Normal form invariants around spin–orbit periodic orbits

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May 31, 2000

Abstract. We consider a model of spin–orbit interaction, describing the motion of an oblate satellite rotating about an internal spin–axis and orbiting about a central planet. The resulting second order differential equation depends upon the parameters provided by the equatorial oblateness of the satellite and its orbital eccentricity. Normal form transformations around the main spin–orbit resonances are carried out explicitely. As an outcome, one can compute some invariants; the fact that these quantities are not identically zero is a necessary condition to prove the existence of nearby periodic orbits (Birkhoff fixed point theorem). Moreover, the nonvanishing of the invariants provides also the stability of the spin–orbit resonances, since it guarantees the existence of invariant curves surrounding the periodic orbit.

Keywords: Stability, normal form, spin–orbit resonance

1. Introduction

We perform nonlinear analysis of periodic orbits in Celestial Mechanics. More precisely, we consider an oblate satellite orbiting around a central planet with revolutional period T_{rev}. The satellite rotates about an internal spin–axis with rotational period T_{rot}. A spin–orbit resonance occurs whenever the ratio of the main periods is a rational number, i.e. $T_{rev}/T_{rot} = p/q$, for some positive integers p, q. Most of the evolved bodies of the solar system (including the Moon) are trapped in a synchronous spin–orbit resonance, namely $T_{rev}/T_{rot} = 1/1$, which implies that the satellite always points the same face to the host planet. The only exception is provided by Mercury for which $T_{rev}/T_{rot} = 3/2$. We consider the periodic orbits associated to the 1:1 and 3:2 resonances and we derive a simplified model of spin–orbit interaction described by a non–autonomous second–order differential equation. The model depends on two parameters denoted by ε (the equatorial oblateness of the satellite) and by e (the orbital eccentricity).

We reduce to the study of the Poincaré mapping and shift the fixed point (corresponding to a periodic orbit with frequency p/q) to the origin. Next we expand in polynomial series of the coordinates up to the third degree and according to (Siegel and Moser, 1971) we implement a normal form transformation. After some auxiliary changes of variables, we reduce the original mapping to the form

$$\begin{cases} r' = r + R(r,\varphi) \\ \varphi' = \varphi + \gamma_0 + \gamma_1 r^2 + S(r,\varphi) \ , \end{cases}$$

where (r,φ) are polar coordinates, while $R(r,\varphi)$, $S(r,\varphi)$ are error terms. Here the quantities γ_0, γ_1 depend, respectively, on the linear and non-linear parts of the mapping. With the aid of the algebraic manipulator "Mathematica", we compute explicitely the quantities γ_0, γ_1 (called *invariants*) for the 1:1 and 3:2 periodic orbits within a precision of $O(\varepsilon^2)$. The fact that γ_1 is not zero is a necessary condition for the applicability of the Birkhoff fixed point theorem and of the determination of the stability of the 1:1 or 3:2 periodic orbits, since one can prove the existence of librational invariant curves surrounding the fixed point (Siegel and Moser, 1971).

However, we remark that the behaviour of γ_1 is different for the 1:1 and 3:2 resonances, since in the first case one has a non trivial first order dependence upon ε, e, while for the 3:2 periodic orbit γ_1 is constant up to $O(\varepsilon^2)$, namely up to the degree of approximation we performed our computations. The same method can be implemented to get a higher order precision, though the computations to $O(\varepsilon^3)$ are extremely laborious.

2. The spin–orbit model

We derive a model of spin–orbit interaction, assuming that an oblate satellite (for example, the Moon) orbits on an eccentric Keplerian orbit around a central planet (the Earth). As is well known, the Moon–Earth system is trapped in a synchronous (or 1:1) spin–orbit resonance, due to the fact that the period of revolution T_{rev} of the Moon about the Earth equals the period of rotation T_{rot} around its spin–axis (namely, $\frac{T_{rev}}{T_{rot}} = \frac{1}{1}$). Since the periods of revolution and rotation are the same, the Moon always points the same face to the Earth. More generally, a *spin–orbit resonance of order* $p:q$ is defined whenever $\frac{T_{rev}}{T_{rot}} = \frac{p}{q}$, for p, $q \in \mathbf{Z}_+$. We introduce a simplified model assuming that the spin–axis of the satellite is perpendicular to the orbit plane and that it coincides with its shortest physical axis. We also neglect any other gravitational or dissipative contribution. Under these assumptions, the equation of motion can be derived from rigid body theory (compare with (Celletti, 1990)). Denoting by $A < B < C$ the principal moments of inertia and by a, e the semimajor axis and the eccentricity of the Keplerian orbit, the equation of motion has the form

$$\ddot{x} \ + \ \frac{3}{2}\ \frac{B-A}{C}(\frac{a}{r})^3 \ \sin(2x - 2f) = 0 \ , \tag{1}$$

where r, f are the instantaneous orbital radius and the true anomaly, while x is the angle between the longest axis of the ellipsoidal satellite and a preassigned reference line. Since we assumed that the orbit is Keplerian, the quantities r and f are known functions of the time. Therefore, (1) can be expanded in Fourier series providing the equation

$$\ddot{x} + \varepsilon \sum_{m\neq 0, m=-\infty}^{\infty} W(\frac{m}{2}, e)\ \sin(2x - mt) = 0\ , \tag{2}$$

where we defined the *perturbing parameter* $\varepsilon \equiv \frac{3}{2}\frac{B-A}{C}$ and where the coefficients W decay as powers of e, being $W(\frac{m}{2}, e) \propto e^{|m-2|}$. For example, the first few coefficients are given by the following expressions:

$$\begin{aligned}
W(\frac{1}{2}, e) &= -\frac{e}{2} + \frac{e^3}{16}\ , & W(1, e) &= 1 - \frac{5}{2}e^2 + \frac{13}{16}e^4 \\
W(\frac{3}{2}, e) &= \frac{7}{2}e - \frac{123}{16}e^3\ , & W(2, e) &= \frac{17}{2}e^2 - \frac{115}{6}e^4 \\
W(\frac{5}{2}, e) &= \frac{845}{48}e^3 - \frac{32525}{768}e^5\ , & W(3, e) &= \frac{533}{16}e^4\ .
\end{aligned}$$

In what follows we shall retain only the leading terms of (2), which are provided by the first order terms in the eccentricity. More precisely, we consider the equation

$$\begin{aligned}
\ddot{x} &+ \varepsilon\Big[-\frac{e}{2}\ \sin(2x - t) + \sin(2x - 2t) \\
&+ \frac{7}{2}e \sin(2x - 3t)\Big] = 0\ ,
\end{aligned} \tag{3}$$

where $O(e^2)$ have been neglected. We will see that the above equation provides already the essential features for the computation of the normal form and of the associated inviariants. Higher orders computations can be performed in a similar way to that used to analyze (3). We remark that (3) is associated to a one–dimensional, time–dependent Hamiltonian system of the form: $H(y, x, t) = \frac{y^2}{2} + \varepsilon V(x, t)$, where $y = \dot{x}$. In particular, the Poincaré map obtained from (3) shows a pendulum–like structure. A generic periodic orbit (which corresponds to a fixed point of the Poincaré map) is surrounded by librational islands; a chaotic separatrix divides the librational zone from the region where rotational motion takes place.

3. Normal form around a periodic orbit

We consider the periodic orbit corresponding to a $p : q$ spin–orbit resonance, which is defined as a solution of (3) such that $x(t+2\pi q) = x(t)+2\pi p$. In this chapter, normal form transformations are carried out explicitely for the Poincaré mapping associated to (3). After shifting the fixed point to the origin, we expand the map in polynomial series of the coordinates and as power series of the perturbing parameter ε. By means of several changes of variables we reduce to the study of a mapping of the form

$$\begin{cases} r' = r + R(r,\varphi) \\ \varphi' = \varphi + \gamma_0 + \gamma_1 r^2 + S(r,\varphi) \ , \end{cases}$$

where (r,φ) are polar coordinates around the fixed point and $R(r,\varphi)$, $S(r,\varphi)$ are the *error terms*. The quantities γ_0, γ_1 are called the *invariants* related, respectively, to the linear and nonlinear parts of the original mapping. Their computation is carried out explicitely up to the order ε^2 for the 1:1 and 3:2 resonances.

3.1. Polynomial expansions

Let us write the equation of motion (3) as

$$\begin{cases} \dot{x} = y \\ \dot{y} = \varepsilon g(x,t) \ . \end{cases}$$

The solution of the above differential system is given in integral form by

$$\begin{aligned} x(t) &= \hat{x} + \int_0^t y(s)ds = \hat{x} + \hat{y}t + \varepsilon \int_0^t \int_0^s g(x(\tau),\tau) \ d\tau ds \\ y(t) &= \hat{y} + \varepsilon \int_0^t g(x(s),s)ds \end{aligned} \tag{4}$$

for some initial conditions $(\hat{x},\hat{y})$. A $p : q$ spin–orbit resonance is characterized by the *periodicity conditions*

$$\begin{aligned} x(t+T) &= x(t) + 2\pi p \\ y(t+T) &= y(t) \ , \end{aligned} \tag{5}$$

where $T \equiv 2\pi q$ is the period of the motion. We consider the Poincaré mapping at time T defined as

$$\begin{aligned} x' &= \hat{x} + \hat{y}T + \varepsilon \int_0^T \int_0^s g(x(\tau),\tau) d\tau ds \\ y' &= \hat{y} + \varepsilon \int_0^T g(x(s),s)ds \ . \end{aligned} \tag{6}$$

Exact initial conditions $(\overline{x}, \overline{y})$ on the periodic orbit define a fixed point of the Poincaré map. We expand the r.h.s. of (6) up to $O(\varepsilon^2)$ as

$$\begin{cases} x' = \hat{x} + \hat{y}T + \varepsilon x_1(T; \hat{x}, \hat{y}) + \varepsilon^2 x_2(T; \hat{x}, \hat{y}) \\ y' = \hat{y} + \varepsilon y_1(T; \hat{x}, \hat{y}) + \varepsilon^2 y_2(T; \hat{x}, \hat{y}) \ , \end{cases}$$

where

$$\begin{aligned} x_1(T; \hat{x}, \hat{y}) &= \int_0^T \int_0^t g(\hat{x} + \hat{y}\tau, \tau) d\tau dt \\ x_2(T; \hat{x}, \hat{y}) &= \int_0^T \int_0^t g_x(\hat{x} + \hat{y}\tau, \tau) x_1(\tau; \hat{x}, \hat{y}) d\tau dt \\ y_1(T; \hat{x}, \hat{y}) &= \int_0^T g(\hat{x} + \hat{y}s, s) ds \\ y_2(T; \hat{x}, \hat{y}) &= \int_0^T g_x(\hat{x} + \hat{y}s, s) x_1(s; \hat{x}, \hat{y}) ds \ . \end{aligned}$$

We next translate the fixed point to the origin by means of the transformation

$$\begin{cases} \eta_1 = \hat{x} - \overline{x} \\ \eta_2 = \hat{y} - \overline{y} \ . \end{cases}$$

The new variables (η_1, η_2) satisfy the evolution equations

$$\begin{aligned} \eta_1' &= \eta_1 + \eta_2 T + \overline{y}T + \varepsilon x_1(T; \eta_1 + \overline{x}, \eta_2 + \overline{y}) + \varepsilon^2 x_2(T; \eta_1 + \overline{x}, \eta_2 + \overline{y}) \\ \eta_2' &= \eta_2 + \varepsilon y_1(T; \eta_1 + \overline{x}, \eta_2 + \overline{y}) + \varepsilon^2 y_2(T; \eta_1 + \overline{x}, \eta_2 + \overline{y}) \ . \end{aligned} \tag{7}$$

By (4) and (5) one easily gets that the solution $(x(t), y(t))$ must satisfy the relations

$$\begin{aligned} &\textstyle\int_0^T \ g(x(s), s) ds = 0 \\ &\textstyle\int_0^T \ y(s) ds = 2\pi p \ . \end{aligned} \tag{8}$$

Let us write the initial conditions as series expansions in the parameter ε setting $\overline{x} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$, $\overline{y} = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + ...$, so that

$$\begin{cases} x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2)t + \varepsilon \tilde{x}_1(t) + \varepsilon^2 \tilde{x}_2(t) + O(\varepsilon^3) \\ y(t) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon \tilde{y}_1(t) + \varepsilon^2 \tilde{y}_2(t) + O(\varepsilon^3) \ , \end{cases}$$

where the functions $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ are defined as

$$\begin{aligned} \tilde{x}_1(t) &= \int_0^t y_1(s) ds \\ \tilde{x}_2(t) &= \int_0^t y_2(s) ds \end{aligned}$$

$$\begin{aligned}
\tilde{y}_1(t) &= \int_0^t g(x_0+y_0s,s)ds \\
\tilde{y}_2(t) &= \int_0^t g_x(x_0+y_0s,s)(x_1+y_1s+\tilde{x}_1(s))ds\ ,
\end{aligned}$$

where $y_0=\frac{p}{q}$, x_0 is a solution of $\int_0^T g(x_0+y_0t,t)dt=0$, while

$$\begin{aligned}
y_1 &= -\frac{1}{T}\int_0^T\int_0^t g^0 dsdt \\
x_1 &= -\frac{1}{\int_0^T g_x^0 dt}\ (y_1\int_0^T g_x^0 t\ dt+\int_0^T g_x^0\tilde{x}_1(t)\ dt) \\
y_2 &= -\frac{1}{T}\int_0^T\int_0^t g_x^0\cdot(x_1+y_1+\tilde{x}_1(s))dsdt \\
x_2 &= -\frac{1}{\int_0^T g_x^0 dt}\int_0^T [y_2g_x^0t+g_x^0\tilde{x}_2(t)+\frac{1}{2}g_{xx}^0\cdot(x_1+y_1t+\tilde{x}_1(t))^2]dt\ ,
\end{aligned}$$

having set $g^0\equiv g(x_0+y_0t,t)$ and similarly for g_x^0, g_{xx}^0. By means of the definitions of x_0, x_1, y_0, y_1, we rewrite the first equation in (7) up to the second order in ε as

$$\begin{aligned}
\eta_1' &= \eta_1+\eta_2T+y_0T+\varepsilon\int_0^T\int_0^\tau [g(\eta_1+x_0+(\eta_2+y_0)s,s)-g(x_0+y_0s,s)]dsd\tau \\
&+ \varepsilon^2\int_0^T\int_0^\tau [g_x(\eta_1+x_0+(\eta_2+y_0)s,s)\cdot x_1(s;\eta_1+x_0,\eta_2+y_0) \\
&+ g_x(\eta_1+x_0+(\eta_2+y_0)s,s)(x_1+y_1s)-g_x(x_0+y_0s,s) \\
&\cdot (x_1+y_1s+x_1(s;x_0,y_0))]\ dsd\tau\ .
\end{aligned}$$

Upon expansion in Taylor series up to the third order around $\eta_1=0$, $\eta_2=0$ one gets

$$\begin{aligned}
\eta_1' &= \eta_1+\eta_2T+\varepsilon\int_0^T\int_0^\tau [g_x^0(\eta_1+\eta_2s)+\frac{1}{2}g_{xx}^0(\eta_1+\eta_2s)^2+\frac{1}{6}g_{xxx}^0(\eta_1+\eta_2s)^3]dsd\tau \\
&+ \varepsilon^2\int_0^T\int_0^\tau \{[g_{xx}^0(\eta_1+\eta_2s)+\frac{1}{2}g_{xxx}^0(\eta_1+\eta_2s)^2+\frac{1}{6}g_{xxxx}^0(\eta_1+\eta_2s)^3](x_1^0+x_1+y_1s) \\
&+ (x_{1x}^0\eta_1+x_{1y}^0\eta_2)[g_x^0+g_{xx}^0(\eta_1+\eta_2s)+\frac{1}{2}g_{xxx}^0(\eta_1+\eta_2s)^2] \\
&+ (\frac{1}{2}x_{1xx}^0\eta_1^2+\frac{1}{2}x_{1yy}^0\eta_2^2+x_{1xy}^0\eta_1\eta_2)[g_x^0+g_{xx}^0(\eta_1+\eta_2s)] \\
&+ (\frac{1}{6}x_{1xxx}^0\eta_1^3+\frac{1}{2}x_{1xxy}^0\eta_1^2\eta_2+\frac{1}{2}x_{1xyy}^0\eta_1\eta_2^2+\frac{1}{6}x_{1yyy}^0\eta_2^3)\ g_x^0\}ds\ d\tau\ ,
\end{aligned}$$

where $g_x^0\ \equiv\ g_x(x_0+y_0s,s)$, $x_1^0\ \equiv\ x_1(s;x_0,y_0)$, $x_{1x}^0\ \equiv\ \frac{\partial x_1}{\partial x}(s;x_0,y_0)$ and similarly for higher order derivatives. Analogously, for the second equation one obtains

$$\begin{aligned}
\eta_2' &= \eta_2 + \varepsilon \int_0^T [g(\eta_1 + x_0 + (\eta_2 + y_0)s, s) - g(x_0 + y_0 s, s)]ds \\
&+ \varepsilon^2 \int_0^T [g_x(\eta_1 + x_0 + (\eta_2 + y_0)s, s)\ (x_1 + y_1 s + x_1(s; \eta_1 + x_0, \eta_2 + y_0)) \\
&- g_x(x_0 + y_0 s, s)(x_1 + y_1 s + x_1(s; x_0, y_0))]ds
\end{aligned}$$

and expanding in Taylor series up to third order, one gets

$$\begin{aligned}
\eta_2' &= \eta_2 + \varepsilon \int_0^T [g_x^0(\eta_1 + \eta_2 s) + \frac{1}{2} g_{xx}^0 (\eta_1 + \eta_2 s)^2 + \frac{1}{6} g_{xxx}^0 (\eta_1 + \eta_2 s)^3]ds \\
&+ \varepsilon^2 \int_0^T \{[g_{xx}^0(\eta_1 + \eta_2 s) + \frac{1}{2} g_{xxx}^0 (\eta_1 + \eta_2 s)^2 + \frac{1}{6} g_{xxxx}^0 (\eta_1 + \eta_2 s)^3](x_1^0 + x_1 + y_1 s) \\
&+ (x_{1x}^0 \eta_1 + x_{1y}^0 \eta_2)[g_x^0 + g_{xx}^0(\eta_1 + \eta_2 s) + \frac{1}{2} g_{xxx}^0 (\eta_1 + \eta_2 s)^2] \\
&+ (\frac{1}{2} x_{1xx}^0 \eta_1^2 + \frac{1}{2} x_{1yy}^0 \eta_2^2 + x_{1xy}^0 \eta_1 \eta_2)[g_x^0 + g_{xx}^0(\eta_1 + \eta_2 s)] \\
&+ (\frac{1}{6} x_{1xxx}^0 \eta_1^3 + \frac{1}{2} x_{1xxy}^0 \eta_1^2 \eta_2 + \frac{1}{2} x_{1xyy}^0 \eta_1 \eta_2^2 + \frac{1}{6} x_{1yyy}^0 \eta_2^3)\ g_x^0\}ds \ .
\end{aligned}$$

Let us rewrite the final system as

$$\begin{aligned}
\eta_1' &= a\eta_1 + b\eta_2 + p_2(\eta_1, \eta_2; \varepsilon) + p_3(\eta_1, \eta_2; \varepsilon) \\
\eta_2' &= c\eta_1 + d\eta_2 + q_2(\eta_1, \eta_2; \varepsilon) + q_3(\eta_1, \eta_2; \varepsilon) \ ,
\end{aligned} \tag{9}$$

where p_n, q_n are n–th order polynomials in η_1, η_2. As an example, we provide in the Appendix explicit formulae for the 1:1 resonance of the coefficients a, b, c, d and the polynomials p_2, p_3, q_2, q_3. An explicit computation shows that for the elliptic 1:1 and 3:2 periodic orbits the linear coefficients are related by $a = d$ and $bc < 0$; therefore, the eigenvalues of the linearized part are complex conjugates with $\lambda = \lambda_1 \pm i\lambda_2$, where $\lambda_1 = a$, $\lambda_2 = \sqrt{-bc}$.

Remark: Let us rewrite the first periodicity condition in (8) for the zero–order periodic orbit $x(t) = x_0 + y_0 t$, $y(t) = y_0 = const$ as

$$\int_0^T g(x_0 + y_0 s, s)\ ds \ = \ 0 \ . \tag{10}$$

One can easily observe that a necessary condition for the application of an implicit function theorem on the Poincaré map associated to the periodic orbit is (cfr. (Celletti and Chierchia, 1998))

$$\int_0^T g_x(x_0 + y_0 s, s)\ ds \ \neq \ 0 \ . \tag{11}$$

Conditions (10) and (11) correspond to the requirement that the subharmonic Melnikov integral (Guckenheimer and Holmes, 1983) has a simple zero, whenever the initial phase, say t_0, of the periodic orbit is set to zero. Moreover, if (11) is not satisfied, then the coefficient c in (9) is identically zero up to $O(\varepsilon^2)$ leading to real eigenvalues ($\lambda_2 = 0$).

3.2. Normal form expansion

Once reduced the initial system to (9), we proceed to transform it in normal form. Following (Siegel and Moser, 1971), we perform a coordinate transformation

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = C \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} , \qquad C \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ,$$

so that the linear part of the transformed system becomes

$$\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = T \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} , \qquad T \equiv \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} .$$

Denoting by S the linearized matrix associated to (9), i.e. $S \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we determine C by the condition

$$C^{-1} S C \; = \; T \; ,$$

with the further requirement that det $C = 1$ according to the area–preservation property. More precisely, the matrix C takes the form

$$C \; = \; \begin{pmatrix} 0 & \frac{i}{k} \\ ik & 0 \end{pmatrix} \; ,$$

where $k \equiv \sqrt{-\frac{c}{\lambda_2}}$. Finally, the system including nonlinear terms is given by

$$\begin{cases} \xi_1' = \lambda_1 \xi_1 - \lambda_2 \xi_2 - \frac{i}{k} \, [q_2(\frac{i}{k}\xi_2, ik\xi_1) + q_3(\frac{i}{k}\xi_2, ik\xi_1)] \\ \xi_2' = \lambda_2 \xi_1 + \lambda_1 \xi_2 - ik \, [p_2(\frac{i}{k}\xi_2, ik\xi_1) + p_3(\frac{i}{k}\xi_2, ik\xi_1)] \; . \end{cases}$$

Next we transform to complex variables setting

$$\begin{cases} Y = \xi_1 + i\xi_2 \\ X = \xi_1 - i\xi_2 \; , \end{cases}$$

where (Y, X) satisfy the evolutionary equations

$$\begin{aligned} Y' \; &= \; \lambda Y - \frac{i}{k} \, [q_2(\frac{Y-X}{2k}, ik\frac{Y+X}{2}) + q_3(\frac{Y-X}{2k}, ik\frac{Y+X}{2})] \\ &+ \; k \, [p_2(\frac{Y-X}{2k}, ik\frac{Y+X}{2}) + p_3(\frac{Y-X}{2k}, ik\frac{Y+X}{2})] \\ X' \; &= \; \overline{\lambda} X - \frac{i}{k} \, [q_2(\frac{Y-X}{2k}, ik\frac{Y+X}{2}) + q_3(\frac{Y-X}{2k}, ik\frac{Y+X}{2})] \\ &- \; k \, [p_2(\frac{Y-X}{2k}, ik\frac{Y+X}{2}) + p_3(\frac{Y-X}{2k}, ik\frac{Y+X}{2})] \; . \end{aligned} \tag{12}$$

Let us rewrite (12) as

$$\begin{aligned} Y' &= \lambda Y + P_2(Y,X) + P_3(Y,X) \\ X' &= \overline{\lambda} X + Q_2(Y,X) + Q_3(Y,X) \ , \end{aligned} \tag{13}$$

where P_n, Q_n are n–th order homogeneous polynomials in (Y,X). The last coordinate transformation, $(Y,X) \to (\xi,\eta)$, is defined by

$$\begin{aligned} Y &= \xi + \Phi_2(\xi,\eta) + \Phi_3(\xi,\eta) \\ X &= \eta + \Psi_2(\xi,\eta) + \Psi_3(\xi,\eta) \ , \end{aligned} \tag{14}$$

so that (ξ,η) satisfy the normal form equations

$$\begin{aligned} \xi' &= exp(i(\gamma_0 + \gamma_1 \xi\eta + \gamma_2(\xi\eta)^2 + ...))\xi = \lambda\ \xi + i\gamma_1\lambda\ \xi^2\eta + ... \\ \eta' &= exp(-i(\gamma_0 + \gamma_1 \xi\eta + \gamma_2(\xi\eta)^2 + ...))\eta = \overline{\lambda}\ \eta - i\gamma_1\overline{\lambda}\ \xi\eta^2 + .. \end{aligned} \tag{15}$$

We notice that γ_0 is related to the eigenvalues of the linearized part by $e^{i\gamma_0} = \lambda$, namely $\gamma_0 = \mathrm{arctg}\frac{\lambda_2}{\lambda_1}$. The generating functions Φ_j, Ψ_j are determined inserting (14) into (13) and equating same polynomial orders in (ξ,η). One easily obtains the functional equations:

$$\begin{aligned} \Phi_2(\lambda\xi,\overline{\lambda}\eta) &= \lambda\Phi_2(\xi,\eta) + P_2(\xi,\eta) \\ \Psi_2(\lambda\xi,\overline{\lambda}\eta) &= \overline{\lambda}\Psi_2(\xi,\eta) + Q_2(\xi,\eta) \\ i\lambda\gamma_1\xi^2\eta + \Phi_3(\lambda\xi,\overline{\lambda}\eta) &= \lambda\Phi_3(\xi,\eta) + P_{2x}(\xi,\eta)\Phi_2(\xi,\eta) \\ &+ P_{2y}(\xi,\eta)\Psi_2(\xi,\eta) + P_3(\xi,\eta) \\ -i\overline{\lambda}\gamma_1\xi\eta^2 + \Psi_3(\lambda\xi,\overline{\lambda}\eta) &= \overline{\lambda}\Psi_3(\xi,\eta) + Q_{2x}(\xi,\eta)\Phi_2(\xi,\eta) \\ &+ Q_{2y}(\xi,\eta)\Psi_2(\xi,\eta) + Q_3(\xi,\eta) \ . \end{aligned} \tag{16}$$

Using the expansions

$$\begin{aligned} P_2(\xi,\eta) &= P_{20}\xi^2 + P_{11}\xi\eta + P_{02}\eta^2 \\ Q_2(\xi,\eta) &= Q_{20}\xi^2 + Q_{11}\xi\eta + Q_{02}\eta^2 \\ \Phi_2(\xi,\eta) &= \Phi_{20}\xi^2 + \Phi_{11}\xi\eta + \Phi_{02}\eta^2 \\ \Psi_2(\xi,\eta) &= \Psi_{20}\xi^2 + \Psi_{11}\xi\eta + \Psi_{02}\eta^2 \\ P_3(\xi,\eta) &= P_{30}\xi^3 + P_{21}\xi^2\eta + P_{12}\xi\eta^2 + P_{03}\eta^3 \\ Q_3(\xi,\eta) &= Q_{30}\xi^3 + Q_{21}\xi^2\eta + Q_{12}\xi\eta^2 + Q_{03}\eta^3 \ , \end{aligned}$$

one gets

$$\begin{cases} i\lambda\gamma_1 = R_+ \\ -i\overline{\lambda}\gamma_1 = R_- \ , \end{cases}$$

where

$$\begin{aligned} R_+ &\equiv P_{21} + 2P_{20}\Phi_{11} + P_{11}\Phi_{20} + P_{11}\Psi_{11} + 2P_{02}\Psi_{20} \\ R_- &\equiv Q_{12} + 2Q_{20}\Phi_{02} + Q_{11}\Phi_{11} + Q_{11}\Psi_{02} + 2Q_{02}\Psi_{11} \ , \end{aligned}$$

so that we can determine γ_1 by

$$\gamma_1 = \frac{R_+ + R_-}{-2\lambda_2} \tag{17}$$

or, equivalently, by

$$\gamma_1 = \frac{R_+ - R_-}{2i\lambda_1} . \tag{18}$$

What remains to do is to compute explicitely R_+ and R_-; we recall that the coefficients P_{ij}, Q_{ij} are explicitely known from the expression of the initial mapping we started with. The terms Φ_{ij}, Ψ_{ij} of the generating function can be derived from the first two functional equations in (16). Precisely, one obtains

$$\Phi_{20} = \frac{P_{20}}{\lambda^2 - \lambda} , \qquad \Phi_{11} = \frac{P_{11}}{1-\lambda} , \qquad \Phi_{02} = \frac{P_{02}}{\overline{\lambda}^2 - \lambda}$$
$$\Psi_{20} = \frac{Q_{20}}{\lambda^2 - \overline{\lambda}} , \qquad \Psi_{11} = \frac{Q_{11}}{1-\overline{\lambda}} , \qquad \Psi_{02} = \frac{Q_{02}}{\overline{\lambda}^2 - \overline{\lambda}} .$$

These formulae provide the necessary elements to compute the invariant γ_1. In particular, having expanded the initial system including terms of order ε^2 (compare with (7)), we can compute the invariant γ_1 within an error of order $O(\varepsilon^2)$.

4. Results

Once obtained the invariant γ_1 as in (17) (or in (18)), it is convenient to retransform (15) into real variables $(\xi,\eta) \to (z_1 = \frac{\xi+\eta}{2}, z_2 = \frac{\xi-\eta}{2i})$, where

$$\begin{cases} z_1' = z_1 \cos(\gamma_0 + \gamma_1(z_1^2+z_2^2)) - z_2 \sin(\gamma_0 + \gamma_1(z_1^2+z_2^2)) + R_1 \\ z_2' = z_1 \sin(\gamma_0 + \gamma_1(z_1^2+z_2^2)) + z_2 \cos(\gamma_0 + \gamma_1(z_1^2+z_2^2)) + R_2 , \end{cases}$$

for given error terms R_1, R_2 (see (Celletti, 1995)). Consequently, one can introduce polar coordinates (r,φ) defined by

$$\begin{cases} z_1 = r\cos\varphi \\ z_2 = r\sin\varphi . \end{cases}$$

The variables (r,φ) satisfy

$$\begin{aligned} r' &= r + R(r,\varphi) \\ \varphi' &= \varphi + \gamma_0 + \gamma_1 r^2 + S(r,\varphi) , \end{aligned} \tag{19}$$

where $R(r,\varphi)$, $S(r,\varphi)$ are error terms. We next consider the equation (3) describing the spin–orbit interaction and we limit the study to the main resonances $p/q = 1/1$ and $p/q = 3/2$. The computation of the invariant γ_1 up to $O(\varepsilon^2)$ has been obtained implementing the formulae of §3 using the algebraic manipulator "Mathematica". After every transformation, the area–preserving property has always been controlled up to the order of approximation we carried out in our computations. Taking $p = q = 1$ one obtains $\gamma_1 = \frac{T}{4} - \frac{21}{4}Te^2\varepsilon + O(\varepsilon^2)$ $(T = 2\pi)$, while for the 3:2 resonance (i.e., $p = 3$, $q = 2$) one gets $\gamma_1 = \frac{T}{4} + O(\varepsilon^2)$ $(T = 4\pi)$. More precisely, the first order term in the ε–expansion of γ_1 for the 3:2 resonance is zero, indicating a *degenerate* behaviour of the associated periodic orbit as a function of ε. The dependence on the perturbing parameter appears just in the coefficient γ_0, being $\gamma_0 = \mathrm{arctg}[\sqrt{7e\varepsilon}\ T(1 + \frac{7}{3}e\varepsilon T^2)]$. On the contrary, for the 1:1 resonance we have obtained a nontrivial dependence of γ_1 upon ε, while γ_0 is defined as $\gamma_0 = \mathrm{arctg}[\sqrt{\frac{1}{2}\varepsilon}\ T(2 - 7e^2\varepsilon + \frac{4}{3}\varepsilon T^2)]$.

For a $p:q$ periodic orbit, the result that the invariant γ_1 is non zero allows to state theorems about the existence of periodic orbits around the $p:q$ resonance and its stability, in the sense of existence of invariant curves around the fixed point, corresponding to the periodic orbit with frequency p/q. The first statement is known as the Birkhoff fixed point theorem (Siegel and Moser, 1971). Let $\overline{\eta} = (\eta_1, \eta_2)$ and rewrite (9) as $\overline{\eta}' = M\overline{\eta}$; then, one has the following result.

Birkhoff theorem: Under the assumption that one of the invariants γ_l is not zero $(l \geq 1)$, in a sufficiently small neighbourhood U of the origin and for all sufficiently large integers $n > n_0(U)$, there exist fixed points $\overline{\eta} \neq 0$ of M^n, such that $M^j\overline{\eta} \in U$ for $j = 0, ..., n-1$.

As for the stability of the $p:q$ periodic orbit we can invoke the following result (Siegel and Moser, 1971):

Stability theorem: For sufficiently small values of ε, if one of the invariants γ_l is not zero $(l \geq 1)$, the mapping (9) is stable at the origin, in the sense that there exist closed invariant curves surrounding the fixed point.

Indeed, we remark that neglecting the error terms R, S in (19), one has the invariant circles $z_1^2 + z_2^2 =$ const., which can be continued into invariant curves for sufficiently small values of ε, as far as γ_1 is not zero. This condition implies that the angle of rotation $\gamma_0 + \gamma_1(z_1^2 + z_2^2)$ varies with the radius of the circle. The question of how small should ε be is left open, in the sense that the proofs of the above theorems as stated in (Siegel and Moser, 1971) provide explicit estimates on the perturbing parameter ε. We demand this task to a future work. However, we underline that the aim of this paper was to check that the requirements for the applicability of the above theorems are satisfied. Indeed, both the 1:1 and 3:2 spin–orbit resonances fulfill the condition of a non vanishing invariant γ_1, though the dependence upon ε is at the first order in the perturbing parameter for the 1:1 resonance and *at least* of order ε^2 for the 3:2 resonance.

Acknowledgements: We are grateful to the anonymous referee for having suggested the relation between Birkhoff and Melnikov theories.

Appendix

We give explicit formulae for the coefficients a, b, c, d and of the polynomials p_2, p_3, q_2, q_3 appearing in (9). The following formulae are given for the 1:1 resonance:

$$\begin{aligned}
a\ &= 1 - \varepsilon T^2 + \varepsilon^2 (7e^2T^2 + \frac{T^4}{6}) \\
b\ &= T + \varepsilon(12eT - \frac{T^3}{3}) + \varepsilon^2(96eT + 115e^2T - 4eT^3 + \frac{7}{3}e^2T^3 + \frac{T^5}{30}) \\
c\ &= -2\varepsilon T + \varepsilon^2(24eT + 14e^2T + \frac{2}{3}T^3) \\
d\ &= a \\
p_2\ (\eta_1,\eta_2) &= B_{11}\eta_1^2 + B_{12}\eta_1\eta_2 + B_{22}\eta_2^2 \\
p_3\ (\eta_1,\eta_2) &= C_{111}\eta_1^3 + C_{112}\eta_1^2\eta_2 + C_{122}\eta_1\eta_2^2 + C_{222}\eta_2^3 \\
q_2\ (\eta_1,\eta_2) &= E_{11}\eta_1^2 + E_{12}\eta_1\eta_2 + E_{22}\eta_2^2 \\
q_3\ (\eta_1,\eta_2) &= F_{111}\eta_1^3 + F_{112}\eta_1^2\eta_2 + F_{122}\eta_1\eta_2^2 + F_{222}\eta_2^3\ ,
\end{aligned}$$

where

$$\begin{aligned}
B_{11} &= -8e\varepsilon T + \varepsilon^2(-128eT - 48e^2 + \frac{8}{3}eT^3) \\
B_{12} &= -96e\varepsilon^2T^2 \\
B_{22} &= 48e\varepsilon T + \varepsilon^2(1056eT + 1008e^2T - 48eT^3) \\
C_{111} &= \frac{2}{3}\varepsilon T^2 + \varepsilon^2(-\frac{56}{3}e^2T^2 - \frac{4}{9}T^4) \\
C_{112} &= \varepsilon(-24eT + \frac{2}{3}T^3) + \varepsilon^2(-1056eT - 656e^2T + 32eT^3 - \frac{56}{3}e^2\ T^3 - \frac{8}{15}T^5) \\
C_{122} &= \varepsilon(-12eT^2 + \frac{T^4}{3}) + \varepsilon^2(-816eT^2 - 88e^2T^2 + 24eT^4 - \frac{28}{3}e^2T^4 - \frac{11}{45}T^6) \\
C_{222} &= \varepsilon(96eT - 4eT^3 + \frac{T^5}{15}) + \varepsilon^2(4992eT + 4496e^2T - 304eT^3 - \frac{232}{3}e^2T^3
\end{aligned}$$

$$
\begin{aligned}
&+ \ \frac{28}{5}eT^5 - \frac{28}{15}e^2T^5 - \frac{11}{315}\varepsilon^2T^7) \\
E_{11} &= -8e\varepsilon^2T^2 \\
E_{12} &= 16e\varepsilon T + \varepsilon^2(256eT + 96e^2T - \frac{64}{3}eT^3) \\
E_{22} &= 8e\varepsilon T^2 + \varepsilon^2(80eT^2 + 144e^2T^2 - \frac{16}{3}eT^4) \\
F_{111} &= \frac{4}{3}\varepsilon T - \varepsilon^2(64eT + \frac{496}{3}e^2T + \frac{16}{9}T^3) \\
F_{112} &= 2\varepsilon T^2 - \varepsilon^2(96eT^2 + 184e^2T^2 + \frac{8}{3}T^4) \\
F_{122} &= \varepsilon(24eT + \frac{4}{3}T^3) + \varepsilon^2(1056eT + 656e^2T - 80eT^3 - \frac{112}{3}e^2T^3 - \frac{22}{15}T^5) \\
F_{222} &= \varepsilon(12eT^2 + \frac{T^4}{3}) + \varepsilon^2(432eT^2 + 856e^2T^2 - 16eT^4 - \frac{28}{3}e^2T^4 - \frac{11}{45}T^6) \ .
\end{aligned}
$$

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