Positive Commutators in Non-Equilibrium Quantum Statistical Mechanics *

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Abstract

The method of positive commutators, developed for zero temperature problems over the last twenty years, has been powering progress in the spectral analysis of Hamiltonians in quantum mechanics. We extend this method to positive temperatures, i.e. to non-equilibrium quantum statistical mechanics.

We use the positive commutator technique to give an alternative proof of a fundamental property of large quantum systems, called *Return to Equilibrium*. This property says that equilibrium states are (asymptotically) stable: if a system is slightly perturbed from its equilibrium state, then it converges back to that equilibrium state as time goes to infinity.

Keywords: positive commutator, Mourre estimate, return to equilibrium, virial theorem, Fermi golden rule
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1 Introduction

In this paper, we study a class of open quantum systems consisting of two interacting subsystems: a finite system, called the particle system coupled to a reservoir (large system), described by the spatially infinitely extended photon-field (a massless Bose field). In the GNS representation of the C^*-algebra of observables corresponding to the equilibrium state of the coupled system, the dynamics is generated by a *Liouville operator*, also called thermal Hamiltonian, acting on a positive temperature Hilbert space (the GNS Hilbert space). Many key properties of the system, such as return to equilibrium, RTE (asymptotic stability of the equilibrium state), can be expressed in

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terms of the spectral characteristics of this operator.

Applying the positive commutator (PC) method to the Liouville operator of systems in question, we obtain rather detailed information on the spectrum of these operators. This allows us to recover, with a partial improvement, a recent fundamental result by several authors on RTE.

Our main technical result is a positive commutator estimate (also called a Mourre estimate) for the Liouville operator. This result holds for a wider class of systems than previously considered.

Spectral information on the Liouville operator, and hence the property of RTE, is extracted from the PC estimate through Virial Theorem type arguments. It turns out that the existing Virial Theorem techniques are too restrictive to apply to positive temperature systems, and we need to extend them beyond their traditional range of application.

There is a restriction on the class of systems for which we prove RTE, due to our Virial Theorem type result mentioned above. This is the first result of this kind, and we expect that it will be improved to yield the RTE result for a considerably wider class of systems.

2 Spectral approach to RTE and PC method

In this section, we give a mathematical picture of the problem of RTE, including an outline of the strategy (the PC technique) we use to show RTE.

2.1 Spectral characterization of RTE

Let us recall some basic facts and definitions about quantum statistical systems. For more details, we refer to [BRII], [H], [BFS4].

The *observables* of a quantum statistical system form a C^*-algebra $\mathfrak{A}$. The *dynamics* is given by a strongly continuous one-parameter group of $*$-automorphisms α_t on the algebra of observables: $\mathbb{R} \ni t \mapsto \alpha_t \in \mathrm{Aut}(\mathfrak{A})$. *States* of the system are positive linear normalized functionals on the algebra of observables, $\omega : \mathfrak{A} \to \mathbb{C}$. The evolution of a state ω is given by its composition with the dynamics: $\omega_t = \omega \circ \alpha_t$. A state is called *stationary* iff $\omega_t = \omega$, $\forall t$.

Given a dynamics α_t, the equilibrium states at temperature $T > 0$ are identified with the β-KMS states ω_β (w.r.t. α_t), with $\beta = 1/T$. Some properties of KMS states are:

- *stationarity*: every β-KMS state is stationary: $\omega_\beta \circ \alpha_t = \omega_\beta$, $\forall t$;
- *structural stability*: suppose ω_β is a β-KMS state w.r.t. a given dynamics α_t, and that $\tilde{\alpha}_t$ is another dynamics obtained by slightly perturbing α_t. Then there exists a β-KMS state $\tilde{\omega}_\beta$ w.r.t. the perturbed dynamics $\tilde{\alpha}_t$.

The term equilibrium refers, among other, to the *conjectured* dynamical stability of KMS states in the sense that if ω_β is a β-KMS state, then $\omega' \circ \alpha_t \to \omega_\beta$ as $t \to \infty$, for states ω' that are close to ω_β, see (1) and (2). This is called the property of return to equilibrium.

Using the GNS (Gelfand-Naimark-Segal) construction, one reformulates the C^*-algebra setup described above in a Hilbert space framework. Namely, to any pair $(\mathfrak{A}, \omega)$, one associates a unique triple $(\mathcal{H}, \pi, \Omega)$, where $\mathcal{H}$ is a Hilbert space (the GNS Hilbert space), $\Omega \in \mathcal{H}$ is a distinguished cyclic vector, and $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ (bounded operators) is a $*$-homomorphism s.t. $\forall a \in \mathfrak{A}$: $\omega(a) = \langle \Omega, \pi(a)\Omega \rangle$. Here, $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$. To complete the Hilbert space picture, we need to describe the dynamics on $\mathcal{H}$. One shows that if the state ω is stationary w.r.t. α_t, then there is a unique unitary group $U(t)$ on $\mathcal{H}$ s.t. $\omega(\alpha_t(a)) = \langle \Omega, U^*(t)\pi(a)U(t)\Omega \rangle$ and $U(t)\Omega = \Omega$, $\forall t$. We say the dynamics α_t is unitarily implemented on $\mathcal{H}$ by $U(t)$. If $t \mapsto U(t)$ is strongly continuous, then it has a selfadjoint generator L (Stone's theorem), called the Liouville operator or *Liouvillian*, and we write $U(t) = e^{-itL}$. Notice that $L\Omega = 0$.

Remark that L is constructed from an stationary state ω. There is a natural neighbourhood around ω in which the dynamics of states is also determined by L: the set of all *normal* states ω' w.r.t. ω. By definition, ω' is normal w.r.t. ω, iff

$$\forall a \in \mathfrak{A} : \quad \omega'(a) = \mathrm{tr}\,(\rho\pi(a)), \tag{1}$$

where $\mathrm{tr}\,(\cdot)$ is the trace on the GNS Hilbert space $\mathcal{H}$ of ω, ρ is a trace class operator on $\mathcal{H}$, normalized as $\mathrm{tr}\,\rho = 1$, and π is the GNS representation of ω.

Representing the dynamics of the system on a *Hilbert space* (as opposed to a C^*-algebra) allows us to connect the spectral properties of the Liouville operator with the dynamical behaviour of the system in the following way:

Proposition 2.1 (spectral characterization of RTE). *Let ω_β be a β-KMS state with respect to the dynamics α_t, and denote the GNS representation of ω_β by $(\mathcal{H}, \pi, \Phi)$. Suppose that the Liouvillian L generating the dynamics on $\mathcal{H}$ has no eigenvalues except for a simple one at zero, so that the only eigenvector of L is Φ. Then, for any normal state ω' w.r.t. π, and for any observable $a \in \mathfrak{A}$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \omega'(\alpha_t(a))dt = \omega_\beta(a). \tag{2}$$

This means that the system exhibits return to equilibrium in an ergodic mean sense.

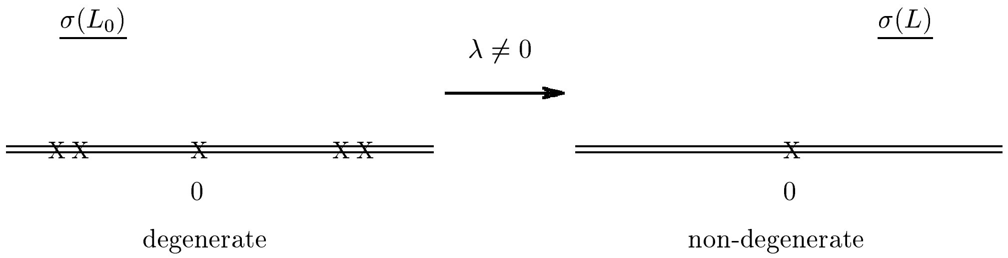
The proof is given e.g. in [BFS4], [M]. Better information on the spectrum of L yields stronger convergence; if L has absolutely continuous spectrum, except a simple eigenvalue at 0, then (2) can be replaced by $\lim_{t\to\infty} \omega'(\alpha_t(a)) = \omega_\beta(a)$, $\forall a \in \mathfrak{A}$.

2.2 The PC method

This section introduces the general idea of the PC method. The Liouville operators in the class of systems we consider consist of two parts:

$$L = L_0 + \lambda I,$$

where L_0 is the uncoupled Liouville operator, describing the two subsystems (particles and field) when they do not interact. I is the interaction, and λ is a real (small) coupling parameter. The spectrum of L_0 consists of a continuum covering the whole real axis, and it has embedded eigenvalues, arranged symmetrically w.r.t. zero. Moreover, zero is a degenerate eigenvalue. We would like to show that for $\lambda \neq 0$, the spectrum of L has no eigenvalues, except for a simple one at zero, because then Proposition 2.1 tells us that the system exhibits RTE!



In other words, we want to show that all nonzero eigenvalues of L_0 are unstable under the perturbation λI, and that this perturbation removes the degeneracy of the zero eigenvalue. Due to the structural stability of KMS states, one can construct a β-KMS state w.r.t. the dynamics of the coupled system. Hence L has a zero eigenvalue (the eigenvector being the KMS state). This means that our task reduces to showing instability of all nonzero eigenvalues, and that the dimension of the nullspace of L is at most one.

It is conventional wisdom that embedded eigenvalues are unstable under generic perturbations, turning into resonances. We now outline the technique we use to show instability of embedded eigenvalues: the PC technique.

To do so, we concentrate first on a nonzero (isolated) eigenvalue e of L_0 whose instability we want to show. The main idea is to construct an antiselfajoint operator A, called the *adjoint operator*, s.t. we have the following PC estimate:

$$E_\Delta(L)[L, A]E_\Delta(L) \geq \theta E_\Delta^2(L), \tag{3}$$

where $\theta > 0$ is a srictly positive number, $E_\Delta(L)$ denotes the spectral projector of L onto the interval Δ, and $[\cdot, \cdot]$ is the commutator. Here, Δ is chosen to contain the eigenvalue e but no other eigenvalues of L_0. Equation (3) is also called a (strict)

Mourre estimate. If it is satisfied, then one sees that L has no eigenvalues in Δ by using the following argument by contradiction: suppose that $L\psi = e'\psi$, with $e' \in \Delta$ and $\|\psi\| = 1$. Then we have $E_\Delta(L)\psi = \psi$, and the PC estimate (3) gives on one hand $\langle\psi, [L, A]\psi\rangle \geq \theta$. On the other hand, formally expanding the commutator yields

$$\langle\psi, [L, A]\psi\rangle = \langle\psi, [L - e', A]\psi\rangle = 2\mathrm{Re}\,\langle(L - e')\psi, A\psi\rangle = 0, \tag{4}$$

which gives the contradiction $\theta \leq 0$, hence showing that there cannot be any eigenvalue of L in Δ.

This formal proof is in general wrong. Indeed, both operators L and A are unbounded, and one has to take great care of domain questions, including the very definition of the commutator $[L, A]$.

Relation (4) is called the *Virial Theorem*, and it can be made in many concrete cases rigorous by approximating the hypothetical eigenfunction ψ by "nice" vectors. The situation in which this works is quite generally given by the case where $[L, A]$ is bounded relatively to L, which is in particular satisfied for N-body Schrödinger systems, and systems of particles coupled to a field *at zero temperature*. However, in our case the condition is not satisfied, and as mentioned above, we have to develop a more general argument of this type.

The treatment of the zero eigenvalue is similar, except that we must not try to prove (3) on the whole space $\mathrm{Ran}\, E_\Delta(L)$, since we know that L has a zero eigenvalue, but only on $\mathrm{Ran}\, E_\Delta(L)P^\perp$, where P is the rank-one projector onto the (known) eigenvector, and $P^\perp$ is its orthogonal complement.

3 A class of open quantum systems

The choice of the class of systems we analyze is motivated by the quantum mechanical models of nonrelativistic matter coupled to the radiation field, or matter interacting with a phonon field (quantized modes of a lattice), or a generalized spin-boson system. A good review of physical models leading to the class of Hamiltonians considered here is found in [HSp].

The particle system is described by a Hilbert space $\mathcal{H}_p$ as its state space, and by a Hamiltonian H_p which governs the evolution. H_p is a selfadjoint operator on $\mathcal{H}_p$ which has purely discrete spectrum:

$$\sigma(H_p) = \{E_j\}_{j=0}^\infty, \tag{5}$$

(where multiplicities are included, i.e. for a degenerate eigenvalue E_i, we have $E_i = E_j$ for some $j \neq i$), and we denote the orthonormal basis diagonalizing H_p by $\{\varphi_j\}$. We further assume that

$$Z_p(\beta) := \mathrm{tr}\, e^{-\beta H_p} < \infty, \quad \forall\beta > 0. \tag{6}$$

We do not need to further specify the particle system. As a concrete example, one may think of a system of finitely many Schrödinger particles in a box (hence the name particle system), or a spin system. In some of our results (see Theorem 4.4 on the Fermi Golden Rule Condition), we shall assume that the spectrum of H_p is finite (N-level system).

The reservoir is given by the spatially infinitely extended free massless Bosonic field. Physical examples for the reservoir are the quantized electromagnetic field (photons), or the quantized vibrations of a lattice (phonons). In the case of photons, we are dealing with a field with two components (Coulomb gauge), while the phonons are described by a scalar field. For notational simplicity, we take the field to be scalar, i.e. the one particle space of the field is given by $\mathfrak{H} = L^2(\mathbb{R}^3, d^3k)$, and the state space of the field is the Bosonic Fock space over $\mathfrak{H}$: $\mathcal{H}_f = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes_{\rm sym}^n}$.

The Hamiltonian of the field is the second quantization of the multiplication operator by $\omega = |k|$, and if $a^*(k)$, $a(k)$ denote the (distribution valued) creation and annihilation operators, then we can express the Hamiltonian as $H_f = \mathrm{d}\Gamma(\omega) = \int d^3k\omega(k)a^*(k)a(k)$. For details on creation and annihilation operators and on the second quantization, we refer to [BRII].

The combined, uncoupled system at zero temperature is described by the Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$, and its dynamics is determined by the selfadjoint Hamiltonian $H_0 = H_p \otimes \mathbb{1}_f + \mathbb{1}_p \otimes H_f$. The full Hamiltonian of the interacting system is

$$H = H_0 + \lambda v, \tag{7}$$

where the coupling constant λ is a small real number, and

$$v = G \otimes (a(g) + a^*(g)). \tag{8}$$

Here, G is a bounded selfadjoint operator on $\mathcal{H}_p$. The function $g \in \mathfrak{H}$ is called the *form factor* and the smoothed out creator is given by $a^*(g) = \int d^3k\, g(k)a^*(k)$. We assume g to be a bounded C^1-function, satisfying the following infra-red (IR) and ultra-violet (UV) conditions (recall that $\omega = |k|$):

$$\begin{array}{ll} \text{IR:} & |g(k)| \leq C\omega^p, \text{ for some } p > 0, \text{ as } \omega \to 0, \\ & \text{for some results, we assume } p > 2, \\ \text{UV:} & |g(k)| \leq C\omega^{-q}, \text{ for some } q > 5/2, \text{ as } \omega \to \infty. \end{array} \tag{9}$$

In addition, we assume that conditions (9) hold for the derivative $\partial_\omega g$, if p, q are replaced by $p-1, q+1$.

We point out that the physical value coming from the model of an atom coupled to the radiation field is $p = 1/2$.

Before stating our main result, we introduce a condition on the interaction, called the *Fermi Golden Rule Condition.* In the language of quantum resonances, this condition expresses the fact that the bifurcation of complex eigenvalues (resonance poles) of the spectrally deformed Liouvillian takes place at second order in the perturbation (i.e. the lifetime of the resonance is of the order λ^{-2}).

We mention that the Liouvillian corresponding to the particle system at positive temperature is given by $L_p = H_p \otimes \mathbb{1} - \mathbb{1} \otimes H_p$, acting on the Hilbert space $\mathcal{H}_p \otimes \mathcal{H}_p$ (see sentence after (21)). Thus L_p has discrete spectrum given by $\sigma(L_p) = \{e = E_i - E_j : E_i, E_j \in \sigma(H_p)\}$. For every eigenvalue e of L_p, we define an operator $\Gamma(e)$ acting on the subspace $\operatorname{Ran} P(L_p = e) \subset \mathcal{H}_p \otimes \mathcal{H}_p$ by

$$\Gamma(e) = \int_{\mathbb{R}\times S^2} m^*(u,\alpha) P(L_p \neq e) \delta(L_p - e + u) m(u,\alpha), \tag{10}$$

where δ denotes the Dirac function, and where the operator m is given by

$$m(u,\alpha) = G_l \, g_1(u,\alpha) - G_r \, g_2(u,\alpha). \tag{11}$$

Here, $g_{1,2}$ are two functions on $\mathbb{R} \times S^2$, constructed from g, see (23), and $G_l := G \otimes \mathbb{1}_p$, $G_r := \mathbb{1}_p \otimes \mathcal{C}G\mathcal{C}$, where $\mathcal{C}$ is the antilinear map on $\mathcal{H}_p$ that, in the basis that diagonalizes H_p, has the effect of complex conjugation of coordinates.

It is clear from (10) that $\Gamma(e)$ is a non-negative selfadjoint operator. The Fermi Golden Rule Condition is used to show instability of embedded eigenvalues. For nonzero eigenvalues, the condition says that $\Gamma(e)$ is *strictly positive*:

$$\text{for } e \neq 0, \quad \gamma_e := \inf \sigma \left(\Gamma(e) \restriction \operatorname{Ran} P(L_p = e) \right) > 0. \tag{12}$$

We show in Theorem 4.4 that $\Gamma(0)$ has a simple eigenvalue at zero, the eigenvector being the Gibbs state of the particle system, Ω_β^p (see (17)). This reflects the fact that the zero eigenvalue of L_0 survives the perturbation, however, its degeneracy is removed, i.e. the zero eigenvalue of L is simple. The Fermi Golden Rule Condition for $e = 0$ requires strict positivity on the complement of the zero eigenspace of $\Gamma(0)$, i.e.

$$\gamma_0 := \inf \sigma \left(\Gamma(0) \restriction \operatorname{Ran} P(L_p = 0) P^\perp_{\Omega_\beta^p} \right) > 0. \tag{13}$$

Here, $P_{\Omega_\beta^p}$ is the projection onto $\mathbb{C}\Omega_\beta^p$, and $P^\perp_{\Omega_\beta^p} = \mathbb{1} - P_{\Omega_\beta^p}$. We give in Theorem 4.4 below explicit conditions on G and $g(k)$ s.t. (12) and (13) hold.

4 Main results

Our main technical result is the abstract PC estimate, Theorem 4.1. This result is the basis for the spectral analysis of the Liouvillian, as explained above. We point out that the PC estimate holds for infra-red behaviour of the form factor (see (9)) characterized by $p > 0$, which covers the physical case $p = 1/2$.

In Theorem 4.2, we characterize the spectrum of the Liouvillian in view of the property of RTE. To prove this result, we combine the PC estimate with a Virial Theorem type argument. It is for the latter that we need presently the more restricting infra-red behaviour $p > 2$. We think that our method can be improved so that the physical case is included.

A direct consequence of Theorem 4.2 is Corollary 4.3 which says that the system exhibits RTE (recall also Proposition 2.1).

All the results hold under assumption of the Fermi Golden Rule Condition, (12) and (13). In Theorem 4.4, we give explicit conditions on the operator G and the form factor g so that the Fermi Golden Rule Condition holds.

Theorem 4.1 (Positive Commutator Estimate). *Assume the IR and UV behaviour (9), with $p > 0$. Let Δ be an interval containing exactly one eigenvalue e of L_0 and let $h \in C_0^\infty$ be a smooth function s.t. $h = 1$ on Δ and $\mathrm{supp} h \cap \sigma(L_p) = \{e\}$. Assume the Fermi Golden Rule Condition (12) (or (13)) holds. Let $\beta \geq \beta_0$, for any fixed $0 < \beta_0 < \infty$, and assume $|\lambda| \ll 1$, independent of $\beta \geq \beta_0$. Then we have in the sense of quadratic forms on $\mathcal{D}(N^{1/2})$, for some explicitely constructed antiselfadjoint operator A:*

$$h(L)[L, A]h(L) \geq \tfrac{1}{2}\lambda^{91/50} h(L) \left(\gamma_e(1 - 5\delta_{e,0} P_{\Omega_{\beta,0}}) - O(\lambda^{1/200})\right) h(L). \qquad (14)$$

Remarks. 1. N is the number operator in the positive temperature Hilbert space (see (18) and (24)), and $P_{\Omega_{\beta,0}}$ is the projector onto the span of $\Omega_{\beta,0}$, the β-KMS state of the uncoupled system (see (19)). Also, $\delta_{e,0}$ is the Kronecker symbol, equal to one if $e = 0$ and zero else.
2. We show in Theorem 6.2 that L is essentially selfadjoint on a dense domain in the positive temperature Hilbert space.
3. The commutator $[L, A]$ is by construction in first approximation equal to N (see Section 7), and $h(L)$ leaves the domain $\mathcal{D}(N^{1/2})$ invariant (see e.g. [M]), so that (14) is well defined.

Theorem 4.2 (Spectrum of L). *Assume the IR condition $p > 2$ (see (9)). Then the Liouvillian L has the following spectral properties:*

1) *Let $e \neq 0$ be a nonzero eigenvalue of L_0, and suppose that the Fermi Golden Rule Condition (12) holds for e. Let $\beta \geq \beta_0$, for any fixed $0 < \beta_0 < \infty$, and $|\lambda| \ll 1$, independent of $\beta \geq \beta_0$.*
   Then L has no eigenvalues in the open interval (e_-, e_+), where e_- is the biggest eigenvalue of L_0 smaller than e, and e_+ is the smallest eigenvalue of L_0 bigger than e.

2) *Assume the Fermi Golden Rule Condition (13) holds for $e = 0$. If $|\lambda| \ll \min\{1, \beta^{-1}\}$, then L has a simple eigenvalue at zero.*

Remark. Theorem 4.2 shows that if the Fermi Golden Rule Condition holds for all eigenvalues of L_0, then L has no eigenvalues, except a simple one at zero.

Corollary 4.3 (Return to Equilibrium). *Suppose the IR condition as in Theorem 4.2, and that the Fermi Golden Rule Condition is satisfied for all eigenvalues of L_0. Then every normal state w.r.t. the β-KMS state $\Omega_{\beta,\lambda}$ (the zero eigenvector of L) exhibits return to equilibrium in an ergodic mean sense.*

The Corollary follows immediately from Theorem 4.2 and Proposition 2.1, where the ergodic mean convergence is defined by (2).

Now we give an explicit formulation of the Fermi Golden Rule Condition. To do so, we introduce the following notation. For a fixed eigenvalue $e \neq 0$ of L_0, define the subsets of $\mathbb{N}$:

$$\begin{aligned}
\mathcal{N}_r^{(i)} &:= \{j | E_i - E_j = e\}, \\
\mathcal{N}_l^{(j)} &:= \{i | E_i - E_j = e\}, \\
\mathcal{N}_r &:= \cup_i \mathcal{N}_r^{(i)} = \{j | E_i - E_j = e \text{ for some } i\}, \\
\mathcal{N}_l &:= \cup_j \mathcal{N}_l^{(j)} = \{i | E_i - E_j = e \text{ for some } j\}.
\end{aligned}$$

We also let P_i denote the rank-one projector onto $\mathbb{C}\varphi_i$, where we recall that $\{\varphi_i\}$ is the orthonormal basis diagonalizing H_p. For any nonempty subset $\mathcal{N} \subset \mathbb{N}$, put

$$P_{\mathcal{N}} := \sum_{j \in \mathcal{N}} P_j, \quad \text{and} \quad P_{\mathcal{N}} := 0 \text{ if } \mathcal{N} \text{ is empty.}$$

Set $E_{mn} := E_m - E_n$, and for $e \in \sigma(L_p)\backslash\{0\}$, $m \in \mathcal{N}_l$ and $n \in \mathcal{N}_r$, define:

$$\delta_m := \inf \sigma\left(P_{\mathcal{N}_r^{(m)}} G P_{\mathcal{N}_r^c} G P_{\mathcal{N}_r^{(m)}} \restriction P_{\mathcal{N}_r^{(m)}}\right) \geq 0, \tag{15}$$

$$\delta'_n := \inf \sigma\left(P_{\mathcal{N}_l^{(n)}} G P_{\mathcal{N}_l^c} G P_{\mathcal{N}_l^{(n)}} \restriction P_{\mathcal{N}_l^{(n)}}\right) \geq 0. \tag{16}$$

Here, the superscript ^{c} denotes the complement. Notice that if $e = 0$, then $\mathcal{N}_r^c = \mathcal{N}_l^c$ are empty, and $\delta_m, \delta'_n = 0$.

Theorem 4.4 (Spectrum of $\Gamma(e)$). *Set $\Gamma_p(e) := P(L_p = e)\Gamma(e)P(L_p = e)$.*

1) *Let $e \neq 0$ and set $\delta_0 := \inf_{m \in \mathcal{N}_l}\{\delta_m\} + \inf_{n \in \mathcal{N}_r}\{\delta'_n\}$. Then we have for all $\beta \geq \beta_0$ (for any $0 < \beta_0 < \infty$ fixed):*

$$\Gamma_p(e) \geq \delta_0 \inf_{\{E_{ij} \neq 0\}} \left(|E_{ij}| \int_{S^2} dS \, |g(|E_{ij}|, \alpha)|^2 \right) P(L_p = e).$$

In particular, the Fermi Golden Rule Condition (12) is satisfied if the r.h.s. is not zero.

2) $\Gamma_p(0)$ *has an eigenvalue at zero, with the particle Gibbs state* Ω^p_β *as eigenvector:*

$$\Omega^p_\beta = Z_p(\beta)^{-1/2} \sum_i e^{-\beta E_i/2} \varphi_i \otimes \varphi_i, \tag{17}$$

where we recall that $Z_p(\beta)$ *was defined in (6). Moreover, if*

$$g_0 := \inf_{\{E_{mn}<0\}} |\langle \varphi_n, G\varphi_m\rangle|^2 \frac{e^{\beta E_n}}{e^{-\beta E_{mn}}-1} \int_{\mathbb{R}^3} \delta(E_{mn}+\omega)|g|^2 \geq 0$$

is strictly positive, then zero is a simple eigenvalue of $\Gamma_p(0)$ *with unique eigenvector* Ω^p_β *and the spectrum of* $\Gamma_p(0)$ *has a gap at zero:* $(0, 2g_0 Z_p) \cap \sigma(\Gamma_p(0)) = \emptyset$*. In particular, the Fermi Golden Rule Condition (13) holds.*

Remarks. 1. If $e \neq 0$ is finitely degenerate, then the infimum in the definition of δ_0 is a minimum. If e is nondegenerate, i.e. if $e = E_{m_0 n_0}$ for a unique pair (m_0, n_0), then δ_0 reduces to $\delta_0 = \sum_{n\neq n_0} |\langle \varphi_n, G\varphi_{n_0}\rangle|^2 + \sum_{m \neq m_0} |\langle \varphi_m, G\varphi_{m_0}\rangle|^2$.
2. If H_p is unbounded, then $g_0 = 0$. Indeed, let m be fixed, and take $n \to \infty$, then $E_{mn} < 0$ and $\langle \varphi_n, G\varphi_m\rangle \to 0$, since φ_n goes weakly to zero. Notice though that $g_0 > 0$ is only a sufficient condition for the Fermi Golden Rule Condition to hold at zero.
3. The size of the gap, $2g_0 Z_p$, is bounded away from zero uniformly in $\beta \geq \beta_0$, since

$$\lim_{\beta\to\infty} \inf_{\{E_m<E_n\}} \frac{\operatorname{tr} e^{-\beta H_p}}{e^{-\beta E_m} - e^{-\beta E_n}} = \lim_{\beta\to\infty} \inf_{\{\hat{E}_m<\hat{E}_n\}} \frac{\operatorname{tr} e^{-\beta \hat{H}_p}}{e^{-\beta \hat{E}_m} - e^{-\beta \hat{E}_n}},$$

where $\hat{E}_i := E_i - E_0 \geq 0$ (E_0 is the smallest eigenvalue of H_p) and $\hat{H}_p := H_p - E_0 \geq 0$ (the smallest eigenvalue of $\hat{H}_p$ is zero).

5 Review of previous results

Proving the RTE property is one of the key problems of non-equilibrium statistical mechanics. Until recently, this property was proven for specially designed abstract models (see [BRII]). The first result for realistic systems came in the pioneering work of Jakšić and Pillet [JP1,2] in 1996.

In their work, Jakšić and Pillet prove return to equilibrium, with exponential rate of convergence in time, for spin systems (i.e. two-level systems) for sufficiently high temperatures. Their work introduces the spectral approach to RTE. Jakšić and Pillet examine an two-level system coupled to the massless bosonic field, whose Liouvillian they construct using the work of Araki and Woods [AW] and Haag, Hugenholtz and Winnink [HHW]. The spectrum of the Liouvillian, acting on a suitably transformed positive temperature Hilbert space, which we call the *Jakšić-Pillet glued space*, is analyzed in the spirit of the theory of quantum resonances, using *spectral deformation* techniques, and where the deformation is generated by energy-translation. The IR condition on the form factor is $g(\omega) \sim \omega^p$, $\omega \to 0$, with $p > -1/2$, hence includes the physical case $p = 1/2$. However, the complex deformation pushes the continuous spectrum of the Liouvillian below the real axis by a distance only proportional to $1/\beta = T$, which results in the condition $|\lambda| < 1/\beta$. The spectral deformation technique imposes certain analyticity conditions on the form factor.

The N-level system coupled to the massless bosonic field is treated in [BFS4], but the spectrum of the Liouvillian is analyzed using complex dilation instead of translation. RTE with exponentially fast rate in convergence in time is established for small coupling constant λ *independent* of β. Bach, Fröhlich and Sigal adapt in this work their Renormalization Group method developed in [BFS1,2,3] to the positive temperature case. The IR condition is $p > 0$, which includes the physical case.

In a recent work, Dereziński and Jakšić [DJ] consider the Liouvillian of the N-level system interacting with the massless bosonic field. Their analysis of the spectrum of the Liouvillian uses the *Feshbach method* which is justified with the help of the *Mourre Theory* (see [HS2, Section IV] and references therein), applied to the reduced Liouvillian (away from the vacuum sector) and based on a global positive commutator estimate. The IR condition for instability of nonzero eigenvalues is $p > 0$, and for the lifting of the degeneracy of the zero eigenvalue, it is $p > 1$, which does not include the physical case.

The method for the spectral analysis of the Liouvillian we use employs the energy-translation generator in the Jakšić-Pillet glued positive temperature Hilbert space, as in [JP1,2] and [DJ]. We prove a Mourre estimate for the original Liouvillian with a conjugate operator which is a deformation of the enrgy shift generator mentioned above. This method has been developed in the zero-temperature case in [BFSS] (for the dilation generator though). We believe that this method is simple and robust.

Our construction of the PC works for the IR condition $p > 0$, which includes the physical case. In order to conclude absence of eigenvalues from the PC estimate, the Virial Theorem is needed. It turns out that the systems for which the Virial Theorem was applied so far have always satisfied the condition that $[L, A]$ is relatively bounded with respect to L, in which case a general theory has been developed, see [ABG] (for specific systems, see also [BFSS] for particle-field at zero temperature, [HS1,2] for N-body systems). It should be pointed out that the Virial Theorem is an important tool of interest on its own, still currently under research, see e.g. [GG].

In this work, we develop a Virial Theorem type argument in our case where the commutator $[L, A]$ is not relatively L-bounded. But this comes at the price that our result involves the triple commutator $[[[L, A], A], A]$, and consequently, we need a restrictive IR behaviour of the form factor, namely $p > 2$. We think that this restriction coming from the part of the proof using the Virial Theorem (not the PC estimate), can be improved by a better understanding of the Virial Theorem.

We finish this comparison of results by mentioning that in order to show the lifting of the zero eigenvalue degeneracy, we need the condition $|\lambda| < 1/\beta$.

6 The particle-field system at positive temperature

In this section, we review the GNS construction for the particle-field system at positive temperature. In particular, we introduce the Liouvillian implementing the dynamics determined by the coupled Hamiltonian (7).

The *algebra of observables* of the particle-field system is given by the C^*-algebra $\mathfrak{A} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{W}(\mathfrak{H})$, where $\mathcal{B}(\mathcal{H})$ denotes the bounded operators on $\mathcal{H}$, and $\mathcal{W}(\mathfrak{H})$ is the Weyl CCR-algebra, i.e. the C^*-algebra generated by $\{e^{i(a^*(f)+a(f))/\sqrt{2}} : f \in \mathfrak{H}\}$ (consult e.g. [BRII] for more details on the Weyl CCR-algebra). The *time evolution* on $\mathfrak{A}$ is given by the Heisenberg evolution generated by H, $a \mapsto \alpha_t(a) = e^{itH} a e^{-itH}$.

The equilibrium state at temperatrue $T = 1/\beta$ for the non-interacting system (i.e. with respect to the Heisenberg dynamics generated by the non-interacting Hamiltonian), is given by the product $\omega_{\beta,0} = \omega_\beta^p \otimes \omega_\beta^f$. Here, $\omega_\beta^p(\cdot) = \frac{\mathrm{tr}\,(e^{-\beta H_p}\,\cdot)}{\mathrm{tr}\,e^{-\beta H_p}}$ is the particle-Gibbs state at temperature β, and ω_β^f is a field KMS state at temperature β. By the property of structural stability, one can construct a β-KMS state $\omega_{\beta,\lambda}$ for the interacting system, and we now give the GNS construction. Since we will not need the representation map π in what follows, we omit it from our presentation. For more details, we refer to [BFS4], [JP1,2]. The GNS Hilbertspace $\mathcal{H}$ and the cyclic vector $\Omega_{\beta,0}$ corresponding to the non-interacting KMS state are given respectively by

$$\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_p \otimes \mathcal{F}(L^2(\mathbb{R} \times S^2)), \tag{18}$$

$$\Omega_{\beta,0} = \Omega_\beta^p \otimes \Omega, \tag{19}$$

where $\mathcal{F}(L^2(\mathbb{R} \times S^2))$ is the Fock space over $L^2(\mathbb{R} \times S^2)$ with vaccum Ω, which we call the *Jakšić-Pillet glued space*, and Ω_β^p is the particle Gibbs state at temperature β given in (17). The cyclic vector $\Omega_{\beta,\lambda}$ corresponding to the interacting KMS state satisfies (see [BFS4]) for $\beta|\lambda|$ small:

$$\|\Omega_{\beta,\lambda} - \Omega_{\beta,0}\| = O(\beta|\lambda|). \tag{20}$$

To complete the GNS picture, we give the Liouvillians L_0, L implementing the non-interacting and interacting dynamics (generated by the non-interacting and interacting Hamiltonians) respectively:

$$L = L_0 + \lambda I, \tag{21}$$

$$L_0 = L_p + L_f,$$

$$I = G_l \otimes (a^*(g_1) + a(g_1)) - G_r \otimes (a^*(g_2) + a(g_2)). \tag{22}$$

Here, we have set $L_p = H_p \otimes \mathbb{1}_p - \mathbb{1}_p \otimes H_p$ and $H_f = \mathrm{d}\Gamma(u)$, where u is the first (the radial) variable in $\mathbb{R} \times S^2$. We have also set for $g \in L^2(\mathbb{R}_+ \times S^2)$:

$$g_1(u,\alpha) = \begin{cases} \sqrt{1+\mu(u)}\ u\ g(u,\alpha), & u \geq 0 \\ \sqrt{\mu(-u)}\ u\ \overline{g}(-u,\alpha), & u < 0 \end{cases} \tag{23}$$

where $g_2(u,\alpha) = -g_1(-u,\alpha)$, $G_{l,r}$ were defined after (11) and the function $\mu = \mu(k)$ is the momentum density distribution, given by Planck's law describing black body radiation: $\mu(k) = (e^{\beta\omega} - 1)^{-1}$, $\omega = |k|$.

Our next task is to show selfadjointness of the Liouvillian L acting on the positive temperature Hilbert space (18). To do so, we introduce the positive operator $\Lambda = \mathrm{d}\Gamma(|u|)$ with domain $\mathcal{D}(\Lambda) = \{\psi \in \mathcal{H} : \|\Lambda\psi\| < \infty\}$ and the number operator

$$N = \mathrm{d}\Gamma(1) \tag{24}$$

with natural domain $\mathcal{D}(N) = \{\psi \in \mathcal{H} : \|N\psi\| < \infty\}$.

Proposition 6.1 (Relative Bounds). *Set $L^2 = L^2(\mathbb{R}\times S^2)$, and let $0 < \beta_0 < \infty$ be a fixed number.*

1) *If $f \in L^2$, then $||a(f)N^{-1/2}|| \le ||f||_{L^2}$.*

2) *If $|u|^{-1/2} f \in L^2$, then $||a(f)\Lambda^{-1/2}|| \le || \, |u|^{-1/2} f||_{L^2}$.*

3) *For $\psi \in \mathcal{D}(N^{1/2})$ and $\psi \in \mathcal{D}(\Lambda^{1/2})$ respectively, we have the following bounds, uniformly in $\beta \ge \beta_0$:*

$$\begin{aligned} ||I\psi||^2 &\le C||G|| \left(\|N^{1/2}\psi\|^2 + \|\psi\|^2\right), \\ ||I\psi||^2 &\le C||G|| \left(\|\Lambda^{1/2}\psi\|^2 + \|\psi\|^2\right). \end{aligned}$$

Here, $C \le C'(1+\beta_0^{-1})$, where C' is independent of β, β_0.

4) *For $\psi \in \mathcal{D}(N^{1/2})$, any $c > 0$, and uniformly in $\beta \ge \beta_0$, we have*

$$|\langle\psi, \lambda I\psi\rangle| \le c||N^{1/2}\psi||^2 + \frac{16\lambda^2}{c}||G||^2||\psi||^2 \int_{\mathbb{R}^3} (1+\beta_0^{-1}\omega^{-1})|g|^2 d^3k.$$

5) *For $\psi \in \mathcal{D}(\Lambda^{1/2})$, any $c > 0$, and uniformly in $\beta \ge \beta_0$, we have*

$$|\langle\psi, \lambda I\psi\rangle| \le c||\Lambda^{1/2}\psi||^2 + \frac{32\lambda^2}{c}||G||^2||\psi||^2 \int_{\mathbb{R}^3} (1+\beta_0^{-1}\omega^{-1})\frac{|g|^2}{\omega} d^3k.$$

Remark. The parameter β_0 gives the highest temperature, $T_0 = 1/\beta_0$, s.t. our estimates 3)-5) are valid uniformly in $T \le T_0$. T_0 can be fixed at any arbitrary large value. Since we are not interested in the large temperature limit $T \to \infty$, we set from now on for notational convenience $T_0 = 1$.

Notice that 4) and 5) tell us that $\forall c > 0$,

$$|\lambda I| \le cN + O(\lambda^2/c), \quad |\lambda I| \le c\Lambda + O(\lambda^2/c),$$

where we understand these inequalities holding in a sense of quadratic forms on $\mathcal{D}(N^{1/2})$ and $\mathcal{D}(\Lambda^{1/2})$ respectively.

Proof of Proposition 6.1. The proof is standard (see e.g. [BFS4], [JP1,2]), we only present the proof of 3), as an example of how to keep track of β.

From $\|I\psi\|^2 \leq 4\|G\|^2 \left(\|a^*(g_1)\psi\|^2 + \|a^*(g_2)\psi\|^2 + \|a(g_1)\psi\|^2 + \|a(g_2)\psi\|^2\right)$, and using the CCR $[a^*(f), a(g)] = \langle f, g\rangle$, we get

$$\|a^*(g_{1,2})\psi\|^2 = \langle \psi, a(g_{1,2})a^*(g_{1,2})\psi\rangle = \|a(g_{1,2})\psi\|^2 + \|g_{1,2}\|^2_{L^2}\|\psi\|^2,$$

so $\|I\psi\|^2 \leq 8\|G\|^2 \left(\|a(g_1)\psi\|^2 + \|a(g_2)\|^2 + 2\|g_1\|^2_{L^2}\|\psi\|^2\right)$, where we used $\|g_1\|_{L^2} = \|g_2\|_{L^2}$, since $g_1(u, \alpha) = -g_2(-u, \alpha)$. Using 1) and 2) above, we get

$$\begin{aligned} \|I\psi\|^2 &\leq 16\|G\|^2\|g_1\|^2_{L^2}\left(\|N^{1/2}\psi\|^2 + \|\psi\|^2\right), \\ \|I\psi\|^2 &\leq 16\|G\|^2 \left\| |u|^{-1/2} g_1\right\|^2_{L^2}\left(\|\Lambda^{1/2}\psi\|^2 + \|\psi\|^2\right). \end{aligned}$$

Next, we show that $\|g_1\|_{L^2} \leq C$ and $\| \, |u|^{-1/2} g_1\|_{L^2} \leq C$, uniformly in $\beta \geq \beta_0$. Indeed, notice that $\|g_1\|^2_{L^2} = \int_{\mathbb{R}^3}(1+2\mu)|g(\omega,\alpha)|^2 d\omega dS(\alpha) = \|g_2\|^2_{L^2}$, where we represented g in the integral in spherical coordinates. Since we have $1 + 2\mu = 1 + 2(e^{\beta\omega} - 1)^{-1} \leq 1 + 2\beta^{-1}\omega^{-1} \leq 1 + 2\beta_0^{-1}\omega^{-1}$, uniformly in $\beta \geq \beta_0$, we get with (9) (for $p > 0$) the following uniform bound in $\beta \geq \beta_0$:

$$\|g_{1,2}\|^2_{L^2} \leq 2\int_{\mathbb{R}^3}(1 + \beta_0^{-1}\omega^{-1})|g(k)|^2 d^3k = C < \infty. \tag{25}$$

Similarly, $\| \, |u|^{-1/2} g_1\|^2_{L^2} \leq 2\int_{\mathbb{R}^3}(1 + \beta_0^{-1}\omega^{-1})\omega^{-1}|g(\omega,\alpha)|^2 d^3k = C < \infty$, uniformly in $\beta \geq \beta_0$. It is clear from the last two estimates that C satisfies the bound indicated in the proposition. ∎

These relative bounds and Nelson's commutator theorem yield essential selfadjointness of the Liouvillian:

Theorem 6.2 (Selfadjointness of the Liouvillian). *Since H_p is bounded below, there is a $C > 0$ s.t. $H_p > -C$. Suppose that $[G, H_p](H_p + C)^{-1/2}$ is bounded in the sense that the quadratic form $\psi \mapsto 2i\mathrm{Im}\,\langle G\psi, H_p\psi\rangle$, defined on $\mathcal{D}(H_p)$, is represented by an operator denoted $[G, H_p]_{\mathrm{o}}$, s.t. $[G, H_p]_{\mathrm{o}}(H_p + C)^{-1/2}$ is bounded. Then $\forall\lambda \in \mathbb{R}$, L is essentially selfadjoint on*

$$\mathcal{D}_0 := \mathcal{D}(H_p) \otimes \mathcal{D}(H_p) \otimes \mathcal{D}(\Lambda) \subset \mathcal{H}_p \otimes \mathcal{H}_p \otimes \mathcal{F}(L^2(\mathbb{R} \times S^2)).$$

Proof. The proof uses Nelson's commutator theorem (see [RS], Theorem X.37). Let $\mathcal{N} = (H_p + C) \otimes \mathbb{1}_p + \mathbb{1}_p \otimes (H_p + C) + \Lambda + 1$, then $\mathcal{N}$ is selfadjoint on $\mathcal{D}_0$ and $\mathcal{N} \geq 1$. Also, L is defined and symmetric on $\mathcal{D}_0$.

According to Nelson's commutator theorem, in order to prove Theorem 1.2, we have to show that $\forall\psi \in \mathcal{D}_0$ and some constant $d > 0$,

$$\begin{aligned} ||L\psi|| &\leq d||\mathcal{N}\psi||, &(26)\\ |\langle L\psi, \mathcal{N}\psi\rangle - \langle \mathcal{N}\psi, L\psi\rangle| &\leq d||\mathcal{N}^{1/2}\psi||^2. &(27) \end{aligned}$$

Estimate (26) easily follows from $||L_p\mathcal{N}^{-1}|| \leq 1$, $||L_f\mathcal{N}^{-1}|| \leq 1$ and $||I\mathcal{N}^{-1}|| \leq ||I(\Lambda+1)^{-1/2}||\,||(\Lambda+1)^{1/2}(\Lambda+1)^{-1}|| \leq d$ (by 3) of Proposition 6.1).

To show (27), notice that L_0 commutes with $\mathcal{N}$, so the l.h.s. of (27) reduces to

$$|\langle I\psi, \mathcal{N}\psi\rangle - \langle \mathcal{N}\psi, I\psi\rangle| \leq |\langle I\psi, \Lambda\psi\rangle - \langle \Lambda\psi, I\psi\rangle| + K, \tag{28}$$

where

$$\begin{aligned} K &= |\langle I\psi, ((H_p+C)\otimes \mathbb{1} + \mathbb{1}\otimes (H_p+C))\psi\rangle \\ &\quad - \langle ((H_p+C)\otimes \mathbb{1} + \mathbb{1}\otimes (H_p+C))\psi, I\psi\rangle| \,. \end{aligned} \tag{29}$$

Let us examine the first term on the r.h.s. of (28). It is easily shown that since $|u|g_{1,2} \in L^2(\mathbb{R}\times S^2)$, then $a^*(g_{1,2})\Lambda = \Lambda a^*(g_{1,2}) + a^*(|u|g_{1,2})$ on $\mathcal{D}(\Lambda)$. This shows that $a^\#(g_{1,2})$ leave $\mathcal{D}(\Lambda)$ invariant and so we have $\forall \psi\in\mathcal{D}_0$:

$$\begin{aligned} &|\langle I\psi, \Lambda\psi\rangle - \langle \Lambda\psi, I\psi\rangle| \\ &\quad = |\langle \psi, (I\Lambda - \Lambda I)\psi\rangle| \\ &\quad = \big|\big\langle \psi, \big(G_l\otimes(a^*(|u|g_1) - a(|u|g_1)) - G_r\otimes(a^*(|u|g_2) - a(|u|g_2))\big)\psi\big\rangle\big| \\ &\quad \leq c||\psi||\,||(\Lambda+1)^{1/2}\psi|| \leq c||\mathcal{N}^{1/2}\psi||^2, \end{aligned}$$

where we used Proposition 6.1 in the third step.

Now we look at K given in (29). Using the specific form of I (see (22)), we can write $K \leq |K_1| + |K_2|$, where

$$\begin{aligned} K_1 &= \langle G_l\otimes(a(g_1)+a^*(g_1))\psi, (H_p+C)\otimes\mathbb{1}\psi\rangle \\ &\quad - \langle (H_p+C)\otimes\mathbb{1}\psi, G_l\otimes(a(g_1)+a^*(g_1))\psi\rangle\,, \\ K_2 &= \langle G_r\otimes(a(g_2)+a^*(g_2))\psi, \mathbb{1}\otimes(H_p+C)\psi\rangle \\ &\quad - \langle \mathbb{1}\otimes(H_p+C)\psi, G_r\otimes(a(g_2)+a^*(g_2))\psi\rangle\,. \end{aligned}$$

We examine K_1. Let $\psi\in\mathcal{D}_0$, then $(H_p+C)^{1/2}\psi \in \mathcal{H}$, and so

$$\begin{aligned} K_1 &= 2i\mathrm{Im}\,\langle G_l\otimes(a(g_1)+a^*(g_1)), (H_p+C)\otimes\mathbb{1}\psi\rangle \\ &= 2i\mathrm{Im}\,\langle (a(g_1)+a^*(g_1))\psi, [G, H_p]_o\psi\rangle\,, \end{aligned}$$

so we obtain $|K_1| \leq c\|(\Lambda+1)^{1/2}\psi\|\,\|(H_p+C)^{1/2}\otimes\mathbb{1}\psi\| \leq c\|\mathcal{N}^{1/2}\psi\|^2$. The same estimate is obtained for $|K_2|$ in a similar way. This shows (27) and completes the proof. ■

7 Proof of Theorem 4.1

The proof consists of two main steps: the PC estimate w.r.t. spectral localization in the uncoupled Liouvillian L_0 and a passage from this estimate to the one localized w.r.t. the full Liouvillian L.

Our estimates are uniform in $\beta \geq \beta_0$ (for any $0 < \beta_0 < \infty$ fixed), but as we have mentioned in the remark after Proposition 6.1, we set $\beta_0 = 1$.

7.1 Positive commutator with respect to spectral localization in L_0

We construct an operator B (see (35)) which is positive on spectral subspaces of L_0. The main result of this section is Theorem 7.3, which shows positivity under assumption of the Fermi Golden Rule Condition.

On $L^2(\mathbb{R}\times S^2)$ and for $t\in\mathbb{R}$, we define the unitary transformation $\left(\tilde{U}_t\psi\right)(u,\alpha)=\psi(u-t,\alpha)$, which induces a unitary transformation U_t on Fock space $\mathcal{F}=\mathcal{F}(L^2(\mathbb{R}\times S^2))$: $U_t=\Gamma(\tilde{U}_t)$, i.e. for $\psi\in\mathcal{F}$, the projection onto the n-sector of $U_t\psi$ is given by

$$(U_t\psi)_n(u_1,\dots,u_n)=\psi_n(u_1-t,\dots,u_n-t).$$

Here and often in the future, we do not display the angular variables $\alpha_1,\dots,\alpha_n$ in the argument of ψ_n. U_t is a strongly continuous unitary one-parameter ($t\in\mathbb{R}$) group on $\mathcal{F}$. Its anti-selfadjoint generator A_0, defined in the strong sense by $\partial_t|_{t=0}U_t=A_0$, is

$$A_0=-\mathrm{d}\Gamma(\partial_u).$$

The domain of the unbounded operator A_0, $\mathcal{D}(A_0)=\{\psi\in\mathcal{F}:\partial_t|_{t=0}U_t\psi\in\mathcal{F}\}$, is dense in $\mathcal{F}$, which simply follows from the fact that A_0 is the generator of a strongly continuous group. From now on, we write

$$U_t=e^{tA_0},\quad t\in\mathbb{R}.$$

Proposition 7.1 *On the dense set $\mathcal{D}(L_0)\cap\mathcal{D}(N)$, we have $e^{-tA_0}Le^{tA_0}=L_0+tN+\lambda I_t$, where I_t is obtained from I by replacing the form factor g by its translate g^t, and $g^t(u,\alpha)=g(u+t,\alpha)$. We obtain therefore*

$$\partial_t|_{t=0}e^{-tA_0}Le^{tA_0}=N+\lambda\tilde{I}, \tag{30}$$

where $\tilde{I}=G_l\otimes(a^*(\partial_u g_1)+a(\partial_u g_1))-G_r\otimes(a^*(\partial_u g_2)+a(\partial_u g_2))$. The derivative in (30) is understood in the strong topology.

We give the proof in Appendix A.2. On a formal level, we have $\partial_t|_{t=0}\,e^{-tA_0}Le^{tA_0}=-A_0L+LA_0=[L,A_0]$, which suggests the *definition of the unbounded operator* $[L,A_0]$ with domain $\mathcal{D}([L,A_0])=\mathcal{D}(N)$ as

$$[L,A_0]:=N+\lambda\tilde{I}. \tag{31}$$

We point out that the operator $[L,A_0]$ is defined as the r.h.s. of (31), and not as a commutator in the sense of LA_0-A_0L. Remark that $[L,A_0]$ is positive on $\mathcal{D}(N)\cap\mathrm{Ran}\,P_\Omega^\perp$, where Ω is the vacuum in $\mathcal{F}$. Indeed, from Proposition 6.1, it follows (take e.g. $c = 1/4$) $[L, A_0] \geq \frac{3}{4}N - O(\lambda^2)$, so that $P_\Omega^\perp[L, A_0]P_\Omega^\perp \geq (3/4 - O(\lambda^2))\, P_\Omega^\perp$. On the other hand, $P_\Omega[L, A_0]P_\Omega = 0$, so if we want to find an operator that is positive also on $\mathbb{C}\Omega$, then we need to modify A_0.

For a fixed eigenvalue $e \in \sigma(L_p)$, define

$$\begin{aligned} b(e) &= \theta\lambda\left(\overline{Q}R_\epsilon^2 IQ - QIR_\epsilon^2\overline{Q}\right), \\ R_\epsilon &= \left((L_0 - e)^2 + \epsilon^2\right)^{-1/2}. \end{aligned} \tag{32}$$

Here, θ and ϵ are positive parameters, and Q, $\overline{Q}$ are projection operators on $\mathcal{H}$ defined as

$$Q = P(L_p = e) \otimes P_\Omega, \quad \overline{Q} = \mathbb{1} - Q. \tag{33}$$

In what follows, we denote $\overline{R}_\epsilon := \overline{Q}R_\epsilon$.

Proposition 7.2 *The operator $b = b(e)$ is bounded and $[L, b] = Lb - bL$ is well defined on $\mathcal{D}_0$ and it extends to a bounded operator on the whole space. We denote the extended operator again by $[L, b]$.*

Proof. The operator b is bounded since both IQ and QI are bounded. Furthermore, since $||L_0R_\epsilon|| \leq 1 + |e|/\epsilon$ and $||L_0Q|| = |e|$, then $[L_0, b]$ is bounded. Moreover, since $||IQ|| \leq C$ and $||I\overline{R}_\epsilon^2 IQ|| \leq C\epsilon^{-2}||(N+1)IQ|| \leq 2C\epsilon^{-2}||IQ|| \leq C\epsilon^{-2}$, then also $||[I, b]|| < \infty$. We used the fact that $\mathrm{Ran}\, IQ \subset \mathrm{Ran}\, P(N \leq 1)$, since I is linear in creators and $NQ = 0$. ■

We define the operator $[L, A]$ by $\mathcal{D}([L, A]) = \mathcal{D}(N)$ and

$$[L, A] := [L, A_0] + [L, b] = N + \lambda\tilde{I} + [L, b], \tag{34}$$

and the operator B by $\mathcal{D}(B) = \mathcal{D}(N)$ and

$$B := [L, A] - \frac{1}{10}N = \frac{9}{10}N + \lambda\tilde{I} + [L, b]. \tag{35}$$

Here is the main result of this section:

Theorem 7.3 *Let $e \in \sigma(L_p)$ and let Δ be an interval around e not containing any other eigenvalue of L_p. Let E_Δ be the (sharp) indicator function of Δ and set $E_\Delta^0 = E_\Delta(L_0)$. Assume that the Fermi Golden Rule Condition (12) (or (13)) holds, and that the parameters satisfy $\theta, \epsilon \ll 1$ and $\epsilon \ll \theta, \theta\lambda^2 \ll \epsilon^3$. Then we have on $\mathcal{D}(N^{1/2})$, in the sense of quadratic forms:*

$$E_\Delta^0 B E_\Delta^0 \geq \frac{\theta\lambda^2}{\epsilon}\gamma_e E_\Delta^0(1 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}})E_\Delta^0, \tag{36}$$

where $P_{\Omega_{\beta,0}}$ is the projector onto the span of $\Omega_{\beta,0}$ defined in (19).

An essential ingredient of the proof of Theorem 7.3 is the *Feshbach method*, which we explain now.

7.2 The Feshbach method

The main idea of the Feshbach method is to use an isospectral correspondence between operators acting on a Hilbert space and operators acting on some subspace. We explain this method adapted to our case. For a more general exposition, see e.g. [BFS2].

Consider the Hilbert spaces $\mathcal{H}_e$ defined by $\mathcal{H}_e = \mathrm{Ran}\, \chi_\nu E^0_\Delta$, where $\chi_\nu = \chi(N \leq \nu)$ is a cutoff in N, and ν is a positive integer. With our definitions of $Q, \overline{Q}$, (see (33)) we have

$$\mathcal{H}_e = \mathrm{Ran}\, \chi_\nu E^0_\Delta Q \oplus \mathrm{Ran}\, \chi_\nu E^0_\Delta \overline{Q}. \tag{37}$$

Define $Q_1 = \chi_\nu E^0_\Delta Q$ and $Q_2 = \chi_\nu E^0_\Delta \overline{Q}$ and set $B_{ij} = Q_i B Q_j$, $i, j = 1, 2$. The operators B_{ij} are bounded due to the cutoff in N. Notice that $Q_{1,2}$ are projection operators (i.e. $Q^2_{1,2} = Q_{1,2}$) since χ_ν commutes with E^0_Δ and Q.

By saying that $z \in \mathbb{C}$ is in the resolvent set of B_{22}, we mean that $(B_{22} - z)^{-1} \restriction \mathrm{Ran}\, Q_2$ exists as a bounded operator. The main ingredient of the Feshbach method is the following observation:

Proposition 7.4 (isospectrality of the Feshbach map). *If z is in the resolvent set of B_{22} and if*

$$\left\| Q_2 (B_{22} - z)^{-1} Q_2 B Q_1 \right\| < \infty, \quad \left\| Q_1 B Q_2 (B_{22} - z)^{-1} Q_2 \right\| < \infty, \tag{38}$$

then we have $z \in \sigma_\#(B) \Longleftrightarrow z \in \sigma_\#(\mathcal{E}_z)$, where the Feshbach map $\mathcal{E}_z = \mathcal{E}_z(B)$ is defined by $B \mapsto \mathcal{E}_z = B_{11} - B_{12}(B_{22} - z)^{-1} B_{21}$, and $\sigma_\#$ stands for σ or σ_{pp} (spectrum or pure point spectrum).

The proof of Proposition 7.4 is given in a more general setting e.g. in [BFS2], we do not repeat it here. We use the isospectrality of the Feshbach map to show positivity of B in the following way (see also [BFSS]):

Corollary 7.5 *Let $\vartheta_0 = \inf \sigma(B \restriction \mathcal{H}_e)$ and suppose that $B_{22} \geq \tilde{\vartheta} Q_2$ for some $\tilde{\vartheta} > -\infty$, and that $\inf \sigma(\mathcal{E}_\vartheta) \geq \Sigma_0$ uniformly in ϑ for $\vartheta \leq \vartheta_1$, where Σ_0 and ϑ_1 are two fixed (finite) numbers. Then we have $\vartheta_0 \geq \min\{\tilde{\vartheta}, \inf \sigma(\mathcal{E}_{\vartheta_0})\}$.*

Remarks. 1. All our estimates in this section will be independent of the N-cutoff introduced in (37). In particular, $\tilde{\vartheta}, \vartheta_0, \vartheta_1, \Sigma_0$ are independent of ν. This will allow us to obtain inequality (36) on $\mathcal{D}(N^{1/2})$ from the corresponding estimate on $\operatorname{Ran}\chi(N \leq \nu)$ by letting $\nu \to \infty$ (see (59) below).

2. The condition $\inf \sigma(\mathcal{E}_\vartheta) \geq \Sigma_0$ uniformly in ϑ for $\vartheta \leq \vartheta_1$, implies that $\vartheta_0 \neq -\infty$. Indeed, if $\vartheta_0 = -\infty$, then we have a sequence $\vartheta_n \to -\infty$, $\vartheta_n \in \sigma(B \restriction \mathcal{H}_e)$, so by Proposition 3.5 (we show below that (38) is satisfied for $z = \vartheta_n$, $\forall n$), we would have $\inf \sigma(\mathcal{E}_{\vartheta_n}) \to -\infty$ as $n \to \infty$, which is in contradiction with the lower bound condition $\inf \sigma(\mathcal{E}_\vartheta) \geq \Sigma_0$.

Proof of Corollary 7.5. If $\vartheta_0 < \tilde{\vartheta}$, then ϑ_0 is in the resolvent set of B_{22}, and it is easy to show that (38) holds for $z = \vartheta_0$.

Now since $\vartheta_0 = \inf \sigma(B \restriction \mathcal{H}_e)$ and $\vartheta_0 < \tilde{\vartheta}$, there is a sequence $\vartheta_n \to \vartheta_0$, $\vartheta_n < \tilde{\vartheta}$, $\vartheta_n \in \sigma(B \restriction \mathcal{H}_e)$, and such that $\forall n$, (38) is satisfied for $z = \vartheta_n$.

By the above remark 2, we have $\vartheta_0 > -\infty$, and therefore $\vartheta_0 \in \sigma(B \restriction \mathcal{H}_e)$, because the spectrum is a closed set. By Proposition 7.4, $\vartheta_0 \in \sigma(\mathcal{E}_{\vartheta_0})$, which finishes the proof. ■

7.3 Proof of Theorem 7.3

The idea is to apply Corollary 7.5 to the operator

$$B' = B - \delta_{e,0}\delta P^{\perp}_{\Omega_{\beta,0}}, \tag{39}$$

where $\delta_{e,0}$ is the Kronecker symbol, i.e. it is one if $e = 0$ and zero else. The positive number δ will be chosen appropriately below, see (57).

First, we show that $B'_{22} \geq (3/4 - \delta_{e,0}\,\delta)Q_2$ (see (41)), then we show that $\mathcal{E}_\vartheta \geq -1 - \delta_{e,0}\,\delta =: \Sigma_0$ (Proposition 7.6), uniformly in ϑ for $\vartheta \leq 1/2 - \delta_{e,0}\,\delta$. Invoking Corollary 7.5 will then yield the result. Notice that due to the cutoff χ_ν in (37), B_{ij}, $i, j \in \{1, 2\}$ are bounded operators. All the following estimates are independent of ν.

We first calculate $B'_{22} = Q_2 B' Q_2$. Using $QQ_2 = 0$, and $\delta_{e,0} P^{\perp}_{\Omega_{\beta,0}} Q_2 = \delta_{e,0} Q_2$, we obtain from (39) and (35)

$$B'_{22} = Q_2 \left(\frac{9}{10} N + \lambda \tilde{I} + \theta\lambda^2 (\overline{R}_\epsilon^2 IQI - IQI\overline{R}_\epsilon^2) - \delta_{e,0}\delta \right) Q_2. \tag{40}$$

Proceeding as in the proof of Proposition 6.1, one shows that $\forall c > 0$,

$$\left| \left\langle \psi, \lambda \tilde{I} \psi \right\rangle \right| \leq c||N^{1/2}\psi||^2 + C\tfrac{\lambda^2}{c}||\partial_u g_1||^2_{L^2}||\psi||^2.$$

With our assumptions on g, $||\partial_u g_1||^2_{L^2} < \infty$, uniformly in $\beta \geq 1$. Using the inequality above with $c = 1/10$ and $\|\overline{R}_\epsilon^2 IQI\| \leq C\epsilon^{-2}$, we obtain

$$B'_{22} \geq Q_2 \left(\frac{8}{10} N - O(\lambda^2 + \theta\lambda^2\epsilon^{-2}) - \delta_{e,0}\delta \right) Q_2.$$

As can be easily checked, $Q_2 = Q_2 P_\Omega^\perp$, so we have $NQ_2 \geq Q_2$, and we conclude that

$$B'_{22} \geq \left(\frac{8}{10} - \delta_{e,0}\delta - O(\lambda^2 + \theta\lambda^2\epsilon^{-2}) \right) Q_2 \geq \left(\frac{3}{4} - \delta_{e,0}\delta \right) Q_2, \tag{41}$$

provided $\lambda^2 \ll 1$ and $\theta\lambda^2 \ll \epsilon^2$. In the language of Corollary 7.5, this means we can take $\tilde{\vartheta} = 3/4 - \delta_{e,0}\delta$.

In a next step, we calculate a lower bound on $\mathcal{E}_\vartheta$ for $\vartheta \leq 1/2 - \delta_{e,0}\delta$.

Proposition 7.6 *We have, uniformly in ϑ for $\vartheta \leq 1/2 - \delta_{e,0}\delta$:*

$$\mathcal{E}_\vartheta \geq 2\pi \frac{\theta\lambda^2}{\epsilon}(1-5\theta) Q_1 \left(\Gamma(e) - \frac{\epsilon\delta_{e,0}\delta}{2\theta\lambda^2} P^\perp_{\Omega^p_\beta} - O(\epsilon^{1/4} + \epsilon\theta^{-1} + \theta\lambda^2\epsilon^{-3}) \right) Q_1, \tag{42}$$

where the error term is independent of δ. Recall that Ω^p_β is the particle Gibbs state defined in (17).

Proof of Proposition 7.6 By definition, $\mathcal{E}_\vartheta = B'_{11} - B'_{12}(B'_{22} - \vartheta)^{-1} B'_{21}$. We show that B'_{11} is positive and $B'_{12}(B'_{22} - \vartheta)^{-1}B'_{21}$ is small compared to B'_{11}.

With $\overline{Q}Q_1 = 0$, $QQ_1 = Q_1$ and $\delta_{e,0} P^\perp_{\Omega_{\beta,0}} Q_1 = \delta_{e,0} P^\perp_{\Omega^p_\beta} Q_1$, we obtain from (39) and (35):

$$B'_{11} \geq 2\theta\lambda^2 Q_1 \left(I\overline{R}^2_\epsilon I - \frac{\delta_{e,0}\delta}{2\theta\lambda^2} P^\perp_{\Omega^p_\beta} \right) Q_1 - O(\lambda^2), \tag{43}$$

where we used $\lambda\tilde{I} \geq -\frac{1}{10}N - O(\lambda^2)$ and $Q_1 N = 0$.

Let us now examine $B'_{12}(B'_{22} - \vartheta)^{-1}B'_{21}$. Notice that from (40), we get

$$Q_2(B'_{22} - \vartheta)Q_2 = \tfrac{9}{10} Q_2 N^{1/2}(1 - \tfrac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1} + K_1)N^{1/2}Q_2, \tag{44}$$

where we defined the bounded selfadjoint operator K_1 acting on $\operatorname{Ran} Q_2$ as

$$K_1 = \frac{10}{9} N^{-1/2} \left(\lambda\tilde{I} + \theta\lambda^2(\overline{R}^2_\epsilon IQI - IQI\overline{R}^2_\epsilon) \right) N^{-1/2}. \tag{45}$$

Since $||Q_2 N^{-1/2}|| \leq 1$ and $||\tilde{I}(N+1)^{-1/2}|| \leq C$, we get $||K_1|| \leq C(\lambda + \theta\lambda^2\epsilon^{-2})$. Now on $\operatorname{Ran} P^\perp_\Omega$, we have $N \geq 1$, so since we look at ϑ s.t. $\vartheta + \delta_{e,0}\delta \leq 1/2$, we obtain

$$1 - \frac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1} \geq 1 - \frac{10}{9}\frac{1}{2} = \frac{4}{9}. \tag{46}$$

Therefore we can rewrite (44) as

$$\begin{aligned} Q_2(B'_{22} - \vartheta)Q_2 \;&=\; \tfrac{9}{10} Q_2 N^{1/2} \big(1 - \tfrac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{1/2} (\mathbb{1} + K_2) \\ &\quad \times \big(1 - \tfrac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{1/2} N^{1/2} Q_2, \end{aligned} \tag{47}$$

where $K_2 = \big(1 - \frac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{-1/2} K_1 \big(1 - \frac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{-1/2}$, and $||K_2|| \leq \frac{9}{4}||K_1|| = O(\lambda + \theta\lambda^2\epsilon^{-2}) \ll 1$. We have thus from (47):

$$Q_2(B'_{22} - \vartheta)^{-1}Q_2 = \frac{10}{9} Q_2 N^{-1/2}\big(\mathbb{1} - \tfrac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{-1/2} K^2 \times \big(\mathbb{1} - \tfrac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{-1/2} N^{-1/2}Q_2, \tag{48}$$

where $K = (\mathbb{1} + K_2)^{-1/2}$ is bounded and selfadjoint with $||K||^2 = ||K^2|| = ||(\mathbb{1} + K_2)^{-1}|| \leq \frac{1}{1-||K_2||} < 2$. We have therefore, from (48) and (46), and uniformly in ϑ for $\vartheta \leq 1/2 - \delta_{e,0}\delta$:

$$\begin{aligned}\big\langle \psi, B'_{12}(B'_{22} - \vartheta)^{-1}B'_{21}\psi\big\rangle &= \frac{10}{9}||K\big(\mathbb{1} - \tfrac{10}{9}(\vartheta + \delta_{e,0}\delta)N^{-1}\big)^{-1/2}N^{-1/2}B'_{21}\psi||^2 \\ &\leq 2\frac{10}{9}\frac{9}{4}||N^{-1/2}B'_{21}\psi||^2 = 5||N^{-1/2}B_{21}\psi||^2. \end{aligned}\tag{49}$$

Notice that $B'_{12} = B_{12}$ and $B'_{21} = B_{21}$. Now, remembering (35), and since $NQ_1 = 0$ and $Q_2 Q = 0 = \overline{Q}Q_1$,

$$N^{-1/2}B_{21} = N^{-1/2}Q_2\big[\lambda\tilde{I} + \theta\lambda(L_0 - e)\overline{R}_\epsilon^2 I - \theta\lambda\overline{R}_\epsilon^2 I(L_0 - e) + \theta\lambda^2(I\overline{R}_\epsilon^2 I - \overline{R}_\epsilon^2 IQI)\big]Q_1.$$

Using $||N^{-1/2}Q_2|| \leq 1$, $||\tilde{I}Q_1|| \leq C$, $||IQ_1|| \leq C$, $\|N^{-1/2}I\| \leq C$, $(L_0 - e)Q_1 = 0$, $||(L_0 - e)R_\epsilon|| \leq 1$, we get

$$||N^{-1/2}B_{21}\psi||^2 \leq C(\lambda^2 + \theta^2\lambda^4\epsilon^{-4})||\psi||^2 + 2\theta^2\lambda^2||\overline{R}_\epsilon IQ_1\psi||^2,$$

thus with (49), we obtain

$$-\big\langle \psi, B'_{12}(B'_{22} - \vartheta)^{-1}B'_{21}\psi\big\rangle \geq -10\theta^2\lambda^2\left\langle \psi, Q_1 I\overline{R}_\epsilon^2 IQ_1\psi\right\rangle - O(\lambda^2 + \theta^2\lambda^4\epsilon^{-4})||\psi||^2,$$

and so, together with (43), we get, uniformly in ϑ for $\vartheta \leq 1/2 - \delta_{e,0}\delta$:

$$\mathcal{E}_\vartheta \geq 2\theta\lambda^2(1 - 5\theta)Q_1(I\overline{R}_\epsilon^2 I - \frac{\delta_{e,0}\delta}{2\theta\lambda^2}P^\perp_{\Omega^p_\beta})Q_1 - O(\lambda^2 + \theta^2\lambda^4\epsilon^{-4}). \tag{50}$$

We point out that the error term in the last inequality does not depend on δ. With the choice of parameters we will make (see (77)), (50) shows that $\mathcal{E}_\vartheta \geq -1 - \delta_{e,0}\delta$ uniformly in ϑ for $\vartheta \leq 1/2 - \delta_{e,0}\delta$, i.e. in the language of Corollary 7.5, $\Sigma_0 = -1 - \delta_{e,0}\delta$.

The remaining part of the proof consists in relating the strict positivity of the nonnegative operator $Q_1 I\overline{R}_\epsilon^2 IQ_1$ to the Fermi Golden Rule Condition. We let I_a and $I_c = I_a^*$ denote the parts of I containing annihilators and creators only, so that $I = I_a + I_c$. Thus

$$Q_1 I\overline{R}_\epsilon^2 IQ_1 = Q_1 I_a\overline{R}_\epsilon^2 I_c Q_1 = Q_1 I_a R_\epsilon^2 I_c Q_1. \tag{51}$$

In the first step, we used $I_aQ_1 = 0$ and $Q_1I_c = 0$ (since $I_aP_\Omega = 0$) and in the second step, we used $Q_1I_a\overline{Q} = Q_1I_a$ (since $I_aQ = 0$). Now write

$$Q_1I_aR_\epsilon^2I_cQ_1 = Q_1\int\int m^*(u,\alpha)a(u,\alpha)R_\epsilon^2(e)a^*(u',\alpha')m(u',\alpha')\ Q_1, \tag{52}$$

where m is defined (11), and where we display the dependence of R_ϵ^2 on e. The operator-valued distributions (a and a^*) satisfy the canonical commutation relations $[a(u,\alpha), a^*(u',\alpha')] = \delta(u-u')\delta(\alpha-\alpha')$. Next, we notice that the pull-through formula $a(u,\alpha)L_f = (L_f + u)a(u,\alpha)$ implies

$$a(u,\alpha)R_\epsilon^2(e) = R_\epsilon^2(e-u)a(u,\alpha). \tag{53}$$

Using the CCR and formula (53) together with the fact that $a(u,\alpha)Q_1 = 0$, we commute $a(u,\alpha)$ in (52) to the right and arrive at

$$(52) = Q_1\int m^*(u,\alpha)R_\epsilon^2(e-u)m(u,\alpha)\ Q_1. \tag{54}$$

We can pull a factor P_Ω out of Q_1 and place it inside the integral next to $R_\epsilon^2(e-u)$ and thus replace $R_\epsilon^2(e-u)$ by $((L_p - e + u)^2 + \epsilon^2)^{-1}$. Notice that $\epsilon((L_p - e + u)^2 + \epsilon^2)^{-1} \to \delta(L_p - e + u)$ as $\epsilon \to 0$. More precisely, we have

Proposition 7.7 *For $\epsilon \ll 1$, we have*

$$Q_1\int m^*(u,\alpha)\left((L_p - e + u)^2 + \epsilon^2\right)^{-1} m(u,\alpha)\ Q_1 \geq Q_1\frac{\pi}{\epsilon}\left(\Gamma(e) - O(\epsilon^{1/4})\right)Q_1.$$

Proposition 7.7, which we prove in Appendix A.3, together with (50)-(52) and (54) yields (42), proving Proposition 7.6. ■

Now we finish the proof of Theorem 7.3. If the Fermi Golden Rule Condition (12) holds, then for $e \neq 0$, we have $\Gamma(e) \geq \gamma_e > 0$ on $\mathrm{Ran}\, Q_1$, so we obtain from (42), and under the conditions on the parameters stated in Theorem 7.3: $\mathcal{E}_\vartheta \geq \pi\frac{\theta\lambda^2}{\epsilon}\gamma_e$, so by Corollary 7.5:

$$\inf\sigma(B \restriction \mathcal{H}_e) \geq \min\{1/2, \pi\theta\lambda^2\epsilon^{-1}\gamma_e\} = \pi\frac{\theta\lambda^2}{\epsilon}\gamma_e, \tag{55}$$

since by our choice of the parameters (see (77)), we have $\frac{\theta\lambda^2}{\epsilon} \ll 1$.

For $e = 0$, we have $\Gamma(0) = \Gamma(0)P^\perp_{\Omega^p_\beta}$, since $\Gamma(0)\Omega^p_\beta = 0$ (see Theorem 4.4), so Proposition 7.6 gives

$$\mathcal{E}_\vartheta \geq \pi\frac{\theta\lambda^2}{\epsilon}Q_1\left(\left\{\gamma_0 - \frac{\epsilon\delta}{2\theta\lambda^2}\right\}P^\perp_{\Omega^p_\beta} - O(\epsilon^{1/4} + \epsilon\theta^{-1} + \theta\lambda^2\epsilon^{-3})\right)Q_1. \tag{56}$$

For some fixed $0 < a < \frac{\gamma_0}{2(\pi-1)}$ (independent of $\theta, \lambda, \epsilon$), we choose

$$\delta = \frac{2\theta\lambda^2}{\epsilon}(\gamma_0 + a) \ll 1, \tag{57}$$

which gives with (56):

$$\begin{aligned}
\mathcal{E}_\vartheta &\geq \pi\frac{\theta\lambda^2}{\epsilon}Q_1\left(-aP^{\perp}_{\Omega^p_\beta} - O(\epsilon^{1/4} + \epsilon\theta^{-1} + \theta\lambda^2\epsilon^{-3})\right)Q_1 \\
&\geq \pi\frac{\theta\lambda^2}{\epsilon}\left(-a - O(\epsilon^{1/4} + \epsilon\theta^{-1} + \theta\lambda^2\epsilon^{-3})\right)Q_1 \\
&\geq -2\pi\frac{\theta\lambda^2}{\epsilon}a\,Q_1.
\end{aligned}$$

Remembering that $B' = B - \delta P^{\perp}_{\Omega_{\beta,0}}$, we obtain from Corollary 7.5

$$\inf\sigma\left((B - \delta P^{\perp}_{\Omega_{\beta,0}}) \upharpoonright \mathcal{H}_0\right) \geq \min\{1/2, -2\pi a\theta\lambda^2/\epsilon\} = -2\pi a\frac{\theta\lambda^2}{\epsilon},$$

from which we conclude that

$$\begin{aligned}
\chi_\nu E^0_\Delta B E^0_\Delta\,\chi_\nu &\geq \chi_\nu E^0_\Delta\left(-2\pi a\frac{\theta\lambda^2}{\epsilon} + \delta P^{\perp}_{\Omega_{\beta,0}}\right)E^0_\Delta\,\chi_\nu \\
&= 2\frac{\theta\lambda^2}{\epsilon}\gamma_0\chi_\nu E^0_\Delta\left(1 - a(\pi-1)/\gamma_0 - (1 + a/\gamma_0)P_{\Omega_{\beta,0}}\right)E^0_\Delta\,\chi_\nu \\
&\geq \frac{\theta\lambda^2}{\epsilon}\gamma_0\chi_\nu E^0_\Delta(1 - \tfrac{5}{2}P_{\Omega_{\beta,0}})E^0_\Delta\,\chi_\nu,
\end{aligned} \tag{58}$$

where we used $a/\gamma_0 \leq \frac{1}{2(\pi-1)}$.

Estimates (55) and (58) yield $\forall\psi$:

$$\left\langle\psi, \chi_\nu E^0_\Delta B E^0_\Delta\chi_\nu\psi\right\rangle \geq \frac{\theta\lambda^2}{\epsilon}\gamma_e\left\langle\psi, \chi_\nu E^0_\Delta(1 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}})E^0_\Delta\chi_\nu\psi\right\rangle. \tag{59}$$

Suppose now $\psi \in \mathcal{D}(N^{1/2})$. Then, since $(N+1)^{-1/2}B(N+1)^{-1/2}$ is bounded (see the definition of B, (35)), and since $\chi_\nu \to \mathbb{1}$ strongly as $\nu \to \infty$, we conclude that $\forall\psi \in \mathcal{D}(N^{1/2})$:

$$\left\langle\psi, E^0_\Delta B E^0_\Delta\psi\right\rangle \geq \frac{\theta\lambda^2}{\epsilon}\gamma_e\left\langle\psi, E^0_\Delta(1 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}})E^0_\Delta\psi\right\rangle,$$

which proves Theorem 7.3. ■

8 Positive Commutator with respect to spectral localization in L

We pass from the positive commutator estimate w.r.t. L_0 given in Theorem 7.3 to one w.r.t. the full Liouvillian L, hence proving Theorem 4.1. The essential ingredient of this procedure is the IMS localization formula, which we apply to a partition of unity w.r.t. N. Then, we carry out the estimates on each piece of the partition separately. Let $1 = \hat{\chi}_1^2(x) + \hat{\chi}_2^2(x)$, $x \in \mathbb{R}_+$, $\hat{\chi}_1^2 \in C_0^\infty([0,1])$, be a C^∞-partition of unity. For some scaling parameter $\sigma \gg 1$, define $\chi_i = \chi_i(N) = \hat{\chi}_i(N/\sigma)$, $\quad i = 1, 2$. The reason why we introduce the partition of unity is that $I\chi_1 = O(\sigma^{1/2})$ is bounded. Since the χ_i leave $\mathcal{D}(N^{1/2})$ invariant, then $[\chi_i, [\chi_i, B]] = \chi_i^2 B - 2\chi_i B \chi_i + B\chi_i^2$ is well defined on $\mathcal{D}(N^{1/2})$ in the sense of quadratic forms, and by summing over $i = 1, 2$, we get the so-called IMS localization formula (see also [CFKS]):

$$B = \sum_{1,2} \chi_i B \chi_i + \frac{1}{2}[\chi_i, [\chi_i, B]]. \tag{60}$$

Furthermore, we obtain from (60) and (35), in the sense of quadratic forms on $\mathcal{D}(N^{1/2})$:

$$h(L)[L, A]h(L) = \frac{1}{10}h(L)Nh(L) + \sum_{1,2} h(L)\chi_i B \chi_i h(L) + \frac{1}{2}h(L)[\chi_i, [\chi_i, B]]h(L). \tag{61}$$

The proof of Theorem 4.1 is now carried out in four parts: 1) in Proposition 8.1, we estimate $h(L)\chi_2 B \chi_2 h(L)$, 2) in Proposition 8.2, we estimate $h(L)\chi_1 B \chi_1 h(L) + \frac{1}{10}h(L)Nh(L)$, 3) in Proposition 8.3, we estimate $\frac{1}{2}\sum_{1,2} h(L)[\chi_i, [\chi_i, B]]h(L)$, and 4) to complete the proof, we choose relations among the parameters $\theta, \lambda, \epsilon, \sigma$ (see (77)).

Proposition 8.1. *If $\lambda^2\theta \ll \epsilon^2\sigma$ and $\lambda^2 \ll \sigma$, then we have*

$$h\chi_2 B \chi_2 h \geq \frac{\sigma}{2} h\chi_2^2 h. \tag{62}$$

Proof. Recall that $B = \frac{9}{10}N + \lambda\tilde{I} + [L, b]$. Since $Q\chi_2 = 0$, we have $\forall\psi$:

$$\langle\psi, \chi_2[L, b]\chi_2\psi\rangle = -\theta\lambda^2 \left\langle\psi, \chi_2(\overline{R}_\epsilon^2 IQI + IQI\overline{R}_\epsilon^2)\chi_2\psi\right\rangle \geq -C\theta\lambda^2\epsilon^{-2}\left\langle\psi, \chi_2^2\psi\right\rangle.$$

Furthermore, Proposition 6.1 gives $\forall c > 0$, $\lambda\tilde{I} \geq cN - O(\lambda^2/c)$, so

$$\begin{aligned} \left\langle\psi, \chi_2(9N/10 + \lambda\tilde{I})\chi_2\psi\right\rangle &\geq \left\langle\psi, \chi_2\left[(9/10 - c)N - O(\lambda^2/c)\right]\chi_2\psi\right\rangle \\ &\geq \frac{3}{4}\sigma\left\langle\psi, \chi_2^2\psi\right\rangle, \end{aligned}$$

where we picked the value $c = 1/10$, and used $\chi_2 N \chi_2 \geq \sigma \chi_2^2$. ■

Proposition 8.2. *We have*

$$\begin{aligned} h\chi_1 B \chi_1 h + \frac{1}{10} hNh &\geq \frac{\theta\lambda^2}{\epsilon}\gamma_e \left(1 - O(\lambda\sigma^{1/2})\right) h\chi_1^2 h - \frac{5}{2}\frac{\theta\lambda^2}{\epsilon}\gamma_0 \delta_{e,0} h P_{\Omega_{\beta,0}} h \\ &\quad -\frac{\theta\lambda^2}{\epsilon} O\left(\epsilon\theta^{-1} + \epsilon\sigma^{1/2} + \lambda\sigma\epsilon^{-1}\right) h^2. \end{aligned}$$

Proof. Let $F^0_{\Delta'} := F_{\Delta'}(L_0)$, where Δ' is an interval whose interior contains the closure of Δ, and $F_{\Delta'}$ is a smooth characteristic function with support in Δ', s.t. $E_\Delta(L_0)\overline{F^0_{\Delta'}} = 0$, where we denoted $\mathbb{1} - F^0_{\Delta'} =: \overline{F^0_{\Delta'}}$. We take Δ' to contain only one eigenvalue of $\sigma(L_0)$, namely e, so that (36) in Theorem 3.4 holds, with E^0_Δ replaced by $E^0_{\Delta'}$. We have

$$h\chi_1 B \chi_1 h + \frac{1}{10} hNh = h\chi_1 F^0_{\Delta'} B F^0_{\Delta'} \chi_1 h \tag{63}$$

$$+\frac{1}{20} hNh + h\chi_1 F^0_{\Delta'} B \overline{F^0_{\Delta'}} \chi_1 h + \text{ adjoint} \tag{64}$$

$$+h\chi_1 \overline{F^0_{\Delta'}} B \overline{F^0_{\Delta'}} \chi_1 h. \tag{65}$$

First, we show that (64) and (65) are bounded below by small terms. To treat (64), notice that

$$\begin{aligned} \chi_1 F^0_{\Delta'} B \overline{F^0_{\Delta'}} \chi_1 &= \chi_1 F^0_{\Delta'} (9N/10 + \lambda \tilde{I} + [L, b]) \overline{F^0_{\Delta'}} \chi_1 \\ &= \frac{9}{10} \chi_1^2 F^0_{\Delta'} \overline{F^0_{\Delta'}} N + \chi_1 F^0_{\Delta'} (\lambda \tilde{I} + [L, b]) \overline{F^0_{\Delta'}} \chi_1 \\ &\geq \chi_1 F^0_{\Delta'} (\lambda \tilde{I} + [L, b]) \overline{F^0_{\Delta'}} \chi_1. \end{aligned} \tag{66}$$

Now for $\phi_{1,2} \in \mathcal{D}(N^{1/2})$, we have for any $c > 0$ (see Proposition 6.1)

$$\begin{aligned} \left|\left\langle \phi_1, \lambda\tilde{I}\phi_2 \right\rangle\right| &\leq \lambda\left(\left|\left\langle \phi_1, \tilde{I}_a \phi_2\right\rangle\right| + \left|\left\langle \phi_2, \tilde{I}_a \phi_1\right\rangle\right|\right) \\ &\leq C\lambda \left(||\phi_1||\, ||N^{1/2}\phi_2|| + ||\phi_2||\, ||N^{1/2}\phi_1||\right) \\ &\leq C\lambda^2 c^{-1}\left(||\phi_1||^2 + ||\phi_2||^2\right) + c\left(||N^{1/2}\phi_1||^2 + ||N^{1/2}\phi_2||^2\right). \end{aligned}$$

With $\phi_1 = F^0_{\Delta'}\chi_1\psi$, $\phi_2 = \overline{F^0_{\Delta'}}\chi_1\psi$, this yields $\forall c > 0$:

$$\left|\left\langle \psi, \chi_1 F^0_{\Delta'} \lambda \tilde{I} \overline{F^0_{\Delta'}} \chi_1 \psi \right\rangle\right| \leq C\frac{\lambda^2}{c} 2||\chi_1\psi||^2 + 2c||N^{1/2}\chi_1\psi||^2,$$

so $\chi_1 F^0_{\Delta'} \lambda \tilde{I} \overline{F^0_{\Delta'}} \chi_1 + \text{adjoint} \geq -4\left(C\frac{\lambda^2}{c}\chi_1^2 + cN\right)$. Taking $c < 1/40$ gives then

$$\begin{aligned} \frac{1}{20} hNh + h\chi_1 F^0_{\Delta'} \lambda \tilde{I} \overline{F^0_{\Delta'}} \chi_1 h + \text{adjoint} &\geq (1/10 - 4c) hNh - C\lambda^2 h\chi_1^2 h \\ &\geq -C\lambda^2 h\chi_1^2 h. \end{aligned} \tag{67}$$

Next, using $Q\overline{F^0_{\Delta'}} = 0$ and $(L_0 - e)Q = 0$, we calculate

$$\begin{aligned}
\chi_1 F^0_{\Delta'}[L,b]\overline{F^0_{\Delta'}}\chi_1 &= \chi_1 F^0_{\Delta'}[L_0 - e, b]\overline{F^0_{\Delta'}}\chi_1 + \lambda\chi_1 F^0_{\Delta'}[I,b]\overline{F^0_{\Delta'}}\chi_1 \\
&= \theta\lambda\chi_1 F^0_{\Delta'}QI\overline{R}^2_\epsilon(L_0 - e)\overline{F^0_{\Delta'}}\chi_1 \\
&\quad +\theta\lambda^2\chi_1 F^0_{\Delta'}\left(-\overline{R}^2_\epsilon IQI - IQI\overline{R}^2_\epsilon + QI\overline{R}^2_\epsilon I\right)\overline{F^0_{\Delta'}}\chi_1 \\
&= O(\theta\lambda + \theta\lambda^2\epsilon^{-2}\sigma^{1/2}),
\end{aligned} \tag{68}$$

where we used $||R_\epsilon\overline{F^0_{\Delta'}}|| \leq |\Delta'|^{-1} \leq C$ and $||I\chi_1|| \leq C\sigma^{1/2}$. Next, since $\mathrm{supp}h \cap \mathrm{supp}\overline{F^0_{\Delta'}} = \emptyset$, then $\chi_1\overline{F^0_{\Delta'}}h(L) = \chi_1\overline{F^0_{\Delta'}}(h(L) - h(L_0))$, so by using the operator calculus introduced in Appendix B, we obtain

$$\chi_1\overline{F^0_{\Delta'}}h(L) = \chi_1\int d\tilde{F}_{\Delta'}(z)(L_0 - z)^{-1}\lambda I(L - z)^{-1}h(L) = O(\lambda\sigma^{1/2}). \tag{69}$$

¿From (68), we then have $h\chi_1 F^0_{\Delta'}[L,b]\overline{F^0_{\Delta'}}\chi_1 h \geq -C\frac{\theta\lambda^2}{\epsilon}(\epsilon\sigma^{1/2} + \lambda\sigma\epsilon^{-1})h^2$, which, together with (67) and (66) yields

$$(64) \geq -C\frac{\theta\lambda^2}{\epsilon}(\epsilon\theta^{-1} + \epsilon\sigma^{1/2} + \lambda\sigma\epsilon^{-1})h^2. \tag{70}$$

Our next step is estimating (65). Again, using $Q\overline{F^0_{\Delta'}} = 0$, we get

$$\begin{aligned}
\chi_1\overline{F^0_{\Delta'}}B\overline{F^0_{\Delta'}}\chi_1 &= \chi_1\overline{F^0_{\Delta'}}(9N/10 + \lambda\tilde{I})\overline{F^0_{\Delta'}}\chi_1 - \theta\lambda^2\chi_1\overline{F^0_{\Delta'}}\left(\overline{R}^2_\epsilon IQI + IQI\overline{R}^2_\epsilon\right)\overline{F^0_{\Delta'}}\chi_1 \\
&\geq -C(\lambda^2 + \theta\lambda^2),
\end{aligned}$$

where we used $\lambda\tilde{I} \geq -cN - O(\lambda^2/c)$ and $||\overline{F^0_{\Delta'}}R^2_\epsilon|| \leq |\Delta'|^{-2} \leq C$. We thus obtain, since $\theta \ll 1$:

$$(65) = h\chi_1\overline{F^0_{\Delta'}}B\overline{F^0_{\Delta'}}\chi_1 h \geq -C\frac{\theta\lambda^2}{\epsilon}\frac{\epsilon}{\theta}h^2. \tag{71}$$

Finally, we investigate the positive term (63). By sandwiching (36) in Theorem 7.3 (with E^0_Δ replaced by $E^0_{\Delta'}$) with $F^0_{\Delta'}$, and noticing that $F^0_{\Delta'}E^0_{\Delta'} = F^0_{\Delta'}$, we arrive at

$$\begin{aligned}
h\chi_1 F^0_{\Delta'}BF^0_{\Delta'}\chi_1 h &\geq \pi\frac{\theta\lambda^2}{\epsilon}\gamma_e h\chi_1 F^0_{\Delta'}\left(1 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}}\right)F^0_{\Delta'}\chi_1 h \\
&\geq \frac{\theta\lambda^2}{\epsilon}\gamma_e h\left(\chi_1^2(F^0_{\Delta'})^2 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}}\right)h \\
&= \frac{\theta\lambda^2}{\epsilon}\gamma_e h\left(\chi_1^2\left(1 - \overline{F^0_{\Delta'}}\right)^2 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}}\right)h \\
&\geq \frac{\theta\lambda^2}{\epsilon}\gamma_e h\left(\chi_1^2\left(1 - 2\overline{F^0_{\Delta'}}\right) - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}}\right)h \\
&\geq \frac{\theta\lambda^2}{\epsilon}\gamma_e h\left(\chi_1^2(1 - C\lambda\sigma^{1/2}) - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}}\right)h,
\end{aligned} \tag{72}$$

where we used (69) in the last step once again, and $-2\chi_1^2(F^0_{\Delta'})^2P_{\Omega_{\beta,0}} \geq -2P_{\Omega_{\beta,0}}$ in the second step. Combining (72) with (70) and (71) yields Proposition 8.2. ■

Proposition 8.3. *We have* $\sum_{1,2} h[\chi_i,[\chi_i,B]]h = \frac{\theta\lambda^2}{\epsilon}O(\epsilon\theta^{-1}\lambda^{-1}\sigma^{-3/2})h^2$.

Proof. Notice that χ_1 and $1-\chi_2$ have compact supports contained in $[0,2]$. Now in the double commutator, we can replace χ_2 by $1-\chi_2$ without changing its value. So if suffices to estimate $[\chi(N/\sigma),[\chi(N/\sigma),B]]$, where $\chi \in C_0^\infty([0,2])$. We have $[\chi(N/\sigma),[\chi(N/\sigma),B]] = [\chi(N/\sigma),[\chi(N/\sigma),\lambda\tilde{I}+[L,b]]]$. It is not difficult to see that we have in the sense of operators on $\mathcal{D}(N^{1/2})$:

$$\begin{aligned}
[\chi(N/\sigma),[\chi(N/\sigma),\lambda\tilde{I}]] &= \frac{\lambda}{\sigma^2}\int d\tilde{\chi}(z)\int d\tilde{\chi}(\zeta)(N/\sigma-z)^{-1}(N/\sigma-\zeta)^{-1} \\
&\quad \times\tilde{I}(N/\sigma-z)^{-1}(N/\sigma-\zeta)^{-1}. \qquad (73)
\end{aligned}$$

We used the operator calculus introduced in the Appendix. Now since $\|\tilde{I}(N/\sigma - z)^{-1/2}\| \leq C\|(N+1)^{1/2}(N/\sigma-z)^{-1}\| \leq C\sigma^{1/2}|\mathrm{Im}z|^{-1}$, which follows from

$$\sup_{x\geq 0}\frac{\sqrt{x+1}}{|x/\sigma - z|} \leq C\sigma^{1/2}|\mathrm{Im}z|^{-1},$$

we conclude that

$$\left\|[\chi(N/\sigma),[\chi(N/\sigma),\lambda\tilde{I}]]\right\| \leq C\lambda\sigma^{-3/2}. \qquad (74)$$

Next, write for simplicity χ instead of $\chi(N/\sigma)$, and look at

$$[\chi,[\chi,[L,b]]] = \theta\lambda[\chi,[\chi,[L,\overline{R}_\epsilon^2 IQ]]] + \text{adjoint}.$$

We claim that

$$[\chi,[L,\overline{R}_\epsilon^2 IQ]] = 0. \qquad (75)$$

Write first $[L,\overline{R}_\epsilon^2 IQ] = \overline{R}_\epsilon^2[L_0,I]Q + \lambda[I,\overline{R}_\epsilon^2 IQ]$. Then $[\chi,\overline{R}_\epsilon^2[L_0,I]Q] = [\overline{\chi},\overline{R}_\epsilon^2[L_0,I]Q] = \overline{\chi}\overline{R}_\epsilon^2[L_0,I]Q - \overline{R}_\epsilon^2[L_0,I]Q\overline{\chi}$. Here, $\overline{\chi} = 1-\chi$. Notice that $Q\overline{\chi} = 0$, and since $\mathrm{Ran}\overline{R}_\epsilon^2[L_0,I]Q \subset \mathrm{Ran}\,P(N=1)$, we have also $\overline{\chi}\overline{R}_\epsilon^2[L_0,I]Q = 0$, for $\sigma > 2$. Similarly, $[\chi,[I,\overline{R}_\epsilon^2 IQ]] = 0$, so (75) follows.

We obtain thus from (74): $[\chi,[\chi,B]] = O(\lambda\sigma^{-3/2})$, which proves the proposition. ■

Now we finish the proof of Theorem 4.1. The IMS localization formula (61) together with Propositions 8.1, 8.2, 8.3 yields

$$\begin{aligned}
h[L,A]h &\geq \frac{\theta\lambda^2}{\epsilon}\gamma_e\left(1-O(\lambda\sigma^{1/2})\right)h\chi_1^2 h + \frac{\sigma}{2}h\chi_2^2 h - \frac{5}{2}\frac{\theta\lambda^2}{\epsilon}\gamma_0\delta_{e,0}hP_{\Omega_{\beta,0}}h \\
&\quad -\frac{\theta\lambda^2}{\epsilon}O\left(\epsilon\theta^{-1}+\epsilon\sigma^{1/2}+\lambda\sigma\epsilon^{-1}+\epsilon\theta^{-1}\lambda^{-1}\sigma^{-3/2}\right)h^2.
\end{aligned}$$

The sum of the first two terms on the r.h.s. is bounded below by

$$\frac{\theta\lambda^2}{\epsilon}\gamma_e\left(1 - O(\lambda\sigma^{1/2})\right)h^2,$$

so we get

$$\begin{aligned} h[L,A]h \;&\geq\; \frac{\theta\lambda^2}{\epsilon}h\Big[\gamma_e\left(1 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}} - O(\lambda\sigma^{1/2})\right) \\ &\quad -O\left(\epsilon\theta^{-1} + \epsilon\sigma^{1/2} + \lambda\sigma\epsilon^{-1} + \epsilon\theta^{-1}\lambda^{-1}\sigma^{-3/2}\right)\Big]h. \end{aligned} \tag{76}$$

Finally, we choose our parameters. Let $\epsilon = \lambda^{\hat{\epsilon}/100}$, $\sigma = \lambda^{-\hat{\sigma}/100}$, $\theta = \lambda^{\hat{\theta}/100}$, and choose

$$(\hat{\epsilon}, \hat{\sigma}, \hat{\theta}) = (44, 55, 26). \tag{77}$$

It is then easily verified that for small λ, the conditions on the parameters given in Theorem 7.3 and Proposition 8.1 hold, and furthermore, (76) becomes

$$\begin{aligned} h[L,A]h \;&\geq\; \lambda^{182/100}h\left[\gamma_e\left(1 - \tfrac{5}{2}\delta_{e,0}P_{\Omega_{\beta,0}} - O(\lambda^{145/200})\right) - O(\lambda^{1/200})\right]h \\ &\geq\; \lambda^{91/50}h\left(\frac{\gamma_e}{2}(1 - 5\delta_{e,0}P_{\Omega_{\beta,0}}) - O(\lambda^{1/200})\right)h. \quad \blacksquare \end{aligned}$$

9 Proof of Theorem 4.2

The proof is by contradiction using the following idea: assume ψ is an eigenfunction of L with eigenvalue e in Δ, $0 \notin \Delta$. Then on one hand, we formally have $\langle [L, A_0]\rangle_\psi = 2i\mathrm{Im}\,\langle (L-e)\psi, A_0\psi\rangle = 0$, and on the other hand, from the positive commutator estimate (Theorem 4.1), $\langle [L, A_0]\rangle_\psi$ should not be too small, which will give a contradiction. Since we do not know whether ψ is in the (form-)domain of the commutator $[L, A_0]$, we replace it by an approximate vector $\psi_{\alpha,\nu}$ which is in the form domain of the commutator. Here, α, ν stand for cutoffs in A_0 and N respectively. Then the PC estimate shows that $\langle [L, A_0]\rangle_{\psi_{\alpha,\nu}} \geq \theta\|\psi_{\alpha,\nu}\|^2$, where $\theta > 0$ is independent of the cutoff parameters α, ν. But we also show that $\langle [L, A_0]\rangle_{\psi_{\alpha,\nu}} \to 0$ as $\alpha, \nu \to 0$, which yields a contradiction. The upper bound on $\langle [L, A_0]\rangle_{\psi_{\alpha,\nu}}$ we call the Virial Theorem type estimate.

For the zero eigenvalue, the argument goes similarly: assume zero is a degenerate eigenvalue of L (we know that its multiplicity is at least one, since we have a zero eigenvector $\Omega_{\beta,\lambda}$ of L). Then take an eigenvector $\psi \in \mathrm{Ran}\, P^\perp_{\Omega_{\beta,\lambda}}$ s.t. $L\psi = 0$. As above, the Virial Theorem type agument shows that $\langle [L, A_0]\rangle_{\psi_{\alpha,\nu}}$ is very small, while the PC estimate from Theorem 4.1 gives a lower bound on this average value. Remark that here, we need to use that the product $P_{\Omega_{\beta,0}}P^\perp_{\Omega_{\beta,\lambda}}$ is small, which is satisfied provided $\beta|\lambda| < C$, see (20).

9.1 The proof of Theorem 4.2 in the spirit of the Virial Theorem

Assume ψ is a normalized eigenvector of L with eigenvalue e. If $e = 0$, we assume in addition that $\psi \in \mathrm{Ran}\, P^{\perp}_{\Omega_{\beta,\lambda}}$. Let $\alpha > 0$ and set $f_\alpha := \alpha^{-1} f(i\alpha A_0)$, where f is a bounded C^∞-function, such that the derivative f' is positive and s.t. $f'(0) = 1$ (take e.g. $f = \mathrm{Arctan}$). Set

$$f'_\alpha := f'(i\alpha A_0), \text{ and } h_\alpha := \sqrt{f'_\alpha}.$$

Furthermore, set $f''_\alpha := f''(i\alpha A_0)$. For $\nu > 0$ and $g \in C_0^\infty(-1,1)$, define $\psi_\nu = g(\nu N)\psi$. Here, α, ν will be chosen small in an appropriate way. The regularized eigenfunction is taken to be $\psi_{\alpha,\nu} = h_\alpha \psi_\nu$. Notice that

$$\psi_{\alpha,\nu} \to \psi, \quad \text{as } \alpha, \nu \to 0. \tag{78}$$

Furthermore, we set for notational convenience in this subsection

$$K := [L, A_0] = N + \lambda \tilde{I}.$$

We now prove the above mentioned upper bound. Using $(L - e)\psi = 0$, we find that

$$\langle f_\alpha (L - e)\rangle_{\psi_\nu} = \langle g_\nu f_\alpha (L - e) g_\nu \rangle_\psi = \langle f_\alpha g_\nu [\lambda I, g_\nu]\rangle_\psi = O(\lambda \alpha^{-1} \nu^{1/2}). \tag{79}$$

In the last step, we used that $[N, I]$ is $N^{1/2}$-bounded. Next, observe that

$$2\mathrm{Im}\, \langle f_\alpha (L - e)\rangle_{\psi_\nu} = \langle [L, if_\alpha]\rangle_{\psi_\nu} = \mathrm{Re}\, \langle [L, if_\alpha]\rangle_{\psi_\nu} = \mathrm{Re}\, \langle f'_\alpha N + \lambda [I, if_\alpha]\rangle_{\psi_\nu}. \tag{80}$$

In the last step, we used

$$\begin{aligned}
[L_0, if_\alpha] &= \int d\tilde{f}(z)(i\alpha A_0 - z)^{-1}[L_0, A_0](i\alpha A_0 - z)^{-1} \\
&= \int d\tilde{f}(z)(i\alpha A_0 - z)^{-2} N \\
&= f'_\alpha N,
\end{aligned}$$

where we used the fact that A_0 and N commute in the second step, and (115) with $p = 1$ in the last step. The commutator $[I, if_\alpha]$ is examined in

Proposition 9.1. *The following equality holds in the sense of operators on $\mathcal{D}(N^{1/2})$ or in the sense of quadratic forms on $\mathcal{D}(N^{1/4})$:*

$$[I, if_\alpha] = f'_\alpha ad^1_{A_0}(I) - \frac{i}{2}\alpha f''_\alpha ad^2_{A_0}(I) + R, \tag{81}$$

where we assume that the k-fold commutator $ad^k_{A_0}(I) := [\cdots[I, A_0], A_0, \cdots, A_0]$ *is* $N^{1/2}$*-bounded (or* $N^{1/4}$*-form bounded) for* $k = 1, 2, 3$*. The term* R *satisfies the estimate* $RN^{-1/2}$*,* $N^{-1/4}RN^{-1/4} = O(\alpha^2)$*.*

Proof. Using the operator calculus introduced in Appendix A.4, we write

$$\begin{aligned}
[I, if_\alpha] &= \int d\tilde{f}(z)(i\alpha A_0 - z)^{-1}[I, A_0](i\alpha A_0 - z)^{-1} \\
&= f'_\alpha ad^1_{A_0}(I) - i\alpha \int d\tilde{f}(z)(i\alpha A_0 - z)^{-2} ad^2_{A_0}(I)(i\alpha A_0 - z)^{-1} \\
&= f'_\alpha ad^1_{A_0}(I) - \frac{i}{2}\alpha f''_\alpha ad^2_{A_0}(I) - \alpha^2 \int d\tilde{f}(z)(i\alpha A_0 - z)^{-3} ad^3_{A_0}(I)(i\alpha A_0 - z)^{-1}.
\end{aligned}$$

The last integral is defined to be R, and the estimates follow by noticing that A_0 and N commute. ■

Notice that it is here that we need $\|ad^k_{A_0}(I)N^{1/2}\| \le C$, $k = 2, 3$, *hence the more restrictive IR behaviour* $p > 2$. We obtain from (81) and recalling that $\tilde{I} = [I, A_0]$:

$$\begin{aligned}
(80) &= \mathrm{Re}\,\langle f'_\alpha K\rangle_{\psi_\nu} - \frac{\lambda}{2}\mathrm{Re}\,\langle i\alpha f''_\alpha ad^2_{A_0}(I)\rangle_{\psi_\nu} + O(\lambda\alpha^2\nu^{-1/2}) \\
&= \langle K\rangle_{\psi_{\alpha,\nu}} + \lambda \mathrm{Re}\left\langle h_\alpha[h_\alpha, \lambda\tilde{I}] - \frac{i}{2}\alpha f''_\alpha ad^2_{A_0}(I)\right\rangle_{\psi_\nu} + O(\lambda\alpha^2\nu^{-1/2}) \\
&= \langle K\rangle_{\psi_{\alpha,\nu}} + O(\lambda\alpha^2\nu^{-1/2}). \qquad (82)
\end{aligned}$$

We used in the last step that the real part in the second above is

$$\left\langle [h_\alpha, [h_\alpha, \tilde{I}]] - \frac{i}{2}\alpha[f''_\alpha, ad^2_{A_0}(I)]\right\rangle_{\psi_\nu} = O(\alpha^2\nu^{-1/2}),$$

since $ad^3_{A_0}(I)$ is $N^{1/2}$-bounded. Combining (82) and (79), we obtain

$$\langle K\rangle_{\psi_{\alpha,\nu}} \le C\lambda\left(\frac{\nu^{1/2}}{\alpha} + \frac{\alpha^2}{\nu^{1/2}}\right)\|\psi\|^2. \qquad (83)$$

Next, we want to establish a lower bound on the last l.h.s. by using the positive commutator estimate. Let Δ be an interval containing exactly one eigenvalue, e, of L_p. We introduce two partitions of unity. The first one is given by

$$\chi^2_\Delta + \overline{\chi}^2_\Delta = 1,$$

where $\chi_\Delta \in C^\infty(\Delta)$, $\chi_\Delta(e) = 1$. We localize in L, i.e. we set $\chi_\Delta = \chi_\Delta(L)$. The second partition of unity is given by

$$\chi^2 + \overline{\chi}^2 = 1,$$

where $\chi \in C^\infty$ is a "smooth Heaviside function", i.e. $\chi(x) = 0$ if $x \leq 0$ and $\chi(x) = 1$ if $x \geq 1$. We set for $n > 0$: $\chi_n = \chi(N/n)$, $\overline{\chi}_n^2 = 1 - \chi_n^2$. We will choose $n < 1/\nu$, so that $\chi_n \psi_\nu = \chi_n \psi$. The last equation will be used freely in what follows. We are going to use the IMS localization formula (60) with respect to both partitions of unity, and we start with the one localizing in N:

$$\begin{aligned}\langle K\rangle_{\psi_{\alpha,\nu}} &= \langle \chi_n K \chi_n \\ &\quad +\overline{\chi}_n K \overline{\chi}_n + \tfrac{1}{2}[\chi_n, [\chi_n, K]] + \tfrac{1}{2}[\overline{\chi}_n, [\overline{\chi}_n, K]]_{\psi_{\alpha,\nu}} \\ &\geq \langle K\rangle_{\chi_n\psi_{\alpha,\nu}} + \tfrac{n}{2}\|\overline{\chi}_n \psi_{\alpha,\nu}\|^2 - O(\lambda n^{-3/2}), \qquad (84)\end{aligned}$$

where we used that $K \geq n/2$ on $\mathrm{Ran}\, P_\Omega^\perp$, and the estimate (74) with σ replaced by n. Next, from the IMS localization formula for the partition of unity w.r.t. L, we have

$$\begin{aligned}\langle K\rangle_{\chi_n\psi_{\alpha,\nu}} &= \langle \chi_\Delta K \chi_\Delta + \overline{\chi}_\Delta K \overline{\chi}_\Delta + R\rangle_{\chi_n\psi_{\alpha,\nu}} \\ &\geq \langle \chi_\Delta (K + [L, b])\chi_\Delta + \overline{\chi}_\Delta K \overline{\chi}_\Delta + R\rangle_{\chi_n\psi_{\alpha,\nu}} - \lambda^{19/50} O(\alpha n + \lambda n^{-1/2}) \\ &\geq \theta \|\chi_\Delta \chi_n \psi_{\alpha,\nu}\|^2 - C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}\chi_\Delta\chi_n\psi_{\alpha,\nu}\|^2 + \langle \overline{\chi}_\Delta K \overline{\chi}_\Delta + R\rangle_{\chi_n\psi_{\alpha,\nu}} \\ &\quad -\lambda^{19/50} O(\alpha n + \lambda n^{-1/2}). \qquad (85)\end{aligned}$$

Here, several remarks are in order. First, we have set $2R = [\overline{\chi}_\Delta, [\overline{\chi}_\Delta, K]] + [\chi_\Delta, [\chi_\Delta, K]]$, and we have used in the second step the fact that

$$\begin{aligned}\langle [L, b]\rangle_{\chi_\Delta\chi_n\psi_{\alpha,\nu}} &= \langle [L - e, b]\rangle_{\chi_\Delta\chi_n h_\alpha\psi} = 2\mathrm{Re}\,\langle \chi_\Delta (L - e) h_\alpha \chi_n \psi, b\chi_\Delta\chi_n h_\alpha \psi\rangle \\ &= \lambda^{19/50} O(\alpha n + \lambda n^{-1/2}).\end{aligned}$$

We recall that b is a bounded operator (see Proposition 7.2), with $\|b\| = O(\lambda^{19/50})$. In the last step in (85), we used the positive commutator estimate, Theorem 4.1, in the following way. For $e \neq 0$, Theorem 4.1. gives right away $\chi_\Delta(K + [L, b])\chi_\Delta \geq \theta\chi_\Delta^2$, where we recall that $[L, A] = [L, A_0] + [L, b]$, and b is defined in (32). We have set $\theta = C\lambda^2$. In the zero eigenvalue case, $e = 0$, we have

$$\begin{aligned}\langle K + [L, b]\rangle_{\chi_\Delta\chi_n\psi_{\alpha,\nu}} &\geq \frac{\lambda^{91/50}}{2}\left\langle \gamma_0(1 - 5P_{\Omega_{\beta,0}}) - O(\lambda^{1/200})\right\rangle_{\chi_\Delta\chi_n\psi_{\alpha,\nu}} \\ &\geq \frac{\lambda^{91/50}}{4}\gamma_0\|\chi_\Delta\chi_n\psi_{\alpha,\nu}\|^2 - \frac{5\lambda^{91/50}}{2}\gamma_0\|P_{\Omega_{\beta,0}}\chi_\Delta\chi_n\psi_{\alpha,\nu}\|^2.\end{aligned}$$

Setting again $\theta = C\lambda^{91/50}$ yields (85).

We now estimate the remainder term R. Notice that the same observation as at the beginning of the proof of Proposition 8.3 shows that we have the estimate $\langle R\rangle_{\chi_n\psi_{\alpha,\nu}} = 2i\mathrm{Im}\,\langle \overline{\chi}_\Delta\chi_n\psi_{\alpha,\nu}, [\overline{\chi}_\Delta, K]\chi_n\psi_{\alpha,\nu}\rangle$. Therefore,

$$\left|\langle R\rangle_{\chi_n\psi_{\alpha,\nu}}\right| \leq C\|\overline{\chi}_\Delta\chi_n h_\alpha\psi\|\,\|[\overline{\chi}_\Delta, K]\chi_n h_\alpha\psi\|. \qquad (86)$$

Now we have on $\mathcal{D}(N)$: $[\overline{\chi}_\Delta, K] = \int d\tilde{\chi}_\Delta(z)(L-z)^{-1}[K,L](L-z)^{-1}$, where we recall that $(L-z)^{-1}$ leaves $\mathcal{D}(N)$ invariant. Furthermore,

$$[K,L] = \lambda[N,I] + \lambda[\tilde{I}, L_0] + \lambda^2[\tilde{I}, I] = \lambda[N,I] + \lambda I(u\partial_u g) + \lambda^2[\tilde{I}, I], \tag{87}$$

where $I(u\partial_u g)$ is obtained from I by replacing the form factor g by $u\partial_u g$. The last commutator in (87) is bounded, and the other two are $N^{1/2}$-bounded, so we obtain

$$\|[\overline{\chi}_\Delta, K]\chi_n h_\alpha \psi\| = O(\lambda n^{1/2})\|\chi_n \psi_{\alpha,\nu}\|. \tag{88}$$

Next, we estimate the first term on the r.h.s. of (86):

$$\begin{aligned}
\|\overline{\chi}_\Delta \chi_n h_\alpha \psi\| &= \|(L-e)^{-1}\overline{\chi}_\Delta (L-e)\chi_n h_\alpha \psi\| \\
&\leq C\|(L-e)\chi_n h_\alpha \psi\| \\
&\leq C\|n^{-1}\lambda[N,I]\chi_n' h_\alpha \psi\| + O(\lambda n^{-3/2}) + C\|\chi_n (L-e) h_\alpha \psi\| \\
&\leq C\lambda n^{-1/2}\|\chi_n' \psi_{\alpha,\nu}\| + O(\lambda n^{-3/2} + \alpha n).
\end{aligned} \tag{89}$$

Combining this with (88) and (86), we arrive at the estimate

$$\left|\langle R\rangle_{\chi_n \psi_{\alpha,\nu}}\right| \leq C\lambda^2 \|\chi_n' \psi_{\alpha,\nu}\|\,\|\chi_n \psi_{\alpha,\nu}\| + O(\lambda^2 n^{-1} + \lambda \alpha n^{3/2}). \tag{90}$$

There is one more term in (85) we have to estimate: $\langle \overline{\chi}_\Delta K \overline{\chi}_\Delta\rangle_{\chi_n \psi_{\alpha,\nu}}$. Since $P_\Omega^\perp (N + \lambda \tilde{I}) P_\Omega^\perp \geq 0$ and since $P_\Omega \tilde{I} P_\Omega = 0$, we have the bound $K \geq P_\Omega^\perp \lambda \tilde{I} P_\Omega + \text{adj.} \geq -C\lambda$, which implies

$$\langle \overline{\chi}_\Delta K \overline{\chi}_\Delta\rangle_{\chi_n \psi_{\alpha,\nu}} \geq -C\lambda \|\overline{\chi}_\Delta \chi_n \psi_{\alpha,\nu}\|^2. \tag{91}$$

Using (91) and (90), we obtain from (85)

$$\begin{aligned}
\langle K\rangle_{\chi_n \psi_{\alpha,\nu}} \geq\ & \theta\|\chi_n \psi_{\alpha,\nu}\|^2 - (\theta + C\lambda)\|\overline{\chi}_\Delta \chi_n \psi_{\alpha,\nu}\|^2 - C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}\chi_\Delta \chi_n \psi_{\alpha,\nu}\|^2 \\
& -C\lambda^2\|\chi_n' \psi_{\alpha,\nu}\|\,\|\chi_n \psi_{\alpha,\nu}\| - \lambda^{19/50} O(\alpha n + \lambda n^{-1/2}) \\
& -\lambda O(\alpha n^{3/2} + \lambda n^{-1}).
\end{aligned} \tag{92}$$

Next, we have for any $\eta, \epsilon > 0$:

$$\begin{aligned}
\|\chi_n' \psi_{\alpha,\nu}\|\,\|\chi_n \psi_{\alpha,\nu}\| &\leq \eta\|\chi_n \psi_{\alpha,\nu}\|^2 + \eta^{-1}\|\chi_n' \psi_{\alpha,\nu}\|^2 \\
&\leq (\epsilon\eta^{-1} + \eta)\|\chi_n \psi_{\alpha,\nu}\|^2 + \eta^{-1}\epsilon^{-2}\|\overline{\chi}_n \psi_{\alpha,\nu}\|^2.
\end{aligned}$$

In the second step, we used the standard fact that we can choose the partition of unity s.t. $\|\chi_n' \psi\|^2 \leq \epsilon\|\chi_n \psi\|^2 + \epsilon^{-2}\|\overline{\chi}_n \psi\|^2$, for any $\epsilon > 0$. Combining this with (92), we obtain from (84):

$$\begin{aligned}
\langle K\rangle_{\psi_{\alpha,\nu}} \geq\ & (\theta - C\lambda^2(\epsilon\eta^{-1} + \eta))\|\chi_n \psi_{\alpha,\nu}\|^2 + (n/2 - C\lambda^2\eta^{-1}\epsilon^{-2})\|\overline{\chi}_n \psi_{\alpha,\nu}\|^2 \\
& -C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}\chi_\Delta \chi_n \psi_{\alpha,\nu}\|^2 - (\theta + C\lambda)\|\overline{\chi}_\Delta \chi_n \psi_{\alpha,\nu}\|^2 \\
& -O(\lambda \alpha n^{3/2} + \lambda^{19/50}\alpha n + \lambda^{69/50} n^{-1/2}).
\end{aligned}$$

Consider λ small and fixed. Then if

$$\frac{n}{2} - C\eta^{-1}\epsilon^{-2} \geq \theta, \tag{93}$$

we obtain

$$\begin{aligned}\langle K\rangle_{h_\alpha\psi_\nu} \quad &\geq \quad \theta\|h_\alpha\psi_\nu\|^2 - C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}\chi_\Delta\chi_n h_\alpha\psi_\nu\|^2 - O(\epsilon\eta^{-1} + \eta + \alpha n^{3/2} + n^{-1/2}) \\ &\quad -C\theta(n^{-1} + n^{-3} + \alpha^2 n^2). \end{aligned}\tag{94}$$

On the last line, we used (89). Let us choose the parameters as follows:

$$\epsilon = \alpha^{1/10},\ \eta = \alpha^{1/20},\ n = \alpha^{-1/2},$$

then (93) is verified, and furthermore, (94) reduces to

$$\langle K\rangle_{\psi_{\alpha,\nu}} \geq \theta\|\psi_{\alpha,\nu}\|^2 - C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}\chi_\Delta\chi_n\psi_{\alpha,\nu}\|^2 - O(\alpha^{1/20}). \tag{95}$$

On the other hand, recalling (83), we obtain by choosing the parameters ν and α as $\nu = \alpha^3$:

$$\langle K\rangle_{\psi_{\alpha,\nu}} \leq C\alpha^{1/2}. \tag{96}$$

Since $\|\psi_{\alpha,\nu}\| \to \|\psi\| = 1$ as $\alpha, \nu \to 0$, and since

$$-C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}\chi_\Delta\chi_n\psi_{\alpha,\nu}\|^2 \to -C\theta\delta_{e,0}\|P_{\Omega_{\beta,0}}P^\perp_{\Omega_{\beta,\lambda}}\psi\|^2$$

(recall that $\psi = P^\perp_{\Omega_{\beta,\lambda}}\psi$ if $e = 0$), we obtain thus for small α from (95) and (96) the inequality

$$\frac{\theta}{2}\left(1 - C\delta_{e,0}\|P_{\Omega_{\beta,0}}P^\perp_{\Omega_{\beta,\lambda}}\psi\|^2\right) \leq C\alpha^{1/2}. \tag{97}$$

For $e \neq 0$, this is a contradiction, and it shows that there can not be any eigenvalues of L in the interval Δ. Remark that there is no smallness condition on the size of Δ, except that it must not contain more than one eigenvalue of L_0, so we can choose $\Delta = (e_-, e_+)$.

Let us look now at the case $e = 0$. Again, we reach a contradiction from (97), provided $\|P_{\Omega_{\beta,0}}P^\perp_{\Omega_{\beta,\lambda}}\psi\|^2 \ll 1$. In this case, we conclude that zero is a simple eigenvalue of L. Now the fact that $\|P_{\Omega_{\beta,0}}P^\perp_{\Omega_{\beta,\lambda}}\| = O(\beta|\lambda|)$ follows immediately from (20), so taking $\beta|\lambda|$ small enough finishes the proof of Theorem 4.2. ■

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A Appendix

A.1 Proof of Theorem 4.4

We use the notation introduced just before and in Theorem 4.4. From $P(L_p = e) = \sum_{\{i,j:E_{ij}=e\}} P_i \otimes P_j$, we obtain together with the definition of $\Gamma(e)$ given in (10):

$$\Gamma_p(e) = \sum_{m,n} (1 - \delta_{E_{mn},e}) \sum_{\{i,j:E_{ij}=e\}} \sum_{\{k,l:E_{kl}=e\}} \int \delta(E_{mn} - e + u) P_{ij}\ m^*\ P_{mn}\ m\ P_{kl}. \tag{98}$$

The idea here is to get a lower bound on the sum over $(m, n) \in \mathbb{N} \times \mathbb{N}$ by summing only over a convenient subset of $\mathbb{N}\times\mathbb{N}$ (notice that every term in the sum is positive). That subset is chosen such that the summands reduce to simpler expressions.

Using the definition of m (see (11)), we obtain

$$\begin{aligned} & P_{ij} m^* P_{mn} m P_{kl} \\ &= P_{ij} \left(G_l \overline{g}_1 - G_r \overline{g}_2\right) P_{mn} \left(G_l g_1 - G_r g_2\right) P_{kl} \\ &= P_i G P_m G P_k \otimes P_n \delta_{jn} \delta_{nl} |g_1|^2 - P_i G P_m \otimes P_n \mathcal{C} G \mathcal{C} P_l \delta_{jn} \delta_{mk} \overline{g}_1 g_2 \\ &\quad - P_m G P_k \otimes P_j \mathcal{C} G \mathcal{C} P_n \delta_{im} \delta_{nl} \overline{g}_2 g_1 + P_m \otimes P_j \mathcal{C} G \mathcal{C} P_n \mathcal{C} G \mathcal{C} P_l \delta_{im} \delta_{mk} |g_2|^2. \end{aligned}$$

Summing over i, j and k, l according to (98) yields

$$\begin{aligned} & \sum_{\{i,j:E_{ij}=e\}} \sum_{\{k,l:E_{kl}=e\}} P_{ij} m^* P_{mn} m P_{kl} \\ &\qquad = \left(\overline{g}_1 P_{\mathcal{N}_l^{(n)}} G P_m \otimes P_n - \overline{g}_2 P_m \otimes P_{\mathcal{N}_r^{(m)}} \mathcal{C} G \mathcal{C} P_n\right) \cdot \text{ adjoint.} \end{aligned}$$

For $(m, n) \in \mathcal{N}_l \times \mathcal{N}_r^c$, we have $P_{\mathcal{N}_l^{(n)}} = 0$ and $P_{\mathcal{N}_r^{(m)}} \neq 0$, and for $(m, n) \in \mathcal{N}_l^c \times \mathcal{N}_r$, we have $P_{\mathcal{N}_l^{(n)}} \neq 0$ and $P_{\mathcal{N}_r^{(m)}} = 0$. As explained above, we now get a lower bound on the sum (98) by summing only over the disjoint union

$$(m, n) \in \mathcal{N}_l \times \mathcal{N}_r^c \,\dot{\cup}\, \mathcal{N}_l^c \times \mathcal{N}_r.$$

An easy calculation shows that

$$\begin{aligned} \Gamma_p(e) \ \geq\ & \inf_{E_{ij}\neq 0} \left(\int_{S^2} dS\, |g_2(E_{ij}, \alpha)|^2\right) \sum_{m\in\mathcal{N}_l} P_m \otimes \mathcal{C} P_{\mathcal{N}_r^{(m)}} G\ P_{\mathcal{N}_r^c}\ G P_{\mathcal{N}_r^{(m)}} \mathcal{C} \\ & + \inf_{E_{ij}\neq 0} \left(\int_{S^2} dS\, |g_1(E_{ij}, \alpha)|^2\right) \sum_{n\in\mathcal{N}_r} P_{\mathcal{N}_l^{(n)}} G\ P_{\mathcal{N}_l^c}\ G P_{\mathcal{N}_l^{(n)}} \otimes P_n. \end{aligned}$$

Next, we investigate the integrals. From (23), we have

$$\int_{S^2} dS |g_{1,2}(E_{ij}, \alpha)|^2 \geq |E_{ij}| \int_{S^2} dS |g(|E_{ij}|, \alpha)|^2,$$

uniformly in $\beta \geq 1$. With (15), (16) and remarking that $\sigma(\mathcal{C}T\mathcal{C}) = \sigma(T)$ for any selfadjoint T, this yields

$$\Gamma_p(e) \geq \inf_{E_{ij}\neq 0} \left(|E_{ij}| \int_{S^2} dS |g(E_{ij}, \alpha)|^2 \right) \left(\inf_{m \in \mathcal{N}_l} \{\delta_m\} + \inf_{n \in \mathcal{N}_r} \{\delta'_n\} \right) P(L_p = e),$$

since $\sum_{m\in\mathcal{N}_l} P_m \otimes P_{\mathcal{N}_r^{(m)}} = \sum_{n\in\mathcal{N}_r} P_{\mathcal{N}_l^{(n)}} \otimes P_n = P(L_p = e)$. This shows 1) of Theorem 4.4.

Now we look at the zero eigenvalue. A general normalized element of $\operatorname{Ran} P(L_p = 0)$ is of the form $\phi = \sum_i c_i \varphi_i \otimes \varphi_i$, with $\sum_i |c_i|^2 = 1$, so

$$\langle \phi, \Gamma(0)\phi \rangle = \sum_{m,n} (1 - \delta_{E_{mn},0}) \sum_{i,j} \overline{c}_i c_j \int \delta(E_{mn} + u) \left\langle \varphi_i \otimes \varphi_i, m^* P_{mn} m \varphi_j \otimes \varphi_j \right\rangle .$$

Using again the explicit form of m given in (11) and $\langle \varphi_m, \mathcal{C}G\mathcal{C}\varphi_n \rangle = \overline{\langle \varphi_m, G\varphi_n \rangle}$, we obtain

$$\langle \phi, \Gamma(0)\phi \rangle = \sum_{m,n} (1 - \delta_{E_{mn},0}) \int \delta(E_{mn} + u) \left| \langle \varphi_n, G\varphi_m \rangle \right|^2 |c_n g_1 - c_m g_2|^2. \tag{99}$$

We split the domain of integration $\mathbb{R} \times S^2$ into $\mathbb{R}_+ \times S^2 \,\dot\cup\, \mathbb{R}_- \times S^2$ and using (23) and $g_2(u, \alpha) = -g_1(-u, \alpha)$, arrive at

$$\begin{aligned} &\int \delta(E_{mn} + u) |c_n g_1 - c_m g_2|^2 \\ &= \int_{\mathbb{R}^3} \left\{ \delta(E_{mn} + \omega) \left| \sqrt{1+\mu}\, c_n g - \sqrt{\mu}\, c_m g \right|^2 + \delta(E_{mn} - \omega) \left| \sqrt{\mu}\, c_n g - \sqrt{1+\mu}\, c_m g \right|^2 \right\} . \end{aligned}$$

This together with (99) gives

$$\begin{aligned} \langle \phi, \Gamma(0)\phi \rangle &= 2 \sum_{\{m,n: E_{mn}<0\}} \left| \langle \varphi_n, G\varphi_m \rangle \right|^2 \frac{e^{\beta E_n}}{e^{-\beta E_{mn}} - 1} \\ &\qquad \times \left| e^{-\beta E_m/2} c_n - e^{-\beta E_n/2} c_m \right|^2 \int \delta(E_{mn} + \omega) |g|^2, \end{aligned} \tag{100}$$

where we used $\delta(E_{mn} + \omega)\mu = \delta(E_{mn} + \omega)(e^{-\beta E_{mn}} - 1)^{-1}$. Equation (100) shows that if we choose $c_n = Z_p^{-1/2} e^{-\beta E_n/2}$, then each term in the sum is zero. Recall now that the particle Gibbs state is given by (17), so $\left\langle \Omega_\beta^p, \Gamma(0)\Omega_\beta^p \right\rangle = 0$. Since $\Gamma(0) \geq 0$, this implies that Ω_β^p is a zero eigenvector of $\Gamma(0)$.

Finally we show that there is a gap in the spectrum of $\Gamma(0)$ at zero. Indeed, from

(100), we get by the definition of g_0 (see statement of Theorem 4.4):

$$\begin{aligned}
\langle \phi, \Gamma(0)\phi \rangle \;\geq\; & 2g_0 \sum_{\{m,n:E_{mn}<0\}} |e^{-\beta E_m/2}c_n - e^{-\beta E_n/2}c_m|^2 \\
&= g_0 \sum_{m,n} |e^{-\beta E_m/2}c_n - e^{-\beta E_n/2}c_m|^2 \\
&= g_0 \sum_{m,n} \left(e^{-\beta E_m}|c_n|^2 + e^{-\beta E_n}|c_m|^2 - e^{-\beta(E_m+E_n)/2}(\overline{c}_n c_m + c_n \overline{c}_m)\right) \\
&= g_0 \Big(Z_p(\beta) + Z_p(\beta) - 2\Big|\sum_m e^{-\beta E_m/2}c_m\Big|^2\Big) \\
&= 2g_0 Z_p(\beta)\Big(1 - \Big|\left\langle \Omega^p_\beta, \phi \right\rangle\Big|^2\Big),
\end{aligned}$$

where we used $\sum_n |c_n|^2 = 1$. Therefore, we obtain on $\operatorname{Ran} P^{\perp}_{\Omega^p_\beta}$: $\Gamma(0) \geq 2g_0 Z_p(\beta)$. This proves that if $g_0 > 0$, then we have a gap at zero and zero is a simple eigenvalue. ■

A.2 Proof of Proposition 7.1

We first show that $\mathcal{D}(\Lambda) \cap \mathcal{D}(N)$ is dense in $\mathcal{F}$. This implies that $\mathcal{D}(L_0) \cap \mathcal{D}(N)$ is dense in $\mathcal{H}$.

Let $F(C_0^\infty)$ denote the finite particle subspace of the Fock space $\mathcal{F}(C_0^\infty)$, i.e. $F = \{(\psi_n)_{n\geq 0} \in \mathcal{F}(C_0^\infty)$ s.t. $\psi_n = 0$ for all but finitely many $n\}$. Clearly, $F(C_0^\infty) \subset \mathcal{D}(\Lambda) \cap \mathcal{D}(N) \subset \mathcal{F}$, but $F(C_0^\infty)$ is dense in $F(L^2)$ which is dense in $\mathcal{F}$, so $F(C_0^\infty)$ is dense in $\mathcal{F}$.

Next, for $t \in \mathbb{R}$, let $L_t = e^{-tA_0} L e^{tA_0}$. For $\psi \in \mathcal{D}(L_0) \cap \mathcal{D}(N) = \mathcal{D}(L_p) \cap \mathcal{D}(L_f) \cap \mathcal{D}(N)$, it is easily seen that $e^{-tA_0} L_f e^{tA_0} = L_f + tN$, and since e^{tA_0} commutes with L_p, then $e^{-tA_0} L_0 e^{tA_0} = L_0 + tN$.

Next, for $f \in L^2(\mathbb{R} \times S^2)$, an easy calculation yields $e^{-tA_0} a^*(f) e^{tA_0}\psi = a^*(f^t)$, where we defined the shifted function $f^t(u, \alpha) = f(u + t, \alpha)$. Taking the adjoint also gives $e^{-tA_0} a(f) e^{tA_0} = a(f^t)$, so $e^{-tA_0} I e^{tA_0} = I_t$, where I_t is obtained from I by replacing the form factors g by the translated g^t. Notice that $f \in L^2 \Rightarrow f^t \in L^2\ \forall t$. This shows the first part of the proposition.

The derivative is given by the limit as $t \to 0$ of $\frac{1}{t}(L_t - L) = \frac{tN + \lambda(I_t - I)}{t}$. Therefore, if

$$\left\| \frac{g_{1,2}^t - g_{1,2}^0}{t} - \partial_u g_{1,2} \right\|_{L^2} \longrightarrow 0, \quad t \to 0, \tag{101}$$

then we have $\frac{1}{t}(I_t - I) \to \tilde{I}$ strongly on $\mathcal{D}(N^{1/2})$, which shows the proposition. Using some quite standard Mean Value Theorem arguments, one can shows that (101) is satisfied, provided conditions (9) hold with $p > 0$. ■

A.3 Proof of Proposition 7.7

We denote the spectrum of L_p by $\sigma(L_p) = \{e_j\}$, where we include multiplicities, i.e. for degenerate eigenvalues, we have $e_j = e_k$ for different $j \neq k$. Let P_j denote the rank one projector onto span$\{\varphi_i\}$, where $\varphi_j \in \mathcal{H}_p \otimes \mathcal{H}_p$ is the unique eigenvector corresponding to e_j. Let e be a fixed eigenvalue of L_p. Setting $m_j = P_j m$, we have

$$\begin{aligned}&\left\langle \psi, Q_1 \int m^* \left((L_p - e + u)^2 + \epsilon^2\right)^{-1} m \; Q_1 \psi \right\rangle \\ &\quad = \sum_{e_j \in \sigma(L_p)} \left\langle \psi, Q_1 \int m_j^* m_j ((e_j - e + u)^2 + \epsilon^2)^{-1} Q_1 \psi \right\rangle. \end{aligned} \tag{102}$$

First, we estimate the term in the sum coming from $\{j : e_j = e\}$:

$$\sum_{\{e_j = e\}} \left\langle \psi, Q_1 \int m_j^* m_j (u^2 + \epsilon^2)^{-1} Q_1 \psi \right\rangle \leq \sum_{\{e_j = e\}} \int u^{-2} ||m_j Q_1 \psi||^2. \tag{103}$$

Now

$$\sum_{\{e_j = e\}} ||m_j Q_1 \psi||^2 = ||P(L_p = e)(G_l g_1 - G_r g_2) Q_1 \psi||^2 \leq 2||G||^2 (|g_1|^2 + |g_2|^2) ||\psi||^2,$$

so (103) $\leq 2||G||^2 ||\psi||^2 \left(||g_1/u||_{L^2}^2 + ||g_2/u||_{L^2}^2\right) = 4||G||^2 ||g_1/u||_{L^2}^2 ||\psi||^2$. ¿From our assumptions on g (see (9)) and (23), it is clear that $\|g_1/u\|_{L^2} = C < \infty$, uniformly in $\beta \geq 1$, and we conclude that

$$(103) \leq C\|\psi\|^2. \tag{104}$$

Next, we estimate the sum of the terms in (102) with $e_j \neq e$ and write it as

$$\sum_{e_j \neq e} \int_{\mathbb{R}} du ((e_j - e + u)^2 + \epsilon^2)^{-1} \tilde{m}_j(u, \psi), \tag{105}$$

where we put $\tilde{m}_j(u, \psi) = \int_{S^2} dS ||m_j(u, \alpha) Q_1 \psi||^2$. $\forall \xi > 0$, we have

$$\begin{aligned}&\sum_{e_j \neq e} \int_{\{|u - (e - e_j)| \geq \xi\}} du \; ((e_j - e + u)^2 + \epsilon^2)^{-1} \tilde{m}_j(u, \psi) \\ &\leq \; \xi^{-2} \sum_{e_j \neq e} \int_{\mathbb{R}} du \; \tilde{m}_j(u, \psi) \\ &\leq \; \xi^{-2} \int ||m(u, \alpha) Q_1 \psi||^2 \leq 4\xi^{-2} ||G||^2 ||g_1/u||_{L^2}^2 ||\psi||^2 \leq C\xi^{-2} \|\psi\|^2. \end{aligned} \tag{106}$$

Next, with the changes of variables $y = u - (e - e_j)$, we arrive at

$$\begin{aligned}
&\sum_{e_j \neq e} \int_{\{|u-(e-e_j)| \leq \xi\}} du \ ((e_j - e + u)^2 + \epsilon^2)^{-1} \tilde{m}_j(u, \psi) \\
&= \left(\int_{-\xi}^{\xi} dy \ (y^2 + \epsilon^2)^{-1} \right) \sum_{e_j \neq e} \tilde{m}_j(e - e_j, \psi) \\
&\quad + \int_{-\xi}^{\xi} dy \ (y^2 + \epsilon^2)^{-1} \sum_{e_j \neq e} \left[\tilde{m}_j(y + e - e_j, \psi) - \tilde{m}_j(e - e_j, \psi) \right]. \qquad (107)
\end{aligned}$$

The mean value theorem yields for the last sum:

$$y \ \partial_y|_{\tilde{y} \in (-\xi, \xi)} \sum_{e_j \neq e} \tilde{m}_j(y + e - e_j, \psi). \qquad (108)$$

Now

$$\begin{aligned}
&\partial_y \sum_{e_j \neq e} \tilde{m}_j(y + e - e_j, \psi) \\
&= 2 \sum_{e_j \neq e} \int_{S^2} dS \ \mathrm{Re} \left\langle P_j (\partial_u m)(y + e - e_j, \alpha) Q_1 \psi, P_j m(y + e - e_j, \alpha) Q_1 \psi \right\rangle .
\end{aligned}$$

Using the Schwarz inequality for sums, we bound the modulus of the r.h.s. from above by

$$2 \int_{S^2} dS \ \sqrt{\sum_{e_j \neq e} ||P_j (\partial_u m)(y + e - e_j, \alpha) Q_1 \psi||^2} \sqrt{\sum_{e_j \neq e} ||P_j m(y + e - e_j, \alpha) Q_1 \psi||^2}. \qquad (109)$$

Now $m(y + e - e_j, \alpha) = G_l g_1(y + e - e_j, \alpha) - G_r g_2(y + e - e_j, \alpha)$, so

$$\begin{aligned}
&||P_j m(y + e - e_j, \alpha) Q_1 \psi||^2 \qquad (110) \\
&\leq 2|g_1(y + e - e_j, \alpha)|^2 ||P_j G_l Q_1 \psi||^2 + 2|g_2(y + e - e_j, \alpha)|^2 ||P_j G_r Q_1 \psi||^2.
\end{aligned}$$

We have to evaluate this at $y = \tilde{y} \in (-\xi, \xi)$. Clearly, $|e - e_j + \tilde{y}| \geq |e - e_j| - |\tilde{y}| > d_0 - \xi \geq d_0/2$, if we choose $\xi \leq d_0/2$, where

$$d_0 := \inf_{e_i \neq e_j} |e_i - e_j| > 0.$$

The r.h.s. of (110) can thus be estimated from above by

$$2 \sup_{|u| > d_0/2} |g_1(u, \alpha)|^2 ||P_j G_l Q_1 \psi||^2 + 2 \sup_{|u| > d_0/2} |g_2(u, \alpha)|^2 ||P_j G_r Q_1 \psi||^2,$$

hence we arrive at

$$|(108)| \leq 32|y|\, ||G||^2||\psi||^2 \int_{S^2} dS \, \left(\sup_{|u|>d_0/2} |\partial_u g_1| + \sup_{|u|>d_0/2} |g_1| \right). \tag{111}$$

Using the conditions (9) with $p > 0$, one shows that the suprema are bounded, uniformly in $\beta \geq 1$, and so is $|g_1|$, thus (111) gives

$$|(108)| \leq C|y|\, ||\psi||^2. \tag{112}$$

Remark that the constant here depends on d_0, $C \sim d_0^{p-1/2}$. This argument is valid for any p. Going back to the second term on the r.h.s. of (107), we have shown:

$$\begin{aligned} &\left| \int_{-\xi}^{\xi} \frac{dy}{y^2+\epsilon^2} \sum_{e_j \neq e} \left[\tilde{m}_j(y+e-e_j,\psi) - \tilde{m}_j(e-e_j,\psi) \right] \right| \\ &\qquad\qquad \leq \; C||\psi||^2 \int_{-\xi}^{\xi} \frac{|y|}{y^2+\epsilon^2} dy \leq C\frac{|\xi|}{\epsilon}||\psi||^2. \end{aligned} \tag{113}$$

Now we consider the first term on the r.h.s. of (107). We see that, as $\epsilon/\xi \to 0$,

$$\int_{-\xi}^{\xi} \frac{dy}{y^2+\epsilon^2} = \frac{2}{\epsilon}\mathrm{Arctan}(\xi/\epsilon) = \frac{2}{\epsilon}\left(\frac{\pi}{2} + o\left((\epsilon/\xi)^\eta\right)\right),$$

for any $0 < \eta < 1$. This simply follows from the fact that for any such η, we have $\lim_{x\to\infty} x^\eta(\mathrm{Arctan}(x) - \pi/2) = 0$. Also,

$$\sum_{e_j \neq e} \tilde{m}_j(e-e_j,\psi) = \int \langle \psi, Q_1 m^* \delta(u-e+L_p) P(L_p \neq e) m Q_1 \psi \rangle .$$

We conclude that (105) is equal to

$$\frac{\pi}{\epsilon}\Big\{(1-O(\epsilon/\xi)) \int \langle \psi, Q_1 m^* \delta(u-e+L_p) P(L_p \neq e) m Q_1 \psi \rangle - O(\xi + \epsilon\xi^{-2})||\psi||^2 \Big\}.$$

Choose e.g. $\xi = \epsilon^{1/4}$ and η close to 1, then the we arrive at

$$(105) = \frac{\pi}{\epsilon}\left\{ \int \langle \psi, Q_1 m^* \delta(u-e+L_p) P(L_p \neq e) m Q_1 \psi \rangle - O(\epsilon^{1/4})||\psi||^2 \right\}.$$

This together with (104) yields

$$\begin{aligned} &Q_1 \int m^*((L_p - e + u)^2 + \epsilon^2)^{-1} m Q_1 \\ &\geq \; Q_1 \frac{\pi}{\epsilon}\left\{ \int m^* P(L_p \neq e)\delta(L_p - e + u) m - O(\epsilon^{1/4}) \right\} Q_1. \quad \blacksquare \end{aligned}$$

A.4 Operator calculus

We outline an operator calculus for functions of selfadjoint operators, used extensively in this work. For a detailed exposition and more references, we refer to [HS3].

Let $f \in C_0^k(\mathbb{R})$, $k \geq 2$, and define the compactly supported complex measure $d\tilde{f}(z) = -\frac{1}{2\pi}(\partial_x + i\partial_y)\tilde{f}(z)dxdy$, where $z = x + iy$ and $\tilde{f}$ is an almost analytic complex extension of f in the sense that $(\partial_x + i\partial_y)\tilde{f}(z) = 0, \quad z \in \mathbb{R}$. Then, for a selfadjoint operator A, one shows that

$$f(A) = \int d\tilde{f}(z)(A-z)^{-1},$$

where the integral is absolutely convergent. Given f, one can construct explicitely an almost analytic extension $\tilde{f}$ supported in a complex neighbourhood of the support of f. One shows that for $p \leq k-2$,

$$\int \left| d\tilde{f}(z) \right| \, |\mathrm{Im} z|^{-p-1} \leq C \sum_{j=0}^{k} \| f^{(j)} \|_{j-p-1}, \tag{114}$$

where $\|f\|_n = \int dx \langle x \rangle^n |f(x)|$, and $\langle x \rangle = (1+x^2)^{1/2}$. Furthermore, the derivatives of $f(A)$ are given by

$$f^{(p)}(A) = p! \int d\tilde{f}(A)(A-z)^{-p-1}. \tag{115}$$

We finish this outline by mentioning that these results extend by a limiting argument to functions f that do not have compact support, as long as the norms in the r.h.s. of (114) are finite.

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