$ is nothing but the one used throughout this paper. The quotient may $\mathcal{A}_{\mathbb{C}}/\mathcal{L}(K_{\mathbb{C}})$ be expressed in terms of the complex holonomy. For $Z \in \mathcal{A}_{\mathbb{C}}$ we define $h_{\mathbb{C}}(Z) \in K_{\mathbb{C}}$ similarly to (3.2). Then the $\mathcal{L}(K_{\mathbb{C}})$-orbits are labeled precisely by the value of $h_{\mathbb{C}}$.

So the manifold $T^*(K)$ that we have been quantizing is a symplectic quotient of the infinite-dimensional flat Kähler manifold $\mathcal{A}_{\mathbb{C}}$. Looking at $T^*(K)$ in this way we may say that we have first reduced $\mathcal{A}_{\mathbb{C}}$ by the loop group $\mathcal{L}(K)$, and then quantized. One may attempt to do things the other way around: *first* quantize $\mathcal{A}_{\mathbb{C}}$ and *then* reduce by $\mathcal{L}(K)$. Motivated by the results of K. Wren [Wr] (see also [L2, Chap. IV.3.8]), Bruce Driver and I considered precisely this procedure [DH]. Although there are technicalities that must be attended to in order to make sense of this, the upshot is that in this case *quantization commutes with reduction*, as explained in the first paragraph of this section.

In the end we have three different procedures for constructing the generalized Segal-Bargmann space for K and the associated Segal-Bargmann transform. The first is the heat kernel construction of [H1], the second is geometric quantization of $T^*(K)$ with a Kähler polarization, and the third is by reduction from $\mathcal{A}_{\mathbb{C}}$. It is not obvious *a priori* that any two of these constructions should agree. That all three agree is an apparent miracle that should be understood better. I expect that if one replaces the compact group K with some other class of Riemannian manifolds, then these constructions will not agree.

Let me now explain how the quantization of $\mathcal{A}_{\mathbb{C}}$ and the reduction by $\mathcal{L}(K)$ are done in [DH]. (See also the expository article [H8].) In the interests of conveying the main ideas I will permit myself to gloss over various technical issues that are dealt with carefully in [DH]. Although [DH] does not use the language of geometric quantization, it can easily be reformulated in those terms. Now, the constructions of geometric quantization are not directly applicable in the infinite-dimensional setting. On the other hand, $\mathcal{A}_{\mathbb{C}}$ is just a flat Hilbert space and there are by now many techniques for dealing with its quantization. Driver and I want to first perform quantization on $\mathbb{C}^n$ and then let n tend to infinity. If one performs geometric quantization on $\mathbb{C}^n$ with a Kähler polarization and the half-form correction one gets $\mathcal{H}L^2(\mathbb{C}^n, \nu_\hbar)$ where

$$d\nu_\hbar = e^{-(\operatorname{Im} z)^2/\hbar}\, dz.$$

See Section 5 below.

In this form we cannot let the dimension go to infinity because the measure is Gaussian only in the imaginary directions. So we introduce a regularization parameter $s > \hbar/2$ and modify the measure to

$$dM_{s,\hbar} = (\pi r)^{-n/2} (\pi\hbar)^{-n/2} e^{-(\operatorname{Im} z)^2/\hbar} e^{-(\operatorname{Re} z)^2/r},$$

where $r = 2(s - \hbar/2)$. The constants are chosen so that $M_{s,\hbar}$ is a probability measure. If one rescales $M_{s,\hbar}$ by a suitable function of s and then lets s tend to infinity one recovers the measure $\nu_\hbar$. Our Hilbert space is then just $\mathcal{H}L^2(\mathbb{C}^n, M_{s,\hbar})$. Now we can let the dimension tend to infinity, and we get

$$\mathcal{H}L^2\left(\overline{\mathcal{A}_{\mathbb{C}}}, M_{s,\hbar}\right),$$

where $M_{s,\hbar}$ is a Gaussian measure on a certain "extension" $\overline{\mathcal{A}_{\mathbb{C}}}$ of $\mathcal{A}_{\mathbb{C}}$. (See [DH, Sect. 4.1].) This we think of as the (regularized) Kähler-polarized Hilbert space.

Our next task is to perform the reduction by $\mathcal{L}(K)$, which means looking for functions in $\mathcal{H}L^2(\overline{\mathcal{A}_{\mathbb{C}}}, M_{s,\hbar})$ that are "invariant" in the appropriate sense under the action of $\mathcal{L}(K)$. The notion of invariance should itself come from geometric quantization, by "quantizing" the action of $\mathcal{L}(K)$ on $\mathcal{A}_{\mathbb{C}}$. Note that $\mathcal{L}(K)$ acts on $\mathcal{A}$ by a combination of rotations and translations; the action of $\mathcal{L}(K)$ on $\mathcal{A}_{\mathbb{C}}$ is then induced from its action on $\mathcal{A}$. Let us revert temporarily to the finite-dimensional situation as in Section 4. Then the way we have chosen our 1-form θ and our Kähler potential κ means that the rotations and translations of $\mathbb{R}^n$ act in the Kähler-polarized Hilbert space $\mathcal{H}L^2(\mathbb{C}^n, \nu_\hbar)$ in the simplest possible way, namely by rotating and translating the variables. (This is not the case in the conventional form of the Segal-Bargmann space.) We will then formally extend this notion to the infinite-dimensional case, which means that an element l of $\mathcal{L}(K)$ acts on a function $F \in \mathcal{H}L^2(\overline{\mathcal{A}_{\mathbb{C}}}, M_{s,\hbar})$ by $F(Z) \to F\left(l^{-1} \cdot Z\right)$.

We want functions in $\mathcal{H}L^2(\overline{\mathcal{A}_{\mathbb{C}}}, M_{s,\hbar})$ that are invariant under this action, i.e. such that $F\left(l^{-1} \cdot Z\right) = F(Z)$ for all $l \in \mathcal{L}(K)$. Since our functions are holomorphic the must also (at least formally) be invariant under $\mathcal{L}(K_{\mathbb{C}})$. So we expect the invariant functions to be those of the form

$$F(Z) = \Phi\left(h_{\mathbb{C}}(Z)\right)$$

where Φ is a holomorphic function on $K_{\mathbb{C}}$. (Certainly every such function is $\mathcal{L}(K)$-invariant. Although Driver and I did not prove that every $\mathcal{L}(K)$-invariant function is of this form, this is probably the case.) The norm of such a function may be computed as

$$\int_{\overline{\mathcal{A}_{\mathbb{C}}}} |F(Z)|^2 \, dM_{s,\hbar}(Z) = \int_{K_{\mathbb{C}}} |\Phi(g)|^2 \, d\mu_{s,\hbar}(g)$$

where $\mu_{s,\hbar}$ is the push-forward of $M_{s,\hbar}$ to $K_{\mathbb{C}}$ under $h_{\mathbb{C}}$. Concretely $\mu_{s,\hbar}$ is a certain heat kernel measure on $K_{\mathbb{C}}$. See [DH] or [H5] for details.

So our regularized reduced quantum Hilbert space is

$$\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\hbar})$$

At this point we may remove the regularization by letting s tend to infinity. It can be shown that

$$\lim_{s \to \infty} \mu_{s,\hbar} = \nu_\hbar$$

where $\nu_\hbar$ is the K-invariant heat kernel measure of [H1]. So without the regularization our reduced quantum Hilbert space becomes finally

$$\mathcal{H}L^2(K_{\mathbb{C}}, \nu_\hbar),$$

which (up to a constant) is the same as $\mathcal{H}L^2(T^*(K), \gamma_\hbar)$, using our identification of $T^*(K)$ with $K_{\mathbb{C}}$.

Meanwhile the vertically polarized Hilbert space for $\mathbb{C}^n$ also requires a regularization before we let n tend to infinity. So we consider $L^2(\mathbb{R}^n, P_s)$, where P_s is the Gaussian measure given by

$$dP_s(x) = (2\pi s)^{-n/2} e^{-|x|^2/2s}.$$

Rescaling P_s by a function of s and then letting s tend to infinity gives back Lebesgue measure on $\mathbb{R}^n$. We then consider the Segal-Bargmann transform $S_\hbar$, which coincides with the pairing map of geometric quantization (Section 5). This is given by

$$S_\hbar f(z) = (2\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-(z-x)^2/2t} f(x)\, dx.$$

With the constants adjusted as above this map has the property that it is unitary between our regularized spaces $L^2(\mathbb{R}^n, P_s)$ and $\mathcal{H}L^2(\mathbb{C}^n, M_{s,\hbar})$, for all $s > \hbar/2$. (See [DH, Sect. 3.1] or [H5].)

Letting the dimension tend to infinity we get a unitary map [DH, Sect. 4.1]

$$S_\hbar : L^2\left(\overline{\mathcal{A}}, P_s\right) \to \mathcal{H}L^2(\overline{\mathcal{A}_{\mathbb{C}}}, M_{s,\hbar}). \tag{3.3}$$

It seems reasonable to think of this as the infinite-dimensional regularized version of the pairing map for $\mathcal{A}_{\mathbb{C}}$. To reduce by $\mathcal{L}(K)$ we consider functions in $L^2\left(\overline{\mathcal{A}}, P_s\right)$ that are $\mathcal{L}(K)$-invariant. According to an important theorem of Gross [G1] these are (as expected) precisely those of the form

$$f(A) = \phi(h(A)), \tag{3.4}$$

where ϕ is a function on K. The norm of such a function is computed as

$$\int_{\overline{\mathcal{A}}} |f(A)|^2 \, dP_s(A) = \int_K |\phi(x)|^2 \, d\rho_s(x).$$

Thus with the vertical polarization our reduced Hilbert space becomes $L^2(K, \rho_s)$. Since

$$\lim_{s\to\infty} d\rho_s(x) = dx$$

we recover in the limit the vertically polarizes subspace for K. (Compare [Go].)

Theorem 3.1. [DH] *Consider the Segal-Bargmann transform $S_{s,\hbar}$ of (3.3). Then consider a function $f \in L^2\left(\overline{\mathcal{A}}, P_s\right)$ of the form $f(A) = \phi(h(A))$, with ϕ a function on K. Then*

$$(S_\hbar f)(Z) = \Phi(h_{\mathbb{C}}(Z))$$

where Φ is the holomorphic function on $K_{\mathbb{C}}$ given by

$$\Phi = \text{ analytic continuation of } e^{\hbar\Delta_K/2}\phi.$$

Restricting $S_\hbar$ to the $\mathcal{L}(K)$-invariant subspace and then letting $s \to \infty$ gives the unitary map

$$C_\hbar : L^2(K, dx) \to \mathcal{H}L^2(K_{\mathbb{C}}, \nu_\hbar)$$

given by $\phi \to$ analytic continuation of $e^{\hbar\Delta_K/2}\phi$.

If we restrict $S_\hbar$ to the $\mathcal{L}(K)$-invariant subspace but keep s finite, then we get a modified form of the Segal-Bargmann transform for K, a unitary map $L^2(K, \rho_s) \rightarrow \mathcal{H}L^2(K_\mathbb{C}, \mu_{s,\hbar})$, still given by $\phi \rightarrow$ analytic continuation of $e^{\hbar\Delta_K/2}\phi$. This transform is examined from a purely finite-dimensional point of view in [H5].

So if we accept the constructions of [DH] as representing regularized forms of the geometric quantization Hilbert spaces and pairing map, then we have the following conclusions. First, the Kähler-polarized and vertically polarized Hilbert spaces for $\mathcal{A}_\mathbb{C}$, after reducing by $\mathcal{L}(K)$ and removing the regularization, are naturally unitarily equivalent to the Kähler-polarized and vertically polarized Hilbert spaces for $T^*(K) = \mathcal{A}_\mathbb{C}//\mathcal{L}(K)$. (I am including the half-forms in the construction of the Kähler-polarized Hilbert spaces.) Second, the pairing map for $\mathcal{A}_\mathbb{C}$, after restricting to the $\mathcal{L}(K)$-invariant subspace and removing the regularization, coincides with the pairing map for $T^*(K)$. Both of these statements are to be understood "up to a constant."

4. The geodesic flow and the heat equation

This section describes how the complex polarization on $T^*(K)$ can be obtained from the vertical polarization by means of the *imaginary-time geodesic flow.* This description is supposed to make the appearance of the heat equation in the pairing map seem more natural. After all the heat operator is nothing but the *imaginary-time **quantized** geodesic flow.* This point of view is due to T. Thiemann [T1, T2].

Suppose that f is a function on K and let $\pi : T^*(K) \rightarrow K$ be the projection map. Then $f \circ \pi$ is the extension of f to $T^*(K)$ that is constant along the fibers. A function of the form $f \circ \pi$ is a "vertically polarized function," that is, constant along the leaves of the vertical polarization. Now recall the function $\kappa : T^*(K) \rightarrow \mathbb{R}$ given by

$$\kappa(x, Y) = |Y|^2.$$

Let Γ_t be the Hamiltonian flow on $T^*(K)$ generated by the function $\kappa/2$. This is the geodesic flow for the bi-invariant metric on K determined by the inner product on the Lie algebra. The following result gives a way of using the geodesic flow to produce a holomorphic function on $T^*(K)$.

Theorem 4.1. *Let $f : K \rightarrow \mathbb{C}$ be any function that admits an entire analytic continuation to $T^*(K) \cong K_\mathbb{C}$, for example, a finite linear combination of matrix entries. Let $\pi : T^*(K) \rightarrow K$ be the projection map, and let Γ_t be the geodesic flow on $T^*(K)$.*

Then for each $m \in T^*(K)$ the map

$$t \rightarrow f(\pi(\Gamma_t(m)))$$

admits an entire analytic continuation (in t) from $\mathbb{R}$ to $\mathbb{C}$. Furthermore the function $f_\mathbb{C} : T^*(K) \rightarrow \mathbb{C}$ given by

$$f_\mathbb{C}(m) = f(\pi(\Gamma_i(m)))$$

is holomorphic on $T^*(K)$ and agrees with f on $K \subset T^*(K)$.

Note that $f_\mathbb{C}$ is the analytic continuation of f from K to $T^*(K)$, with respect to the complex structure on $T^*(K)$ obtained by identifying it with $K_\mathbb{C}$. So in words: to analytically continue f from K to $T^*(K)$, first extend f by making it constant along the fibers and then compose with the time i geodesic flow. So we can say that the Kähler-polarized functions (i.e. holomorphic) are obtained from the vertically polarized functions (i.e. constant along the fibers) by composition with the time i geodesic flow.

Now if g is any function on $T^*(K)$ then $g \circ \Gamma_t$ may be computed formally as

$$g \circ \Gamma_t = \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} \underbrace{\{\cdots\{\{g,\kappa\},\kappa\},\cdots,\kappa\}}_{n}.$$

Thus formally we have

$$f_{\mathbb{C}} = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!} \underbrace{\{\cdots\{\{f\circ\pi,\kappa\},\kappa\},\cdots,\kappa\}}_{n}. \tag{4.1}$$

(Compare [T1, Eq. (2.3)].) In fact, this series converges provided only that f has an analytic continuation to $T^*(K)$. This series is the "Taylor series in the fibers" of $f_{\mathbb{C}}$; that is, on each fiber the nth term of (4.1) is a homogeneous polynomial of degree n.

Theorem 4.2. *Suppose f is any function on K that admits an entire analytic continuation to $T^*(K)$, denoted $f_{\mathbb{C}}$. Then the series on the right in (4.1) absolutely at every point and the sum is equal to $f_{\mathbb{C}}$.*

As an illustrative example, consider the case $K = \mathbb{R}$ so that $T^*(K) = \mathbb{R}^2$. Then consider the function $f(x) = x^k$ on $\mathbb{R}$, so that $(f\circ\pi)(x,y) = x^k$. Using the standard Poisson bracket on $\mathbb{R}^2$, $\{g,h\} = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}$ it is easily verified that

$$\sum_{n=0}^{\infty} \frac{(i/2)^n}{n!} \underbrace{\{\cdots\{\{x^k,y^2\},y^2\},\cdots,y^2\}}_{n} = (x+iy)^k.$$

(The series terminates after the $n = k$ term.) So $f_{\mathbb{C}}(x+iy) = (x+iy)^k$ is indeed the analytic continuation of x^k.

So "classically" the transition from the vertical polarization (functions constant along the fibers) to the Kähler polarization (holomorphic functions) is accomplished by means of the time i geodesic flow. Let us then consider the quantum counterpart of this, namely the transition from the vertically polarized Hilbert space to the Kähler-polarized Hilbert space. In the position Hilbert space the quantum counterpart of the function $\kappa/2$ is the operator

$$H := -\hbar^2 \Delta_K/2.$$

(Possibly one should add an "author-dependent" multiple of the scalar curvature to this operator [O], but since the scalar curvature of K is constant, this does not substantively affect the answer.) The quantum counterpart of the geodesic flow is then the operator

$$\hat{\Gamma}_t := \exp(itH/\hbar)$$

and so the time i quantized geodesic flow is represented by the operator

$$\hat{\Gamma}_i = e^{\hbar\Delta_K/2}.$$

Since this is precisely the heat operator for K, the appearance of the heat operator in the formula for the pairing map perhaps does not seem quite so strange as at first glance.

This way of thinking about the complex structure and the associated Segal-Bargmann transform is due to T. Thiemann [T1]. The relationship between the complex structure and the imaginary time geodesic flow is also implicit in the work of Guillemin-Stenzel, motivated by the work of L. Boutet de Monvel. (See the discussion between Thm. 5.2 and 5.3 in [GStenz2].) Thiemann proposes a very general scheme for building complex structures and Segal-Bargmann transforms (and their associated "coherent states") based on these ideas. However, there are convergence issues that need to be resolved in general, so it is not yet clear when one can carry this program out.

Although results similar to Theorems 4.1 and 4.2 are established in [T2, Lem. 3.1], I give the proofs here for completeness. Similar results hold for the "adapted complex structure" on the tangent bundle of an real-analytic Riemannian manifold, which will be described elsewhere.

Proof. According to a standard result [He, Sect. IV.6], the geodesics in K are the curves of the form $\gamma(t) = xe^{tX}$, with $x \in K$ and $X \in \mathfrak{k}$. This means that if we identify $T^*(K)$ with $K \times \mathfrak{k}$ by left-translation, then the geodesic flow takes the form

$$\Gamma_t(x, Y) = \left(xe^{tY}, Y\right).$$

Thus if f is a function on K then

$$f(\pi(\Gamma_t(x, Y))) = f\left(xe^{tY}\right).$$

We are now supposed to fix x and Y and consider the map $t \to f\left(xe^{tY}\right)$. If f has an analytic continuation to $K_{\mathbb{C}}$, denoted $f_{\mathbb{C}}$, then the map $t \to f\left(xe^{tY}\right)$ has an analytic continuation (in t) given by

$$t \to f_{\mathbb{C}}\left(xe^{tY}\right), \quad t \in \mathbb{C}.$$

(This because the exponential mapping from $\mathfrak{k}_{\mathbb{C}}$ to $K_{\mathbb{C}}$ is holomorphic.) Thus

$$f(\pi(\Gamma_i(x, Y))) = f_{\mathbb{C}}\left(xe^{iY}\right).$$

Now we simply note that the map $(x, Y) \to f_{\mathbb{C}}\left(xe^{iY}\right)$ is holomorphic on $T^*(K)$, with respect to the complex structure obtained by the map $\Phi(x, Y) = xe^{iY}$. This establishes Theorem 4.1.

To establish the series form of this result, Theorem 4.2, we note that (almost) by the definition of the geodesic flow we have

$$\left(\frac{d}{dt}\right)^n (f \circ \pi) \circ \Gamma_t \bigg|_{t=0} = \frac{1}{2^n} \underbrace{\{\cdots \{\{f \circ \pi, \kappa\}, \kappa\}, \cdots, \kappa\}}_{n}. \tag{4.2}$$

On the other hand, if f has an entire analytic continuation to $T^*(K) \cong K_{\mathbb{C}}$, then as established above, the map $t \to (f \circ \pi) \circ \Gamma_t$ has an entire analytic continuation. This analytic continuation can be computed by an absolutely convergent Taylor series at $t = 0$, where the Taylor coefficients at $t = 0$ are computable from (4.2). Thus

$$f_{\mathbb{C}} = (f \circ \pi) \circ \Gamma_i = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!} \underbrace{\{\cdots \{\{f \circ \pi, \kappa\}, \kappa\}, \cdots, \kappa\}}_{n}.$$

This establishes Theorem 4.2. $\square$

5. The $\mathbb{R}^n$ case

It is by now well known that geometric quantization can be used to construct the Segal-Bargmann space for $\mathbb{C}^n$ and the associated Segal-Bargmann transform. (See for example [Wo, Sect. 9.5].) In this section I repeat that construction, but in a manner that is non-standard in two respects. First, I trivialize the quantum line bundle in such a way that the measure in the Segal-Bargmann space is Gaussian only in the imaginary directions. Although this is preferable from the point of view of generalizing to the group case, it is a simple matter in the $\mathbb{R}^n$ case to convert back to the standard Segal-Bargmann space (see below). Second, I initially compute the pairing map "backward," that is, from the Segal-Bargmann space to $L^2(\mathbb{R}^n)$. I then describe this backward map in terms of the backward heat equation, which leads to a description of the forward map in terms of the forward heat equation. By contrast, Woodhouse uses the reproducing kernel for the Segal-Bargmann space in order to compute the pairing map in the forward direction. I include the half-form correction on the complex side, even though this has no effect on the calculations in the $\mathbb{R}^n$ case.

We consider the phase space $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$. We use the coordinates $q_1, \cdots, q_n$, $p_1, \cdots, p_n$, where the q's are the position variables and the p's are the momentum variables. We consider the *canonical one form*

$$\theta = \sum p_k \, dq_k$$

where here and in the following the sum ranges from 1 to n. Then

$$\omega := -d\theta = \Sigma q_k \wedge p_k$$

is the canonical 2-form. We consider a trivial complex line bundle $L = \mathbb{R}^{2n} \times \mathbb{C}$ with a notion of covariant derivative given by

$$\nabla_X = X - \frac{1}{i\hbar}\theta(X).$$

Here ∇_X acts on smooth sections of L, which we think of as smooth functions on $\mathbb{R}^{2n}$.

The *prequantum Hilbert space* is the space of sections of L that are square-integrable with respect to the canonical volume measure on $\mathbb{R}^{2n}$. The canonical volume measure is the one given by integrating the *Liouville volume form*

$$\varepsilon := \frac{1}{n!}\omega \wedge \cdots \wedge \omega \quad (n \text{ times}).$$

Since our prequantum line bundle is trivial we may identify the prequantum Hilbert space with $L^2\left(\mathbb{R}^{2n}, \varepsilon\right)$.

We now consider the usual complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. We think of this complex structure as defining a *Kähler polarization* on $\mathbb{R}^{2n}$. This means that we define a smooth section s of L to be *polarized* if

$$\nabla_{\partial/\partial \bar{z}_k} s = 0 \tag{5.1}$$

for all k.

Proposition 5.1. *If we think of sections s of L as functions then a smooth section s satisfies (5.1) if and only if s is of the form*

$$s(q,p) = e^{-p^2/2\hbar} F(q_1 + ip_1, \cdots, q_n + ip_n) \tag{5.2}$$

where F is a holomorphic function on $\mathbb{C}^n$. Here $p^2 = p_1^2 + \cdots + p_n^2$.

Proof. To prove this we first compute $\nabla_{\partial/\partial\bar{z}_k}$ as

$$\begin{aligned}\nabla_{\partial/\partial\bar{z}_k} &= \frac{\partial}{\partial\bar{z}_k} - \frac{1}{i\hbar}\theta\left(\frac{\partial}{\partial\bar{z}_k}\right)\\ &= \frac{\partial}{\partial\bar{z}_k} - \frac{1}{2i\hbar}p.\end{aligned}$$

Then we note that

$$\begin{aligned}\nabla_{\partial/\partial\bar{z}_k} e^{-p^2/2\hbar} &= \left[\frac{1}{2}\left(\frac{\partial}{\partial q_k} + i\frac{\partial}{\partial p_k}\right)\left(-\frac{p^2}{2\hbar}\right) - \frac{1}{2i\hbar}p\right]e^{-p^2/2\hbar}\\ &= \left[-i\frac{p}{2\hbar} - \frac{1}{2i\hbar}p\right]e^{-p^2/2\hbar}\\ &= 0.\end{aligned}$$

Then if s is any section, we can write s in the form $s = e^{-p^2/2\hbar}F$, for some complex-valued function F. Such a section s is polarized if and only if

$$\begin{aligned}0 &= \nabla_{\partial/\partial\bar{z}_k}\left(e^{-p^2/2\hbar}F\right)\\ &= \left(\nabla_{\partial/\partial\bar{z}_k} e^{-p^2/2\hbar}\right)F + e^{-p^2/2\hbar}\frac{\partial F}{\partial\bar{z}_k}\\ &= e^{-p^2/2\hbar}\frac{\partial F}{\partial\bar{z}_k},\end{aligned}$$

for all k, that is, if and only if F is holomorphic. □

We then define the *Kähler-polarized Hilbert space* to be the space of square-integrable Kähler-polarized sections of L. Note that the L^2 norm of the section s in (5.2) is computable as

$$\|s\|^2 = \int_{\mathbb{C}^n} |F(z)|^2 e^{-p^2/\hbar}\, dz,$$

where $z = q + ip$ and dz is $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$. If we identify the polarized section s with the holomorphic function F then we identify the Kähler-polarized Hilbert space as the space

$$\mathcal{H}L^2(\mathbb{C}^n, e^{-p^2/\hbar}dz). \tag{5.3}$$

Here $\mathcal{H}L^2$ denotes the space of square-integrable holomorphic functions with respect to the indicated measure. This space is a form of the *Segal-Bargmann space.*

The conventional description [Wo, Sect. 9.2] of the Segal-Bargmann space is slightly different form what we have here, for two reasons. First, it is conventional to insert a factor of $\sqrt{2}$ into the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Second, it is common to use a different trivialization of L, resulting in a different Gaussian measure on $\mathbb{C}^n$. The map $F \to e^{z^2/4\hbar}F$ maps "my" Segal-Bargmann space unitarily to $\mathcal{H}L^2(\mathbb{C}^n, e^{-|z|^2/2\hbar}dz)$, which is essentially the standard Segal-Bargmann space. The normalization used here for the $\mathbb{R}^n$ case is the one that generalizes to the group case.

We also define the *canonical bundle* (relative to the given complex structure) to be the bundle whose sections are n-forms of type $(n, 0)$. We then define the *half-form bundle* δ_1 to be the square root of the canonical bundle. The polarized sections of δ_1 are objects of the form

$$F(z)\sqrt{dz_1 \wedge \cdots \wedge dz_n},$$

where F is holomorphic. Here the square root is a mnemonic for a polarized section of δ_1 whose square is $dz_1\wedge\cdots\wedge dz_n$. The absolute value of such a section is computed by setting

$$\begin{aligned}\left|\sqrt{dz_1\wedge\cdots\wedge dz_n}\right|^2 &= \left[\frac{d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n\wedge dz_1\wedge\cdots\wedge dz_n}{(2i)^n\,\varepsilon}\right]^{1/2}\\ &= 1.\end{aligned}\tag{5.4}$$

The *half-form-corrected Hilbert space* is then the space of square-integrable polarized sections of $L\otimes\delta_1$. Polarized sections of $L\otimes\delta_1$ may be expressed uniquely as

$$s = F(z)\,e^{-p^2/2\hbar}\otimes\sqrt{dz_1\wedge\cdots\wedge dz_n}.\tag{5.5}$$

In light of (5.4) our Hilbert space may again be identified with the Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^n, e^{-p^2/2}\,dz)$. Although in this flat case the half-form correction does not affect in the description of the Hilbert space, it still has an important effect on certain subsequent calculations, such as the WKB approximation. (See [Wo, Chap. 10].)

Next we consider the *vertically polarized sections.* A vertically polarized section s of L is one for which $\nabla_{\partial/\partial p_k}s = 0$ for all k. Identifying sections with functions and using $\theta = \Sigma p_k dq_k$ we see that $\nabla_{\partial/\partial p_k} = \partial/\partial p_k$. Thus the vertically polarized sections are simply functions $f(q,p)$ that are independent of p. Unfortunately, such a section cannot be square-integrable (over $\mathbb{R}^{2n}$) unless it is zero almost everywhere.

So we now consider the *canonical bundle* (relative to the vertical polarization). This is the *real* line bundle whose sections are n-forms α satisfying $(\partial/\partial p_k)\lrcorner\alpha = 0$ for all k. Concretely such forms are precisely those expressible as

$$\alpha = f(q,p)\;dq_1\wedge\cdots\wedge dq_n$$

where f is real-valued. Such a n-form is called *polarized* if $(\partial/\partial p_k)\lrcorner d\alpha = 0$ for all k. Such forms are precisely those expressible as

$$\alpha = f(q)\;dq_1\wedge\cdots\wedge dq_n.$$

We now choose an orientation on $\mathbb{R}^n$ and we construct a square root δ_2 of the canonical bundle in such a way that the square of a section of δ_2 is a non-negative multiple of $dq_1\wedge\cdots\wedge dq_n$, where $q_1,\cdots,q_n$ is an oriented coordinate system for $\mathbb{R}^n$. There is a natural notion of polarized sections of δ_2, namely those whose squares are polarized sections of the canonical bundle. The polarized sections of δ_2 are precisely those of the form

$$\beta = f(q)\sqrt{dq_1\wedge\cdots\wedge dq_n}.\tag{5.6}$$

We then consider the space of polarized sections of $L\otimes\delta_2$. Every such section may be written uniquely in the form

$$s = f(q)\otimes\sqrt{dq_1\wedge\cdots\wedge dq_n},\tag{5.7}$$

where now f is complex-valued. We define the inner product of two such sections s_1 and s_2 by

$$(s_1,s_2) = \int_{\mathbb{R}^n}\overline{f_1}(q)\,f_2(q)\;dq_1\wedge\cdots\wedge dq_n.\tag{5.8}$$

Note that the integration is over $\mathbb{R}^n$ not $\mathbb{R}^{2n}$. The *vertically polarized Hilbert space* is the space of polarized sections s of $L\otimes\delta_2$ for which $(s,s)<\infty$. (This construction is explained in a more manifestly coordinate-independent way in the general group case, in Section 2.4.)

Finally, we introduce the *pairing map* between the vertically polarized and Kähler-polarized Hilbert spaces. First we define a pointwise pairing between sections of δ_1 and sections of δ_2 by setting

$$\begin{aligned}\left(\sqrt{dz_1 \wedge \cdots \wedge dz_n}, \sqrt{dq_1 \wedge \cdots \wedge dq_n}\right) &= \left[\frac{d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge dq_1 \wedge \cdots \wedge dq_n}{i^{-n}\varepsilon}\right]^{1/2} \\ &= 1.\end{aligned}$$

Then we may pair a section of $L \otimes \delta_1$ with a section of $L \otimes \delta_2$ by applying the above pairing of δ_1 and δ_2 and the Hermitian structure on L, and then integrating with respect to ε. So if s_1 is a polarized section of $L \otimes \delta_1$ as in (5.5) and s_2 is a polarized section of $L \otimes \delta_2$ then we have explicitly

$$\langle F, f\rangle_{pair} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \overline{F(q+ip)} f(q)\, e^{-p^2/2\hbar}\, dq\, dp. \tag{5.9}$$

Here I have expressed things in terms of $F \in \mathcal{H}L^2(\mathbb{C}^n, e^{-p^2/\hbar})$ and $f \in L^2(\mathbb{R}^n)$.

Theorem 5.2. *Let us identify the vertically polarized Hilbert space with $L^2(\mathbb{R}^n)$ as in (5.8) and the Kähler-polarized Hilbert space with $\mathcal{H}L^2(\mathbb{C}^n, e^{-p^2/2\hbar}dz)$ as in (5.5). Then there exists a unique bounded linear operator $\Pi_\hbar : L^2(\mathbb{R}^n) \to \mathcal{H}L^2(\mathbb{C}^n, \nu_\hbar)$ such that*

$$\langle f, F\rangle = \langle \Pi_\hbar f, F\rangle_{L^2(\mathbb{R}^n)} = \langle f, \Pi_\hbar^* F\rangle_{\mathcal{H}L^2(\mathbb{C}^n, e^{-p^2/\hbar})}.$$

We call $\Pi_\hbar$ the ***pairing map****. We then have the following results.*

1) The map $\Pi_\hbar : L^2(\mathbb{R}^n) \to \mathcal{H}L^2(\mathbb{C}^n, e^{-p^2/\hbar}dz)$ is given by

$$\Pi_\hbar f(z) = a_\hbar \int_{\mathbb{R}^n} e^{-(z-q)^2/2\hbar} f(q)\, dq$$

where $a_\hbar = (\pi\hbar)^{-n/2}(2\pi\hbar)^{-n}$.

2) The map $\Pi_\hbar^*$ may be computed as

$$(\Pi_\hbar^* F)(q) = \int_{\mathbb{R}^n} F(q+ip)\, e^{-p^2/2\hbar}\, d^n p.$$

3) The map $b_\hbar \Pi_\hbar$ is unitary, where $b_\hbar = (\pi\hbar)^{n/4}(2\pi\hbar)^{n/2}$.

Note that the formula for $\Pi_\hbar^*$ comes almost directly from the formula (5.9) for the pairing. The unitarity (up to a constant) of the pairing map in this $\mathbb{R}^n$ case is "explained" by the Stone-von Neumann theorem. The map $\Pi_\hbar$, as given in 1), is the "invariant" form of the Segal-Bargmann transform, as described, for example in [H6, Sect. 6.3].

There are doubtless many ways of proving these results. I will explain here simply how the heat equation creeps into the argument, since the heat equation is essential to the proof in the group case. Fix a holomorphic function F on $\mathbb{C}^n$ that is square-integrable over $\mathbb{R}^n$ and that has moderate growth in the imaginary directions. Then define a function $f_\hbar$ on $\mathbb{R}^n$ by

$$f_\hbar(q) = \int_{\mathbb{R}^n} F(q+ip)\left[\frac{e^{-p^2/2\hbar}}{(2\pi\hbar)^{n/2}}\right] d^n p. \tag{5.10}$$

Note that the Gaussian factor in the square brackets is just the standard heat kernel in the p-variable and in particular satisfies the forward heat equation $\partial u/\partial t = (1/2)\Delta u$. If we differentiate under the integral sign, integrate by parts, and use the Cauchy-Riemann equations $\partial F/\partial p_k = i\partial F/\partial q_k$ we see that

$$\frac{df_\hbar}{d\hbar} = -\frac{1}{2}\Delta f_\hbar, \tag{5.11}$$

which is the *backward* heat equation. Furthermore, letting $\hbar$ tend to zero we see that

$$\lim_{\hbar\downarrow 0} f_\hbar(q) = F(q). \tag{5.12}$$

Thus (up to a factor of $(2\pi\hbar)^{n/2}$) $\Pi_\hbar^* F$ is obtained by applying the *inverse* heat operator to the restriction of F to $\mathbb{R}^n$. Turning this the other way around we have

$$(\Pi_\hbar^*)^{-1} f = (2\pi\hbar)^{n/2} \left(\text{analytic continuation of } e^{\hbar\Delta/2} f\right) \tag{5.13}$$

where $e^{\hbar\Delta/2} f$ means the solution to the heat operator at time $\hbar$, with initial condition f. Of course, $e^{\hbar\Delta/2} f$ can be computed by integrating f against a Gaussian, so we have

$$(\Pi_\hbar^*)^{-1} f(z) = \int_{\mathbb{R}^n} e^{-(z-q)^2/2\hbar} f(q)\, dq$$

where the factors of $2\pi\hbar$ in (5.13) have canceled those in the computation of the heat operator on $\mathbb{R}^n$.

We now recognize $(\Pi_\hbar^*)^{-1}$ as coinciding up to a constant with the "invariant" form $C_\hbar$ of the Segal-Bargmann transform, as described in [H6, Sect. 6.3]. The unitarity of $C_\hbar$ then implies that $\Pi_\hbar$ is unitary up to a constant. The argument in the general case goes in much the same way, using the inversion formula [H2] for the generalized Segal-Bargmann transform of [H1].

6. Appendix: Calculations with ζ and κ

We will as always identify $T^*(K)$ with $K \times \mathfrak{k}$ by means of left-translation and the inner product on $\mathfrak{k}$. We choose an orthonormal basis for $\mathfrak{k}$ and we let $y_1, \cdots, y_n$ be the coordinates with respect to this basis. Then all forms on $K \times \mathfrak{k}$ can be expressed in terms of the left-invariant 1-forms $\eta_1, \cdots, \eta_n$ on K and the translation-invariant 1-forms $dy_1, \cdots, dy_n$ on $\mathfrak{k}$. Since the canonical projection $pr : T^*(K) \to K$ in this description is just projection onto the K factor, $pr^*(\eta_k)$ is just identified with η_k. We identify the tangent space at each point in $K \times \mathfrak{k}$ with $\mathfrak{k} + \mathfrak{k}$.

Meanwhile we identify the tangent space of $K_{\mathbb{C}}$ at each point with $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{k} + \mathfrak{k}$. We then consider the map Φ that identifies $T^*(K) \cong K \times \mathfrak{k}$ with $K_{\mathbb{C}}$,

$$\Phi(x, Y) = xe^{iY}.$$

Since we are identifying the tangent space at every point of both $K \times \mathfrak{k}$ and $K_{\mathbb{C}}$ with $\mathfrak{k} + \mathfrak{k}$, the differential of Φ at any point will be described as a linear map of $\mathfrak{k} + \mathfrak{k}$ to itself. Explicitly we have [H3, Eq. (14)] at each point (x, Y)

$$\Phi_* = \begin{pmatrix} \cos adY & \frac{1-\cos adY}{adY} \\ -\sin adY & \frac{\sin adY}{adY} \end{pmatrix}. \tag{6.1}$$

So now let us use Φ to pull back the left-invariant anti-holomorphic forms $\bar{Z}_k$ to $T^*(K)$. To do this we compute the adjoint of the matrix (6.1), keeping in mind that adY is skew, since our inner product is Ad-K-invariant. We then get that

$$\begin{aligned}\Phi^*\left(\bar{Z}_k\right) = &\text{ terms involving } \eta_l \\ &- i\left[\frac{\sin adY}{adY} + i\frac{\cos adY - 1}{adY}\right]_{lk} dy_l.\end{aligned}$$

Thus

$$\bar{Z}_1 \wedge \cdots \wedge \bar{Z}_n \wedge \eta_1 \wedge \cdots \wedge \eta_n = \zeta(Y)^2\, i^n \eta_1 \wedge \cdots \wedge \eta_n \wedge dy_1 \wedge \cdots \wedge dy_n,$$

where

$$\zeta(Y)^2 = \det\left[\frac{\sin adY}{adY} + i\frac{\cos adY - 1}{adY}\right].$$

Computing in terms of the roots we have

$$\begin{aligned}\zeta(Y)^2 &= \prod_{\alpha\in R} \frac{\sinh\alpha(Y) + \cosh\alpha(Y) - 1}{\alpha(Y)} \\ &= \prod_{\alpha\in R} \frac{e^{\alpha(Y)} - 1}{\alpha(Y)} \\ &= \prod_{\alpha\in R^+} \frac{\left(e^{\alpha(Y)} - 1\right)\left(1 - e^{-\alpha(Y)}\right)}{\alpha(Y)^2}\end{aligned}$$

Since $(e^x - 1)(1 - e^{-x}) = 4\sinh^2(x/2)$ we get

$$\zeta(Y)^2 = \prod_{\alpha\in R^+} \frac{\sinh^2\alpha(Y/2)}{\alpha(Y/2)^2}.$$

Taking a square root gives the desired expression for $\zeta(Y)$.

Now we turn to the Kähler potential κ. As usual we identify $T^*(K)$ with $K \times \mathfrak{k}$ by means of left-translation and the inner product on $\mathfrak{k}$. The canonical projection $\pi : T^*(K) \to K$ in this description is simply the map $(x, Y) \to x$. The canonical 1-form θ is defined by setting

$$\theta(X) = \langle Y, \pi_*(X)\rangle,$$

where X is a tangent vector to $T^*(K)$ at the point (x, Y). Choose an orthonormal basis $e_1, \cdots, e_n$ for $\mathfrak{k}$ and let $y_1, \cdots, y_n$ be the coordinates on $\mathfrak{k}$ with respect to this basis. Let $\alpha_1, \cdots, \alpha_n$ be left-invariant 1-forms on K whose values at the identity are the vectors $e_1, \cdots, e_n$ in $\mathfrak{k} \cong \mathfrak{k}^*$. Then it is easily verified that at each point $(x, Y) \in T^*(K)$ we have

$$\theta = \sum_{k=1}^n y_k \alpha_k.$$

Now let κ be the function on $T^*(K)$ given by

$$\kappa(x, Y) = |Y|^2 = \sum_{k=1}^n y_k^2.$$

We want to verify that

$$\mathrm{Im}\left[\bar{\partial}\kappa\right] = \theta.$$

We start by observing that

$$d\kappa = \sum_{k=1}^{n} 2y_k \, dy_k.$$

To compute $\bar{\partial}\kappa$ we need transport $d\kappa$ to $K_{\mathbb{C}}$, where the complex structure is defined. On $K_{\mathbb{C}}$ we express things in terms of left-invariant 1-forms $\eta_1, \cdots, \eta_n$ and $J\eta_1, \cdots, J\eta_n$. We then want to pull back $d\kappa$ to $K_{\mathbb{C}}$ by means of Φ^{-1}. So we need to compute the inverse transpose of the matrix (6.1) describing Φ_*. This may be computed as

$$\left(\Phi_*^{-1}\right)^{tr} = \frac{adY}{\sin adY}\begin{pmatrix} \frac{\sin adY}{adY} & -\sin adY \\ \frac{1-\cos adY}{adY} & \cos adY \end{pmatrix}.$$

In terms of our basis for 1-forms on $T^*(K)$, $d\kappa$ is represented by the vector

$$\begin{bmatrix} 0 \\ Y \end{bmatrix}$$

so we have to apply the matrix above to this vector. But of course $adY(Y) = 0$ and so we get simply

$$\begin{aligned}\left(\Phi^{-1}\right)^*(d\kappa) &= 2\sum_{k=1}^{n} y_k J\eta_k \\ &= 2\sum_{k=1}^{n} y_k \frac{1}{2i}\left((\eta_k + iJ\eta_k) - (\eta_k - iJ\eta_k)\right).\end{aligned}$$

Thus taking only the term involving the anti-holomorphic 1-forms $\eta_k - iJ\eta_k$ we have

$$\bar{\partial}\kappa = \sum_{k=1}^{n} iy_k\left(\eta_k - iJ\eta_k\right).$$

which is represented by the vector

$$\begin{bmatrix} iY \\ Y \end{bmatrix}$$

We now transfer this back to $T^*(K)$ by means of Φ^*. So applying the transpose of the matrix (6.1) we get

$$\bar{\partial}\kappa = \sum_{k=1}^{n}\left(iy_k\alpha_k + y_k dy_k\right)$$

and so

$$\operatorname{Im}\left[\bar{\partial}\kappa\right] = \sum_{k=1}^{n} y_k\alpha_k = \theta.$$

7. Appendix: Lie groups of compact type

In this appendix I give a proof of Proposition 2.2, the structure result for connected Lie groups of compact type. We consider a connected Lie group K of compact type, with a fixed Ad-invariant inner product on its Lie algebra $\mathfrak{k}$. Since the inner product is Ad-invariant, the orthogonal complement of any ideal in $\mathfrak{k}$ will be an ideal. Thus $\mathfrak{k}$ decomposes as a direct sum of subalgebras that are either simple or one-dimensional. Collecting together the simple factors in one group and the one-dimensional factors in another, we obtain a decomposition of $\mathfrak{k}$ as $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{z}$, where $\mathfrak{k}_1$ is semisimple and $\mathfrak{z}$ is commutative. Since $\mathfrak{k}_1$ is semisimple and admits an Ad-invariant inner product, the connected subgroup K_1 of K with Lie algebra $\mathfrak{k}_1$ will be compact. (By Cor. II.6.5 of [He], the adjoint group of K_1 is a closed subgroup of $Gl(\mathfrak{k}_1) \cap O(\mathfrak{k}_1)$ and is therefore compact. Then Thm. II.6.9 of [He] implies that K_1 itself is compact.)

Now let Γ be the subset of $\mathfrak{z}$ given by

$$\Gamma = \left\{ Z \in \mathfrak{z} \middle| e^Z = id \right\},$$

where id is the identity in K. Since $\mathfrak{z}$ is commutative, Γ is a discrete additive subgroup of $\mathfrak{z}$, hence there exist vectors $X_1, \cdots, X_k$, linearly independent over $\mathbb{R}$, such that Γ is the set of integer linear combinations of the X_k's. (See [Wa, Exer. 3.18] or [BtD, Lem. 3.8].)

Now let $\mathfrak{z}_1$ be the real span of $X_1, \cdots, X_k$, and let $\mathfrak{z}_2$ be the orthogonal complement of $\mathfrak{z}_1$ in $\mathfrak{z}$, with respect to the fixed Ad-invariant inner product. Since $\mathfrak{z}_1$ is commutative, the image of $\mathfrak{z}_1$ under the exponential mapping is connected subgroup of K, which is isomorphic to a torus, hence compact. Thus the connected subgroup H of K whose Lie algebra is $\mathfrak{k}_1 + \mathfrak{z}_1$ is a quotient of $K_1 \times (\mathfrak{z}_1/\Gamma)$, hence compact.

Next consider the map $\Psi : H \times \mathfrak{z}_2 \to K$ given by

$$\Psi(h, X) = he^X,$$

which is a homomorphism because $\mathfrak{z}_2$ is central. I claim that this map is injective. To see this, suppose (h, X) is in the kernel. Then $h = e^{-X}$, which means that h is in the center of K, hence in the center of H. Now, H is a quotient of $K_1 \times (\mathfrak{z}_1/\Gamma)$, so there exist $x \in K_1$ and $y \in (\mathfrak{z}_1/\Gamma)$ such that $h = xy$. Since h is central and y is central, x is central as well. But the center of K_1 is finite, so there exists m such that $x^m = id$. Since y and e^X are central, this means that

$$h^m = x^m y^m e^{mX} = y^m e^{mX} = id.$$

But $y = e^Y$ for some $Y \in \mathfrak{z}_1$, so we have $e^{mY} e^{mX} = e^{mY+mX} = id$, which means that $mY + mX \in \Gamma$. This means that $X = 0$, since $\mathfrak{z}$ is the direct sum of the real span of Γ and $\mathfrak{z}_2$, and so also $h = e^{-X} = id$.

Thus Ψ is an injective homomorphism of $H \times Z_2$ into K. The associated Lie algebra homomorphism is clearly an isomorphism ($\mathfrak{k} = (\mathfrak{k}_1 + \mathfrak{z}_1) + \mathfrak{z}_2$). It follows that Ψ is actually a diffeomorphism. To finish the argument, we need to show that the Lie algebra of H (namely, $\mathfrak{k}_1 + \mathfrak{z}_1$) is orthogonal to $\mathfrak{z}_2$. To see this, note that $\mathfrak{k}_1$ and $\mathfrak{z}_2$ are automatically orthogonal with respect to any Ad-invariant inner product (since the orthogonal projection of $\mathfrak{k}_1$ onto $\mathfrak{z}_2$ is a Lie algebra homomorphism of a semisimple algebra into a commutative algebra), and $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are orthogonal with respect to the chosen inner product, by the construction of $\mathfrak{z}_2$.

References

[B] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, Part I, *Comm. Pure Appl. Math.* **14** (1961), 187-214.

[BtD] T. Bröcker and T. tom Dieck, "Representations of compact Lie groups," Graduate Texts in Mathematics 98, Springer-Verlag, New York, 1995.

[Dr] B. Driver, On the Kakutani-Itô-Segal-Gross and Segal-Bargmann-Hall isomorphisms, *J. Funct. Anal.* **133** (1995), 69-128.

[DG] B. Driver and L. Gross, Hilbert spaces of holomorphic functions on complex Lie groups. *In* "New trends in stochastic analysis. Proceedings of a Taniguchi international workshop,"

(K. Elworthy, S. Kusuoka, and I. Shigekawa, Eds.) , pp. 76-106. World Scientific, Singapore, 1997.

[DH] B. Driver and B. Hall, Yang-Mills theory and the Segal-Bargmann transform, *Commun. Math. Phys.* **201** (1999), 249-290.

[F-J] M. Flensted-Jensen, Spherical functions on a real semisimple Lie group. A method of reduction to the complex case, *J. Funct. Anal.* **30** (1978), 106-146.

[F] G. Folland, "Harmonic analysis in phase space," Princeton Univ. Press, Princeton, N.J., 1989.

[FY] K. Furutani and S. Yoshizawa, A Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces and its application to a geometric quantization. II, *Japan. J. Math. (N.S.)* **21** (1995), 355–392.

[Go] M. Gotay, Constraints, reduction, and quantization, *J. Math. Phys.* **27** (1986), 2051–2066.

[G1] L. Gross, Uniqueness of ground states for Schrödinger operators over loop groups, *J. Funct. Anal.* **112** (1993) 373-441.

[G2] L. Gross, Heat kernel analysis on Lie groups, preprint.

[GM] L. Gross and P. Malliavin, Hall's transform and the Segal-Bargmann map, *in* "Itô's stochastic calculus and probability theory," (M. Fukushima, N. Ikeda, H. Kunita, and S. Watanabe, Eds.), pp. 73-116. Springer-Verlag, Berlin/New York, 1996.

[GStenz1] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampère equation, *J. Differential Geom.* **34** (1991), 561–570.

[GStenz2] Grauert tubes and the homogeneous Monge-Ampère equation. II. J. Differential Geom. 35 (1992), no. 3, 627–641

[GStern] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.

[H1] B. Hall, The Segal-Bargmann "coherent state" transform for compact Lie groups, *J. Funct. Anal.* **122** (1994), 103-151.

[H2] B. Hall, The inverse Segal-Bargmann transform for compact Lie groups, *J. Funct. Anal.*, **143** (1997), 98-116.

[H3] B. Hall, Phase space bounds for quantum mechanics on a compact Lie group, *Comm. Math. Phys.*, **184** (1997), 233-250.

[H4] B. Hall, Quantum mechanics in phase space, *in* "Perspectives on quantization" (L. Coburn and M. Rieffel, Eds), pp. 47–62, Contemporary Mathematics, Vol. 214, Amer. Math. Soc., Providence, RI, 1998.

[H5] B. Hall, A new form of the Segal-Bargmann transform for Lie groups of compact type, *Canad. J. Math.* **51** (1999), 816-834.

[H6] B. Hall, Holomorphic methods in analysis and mathematical physics, *in* "First Summer School in Analysis and Mathematical Physics," (S. Pérez-Esteva and C. Villegas-Blas, Eds.), Contemp. Math. Vol. 260, Amer. Math. Soc., Providence, RI, 2000, pp. 1-59.

[H7] B. Hall, Harmonic analysis with respect to heat kernel measure, *Bull. (N.S.) Amer. Math. Soc.* **38** (2001), 43-78.

[H8] B. Hall, Coherent states and the quantization of 1+1-dimensional Yang-Mills theory, *Rep. Math. Phys.*, to appear. [xxx.lanl.gov, quant-ph/0012050]

[HS] B. Hall and A. Sengupta, The Segal-Bargmann transform for path-groups, *J. Funct. Anal.* **152** (1998), 220-254.

[He] S. Helgason, "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, New York, San Diego, 1978.

[Hi1] O. Hijab, Hermite functions on compact Lie groups. I, *J. Funct. Anal.* **125** (1994), 480–492.

[Hi2] O. Hijab, Hermite functions on compact Lie groups. II, *J. Funct. Anal.* **133** (1995), 41–49.

[Ho] G. Hochschild, The structure of Lie groups. Holden-Day, San Francisco, 1965.

[JL] J. Jorgensen and S. Lang, The ubiquitous heat kernel, preprint.

[Ki] A. Kirillov, Geometric quantization, *in* "Dynamical Systems IV" (V. Arnoľd and S. Novikov, Eds.). Encyclopaedia of Mathematical Sciences, Vol. 4, Springer-Verlag, New York, Berlin, 1990.

[L1] N. Landsman, Rieffel induction as generalized quantum Marsden-Weinstein reduction, *J. Geom. Phys.* **15** (1995), 285–319.

[L2] N. Landsman, Mathematical topics between classical and quantum mechanics. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998.

[LS] L. Lempert and R. Szőke, Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds, *Math. Ann.* **290** (1991), 689–712.

[O] E. Onofri, Mathematical Reviews, Review 83b:58038, of the paper "Geometric quantization for the mechanics on spheres," by K. Ii, *Tôhoku Math. J.* **33** (1981), 289–295.

[R] J. Rawnsley, A nonunitary pairing of polarizations for the Kepler problem, *Trans. Amer. Math. Soc.* **250** (1979), 167–180.

[RS] M. Reed and B. Simon, "Methods of Modern Mathematical Physics, I: Functional Analysis," Academic Press, New York, London, 1972.

[S1] I. Segal, Mathematical problems of relativistic physics, Chap. VI, *in* "Proceedings of the Summer Seminar, Boulder, Colorado, 1960, Vol. II." (M. Kac, Ed.). Lectures in Applied Mathematics, American Math. Soc., Providence, Rhode Island, 1963.

[S2] I. Segal, Mathematical characterization of the physical vacuum for a linear Bose-Einstein field, *Illinois J. Math.* **6** (1962), 500-523.

[S3] I. Segal, The complex wave representation of the free Boson field, *in* "Topics in functional analysis: Essays dedicated to M.G. Krein on the occasion of his 70th birthday" (I. Gohberg and M. Kac, Eds). Advances in Mathematics Supplementary Studies, Vol. 3, pp. 321-343. Academic Press, New York, 1978.

[St] M. Stenzel, The Segal-Bargmann transform on a symmetric space of compact type, *J. Funct. Anal.* **165** (1999), 44–58.

[T1] T. Thiemann, Reality conditions inducing transforms for quantum gauge field theory and quantum gravity, *Classical Quantum Gravity* **13** (1996), 1383–1403.

[T2] T. Thiemann, Gauge field theory coherent states I. General properties, preprint. [xxx.lanl.gov, hep-th/0005233]

[Wa] F. Warner, "Foundations of differentiable manifolds and Lie groups," Graduate Texts in Mathematics 94, Springer-Verlag, New York-Berlin, 1983.

[Wo] N. Woodhouse, "Geometric Quantization," Second Edition, Oxford Univ. Press, Oxford, New York, 1991.

[Wr] K. Wren, Constrained quantisation and θ-angles. II. *Nuclear Phys. B* **521** (1998), 471–502.

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