A generalised inductive approach to the lace expansion

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Abstract

The lace expansion is a powerful tool for analysing the critical behaviour of self-avoiding walks and percolation. It gives rise to a recursion relation which we abstract and study using an adaptation of the inductive method introduced by den Hollander and the authors. We give conditions under which the solution to the recursion relation behaves as a Gaussian, both in Fourier space and in terms of a local central limit theorem. These conditions are shown elsewhere to hold for sufficiently spread-out models of networks of self-avoiding walks in dimensions $d > 4$, and for critical oriented percolation in dimensions $d+1 > 5$, providing a unified approach and an essential ingredient for a detailed analysis of the branching behaviour of these models.

1 The lace expansion

1.1 The recursion relation

When applied to self-avoiding walks or oriented percolation, the lace expansion gives rise to a convolution recursion relation of the form

$$f_{n+1}(k;z) = \sum_{m=1}^{n+1} g_m(k;z) f_{n+1-m}(k;z) + e_{n+1}(k;z) \qquad (n \geq 0), \tag{1.1}$$

with $f_0(k;z) = 1$. Here, $k \in [-\pi,\pi]^d$ is a parameter dual to a spatial lattice variable $x \in \mathbb{Z}^d$. The positive parameter z is related to the activity for self-avoiding walks, and to the bond occupation probability for oriented percolation.

Such a recursion arises, in a simple form, already for ordinary random walk. In fact, let D be a non-negative function on $\mathbb{Z}^d$ which respects the lattice symmetries and obeys $\sum_{x\in\mathbb{Z}^d} D(x) = 1$. Let $z > 0$, and let $q_n(x;z)$ denote the n-step transition function of the random walk on $\mathbb{Z}^d$ with

1-step transition function $D(x)$, multiplied by z^n. Then $q_1(x;z) = zD(x)$. Set $q_0(x;z) = \delta_{0,x}$. Then q_n obeys

$$q_n(x;z) = \sum_{y\in\mathbb{Z}^d} zD(y)q_{n-1}(x-y;z) \quad (n \geq 1). \tag{1.2}$$

Given an absolutely summable function h on $\mathbb{Z}^d$, we denote its Fourier transform by

$$\hat{h}(k) = \sum_{x\in\mathbb{Z}^d} h(x)e^{ik\cdot x} \quad (k \in [-\pi,\pi]^d). \tag{1.3}$$

Then (1.2) implies that

$$\hat{q}_n(k;z) = z\hat{D}(k)\hat{q}_{n-1}(k;z). \tag{1.4}$$

The recursion relation (1.4) is in the form (1.1), with $f_n(k;z) = \hat{q}_n(k;z)$, $g_1(k;z) = z\hat{D}(k)$, $g_m = 0$ for $m \geq 2$, and $e_n = 0$ for $n \geq 1$. It is readily solved to give $\hat{q}_n(k;z) = z^n\hat{D}(k)^n$. Assuming that D has a finite second moment $\sigma^2 = \sum_x |x|^2 D(x)$, Taylor expansion of $\hat{D}$ then gives the central limit theorem

$$\lim_{n\to\infty} \hat{q}_n\Big(\frac{k}{\sqrt{\sigma^2 n}};1\Big) = e^{-k^2/2d}. \tag{1.5}$$

The value $z = 1$ plays the role of a critical point, with solutions of the recursion exponentially growing for $z > 1$ and exponentially decaying if $z < 1$.

Our goal here is to prove that the solutions of certain more elaborate recursion relations still exhibit Gaussian behaviour, as in (1.5). The recursion relations of interest arise, as we will explain below in Section 1.4, in applying the lace expansion to self-avoiding walks or oriented percolation. The lace expansion produces a recursion relation of the form (1.1), with $g_1(k;z) = z\hat{D}(k)$, and with explicit expressions for the functions g_m and e_n for $m \geq 2$ and $n \geq 1$. For oriented percolation, g_n and e_n are almost identical, whereas for self-avoiding walk $e_n = 0$ for all n. A small parameter is introduced into the models, with the result that g_m and e_n are small for $m \geq 2$ and $n \geq 1$. They are small uniformly in m, as well as in terms of their rapid decay as m becomes large. The recursion relation (1.1) can thus be considered to be a small perturbation of the simple equation (1.4). The recursion turns out to have a critical value z_c, and, as we have seen above for ordinary random walk, we will set $z = z_c$ to observe Gaussian behaviour.

We analyse the recursion relation (1.1) using induction on n. Our method is a generalisation of the inductive method introduced by den Hollander and the authors [13] to study weakly self-avoiding walk for $d > 4$. Here, we are considering long-range or "spread-out" models. This necessitates a number of modifications to the analysis of [13], as the small parameter is less explicit for the spread-out model. In addition, the analysis of [13] has been simplified in some respects. Our choice of induction hypotheses has been strongly motivated by those used in [13], which in turn were motivated by earlier work. The inductive approach leads to more detailed results than have been obtained previously for self-avoiding walk or oriented percolation using generating functions. In a different vein, it has also been adapted to prove ballistic behaviour for weakly self-avoiding walk for $d = 1$ [12].

We will prove Gaussian behaviour of f_n under certain assumptions on e_n and g_n, as well as some modest assumptions on f_n. Since the assumptions on e_n and g_n are motivated by the lace expansion, we will give a brief review of relevant aspects of the lace expansion, in Section 1.4, to provide some necessary context. Our assumptions are shown elsewhere to apply for sufficiently spread-out models of self-avoiding walks on $\mathbb{Z}^d$ [14] and critical oriented bond percolation on $\mathbb{Z}^d\times\mathbb{Z}_+$ [7], in dimensions $d > 4$. Our results provide an essential ingredient in both cases, and form the basis for a proof of Gaussian behaviour of sufficiently spread-out polymer networks for $d > 4$, and for a proof that the moment measures of sufficiently spread-out critical oriented percolation converge to those of super-Brownian motion for $d > 4$. In this paper, we extract an important model-independent aspect of the proof of these results. The precise form of the functions e_n and g_n is model-dependent, and the verification of the assumptions on them does depend on the model.

1.2 Assumptions on the recursion relation

Before stating the assumptions, we first introduce some notation. For $x = (x_1, \dots, x_d)$ in $\mathbb{Z}^d$ or $\mathbb{R}^d$, we write $\|x\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ for $p \in [1, \infty)$, and $\|x\|_\infty = \max_{i\in\{1,\dots,d\}} |x_i|$. For a function h on $[-\pi, \pi]^d$, we write $\|h\|_p = [(2\pi)^{-d} \int_{[-\pi,\pi]^d} |h(k)|^p d^dk]^{1/p}$ and use $\|h\|_\infty$ to denote the essential supremum of h. Similarly, for a function H on $\mathbb{Z}^d$, we write $\|H\|_p = [\sum_{x\in\mathbb{Z}^d} |H(x)|^p]^{1/p}$ and $\|H\|_\infty = \sup_x |H(x)|$.

Throughout this paper, we make the following four assumptions, Assumptions S, D, E, G, on the quantities appearing in the recursion equation (1.1).

The first assumption, Assumption S, requires that the functions appearing in (1.1) respect the lattice symmetries, and that f_n remains bounded in a weak sense.

Assumption S. For every $n \in \mathbb{N}$ and $z > 0$, the mapping $k \mapsto f_n(k; z)$ is symmetric under replacement of any component k_i of k by $-k_i$, and under permutations of the components of k. The same holds for $e_n(\cdot; z)$ and $g_n(\cdot; z)$. In addition, for each n, $|f_n(k; z)|$ is bounded uniformly in $k \in [-\pi, \pi]^d$ and z in a neighbourhood of 1 (which may depend on n).

Our next assumption incorporates a "spread-out" aspect to the recursion equation. It introduces a function D which defines the underlying random walk model, about which (1.1) is a perturbation. The assumption involves a non-negative parameter L, which will be taken to be large, and which serves to spread out the steps of the random walk over a large set. We write $D = D_L$ in the statement of the assumption to emphasise this dependence, but the subscript will not be retained elsewhere. An example of a family of D's obeying the assumption is given following its statement. The assumption implies, in particular, that D has a finite second moment, and we define

$$\sigma^2 = -\nabla^2 \hat{D}(0). \tag{1.6}$$

Our assumptions will involve a parameter d, which corresponds to the spatial dimension in our applications. Our theorems will assume that $d > 4$. Error terms will involve an arbitrarily chosen $\delta \in (0, 1 \wedge \frac{d-4}{2})$, which should be thought of as being close to $1 \wedge \frac{d-4}{2}$. We will write

$$a(k) = 1 - \hat{D}(k). \tag{1.7}$$

Assumption D. Let D_L be a non-negative function on $\mathbb{Z}^d$. We assume that

$$f_1(k; z) = z\hat{D}_L(k), \quad e_1(k; z) = 0. \tag{1.8}$$

In particular, this implies that $g_1(k;z) = z\hat{D}_L(k)$. The function D_L is assumed to be normalised, so that $\hat{D}_L(0) = 1$, and to have $2+2\epsilon$ moments, for some $\epsilon > \delta$, i.e.,

$$\sum_{x\in\mathbb{Z}^d} |x|^{2+2\epsilon} D_L(x) < \infty. \tag{1.9}$$

We assume that there is a constant C such that

$$\|D_L\|_\infty \leq CL^{-d}, \qquad \sigma^2 = \sigma_L^2 \leq CL^2, \tag{1.10}$$

for all $L \geq 1$. In addition, we assume that there exist constants $\eta, c_1, c_2 > 0$ such that

$$c_1 L^2 k^2 \leq a_L(k) \leq c_2 L^2 k^2 \quad (\|k\|_\infty \leq L^{-1}), \tag{1.11}$$

$$a_L(k) > \eta \quad (\|k\|_\infty \geq L^{-1}), \tag{1.12}$$

$$a_L(k) < 2 - \eta \quad (k \in [-\pi,\pi]^d). \tag{1.13}$$

Example. Let h be a non-negative bounded function on $\mathbb{R}^d$ which is almost everywhere continuous and symmetric under the lattice symmetries. Assume that there is a function H on $\mathbb{R}^d$ with $H(te)$ non-increasing in $t \geq 0$ for every unit vector $e \in \mathbb{R}^d$, such that $h(x) \leq H(x)$ for all $x \in \mathbb{R}^d$. Assume that $\int_{\mathbb{R}^d} h(x)d^dx = 1$ and $\int_{\mathbb{R}^d} |x|^{2+2\epsilon} h(x)d^dx < \infty$ for some $\epsilon > \delta$. The monotonicity hypothesis implies that $\sum_{x\in\mathbb{Z}^d} h(x/L) < \infty$ for all L. We will verify in Appendix A that the function

$$D_L(x) = \frac{h(x/L)}{\sum_{x\in\mathbb{Z}^d} h(x/L)} \tag{1.14}$$

obeys the conditions of Assumption D, when L is large enough. A good example is given by the function $h(x) = 2^{-d}$ for $x \in [-1,1]^d$, $h(x) = 0$ otherwise. In this case, D_L is uniform on the cube $[-L,L]^d \cap \mathbb{Z}^d$.

We will take L large, yielding a small parameter

$$\beta = L^{-d}. \tag{1.15}$$

Each of e_n, f_n and g_n depend on L, but we do not make this dependence explicit in the notation.

The particular coefficients e_m and g_m arising in lace expansion analyses have Feynman diagram representations, as we discuss further in Section 1.4. These Feynman diagrams can be bounded in terms of f_j, for $j < m$. This *strict* inequality $j < m$ means that the right side of (1.1) can be studied in terms of f_m for $m < n+1$. This mechanism drives the induction.

Our next two assumptions summarise the manner in which bounds on f_m imply bounds on e_n and g_n. Verification of these assumptions requires a model-dependent analysis, which is carried out for self-avoiding walks in [14] and for oriented percolation in [7]. A brief indication of how this is done is given in Section 1.4. However, given the assumptions, the remaining inductive analysis, which we present in this paper, is model-independent.

The relevant bounds on f_m, which *a priori* may or may not be satisfied, are

$$\|\hat{D}^2 f_m(\cdot;z)\|_1 \le K\beta m^{-d/2}, \quad |f_m(0;z)| \le K, \quad |\nabla^2 f_m(0;z)| \le K\sigma^2 m, \tag{1.16}$$

for some positive constant K. The factor $\hat{D}(k)^2$ appearing in the first bound can be understood from the fact that $\|\hat{D}^2\|_1$ is the probability that a random walk taking steps with step distribution D returns to the origin after two steps, which is bounded by $\|D\|_\infty \le \mathcal{O}(\beta)$. Because of this, the factor $\hat{D}(k)^2$ is helpful in extracting the small factor β appearing in the upper bound.

Assumption E. Suppose that there is a $K > 1$ and an interval $I \subset [1-\alpha, 1+\alpha]$, for some $\alpha \in (0,1)$, such that the bounds (1.16) hold for $z \in I$ and for m with $1 \le m \le n$. Then, there exist $C_e = C_e(K) < \infty$ and $L_0 > 0$ such that for all $L \ge L_0$, $z \in I$, $k \in [-\pi,\pi]^d$ and $2 \le m \le n+1$, the following bounds hold:

$$|e_m(k;z)| \le C_e(K)\beta m^{-d/2}, \quad |e_m(k;z) - e_m(0;z)| \le C_e(K) a(k) \beta m^{-(d-2)/2}.$$

Assumption G. Suppose that there is a $K > 1$ and an interval $I \subset [1-\alpha, 1+\alpha]$, for some $\alpha \in (0,1)$, such that the bounds (1.16) hold for $z \in I$ and for m with $1 \le m \le n$. Then, there exist $C_g = C_g(K) < \infty$ and $L_0 > 0$ such that for all $L \ge L_0$, $z \in I$, $k \in [-\pi,\pi]^d$ and $2 \le m \le n+1$, the following bounds hold:

$$|g_m(k;z)| \le C_g(K)\beta m^{-d/2}, \quad |\nabla^2 g_m(0;z)| \le C_g(K)\sigma^2\beta m^{-(d-2)/2},$$
$$|\partial_z g_m(0;z)| \le C_g(K)\beta m^{-(d-2)/2},$$
$$|g_m(k;z) - g_m(0;z) - a(k)\sigma^{-2}\nabla^2 g_m(0;z)| \le C_g(K)\beta a(k)^{1+\epsilon'} m^{-(d-2-2\epsilon')/2},$$

with the last bound valid for any $\epsilon' \in [0,\epsilon]$, with ϵ given in (1.9).

We emphasize that it is the assumption that bounds on f_m for $m \le n$ imply bounds on g_m and e_m for $m \le n+1$ that permits an inductive analysis.

1.3 Main results

We assume henceforth that Assumptions S, D, E and G all hold. Our first theorem shows that there is a value z_c for which f_n behaves as a Gaussian. Recall that σ was defined in (1.6).

Theorem 1.1. *Let $d > 4$ and fix an arbitrary $\delta \in (0, 1 \wedge \frac{d-4}{4})$. There exist positive $L_0 = L_0(d)$, $z_c = z_c(d,L)$, $A = A(d,L)$, and $v = v(d,L)$, such that for $L \ge L_0$,*
(a)

$$f_n\Big(\frac{k}{\sqrt{v\sigma^2 n}}; z_c\Big) = A e^{-\frac{k^2}{2d}}[1 + \mathcal{O}(k^2 n^{-\delta}) + \mathcal{O}(n^{-(d-4)/2})], \tag{1.17}$$

(b)

$$-\frac{\nabla^2 f_n(0;z_c)}{f_n(0;z_c)} = v\sigma^2 n[1 + \mathcal{O}(\beta n^{-\delta})], \tag{1.18}$$

(c)

$$\|\hat{D}^2 f_n(\cdot;z_c)\|_1 \le const. \beta n^{-d/2}. \tag{1.19}$$

The error estimate in (a) is uniform in $\{k \in \mathbb{R}^d : a(k/\sqrt{v\sigma^2 n}) \le \gamma n^{-1}\log n\}$, for γ sufficiently small.

Part (b) is an expression of diffusive behaviour. Part (c) is the first bound of (1.16), and the factor $n^{-d/2}$ is associated with the probability of return to the origin of simple random walk after n steps.

In the proof of Theorem 1.1, we will establish the bounds of (1.16) for all $m \in \mathbb{N}$, with z in an m-dependent interval containing z_c. Consequently, all bounds appearing in Assumptions E and G follow as a Corollary, for $z = z_c$ and all $m \in \mathbb{N}$.

We will also show that z_c, A and v obey the equations

$$\begin{aligned} 1 &= \sum_{m=1}^{\infty} g_m(0; z_c), && (1.20) \\ A &= \frac{1 + \sum_{m=1}^{\infty} e_m(0; z_c)}{\sum_{m=1}^{\infty} m g_m(0; z_c)}, && (1.21) \\ v &= -\frac{\sum_{m=1}^{\infty} \nabla^2 g_m(0; z_c)}{\sigma^2 \sum_{m=1}^{\infty} m g_m(0; z_c)}. && (1.22) \end{aligned}$$

It then follows immediately from the bounds of Assumptions E and G that

$$z_c = 1 + \mathcal{O}(\beta), \quad A = 1 + \mathcal{O}(\beta), \quad v = 1 + \mathcal{O}(\beta). \tag{1.23}$$

Equation (1.20) states that at the critical value z_c of the recursion equation (1.1), with $k = 0$, the coefficients sum up to 1 as in renewal theory. However, unlike the standard situation in renewal theory, here the coefficients g_m depend on an additional variable k, and are not assumed to be non-negative. For the case of spread-out self-avoiding walk, the above bound for z_c improves on the error estimates of [8, 16, 20].

With modest additional assumptions, the critical point z_c can be characterised in terms of the *susceptibility*

$$\chi(z) = \sum_{n=0}^{\infty} f_n(0; z). \tag{1.24}$$

Theorem 1.2. *Suppose that $d > 4$ and that L is sufficiently large. Suppose there is a $p_c > 0$ such that the susceptibility is finite and positive for $z \in (0, p_c)$, with $\lim_{z \uparrow p_c} \chi(z) = \infty$. Suppose also that the bounds of (1.16) for $z = z_c$ and all $m \geq 1$ imply the bounds of Assumptions E and G for all $m \geq 2$, uniformly in $z \in [0, z_c]$. Then $z_c = p_c$.*

Next, we state a local central limit theorem. Let

$$p_n(x; z) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} f_n(k; z) \tag{1.25}$$

denote the inverse Fourier transform of $f_n(k; z)$. The pointwise form

$$p_n(x\sqrt{v\sigma^2 n}; z_c) \approx A \left(\frac{d}{2\pi n v \sigma^2} \right)^{d/2} e^{-dx^2/2} \tag{1.26}$$

of the local central limit theorem cannot hold for all solutions of the recursion (1.1). For example, for self-avoiding walks starting at the origin, $p_n(0; z) = 0$ for every $n \geq 1$ (see Section 1.4.1). This local effect will disappear only if we average over a region that grows with n. For the averaging, we denote the cube of radius R centred at $x \in \mathbb{Z}^d$ by

$$C_R(x) = \{y \in \mathbb{Z}^d : \|x - y\|_\infty \leq R\}. \tag{1.27}$$

We use $\lfloor x \rfloor$ to denote the closest lattice point in $\mathbb{Z}^d$ to $x \in \mathbb{R}^d$ (with an arbitrary rule to break ties), and write $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$.

Theorem 1.3. *Let $d > 4$, and let R_n be any sequence with $\lim_{n\to\infty} R_n = \infty$ and $R_n = o(n^{1/2})$. Then for all $x \in \mathbb{R}^d$ with $x^2[\log R_n]^{-1}$ sufficiently small,*

$$\frac{1}{(2R_n + 1)^d} \sum_{y \in C_{R_n}(\lfloor x\sqrt{v\sigma^2 n}\rfloor)} p_n(y; z_c) \sim A \left(\frac{d}{2\pi n v\sigma^2}\right)^{d/2} e^{-dx^2/2}. \tag{1.28}$$

An explicit error bound for (1.28) in terms of R and n is given in (4.19). Note that (1.28) is not a direct consequence of the convergence of the Fourier transform in Theorem 1.1. The statement that sums over sets of volume $n^{d/2}$ converge to integrals of the Gaussian density over the scaled set follows from Theorem 1.1. However, since we allow arbitrarily slow growth of R_n in Theorem 1.3, we are investigating p_n on a smaller scale.

1.4 Self-avoiding walks and oriented percolation

In this section, we discuss two examples where our assumptions, and hence our main results, apply. The examples are spread-out self-avoiding walks on $\mathbb{Z}^d$, and critical spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$, in dimensions $d > 4$. Both models are based on a function $D = D_L$ obeying the assumptions of Assumption D. Assumption S is immediate for both models. We have two goals in this section. The first goal to provide a short sketch of the lace expansion, in order to indicate how a recursion relation of the form (1.1) arises. The second goal is to give some motivation for Assumptions E and G, as well as for our main results.

1.4.1 Self-avoiding walks

An n-step walk is a mapping $\omega : \{0, \dots, n\} \to \mathbb{Z}^d$. An n-step self-avoiding walk is an n-step walk with $\omega(i) \neq \omega(j)$ for all $i \neq j$. Let $\mathcal{R}_n(x)$ denote the set of n-step walks with $\omega(0) = 0$ and $\omega(n) = x$, and let $\mathcal{C}_n(x)$ denote the subset of $\mathcal{R}_n(x)$ consisting of n-step self-avoiding walks. For $\omega \in \mathcal{R}_n(x)$, let

$$W(\omega) = \prod_{i=1}^{n} D(\omega(i) - \omega(i-1)). \tag{1.29}$$

We define $c_0(x) = \delta_{0,x}$, and, for $n \geq 1$,

$$c_n(x) = \sum_{\omega \in \mathcal{C}_n(x)} W(\omega). \tag{1.30}$$

Without loss of generality, we now require that $D(0) = 0$.

If the sum in (1.30) were over $\mathcal{R}_n(x)$, rather than $\mathcal{C}_n(x)$, we would have the recursion $c_{n+1}(x) = \sum_{y\in\mathbb{Z}^d} c_1(y)c_n(x-y)$. However, the right side of this equation includes contributions from walks which visit the origin twice, and these are not present in $c_{n+1}(x)$. There should therefore be a correction term $F^{(1)}_{n+1}(x)$, defined by the equation

$$c_{n+1}(x) = \sum_{y\in\mathbb{Z}^d} c_1(y)c_n(x-y) - F^{(1)}_{n+1}(x). \tag{1.31}$$

The lace expansion is an expansion of $F^{(1)}_{n+1}(x)$ that treats self-avoiding walk as a small perturbation of ordinary random walk for $d > 4$. It was originally introduced in [5] using a graph theoretic notion of "lace." Here, we follow instead the equivalent inclusion-exclusion approach of [23] (see also [16]), for which the laces do not explicitly appear.

To analyse the correction term $F^{(1)}_{n+1}(x)$, we define $\mathcal{P}_{n+1}(x)$ to be the set of $\omega \in \mathcal{R}_{n+1}(x)$ for which there exists an $l \in \{2, \ldots, n+1\}$ (depending on ω) with $\omega(l) = 0$ and $\omega(i) \neq \omega(j)$ for all $i \neq j$ with $\{i,j\} \neq \{0,l\}$. For the special case $x = 0$, $\mathcal{P}_{n+1}(0)$ is the set of $(n+1)$-step self-avoiding polygons. For general x, $\mathcal{P}_{n+1}(x)$ is the set of self-avoiding polygons followed by a self-avoiding walk from 0 to x, with the total length being $n+1$ and with the walk and polygon mutually avoiding. The set $\mathcal{P}_{n+1}(x)$ is exactly the set of walks that contribute to $\sum_{y\in\mathbb{Z}^d} c_1(y)c_n(x-y)$ but do not contribute to $c_{n+1}(x)$. Therefore

$$F^{(1)}_{n+1}(x) = \sum_{\omega\in\mathcal{P}_{n+1}(x)} W(x). \tag{1.32}$$

Equation (1.31) can then be understood as just the inclusion-exclusion relation: The first term on the right side includes all walks from 0 to x which are self-avoiding *after* the first step, and the second subtracts the contribution due to those which are not self-avoiding from the beginning, i.e., walks that return to the origin.

The inclusion-exclusion relation can now be applied to $F^{(1)}_{n+1}(x)$. We neglect the mutual avoidance of the polygon and self-avoiding walk portions of an $\omega \in \mathcal{P}_{n+1}(x)$, and then correct by excluding the configurations which included intersections of these two portions. For $y \in \mathbb{Z}^d$, let

$$\pi^{(1)}_m(y) = \delta_{0,y} \sum_{\omega\in\mathcal{P}_m(0)} W(\omega). \tag{1.33}$$

Then we define $F^{(2)}_{n+1}(x)$ by

$$F^{(1)}_{n+1}(x) = \sum_{y\in\mathbb{Z}^d} \sum_{m=2}^{n+1} \pi^{(1)}_m(y)c_{n+1-m}(x-y) - F^{(2)}_{n+1}(x). \tag{1.34}$$

The correction term $F^{(2)}_{n+1}(x)$ involves configurations consisting of a self-avoiding polygon and a self-avoiding walk from 0 to x, of total length $n+1$, and with an intersection required between the self-avoiding polygon and the self-avoiding walk (in addition to their intersection at the origin). If we define $\Theta_{n+1}(x)$ to be the corresponding subset of $\mathcal{R}_{n+1}(x)$, then we have

$$F^{(2)}_{n+1}(x) = \sum_{\omega\in\Theta_{n+1}(x)} W(\omega). \tag{1.35}$$

The walk and polygon may intersect more than once, and we focus on the first time an intersection occurs, measuring time according to the walk. We then perform inclusion-exclusion again, neglecting the avoidance between the portions of the self-avoiding walk before and after this first intersection, and then subtracting a correction term. The process is continued indefinitely. We define

$$\pi_m^{(2)}(y) = \sum_{\substack{m_1, m_2, m_3 \geq 1 \\ m_1 + m_2 + m_3 = m}} \prod_{j=1}^{3} \sum_{\omega_j \in \mathcal{C}_{m_j}(y)} W(\omega_i) I(\omega_1, \omega_2, \omega_3), \tag{1.36}$$

where $I(\omega_1, \omega_2, \omega_3)$ is equal to 1 if the ω_i are pairwise mutually avoiding apart from their common endpoints, and otherwise equals 0. Then we are led to

$$c_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} c_1(y) c_n(x-y) + \sum_{y \in \mathbb{Z}^d} \sum_{m=2}^{n+1} \pi_m(y) c_{n+1-m}(x-y), \tag{1.37}$$

with

$$\pi_m(y) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(y), \tag{1.38}$$

with expressions for the $\pi^{(N)}$, for $N \geq 3$, which we do not make explicit here. The $\pi^{(N)}$ have convenient representations in terms of Feynman diagrams [5, 16].

By definition, $c_1(x) = D(x)$. Using this in the first term of (1.37), we then multiply through by z^{n+1}, with $z \geq 0$, and take the Fourier transform to obtain

$$\hat{c}_{n+1}(k) z^{n+1} = z\hat{D}(k)\hat{c}_n(k) z^n + \sum_{m=2}^{n+1} \hat{\pi}_m(k) z^m \hat{c}_{n+1-m}(k) z^{n+1-m}. \tag{1.39}$$

Equation (1.39) is a special case of (1.1), with $e_n = 0$ and

$$f_n(k;z) = \hat{c}_n(k) z^n \quad (n \geq 0), \qquad g_1(k;z) = z\hat{D}(k), \qquad g_n(k;z) = \hat{\pi}_n(k) z^n \quad (n \geq 2). \tag{1.40}$$

Assumption S is a consequence of the lattice symmetry of self-avoiding walks and the fact that $\hat{c}_n(k) \leq 1$. We are already assuming that D obeys the requirements of Assumption D, so Assumption D is immediate from (1.40). Assumption E is vacuous, and the remaining assumption, Assumption G, is the substantial one.

An analysis corresponding to the verification of some version of Assumption G is present in one form or another in all lace expansion analyses [5, 6, 9, 10, 11, 13, 15, 16, 18, 22]. In [14], we prove that Assumption G is satisfied, as part of a larger analysis proving Gaussian behaviour for networks of mutually-avoiding spread-out self-avoiding walks in dimensions $d > 4$.

To give an indication how Assumption G can be verified, we briefly discuss this for the bound on g_m arising from the contribution due to $\hat{\pi}_m^{(2)}$. For this, we simply neglect the mutual avoidance of the ω_j in (1.36) to obtain an upper bound

$$|\hat{\pi}_m^{(2)}(k)| \leq \sum_x \pi_m^{(2)}(x) \leq \sum_x \sum_{\substack{m_1, m_2, m_3 \geq 1 \\ m_1 + m_2 + m_3 = m}} \prod_{j=1}^{3} c_{m_j}(x). \tag{1.41}$$

This succeeds in bounding $\hat{\pi}_m^{(2)}$ in terms of c_l with $l < m$, and the verification of Assumption G can then be carried out, as explained in detail in [14]. The term $\hat{\pi}_m^{(1)}$ is simpler, and higher order terms produce powers of β to ensure convergence of the sum over N in (1.38). The restriction $d > 4$ arises in bounding expressions like the right side of (1.41), using (1.16).

As a consequence of the verification of Assumption G in [14], Theorems 1.1 and 1.3 hold for $f_n(k; z_c) = \hat{c}_n(k) z_c^n$, with z_c, A and D given by Equations (1.20)–(1.22). Theorem 1.1(a,b) were proved in [16] with weaker error bounds, via a different approach using generating functions. Theorem 1.3 is the first time a local central limit theorem has been proved for the self-avoiding walk. The limit $\lim_{n\to 0} \hat{c}_n(0)^{1/n}$ exists by a well-known subadditivity argument [16]. Thus the critical value z_c appearing in Theorem 1.1(a) must be given by $z_c = \lim_{n\to 0} \hat{c}_n(0)^{-1/n}$, consistent with the conclusion of Theorem 1.2. For uniform D, z_c is thus the reciprocal of the connective constant.

The first bound assumed in (1.16), which asserts that $\|\hat{D}^2 \hat{c}_m\|_1 \leq \mathcal{O}(\beta z_c^{-m} m^{-d/2})$, actually implies a bound on $\|c_m\|_\infty$. In fact, if $m \geq 2$, by ignoring the self-avoidance constraint for the first two steps we have

$$\|c_m\|_\infty \leq \|D * D * c_{m-2}\|_\infty \leq \|\hat{D}^2 \hat{c}_{m-2}\|_1. \tag{1.42}$$

Since $z_c^{-1} \leq \mathcal{O}(1)$ by (1.23), the first bound of (1.16) implies that

$$\|c_m\|_\infty \leq \mathcal{O}(\beta z_c^{-m} m^{-d/2}). \tag{1.43}$$

The above argument gives (1.43) when $m \geq 2$, but (1.43) clearly also holds for $m = 1$ since $\|c_1\|_\infty = \|D\|_\infty \leq \mathcal{O}(\beta)$ and z_c^{-1} is bounded away from zero. The inequality (1.43) is useful to estimate (1.41). After having verified Assumption G, the bound $\|\hat{D}^2 \hat{c}_m\|_1 \leq \mathcal{O}(\beta z_c^{-m} m^{-d/2})$ then follows from Theorem 1.1(c). We conclude that (1.43), which is of independent interest, does in fact hold for all m. An alternate proof of this bound, using generating functions and a finite memory cut-off, is given in [16] for the uniform D.

1.4.2 Oriented percolation

Our second example is spread-out oriented percolation. The setting is the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and directed bonds $((x, n), (y, n+1))$, for $n \geq 0$ and $x, y \in \mathbb{Z}^d$. Let $z \in [0, \|D\|_\infty^{-1}]$, so that $zD(y-x) \leq 1$. We associate to each directed bond $((x, n), (y, n+1))$ an independent random variable taking the value 1 with probability $zD(y-x)$ and 0 with probability $1 - zD(y-x)$. We say a bond is *occupied* when the corresponding random variable takes the value 1, and *vacant* when the random variable is 0. The joint probability distribution of the bond variables will be denoted $\mathbb{P}_z$. We say that (x, n) is *connected* to (y, m), and write $(x, n) \to (y, m)$, when there is an oriented path from (x, n) to (y, m) consisting of occupied bonds, or if $(x, n) = (y, m)$. Given a bond configuration, we define a bond to be *pivotal* for $(x, n) \to (y, m)$ if $(x, n) \to (y, m)$ when this bond is made occupied, whereas $(x, n) \not\to (y, m)$ when this bond is made vacant. Finally, (x, n) is *doubly connected* to (y, m), denoted $(x, n) \Rightarrow (y, m)$ when $x \to y$ but there is no pivotal bond for the connection from (x, n) to (y, m). We write $(x, n) \not\Rightarrow (y, m)$ for the complement of $(x, n) \Rightarrow (y, m)$.

Given $z \in [0, \|D\|_\infty^{-1}]$, $n \geq 0$ and $x \in \mathbb{Z}^d$, we define the two-point function

$$\tau_n(x; z) = \mathbb{P}_z((0, 0) \to (x, n)). \tag{1.44}$$

The lace expansion is an expansion for the two-point function. It was derived specifically in the context of oriented percolation by Nguyen and Yang in [18, 19], using laces as in [5]. An expansion for ordinary percolation based on inclusion-exclusion was derived in [9], which applies also to oriented percolation and which is utilised in [7]. Here, we briefly outline a derivation of the Nguyen–Yang expansion using inclusion-exclusion. This approach, which is different than the inclusion-exclusion approach of [9], is developed also in [21].

Given a configuration in which $(0,0) \to (x,n)$, either $(0,0) \Rightarrow (x,n)$ or $(0,0) \not\Rightarrow (x,n)$. For $n \geq 0$, let

$$\rho_n^{(0)}(x;z) = \mathbb{P}_z((0,0) \Rightarrow (x,n)), \tag{1.45}$$
$$\sigma_n^{(0)}(x;z) = \mathbb{P}_z(\{(0,0) \to (x,n)\} \cap \{(0,0) \not\Rightarrow (x,n)\}). \tag{1.46}$$

Then

$$\tau_{n+1}(x;z) = \rho_{n+1}^{(0)}(x;z) + \sigma_{n+1}^{(0)}(x;z) \quad (n \geq 0). \tag{1.47}$$

A configuration contributing to $\sigma_{n+1}^{(0)}(x;z)$ contains at least one pivotal bond, and hence a first pivotal bond $((u,m),(v,m+1))$. Let E denote the event that $(0,0) \Rightarrow (u,m)$, that $((u,m),(v,m+1))$ is occupied, and that $(v,m+1) \to (x,n+1)$. Let F be the event that the bond $((u,m),(v,m+1))$ is pivotal for $(0,0) \to (x,n+1)$. Then

$$\sigma_{n+1}^{(0)}(x;z) = \sum_{u,v} \sum_{m=0}^{n-1} \mathbb{P}_z(E \cap F). \tag{1.48}$$

We use inclusion-exclusion to write $\mathbb{P}_z(E \cap F) = \mathbb{P}_z(E) - \mathbb{P}_z(E \cap F^c)$. Oriented percolation enjoys a Markov property (unlike non-oriented percolation) which implies that

$$\mathbb{P}_z(E) = \rho_m^{(0)}(u;z)\, zD(v-u)\, \tau_{n-m}(x-v;z). \tag{1.49}$$

Thus we have

$$\tau_{n+1}(x;z) = \rho_{n+1}^{(0)}(x;z) + \sum_{u,v} \sum_{m=0}^{n-1} \rho_m^{(0)}(u;z)\, zD(v-u)\, \tau_{n-m}(x-v;z) - \sigma_{n+1}^{(1)}(x;z), \tag{1.50}$$

where

$$\sigma_{n+1}^{(1)}(x;z) = \sum_{u,v} \sum_{m=0}^{n-1} \mathbb{P}_z(E \cap F^c). \tag{1.51}$$

For a configuration contributing to $E \cap F^c$, the first pivotal bond for $(0,0) \to (x,n+1)$, if there is one, will by definition occur later than time m. We then perform inclusion-exclusion with respect to the pivotal nature of this first pivotal bond, as above. The procedure is repeated indefinitely. We pay special attention to the fact that a site is always doubly connected to itself, so that $\rho_0(x;z) = \delta_{0,x}$, and we define $\pi_m^{(0)}(x;z) = \rho_m^{(0)}(x;z)$ for $m \geq 2$, and $\pi_m^{(0)}(x;z) = 0$ for $m = 0,1$.

Repeating the inclusion-exclusion procedure outlined above then leads, after taking the Fourier transform, to

$$\hat{\tau}_{n+1}(k;z) = z\hat{D}(k)\tau_n(k;z) + z\hat{D}(k)\sum_{m=2}^{n}\hat{\pi}_m(k;z)\hat{\tau}_{n-m}(k;z) + \hat{\pi}_{n+1}(k;z), \quad \hat{\tau}_0(0;z) = 1. \tag{1.52}$$

The function $\pi_m(x;z)$ is given by an explicit alternating series [7, 18]

$$\pi_m(x;z) = \sum_{N=0}^{\infty}(-1)^N \pi_m^{(N)}(x;z). \tag{1.53}$$

Equation (1.52) is a special case of (1.1) with the choices

$$f_n(k;z) = \hat{\tau}_n(k;z), \quad e_n(k;z) = \hat{\pi}_n(k;z), \tag{1.54}$$

and

$$g_1(k;z) = z\hat{D}(k), \quad g_2(k;z) = 0, \quad g_m(k;z) = z\hat{D}(k)\hat{\pi}_{m-1}(k;z) \quad (m \geq 3). \tag{1.55}$$

For oriented percolation, Assumption S follows from the lattice symmetry of the model and the fact that $\hat{\tau}_n(k;z) \leq (2n+1)^d$. Assumption D follows from (1.54)–(1.55) and the fact that $\hat{\pi}_1(k;z) = 0$. Assumptions E and G are very closely related. They can be verified, for $d > 4$ and L sufficiently large, by using the BK inequality to bound $\pi^{(N)}$ in terms of the two-point function itself. For example, the BK inequality implies that $\pi_m^{(0)}(x;z) \leq \tau_m(x;z)^2$ $(m \geq 2)$. This is carried out in detail in [7], where the verification of the assumptions is an essential part of a larger analysis relating the scaling limit of critical oriented percolation for $d > 4$ to super-Brownian motion.

Given the result of [7] that Assumptions E and G do hold, the conclusions of Theorems 1.1 and 1.3 therefore apply at $z = z_c$. Theorems 1.1(a,b) were proved in [19], with weaker error bounds. Theorems 1.1(c) and 1.3 are new, and the former plays an essential role in the analysis of [7]. As was the case for self-avoiding walks in (1.43), it is possible to conclude from the assumed bounds (1.16) that

$$\|\tau_n(\cdot;z_c)\|_\infty \leq CK\beta n^{-d/2}. \tag{1.56}$$

See [7] for details.

The hypotheses of Theorem 1.2 concerning the susceptibility are well-known [2], and the hypothesis concerning Assumptions E and G is established in [7]. It therefore follows that z_c corresponds to the critical oriented percolation threshold. This can be understood directly from the conclusion of Theorem 1.1(a), which asserts in particular that $\lim_{n\to\infty}\hat{\tau}_n(0;z_c) = A$, with A positive and finite. The quantity $\hat{\tau}_n(0;z)$ represents the expected number of sites in the intersection of the connected cluster of the origin with $\mathbb{Z}^d \times \{n\}$. This goes to zero for z below the percolation threshold, by exponential decay of connectivities [1, 2, 17]. On the other hand, for z above the percolation threshold, it can be expected that $\hat{\tau}_n(0;z) \to \infty$. This latter statement is a consequence of the shape theorem of [4]. The shape theorem has been proven only for the nearest-neighbour model, but it can be expected to hold also for the spread-out models.

The triangle condition [2, 3] is a diagrammatic sufficient condition for the existence of several critical exponents. This is discussed in detail in [18], where the triangle condition is verified, in particular, for sufficiently spread-out models of oriented percolation when $d+1 > 5$. The bound (1.56), together with the bound $\sum_x \tau_n(x;z_c) < \infty$ (which is now proved, as discussed above), can be combined to give an alternate proof of the triangle condition. This is discussed in detail in [7].

1.5 Organisation

The remainder of this paper is organised as follows. In Section 2, we introduce the induction hypotheses on f_n that will be used to prove our main results, and derive some consequences of the induction hypotheses. The induction is advanced in Section 3. In Section 4, the main results stated in Section 1.3 are proved. We conclude in Appendix A, by showing that the function D defined in (1.14) obeys the hypotheses of Assumption D.

2 Induction hypotheses

In this section, we introduce the induction hypotheses, verify that they hold for $n = 1$, discuss their motivation, and derive some of their consequences.

2.1 Statement of induction hypotheses (H1–H4)

The induction hypotheses involve a sequence v_n, which is defined as follows. We set $v_0 = b_0 = 1$, and for $n \geq 1$ we define

$$b_n = -\frac{1}{\sigma^2}\sum_{m=1}^{n} \nabla^2 g_m(0; z), \quad c_n = \sum_{m=1}^{n}(m-1)g_m(0; z), \quad v_n = \frac{b_n}{1+c_n}. \tag{2.1}$$

The z–dependence of b_n, c_n, v_n will usually be left implicit in the notation. We will often simplify the notation by dropping z also from e_n, f_n and g_n, and write, e.g., $f_n(k) = f_n(k; z)$.

As we will explain in more detail in Section 2.3, the diffusion constant $\sigma^2 v$ of Theorem 1.1 will turn out to be given by $\sigma^2 v_\infty(z_c)$. However, at this stage we have not yet proved that the series in the definition of v_∞ converges. Neither have we yet identified z_c, other than as a solution to (1.20), which involves a series whose convergence has not yet been proved.

The induction hypotheses also involve several constants. Let $d > 4$. We fix $\gamma, \delta, \rho > 0$ according to

$$0 < \tfrac{d-4}{2} - \rho < \gamma < \gamma + \delta < 1 \wedge \tfrac{d-4}{2}. \tag{2.2}$$

For example, we may first fix any $\delta \in (0, 1 \wedge \frac{d-4}{2})$ and then choose γ and ρ accordingly. We also introduce constants $K_1, \ldots, K_5$, which are independent of β.

Let $z_0 = z_1 = 1$, and define z_n recursively by

$$z_{n+1} = 1 - \sum_{m=2}^{n+1} g_m(0; z_n), \qquad n \geq 1. \tag{2.3}$$

For $n \geq 1$, we define intervals

$$I_n = [z_n - K_1\beta n^{-(d-2)/2}, z_n + K_1\beta n^{-(d-2)/2}]. \tag{2.4}$$

Recall the definition $a(k) = 1 - \hat{D}(k)$ from (1.7). Our induction hypotheses are that the following four statements hold for all $z \in I_n$ and all $1 \leq j \leq n$.

(H1) $|z_j - z_{j-1}| \leq K_1 \beta j^{-d/2}$.

(H2) $|v_j - v_{j-1}| \leq K_2 \beta j^{-(d-2)/2}$.

(H3) For k such that $a(k) \leq \gamma j^{-1} \log j$, $f_j(k;z)$ can be written in the form

$$f_j(k;z) = \prod_{i=1}^{j} \left[1 - v_i a(k) + r_i(k)\right],$$

with $r_i(k) = r_i(k;z)$ obeying

$$|r_i(0)| \leq K_3 \beta i^{-(d-2)/2}, \quad |r_i(k) - r_i(0)| \leq K_3 \beta a(k) i^{-\delta}.$$

(H4) For k such that $a(k) > \gamma j^{-1} \log j$, $f_j(k;z)$ obeys the bounds

$$|f_j(k;z)| \leq K_4 a(k)^{-2-\rho} j^{-d/2}, \quad |f_j(k;z) - f_{j-1}(k;z)| \leq K_5 a(k)^{-1-\rho} j^{-d/2}.$$

Note that, for $k = 0$, (H3) reduces to $f_j(0) = \prod_{i=1}^{j}[1 + r_i(0)]$.

We define

$$K_4' = \max\{C_e(cK_4), C_g(cK_4), K_4\}, \tag{2.5}$$

where c is a constant determined in Lemma 2.5 below. To advance the induction, we will need to assume that

$$K_3 \gg K_1 > K_4' \geq K_4 \gg 1, \quad K_2 \geq K_1, 2K_4', \quad K_5 \gg K_4. \tag{2.6}$$

Here $a \gg b$ denotes the statement that a/b is sufficiently large. The amount by which, for instance, K_3 must exceed K_1 is independent of β and will be determined during the course of the advancement of the induction in Section 3.

2.2 Initialisation of the induction

We now verify that the induction hypotheses hold when $n = 1$. Fix $z \in I_1$.

(H1) We simply have $z_1 - z_0 = 1 - 1 = 0$.

(H2) We simply have $|v_1 - v_0| = |z - 1|$, so that (H2) is satisfied provided $K_2 \geq K_1$.

(H3) We are restricted to $a(k) = 0$. By (1.11), this means $k = 0$. By Assumption D, $f_1(0;z) = z$, so that $r_1(0) = z - z_1$. Thus (H3) holds provided we take $K_3 \geq K_1$.

(H4) We note that $|f_1(k;z)| \leq z \leq 2$ for β sufficiently small, $|f_1(k;z) - f_0(k;z)| \leq 3$, and $a(k) \leq 2$. The bounds of (H4) therefore hold provided we take $K_4 \geq 2^{3+\rho}$ and $K_5 \geq 3 \cdot 2^{1+\rho}$.

2.3 Discussion of induction hypotheses

(H1) and the critical point. The critical point can be formally identified as follows. We set $k = 0$ in (1.1), then sum over n, and solve for the susceptibility $\chi(z)$ of (1.24). The result is

$$\chi(z) = \frac{1 + \sum_{m=2}^{\infty} e_m(0; z)}{1 - \sum_{m=1}^{\infty} g_m(0; z)}. \tag{2.7}$$

The critical point should correspond to the smallest zero of the denominator and hence should obey the equation

$$1 - \sum_{m=1}^{\infty} g_m(0; z_c) = 1 - z_c - \sum_{m=2}^{\infty} g_m(0; z_c) = 0. \tag{2.8}$$

However, we do not know *a priori* that the series in (2.7) or (2.8) converge. We therefore approximate (2.8) with the recursion (2.3), which bypasses the convergence issue by discarding the $g_m(0)$ for $m > n + 1$ that cannot be handled at the n^{th} stage of the induction argument. The sequence z_n will ultimately converge to z_c. Equation (2.8) is identical to (1.20).

In dealing with the sequence z_n, it is convenient to formulate the induction hypotheses for a small interval I_n approximating z_c. As we will see in Section 2.4, (H1) guarantees that the intervals I_j are decreasing: $I_1 \supset I_2 \supset \cdots \supset I_n$. Because the length of these intervals is shrinking to zero, their intersection $\cap_{j=1}^{\infty} I_j$ is a single point, namely z_c. Hypothesis (H1) drives the convergence of z_n to z_c and gives some control on the rate. The rate is determined from (2.3) and the ansatz that the difference $z_j - z_{j-1}$ is approximately $-g_{j+1}(0, z_c)$, with $|g_j(k; z_c)| = \mathcal{O}(\beta j^{-d/2})$ as in Assumption G.

(H2). The formula for v_n in (2.1) can be motivated by the following rough argument. Differentiating (1.1) twice with respect to k, setting $k = 0$, and using the fact that odd derivatives vanish by Assumption S, we obtain

$$\nabla^2 f_{n+1}(0) = \sum_{m=1}^{n+1} \left[g_m(0) \nabla^2 f_{n+1-m}(0) + \nabla^2 g_m(0) f_{n+1-m}(0) \right] + \nabla^2 e_{n+1}(0). \tag{2.9}$$

We will use '$\approx$' to denote an uncontrolled approximation in a rough argument. In (2.9), we make the approximations $f_{n+1-m}(0) \approx f_n(0)$ and $\nabla^2 e_{n+1} \approx 0$, subtract $\nabla^2 f_n(0)$ from both sides, and recall the definition of b_n from (2.1), to obtain

$$\nabla^2 f_{n+1}(0) - \nabla^2 f_n(0) \approx -\sigma^2 b_{n+1} f_n(0) + \sum_{m=1}^{n+1} g_m(0) \nabla^2 f_{n+1-m}(0) - \nabla^2 f_n(0). \tag{2.10}$$

In view of (2.3), we insert the approximation $1 \approx \sum_{m=1}^{n+1} g_m(0)$ to obtain

$$\nabla^2 f_{n+1}(0) - \nabla^2 f_n(0) \approx -\sigma^2 b_{n+1} f_n(0) + \sum_{m=1}^{n+1} g_m(0) \left[\nabla^2 f_{n+1-m}(0) - \nabla^2 f_n(0) \right]. \tag{2.11}$$

On the other hand, applying ∇^2 to (H3) with $j = n$ and then using $r_n(0) \approx 0$ and $\nabla^2 r_n(0) \approx 0$ gives

$$\nabla^2 f_n(0) - \nabla^2 f_{n-1}(0) \approx -\sigma^2 v_n f_{n-1}(0). \tag{2.12}$$

In view of this, we make the approximation $\nabla^2 f_{n+1-m}(0) - \nabla^2 f_n(0) \approx (m-1)\sigma^2 v_{n+1} f_n(0)$. Recalling the definition of c_{n+1} from (2.1), (2.11) then gives

$$\nabla^2 f_{n+1}(0) - \nabla^2 f_n(0) \approx -\sigma^2 (b_{n+1} - v_{n+1} c_{n+1}) f_n(0). \tag{2.13}$$

Putting the right-hand side equal to $-\sigma^2 v_{n+1} f_n(0)$, as in (2.12), leads to the formula for v_{n+1} of (2.1).

The assumed bound on $|v_j - v_{j-1}|$ can then be guessed by writing v_j and v_{j-1} in terms of the b_i and c_i, and assuming the bounds of Assumption G to estimate the resulting expression. The calculations will be carried out in detail when we advance (H2) in Section 3.2.

(H3). The bound on $r_i(0)$ ensures that the limit $A = \lim_{j\to\infty} f_j(0; z_c) = \prod_{i=1}^{\infty}[1 + r_i(0; z_c)]$ exists. The expression for f_j in (H3) can be approximately rewritten as

$$f_j(k) \approx f_j(0) \exp \Big[\textstyle\sum_{i=1}^{j} \Big(-v_i a(k) + r_i(k) - r_i(0) \Big) \Big].$$

The bound on $r_i(k) - r_i(0)$ indicates that it is a small perturbation of the leading term $-v_i a(k)$. The limit $v = \lim_{n\to\infty} v_n$ exists, if we assume the bounds on g_m of Assumption G. Also, $a(k) \sim \sigma^2 k^2/2d$ as $k \to 0$. Thus we can understand (H3) as a precise version of the approximation

$$f_j(k) \approx f_j(0) \exp \Big[-\frac{v\sigma^2 k^2 j}{2d} \Big], \tag{2.14}$$

which we expect to be valid at least for $\sigma^2 k^2 j$ of order 1. This is consistent with Theorem 1.1. For large j, the restriction $a(k) \leq \gamma j^{-1} \log j$ is essentially the restriction $\sigma^2 k^2 j \leq 2d\gamma \log j$, which includes the region where $\sigma^2 k^2 j$ is order 1, plus some additional room to manoeuvre.

(H4). For $\sigma^2 k^2 j > 2d\gamma \log j$, we require (and can prove) less accurate control of $f_j(k)$, as expressed in (H4). The form of (H4) has been chosen in part to be less stringent than (H3) for $a(k) = \gamma j^{-1} \log j$, where the transition from (H3) to (H4) takes place. In fact, inserting $\sigma^2 k^2/2d = \gamma j^{-1} \log j$ into (2.14) gives an expression which grows like $j^{-\gamma v}$. The bounds on g_m of Assumption G would imply that $z_c = 1 + \mathcal{O}(\beta)$ and $v = 1 + \mathcal{O}(\beta)$, so that $j^{-\gamma v} = j^{-\gamma(1+\mathcal{O}(\beta))}$. On the other hand, putting $a(k) = \gamma j^{-1} \log j$ in the first bound of (H4) gives a bound which grows like $j^{-(d-4)/2+\rho}$ times a power of a logarithm. By (2.2), this is a weaker bound than the (H3) bound.

2.4 Consequences of induction hypotheses

In this section we will derive some important consequences of the induction hypotheses. The key result is that the induction hypotheses imply (1.16) for all $1 \leq m \leq n$, from which the bounds of Assumptions E and G then follow, for $2 \leq m \leq n+1$.

Here, and throughout the rest of this paper,

> C denotes a strictly positive constant that may depend on d, γ, δ, ρ, but *not* on the K_i, *not* on k, *not* on n, and *not* on β (provided β is sufficiently small, possibly depending on the K_i). The value of C may change from line to line.

The first lemma shows that the intervals I_j are nested, assuming (H1).

Lemma 2.1. *Assume (H1) for $1 \leq j \leq n$. Then $I_1 \supset I_2 \supset \cdots \supset I_n$.*

Proof. Suppose $z \in I_j$, with $2 \le j \le n$. Then by (H1) and (2.4),

$$|z - z_{j-1}| \le |z - z_j| + |z_j - z_{j-1}| \le \frac{K_1\beta}{j^{(d-2)/2}} + \frac{K_1\beta}{j^{d/2}} \le \frac{K_1\beta}{(j-1)^{(d-2)/2}}, \tag{2.15}$$

and hence $z \in I_{j-1}$. □

By Lemma 2.1, if $z \in I_j$ for $1 \le j \le n$, then $z \in I_1$ and hence, by (2.4),

$$|z - 1| \le K_1\beta. \tag{2.16}$$

It also follows from (H2) that, for $z \in I_n$ and $1 \le j \le n$,

$$|v_j - 1| \le CK_2\beta. \tag{2.17}$$

The induction hypothesis (H3) has the useful alternate form

$$f_j(k) = f_j(0) \prod_{i=1}^{j} [1 - v_i a(k) + s_i(k)], \tag{2.18}$$

with $s_i(0) = 0$ and

$$|s_i(k)| \le K_3(1 + C(K_2 + K_3)\beta)\beta a(k) i^{-\delta}. \tag{2.19}$$

In fact, (2.18) is an identity if we define

$$s_i(k) = [1 + r_i(0)]^{-1}[v_i a(k) r_i(0) + (r_i(k) - r_i(0))], \tag{2.20}$$

and (2.19) then follows from (2.17) and the bounds on r_i of (H3).

The next lemma provides an important upper bound on $f_j(k;z)$, for k small depending on j, as in (H3).

Lemma 2.2. *Let $z \in I_n$ and assume (H2–H3) for $1 \le j \le n$. Then for k with $a(k) \le \gamma j^{-1} \log j$,*

$$|f_j(k;z)| \le e^{CK_3\beta} e^{-(1 - C(K_2+K_3)\beta) j a(k)}. \tag{2.21}$$

Proof. We use (2.18), and conclude from the bound on $r_i(0)$ of (H3) that $|f_j(0)| \le e^{CK_3\beta}$. Then we use (2.17) and (2.19) to obtain

$$\prod_{i=1}^{j} |1 - v_i a(k) + s_i(k)| \le \prod_{i=1}^{j} \left|1 - (1 - CK_2\beta) a(k) + CK_3\beta a(k) i^{-\delta}\right|. \tag{2.22}$$

The desired bound then follows, once we use $1 + x \le e^x$ for each factor on the right side. □

The middle bound of (1.16) follows, for $1 \le m \le n$ and $z \in I_m$, directly from Lemma 2.2. We next prove two lemmas which provide the other two bounds of (1.16). This will supply the hypothesis (1.16) for Assumptions E and G, and therefore plays a crucial role in advancing the induction.

Lemma 2.3. *Let $z \in I_n$ and assume (H2), (H3) and (H4). Then for $1 \le j \le n$,*

$$\|\hat{D}^2 f_j(\cdot;z)\|_1 \le C(1 + K_4)\beta j^{-d/2}. \tag{2.23}$$

Proof. Fix $z \in I_n$ and $1 \le j \le n$, and define

$$\begin{aligned}
R_1 &= \{k \in [-\pi,\pi]^d : a(k) \le \gamma j^{-1}\log j,\ \|k\|_\infty \le L^{-1}\},\\
R_2 &= \{k \in [-\pi,\pi]^d : a(k) \le \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\},\\
R_3 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty \le L^{-1}\},\\
R_4 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\}.
\end{aligned} \tag{2.24}$$

The set R_2 is empty if j is sufficiently large. Then

$$\|\hat{D}^2 f_j\|_1 = \sum_{i=1}^{4} \int_{R_i} \hat{D}(k)^2 |f_j(k)| \frac{d^d k}{(2\pi)^d}. \tag{2.25}$$

We will treat each of the four terms on the right side separately.

On R_1, we use (1.11) in conjunction with Lemma 2.2 and the fact that $\hat{D}^2 \le 1$, to obtain

$$\int_{R_1} \hat{D}(k)^2 |f_j(k)| \frac{d^d k}{(2\pi)^d} \le \int_{R_1} C e^{-cj(Lk)^2} \frac{d^d k}{(2\pi)^d} \le \frac{C}{L^d j^{d/2}}. \tag{2.26}$$

On R_2, we use Lemma 2.2 and (1.12) to conclude that there is an $\alpha > 1$ such that

$$\int_{R_2} \hat{D}(k)^2 |f_j(k)| \frac{d^d k}{(2\pi)^d} \le \int_{R_2} \alpha^{-j} \frac{d^d k}{(2\pi)^d} = \alpha^{-j} |R_2|, \tag{2.27}$$

where $|R_2|$ denotes the volume of R_2. This volume is maximal when $j = 3$, so that

$$|R_2| \le |\{k : a(k) \le 1 - \tfrac{\gamma \log 3}{3}\}| \le |\{k : \hat{D}(k) \ge \tfrac{\gamma \log 3}{3}\}| \le (\tfrac{3}{\gamma \log 3})^2 \|\hat{D}^2\|_1 \le (\tfrac{3}{\gamma \log 3})^2 \beta. \tag{2.28}$$

Therefore $\alpha^{-j}|R_2| \le C\beta j^{-d/2}$ and

$$\int_{R_2} \hat{D}(k)^2 |f_j(k)| \frac{d^d k}{(2\pi)^d} \le C\beta j^{-d/2}. \tag{2.29}$$

On R_3 and R_4, we use (H4). As a result, the contribution from these two regions is bounded above by

$$\frac{K_4}{j^{d/2}} \sum_{i=3}^{4} \int_{R_i} \frac{\hat{D}(k)^2}{a(k)^{2+\rho}} \frac{d^d k}{(2\pi)^d}. \tag{2.30}$$

On R_3, we use $\hat{D}(k)^2 \le 1$ and (1.11) to obtain the upper bound

$$\frac{CK_4}{j^{d/2} L^{4+2\rho}} \int_{\|k\|_\infty < L^{-1}} \frac{1}{k^{4+2\rho}} d^d k \le \frac{CK_4}{j^{d/2} L^{4+2\rho}} \left(\frac{1}{L}\right)^{d-4-2\rho} = \frac{CK_4 \beta}{j^{d/2}}, \tag{2.31}$$

where the integral is finite since $\rho < \frac{d-4}{2}$ by (2.2). On R_4, we use (1.12) to obtain the bound

$$\frac{CK_4}{j^{d/2}} \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \frac{d^d k}{(2\pi)^d} \le \frac{CK_4 \beta}{j^{d/2}}. \tag{2.32}$$

This completes the proof. $\square$

Lemma 2.4. *Let $z \in I_n$ and assume (H2) and (H3). Then, for $1 \le j \le n$,*

$$|\nabla^2 f_j(0;z)| \le (1 + (K_2 + K_3)\beta)\sigma^2 j. \tag{2.33}$$

Proof. Fix $z \in I_n$ and j with $1 \le j \le n$. By (2.18) and Assumption S,

$$\nabla^2 f_j(0) = f_j(0) \sum_{i=1}^{j} \big[-\sigma^2 v_i + \nabla^2 s_i(0)\big]. \tag{2.34}$$

By (2.17), $|v_i - 1| \le CK_2\beta$. For the second term on the right side, we let $e_1, \dots, e_d$ denote the standard basis vectors in $\mathbb{R}^d$. By (2.19), for all $i \le n$ we have

$$|\nabla^2 s_i(0)| = \Big| \sum_{l=1}^{d} \lim_{t \to 0} \frac{s_i(te_l) - s_i(0)}{t^2} \Big| \le CK_3\beta i^{-\delta} \sum_{l=1}^{d} \lim_{t \to 0} \frac{a(te_l)}{t^2} = CK_3\sigma^2\beta i^{-\delta}. \tag{2.35}$$

Therefore, by Lemma 2.2,

$$|\nabla^2 f_j(0)| \le e^{CK_3\beta}\sigma^2 j \Big(1 + C(K_2 + K_3)\beta\Big). \tag{2.36}$$

This completes the proof. □

The next lemma is the key to advancing the induction, as it provides bounds for e_{n+1} and g_{n+1}.

Lemma 2.5. *Let $z \in I_n$, and assume (H2), (H3) and (H4). For $k \in [-\pi, \pi]^d$, $2 \le j \le n+1$, and $\epsilon' \in [0, \epsilon]$, the following hold:*
(i) $|g_j(k;z)| \le K_4'\beta j^{-d/2}$,
(ii) $|\nabla^2 g_j(0;z)| \le K_4'\sigma^2\beta j^{-(d-2)/2}$,
(iii) $|\partial_z g_j(0;z)| \le K_4'\beta j^{-(d-2)/2}$,
(iv) $|g_j(k;z) - g_j(0;z) - a(k)\sigma^{-2}\nabla^2 g_j(0;z)| \le K_4'\beta a(k)^{1+\epsilon'} j^{-(d-2-2\epsilon')/2}$,
(v) $|e_j(k;z)| \le K_4'\beta j^{-d/2}$,
(vi) $|e_j(k;z) - e_j(0;z)| \le K_4' a(k)\beta j^{-(d-2)/2}$.

Proof. The bounds (1.16) for $1 \le m \le n$ follow from Lemmas 2.2–2.4, with $K = cK_4$ (this defines c), assuming that β is sufficiently small. The bounds of the lemma then follow immediately from Assumptions E and G, with K_4' given in (2.5). □

3 The induction advanced

In this section we advance the induction hypotheses (H1–H4) from n to $n+1$. Throughout this section, in accordance with the uniformity condition on (H2–H4), we fix $z \in I_{n+1}$. We also assume that $\beta \ll 1$.

3.1 Advancement of (H1)

By (2.3) and the mean-value theorem,

$$\begin{aligned} z_{n+1} - z_n &= -\sum_{m=2}^{n} [g_m(0; z_n) - g_m(0; z_{n-1})] - g_{n+1}(0; z_n) \\ &= -(z_n - z_{n-1}) \sum_{m=2}^{n} \partial_z g_m(0; y_n) - g_{n+1}(0; z_n), \end{aligned} \tag{3.1}$$

for some y_n between z_n and z_{n-1}. By (H1) and (2.4), $y_n \in I_n$. Using Lemma 2.5 and (H1), it then follows that

$$\begin{aligned} |z_{n+1} - z_n| &\le K_1 \beta n^{-d/2} \sum_{m=2}^{n} K_4' \beta m^{-(d-2)/2} + K_4' \beta (n+1)^{-d/2} \\ &\le K_4' \beta (1 + C K_1 \beta)(n+1)^{-d/2}. \end{aligned} \tag{3.2}$$

Thus (H1) holds for $n+1$, for β small and $K_1 > K_4'$.

Having advanced (H1) to $n+1$, it then follows from Lemma 2.1 that $I_1 \supset I_2 \supset \cdots \supset I_{n+1}$. For $n \ge 0$, define

$$\zeta_{n+1} = \zeta_{n+1}(z) = \sum_{m=1}^{n+1} g_m(0; z) - 1 = \sum_{m=2}^{n+1} g_m(0; z) + z - 1. \tag{3.3}$$

The following lemma, whose proof makes use of (H1) for $n+1$, will be needed in what follows.

Lemma 3.1. *For all $z \in I_{n+1}$,*

$$|\zeta_{n+1}| \le C K_1 \beta (n+1)^{-(d-2)/2}. \tag{3.4}$$

Proof. By (2.3) and the mean-value theorem,

$$\begin{aligned} |\zeta_{n+1}| &= \left| (z - z_{n+1}) + \sum_{m=2}^{n+1} [g_m(0; z) - g_m(0; z_n)] \right| \\ &= \left| (z - z_{n+1}) + (z - z_n) \sum_{m=2}^{n+1} \partial_z g_m(0; y_n) \right|, \end{aligned} \tag{3.5}$$

for some y_n between z and z_n. Since $z \in I_{n+1} \subset I_n$ and $z_n \in I_n$, we have $y_n \in I_n$. Therefore, by Lemma 2.5,

$$|\zeta_{n+1}| \le K_1 \beta (n+1)^{-(d-2)/2} + K_1 \beta n^{-(d-2)/2} \sum_{m=2}^{n+1} K_4' \beta m^{-(d-2)/2} \le K_1 \beta (1 + C K_4' \beta)(n+1)^{-(d-2)/2}. \tag{3.6}$$

The lemma then follows, for β sufficiently small. $\square$

3.2 Advancement of (H2)

Let $z \in I_{n+1}$. As observed in Section 3.1, this implies that $z \in I_j$ for all $j \leq n+1$. The definitions in (2.1) imply that

$$v_{n+1} - v_n = \frac{1}{1+c_{n+1}}(b_{n+1} - b_n) - \frac{b_n}{(1+c_n)(1+c_{n+1})}(c_{n+1} - c_n), \tag{3.7}$$

with

$$b_{n+1} - b_n = -\frac{1}{\sigma^2}\nabla^2 g_{n+1}(0), \quad c_{n+1} - c_n = n g_{n+1}(0). \tag{3.8}$$

By Lemma 2.5, both differences in (3.8) are bounded by $K_4'\beta(n+1)^{-(d-2)/2}$, and, in addition,

$$|b_j - 1| \leq CK_4'\beta, \quad |c_j| \leq CK_4'\beta \tag{3.9}$$

for $1 \leq j \leq n+1$. Therefore

$$|v_{n+1} - v_n| \leq K_2\beta(n+1)^{-(d-2)/2}, \tag{3.10}$$

provided we assume $K_2 > 2K_4'$. This advances (H2).

3.3 Advancement of (H3)

3.3.1 The decomposition

The advancement of the induction hypotheses (H3–H4) is the most technical part of the proof. For (H3), we fix k with $a(k) \leq \gamma(n+1)^{-1}\log(n+1)$, and $z \in I_{n+1}$. The induction step will be achieved as soon as we are able to write the ratio $f_{n+1}(k)/f_n(k)$ as

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 - v_{n+1}a(k) + r_{n+1}(k), \tag{3.11}$$

with $r_{n+1}(0)$ and $r_{n+1}(k) - r_{n+1}(0)$ satisfying the bounds required by (H3).

To begin, we divide the recursion relation (1.1) by $f_n(k)$, and use (3.3), to obtain

$$\frac{f_{n+1}(k)}{f_n(k)} \quad = \quad 1 + \sum_{m=1}^{n+1}\Big[g_m(k)\frac{f_{n+1-m}(k)}{f_n(k)} - g_m(0)\Big] + \zeta_{n+1} + \frac{e_{n+1}(k)}{f_n(k)}. \tag{3.12}$$

By (2.1),

$$v_{n+1} = b_{n+1} - v_{n+1}c_{n+1} = -\sigma^{-2}\sum_{m=1}^{n+1}\nabla^2 g_m(0) - v_{n+1}\sum_{m=1}^{n+1}(m-1)g_m(0). \tag{3.13}$$

Thus we can rewrite (3.12) as

$$\frac{f_{n+1}(k)}{f_n(k)} = 1 - v_{n+1}a(k) + r_{n+1}(k), \tag{3.14}$$

where

$$r_{n+1}(k) = X(k) + Y(k) + Z(k) + \zeta_{n+1} \tag{3.15}$$

with

$$\begin{aligned} X(k) &= \sum_{m=2}^{n+1} \Big[\big(g_m(k) - g_m(0)\big) \frac{f_{n+1-m}(k)}{f_n(k)} - a(k)\sigma^{-2}\nabla^2 g_m(0) \Big], && (3.16) \\ Y(k) &= \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(k)}{f_n(k)} - 1 - (m-1)v_{n+1}a(k) \right], && (3.17) \\ Z(k) &= \frac{e_{n+1}(k)}{f_n(k)}. && (3.18) \end{aligned}$$

The $m = 1$ terms in X and Y vanish and have not been included.

We will prove that

$$|r_{n+1}(0)| \le \frac{C(K_1 + K_4')\beta}{(n+1)^{(d-2)/2}}, \qquad |r_{n+1}(k) - r_{n+1}(0)| \le \frac{CK_4'\beta a(k)}{(n+1)^{\delta}}. \tag{3.19}$$

This gives (H3) for $n+1$, provided we assume that $K_3 \gg K_1$ and $K_3 \gg K_4'$. To prove the bounds on r_{n+1} of (3.19), it will be convenient to make use of some elementary convolution bounds, as well as some bounds on ratios involving f_j. These preliminary bounds are given in Section 3.3.2, before we present the proof of (3.19) in Section 3.3.3.

3.3.2 Convolution and ratio bounds

The proof of (3.19) will make use of the following elementary convolution bounds. To keep the discussion simple, we do not obtain optimal bounds.

Lemma 3.2. *For $n \ge 2$,*

$$\sum_{m=2}^{n} \frac{1}{m^a} \sum_{j=n-m+1}^{n} \frac{1}{j^b} \le \begin{cases} Cn^{-a\wedge b+1} & \text{for } a, b > 1 \\ Cn^{-(a-2)\wedge b} & \text{for } a > 2, b > 0 \\ Cn^{-(a-1)\wedge b} & \text{for } a > 2, b > 1 \\ Cn^{-a\wedge b} & \text{for } a, b > 2. \end{cases} \tag{3.20}$$

Proof. Since $m + j \ge n$, either m or j is at least $\frac{n}{2}$. Therefore

$$\sum_{m=2}^{n} \frac{1}{m^a} \sum_{j=n-m+1}^{n} \frac{1}{j^b} \le \left(\frac{2}{n}\right)^a \sum_{m=2}^{n} \sum_{j=n-m+1}^{n} \frac{1}{j^b} + \left(\frac{2}{n}\right)^b \sum_{m=2}^{n} \sum_{j=n-m+1}^{n} \frac{1}{m^a}. \tag{3.21}$$

If $a, b > 1$, then the first term is bounded by Cn^{1-a} and the second by Cn^{1-b}.
If $a > 2, b > 0$, then the first term is bounded by Cn^{2-a} and the second by Cn^{-b}.
If $a > 2, b > 1$, then the first term is bounded by Cn^{1-a} and the second by Cn^{-b}.
If $a, b > 2$, then the first term is bounded by Cn^{-a} and the second by Cn^{-b}. □

We also will make use of several estimates involving ratios. We begin with some preparation. Given a vector $x = (x_l)$ with $\sup_l |x_l| < 1$, define $\chi(x) = \sum_l \frac{|x_l|}{1-|x_l|}$. The bound $(1-t)^{-1} \leq \exp[t(1-t)^{-1}]$, together with Taylor's Theorem applied to $f(t) = \prod_l \frac{1}{1-tx_l}$, gives

$$\left| \prod_l \frac{1}{1-x_l} - 1 \right| \leq \chi(x) e^{\chi(x)}, \quad \left| \prod_l \frac{1}{1-x_l} - 1 - \sum_l x_l \right| \leq \frac{3}{2} \chi(x)^2 e^{\chi(x)}. \tag{3.22}$$

We assume throughout the rest of this section that $a(k) \leq \gamma(n+1)^{-1} \log(n+1)$ and $2 \leq m \leq n+1$, and define

$$\psi_{m,n} = \sum_{j=n+2-m}^{n} \frac{|r_j(0)|}{1-|r_j(0)|}, \quad \chi_{m,n}(k) = \sum_{j=n+2-m}^{n} \frac{v_j a(k) + |s_j(k)|}{1 - v_j a(k) - |s_j(k)|}. \tag{3.23}$$

By (2.17) and (2.19),

$$\chi_{m,n}(k) \leq (m-1) a(k) Q(k) \quad \text{with} \quad Q(k) = [1 + C(K_2 + K_3)\beta][1 + Ca(k)]. \tag{3.24}$$

Since $a(k) \leq \gamma(n+1)^{-1}\log(n+1) \leq \gamma(m-1)^{-1}\log(m-1)$, we have $m-1 \leq \gamma a(k)^{-1} \log(m-1)$. Therefore

$$e^{\chi_{m,n}(k)} \leq e^{(m-1)a(k)Q(k)} \leq (m-1)^{\gamma Q(k)} = (m-1)^{\gamma Q(0)} (m-1)^{\gamma(Q(k)-Q(0))}. \tag{3.25}$$

Now $Q(k) - Q(0) \leq Ca(k)$, and hence, for any $q < Q(0)$ we have

$$e^{\chi_{m,n}(k)} \leq (m-1)^{\gamma Q(0)} C a(k)^{-Ca(k)} (\log(m-1))^C \leq C(n+1)^{\gamma q}. \tag{3.26}$$

Here, we have used the fact that $a(k)^{-Ca(k)}$ remains bounded as $k \to 0$, and we have absorbed the logarithmic factor by a small power. We can choose q to be as close to 1 as desired, by taking β to be small.

We now turn to the ratio bounds. It follows from (H3) and the first inequality of (3.22) that

$$\left| \frac{f_{n+1-m}(0)}{f_n(0)} - 1 \right| \leq \psi_{m,n} e^{\psi_{m,n}} \leq \sum_{j=n+2-m}^{m} \frac{CK_3\beta}{j^{(d-2)/2}} \leq \frac{CK_3\beta}{(n+2-m)^{(d-4)/2}}. \tag{3.27}$$

Therefore

$$\left| \frac{f_{n+1-m}(0)}{f_n(0)} \right| \leq 1 + CK_3\beta. \tag{3.28}$$

By (2.18),

$$\left| \frac{f_{n+1-m}(k)}{f_n(k)} - 1 \right| \leq \left| \frac{f_{n+1-m}(0)}{f_n(0)} \right| \left| \prod_{j=n+2-m}^{n} [1 - v_j a(k) + s_j(k)]^{-1} - 1 \right| + \left| \frac{f_{n+1-m}(0)}{f_n(0)} - 1 \right|. \tag{3.29}$$

The first inequality of (3.22), together with (3.24)–(3.28), then gives

$$\left| \frac{f_{n+1-m}(k)}{f_n(k)} - 1 \right| \leq C(m-1)a(k)(n+1)^{\gamma q} + \frac{CK_3\beta}{(n+2-m)^{(d-4)/2}}. \tag{3.30}$$

Similarly,

$$\left| \frac{f_n(0)}{f_n(k)} - 1 \right| \leq Ca(k)(n+1)^{1+\gamma q}. \tag{3.31}$$

Next, we estimate the quantity $R_{m,n}(k)$, which is defined by

$$R_{m,n}(k) = \prod_{j=n+2-m}^{n} [1 - v_j a(k) + s_j(k)]^{-1} - 1 - \sum_{j=n+2-m}^{n} [v_j a(k) - s_j(k)]. \tag{3.32}$$

By the second inequality of (3.22), together with (3.24) and (3.26), this obeys

$$|R_{m,n}(k)| \leq \frac{3}{2}\chi_{m,n}(k)^2 e^{\chi_{m,n}(k)} \leq Cm^2 a(k)^2 (n+1)^{\gamma q}. \tag{3.33}$$

Finally, we apply (H3) to obtain

$$\left| \frac{f_{m-1}(k)}{f_m(k)} - 1 \right| = \left| [1 - v_m a(k) + (r_m(k) - r_m(0)) + r_m(0)]^{-1} - 1 \right| \leq Ca(k) + \frac{CK_3\beta}{m^{(d-2)/2}}. \tag{3.34}$$

3.3.3 The induction step

By definition,

$$r_{n+1}(0) = Y(0) + Z(0) + \zeta_{n+1} \tag{3.35}$$

and

$$r_{n+1}(k) - r_{n+1}(0) = X(k) + \Big(Y(k) - Y(0)\Big) + \Big(Z(k) - Z(0)\Big). \tag{3.36}$$

Since $|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-(d-2)/2}$ by Lemma 3.1, to prove (3.19) it suffices to show that

$$|Y(0)| \leq CK_4'\beta(n+1)^{-(d-2)/2}, \quad |Z(0)| \leq CK_4'\beta(n+1)^{-(d-2)/2} \tag{3.37}$$

and

$$\begin{gathered} |X(k)| \leq CK_4'\beta a(k)(n+1)^{-\delta}, \quad |Y(k) - Y(0)| \leq CK_4'\beta a(k)(n+1)^{-\delta}, \\ |Z(k) - Z(0)| \leq CK_4'\beta a(k)(n+1)^{-\delta}. \end{gathered} \tag{3.38}$$

The remainder of the proof is devoted to establishing (3.37) and (3.38).

Bound on X. We write X as $X = X_1 + X_2$, with

$$X_1 = \sum_{m=2}^{n+1} \Big[g_m(k) - g_m(0) - a(k)\sigma^{-2}\nabla^2 g_m(0)\Big], \tag{3.39}$$

$$X_2 = \sum_{m=2}^{n+1} \Big[g_m(k) - g_m(0)\Big]\Big[\frac{f_{n+1-m}(k)}{f_n(k)} - 1\Big]. \tag{3.40}$$

The term X_1 is bounded using Lemma 2.5(iv) with $\epsilon' \in (\delta, \epsilon \wedge 1 \wedge \frac{d-4}{2})$, and using the fact that $a(k) \leq \gamma(n+1)^{-1} \log(n+1)$, by

$$|X_1| \leq K_4' \beta a(k)^{1+\epsilon'} \sum_{m=2}^{n+1} \frac{1}{m^{(d-2-2\epsilon')/2}} \leq CK_4'\beta a(k)^{1+\epsilon'} \leq \frac{CK_4'\beta a(k)}{(n+1)^{\delta}}. \tag{3.41}$$

For X_2, we first apply Lemma 2.5(ii,iv), with $\epsilon' = 0$, to obtain

$$|g_m(k) - g_m(0)| \leq 2K_4'\beta a(k) m^{-(d-2)/2}. \tag{3.42}$$

Applying (3.30) then gives

$$|X_2| \leq CK_4'\beta a(k) \sum_{m=2}^{n+1} \frac{1}{m^{(d-2)/2}} \left((m-1)a(k)(n+1)^{\gamma q} + \frac{K_3\beta}{(n+2-m)^{(d-4)/2}} \right). \tag{3.43}$$

By an elementary estimate, the contribution from the second term on the right side is bounded above by $CK_3K_4'\beta^2 a(k)(n+1)^{-(d-4)/2}$. The first term is bounded above by

$$CK_4'\beta a(k)(n+1)^{0\vee(6-d)/2}(n+1)^{\gamma q-1}\log(n+1) \tag{3.44}$$

(a harmless factor $\log(n+1)$ should appear for $d=6$). Since we may choose q to be as close to 1 as desired, by (2.2) this is bounded above by $CK_4'\beta a(k)(n+1)^{-\delta}$. With (3.41), this proves the bound on X in (3.38).

Bound on Y. By (2.18),

$$\frac{f_{n+1-m}(k)}{f_n(k)} = \frac{f_{n+1-m}(0)}{f_n(0)} \prod_{j=n+2-m}^{n} [1 - v_j a(k) + s_j(k)]^{-1}. \tag{3.45}$$

Recalling the definition of $R_{m,n}(k)$ in (3.32), we can therefore decompose Y as $Y = Y_1 + Y_2 + Y_3 + Y_4$ with

$$Y_1 = \sum_{m=2}^{n+1} g_m(0) \frac{f_{n+1-m}(0)}{f_n(0)} R_{m,n}(k), \tag{3.46}$$

$$Y_2 = \sum_{m=2}^{n+1} g_m(0) \frac{f_{n+1-m}(0)}{f_n(0)} \sum_{j=n+2-m}^{n} [(v_j - v_{n+1})a(k) - s_j(k)], \tag{3.47}$$

$$Y_3 = \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(0)}{f_n(0)} - 1 \right] (m-1) v_{n+1} a(k), \tag{3.48}$$

$$Y_4 = \sum_{m=2}^{n+1} g_m(0) \left[\frac{f_{n+1-m}(0)}{f_n(0)} - 1 \right]. \tag{3.49}$$

Then

$$Y(0) = Y_4 \quad \text{and} \quad Y(k) - Y(0) = Y_1 + Y_2 + Y_3. \tag{3.50}$$

For Y_1, we use Lemma 2.5, (3.28) and (3.33) to obtain

$$|Y_1| \leq CK_4'\beta a(k)^2(n+1)^{\gamma q}\sum_{m=2}^{n}\frac{1}{m^{(d-4)/2}}. \tag{3.51}$$

As in the analysis of the first term of (3.43), we therefore have

$$|Y_1| \leq \frac{CK_4'\beta a(k)}{(n+1)^{\delta}}. \tag{3.52}$$

For Y_2, we use Lemma 2.5, (3.28), (H2), (2.19) and Lemma 3.2 to obtain

$$|Y_2| \leq \sum_{m=2}^{n+1}\frac{K_4'\beta}{m^{d/2}}C\sum_{j=n+2-m}^{n}\left[\frac{K_2\beta a(k)}{j^{(d-4)/2}}+\frac{K_3\beta a(k)}{j^{\delta}}\right] \leq \frac{CK_4'(K_2+K_3)\beta^2 a(k)}{(n+1)^{\delta}}. \tag{3.53}$$

The term Y_3 obeys

$$|Y_3| \leq \sum_{m=2}^{n+1}\frac{K_4'\beta}{m^{(d-2)/2}}\frac{CK_3\beta}{(n+2-m)^{(d-4)/2}}a(k) \leq \frac{CK_4'K_3\beta^2 a(k)}{(n+1)^{(d-4)/2}}, \tag{3.54}$$

where we used Lemma 2.5, (3.27), (2.17), and an elementary convolution bound. This proves the bound on $|Y(k)-Y(0)|$ of (3.38), if β is sufficiently small.

We bound Y_4 in a similar fashion, using Lemma 3.2 and the intermediate bound of (3.27) to obtain

$$|Y_4| \leq \sum_{m=2}^{n+1}\frac{K_4'\beta}{m^{d/2}}\sum_{j=n+2-m}^{n}\frac{CK_3\beta}{j^{(d-2)/2}} \leq \frac{CK_4'K_3\beta^2}{(n+1)^{(d-2)/2}}. \tag{3.55}$$

Taking β small then gives the bound on $Y(0)$ of (3.37).

Bound on Z. We decompose Z as

$$Z = \frac{e_{n+1}(0)}{f_n(0)} + \frac{1}{f_n(0)}\left[e_{n+1}(k)-e_{n+1}(0)\right] + \frac{e_{n+1}(k)}{f_n(0)}\left[\frac{f_n(0)}{f_n(k)}-1\right] = Z_1+Z_2+Z_3. \tag{3.56}$$

Then

$$Z(0) = Z_1 \quad \text{and} \quad Z(k)-Z(0) = Z_2+Z_3. \tag{3.57}$$

Using Lemma 2.5(v,vi), and (3.28) with $m=n+1$, we obtain

$$|Z_1| \leq CK_4'\beta(n+1)^{-d/2} \quad \text{and} \quad |Z_2| \leq CK_4'\beta a(k)(n+1)^{-(d-2)/2}. \tag{3.58}$$

Also, by (3.28) and (3.31), we have

$$|Z_3| \leq CK_4'\beta(n+1)^{-d/2}a(k)(n+1)^{1+\gamma q} \leq CK_4'\beta a(k)(n+1)^{-(1+\delta)}. \tag{3.59}$$

This completes the proof of (3.19), and hence completes the advancement of (H3) to $n+1$.

3.4 Advancement of (H4)

In this section, we fix $a(k) > \gamma(n+1)^{-1}\log(n+1)$. To advance (H4) to $j = n+1$, we first recall the definitions of b_{n+1}, ζ_{n+1} and X_1 from (2.1), (3.3) and (3.39). After some algebra, (1.1) can be rewritten as

$$f_{n+1}(k) = f_n(k)\Big(1 \ - a(k)b_{n+1} + X_1 + \zeta_{n+1}\Big) + W + e_{n+1}(k), \tag{3.60}$$

with

$$W = \sum_{m=2}^{n+1} g_m(k)\left[f_{n+1-m}(k) - f_n(k)\right]. \tag{3.61}$$

We already have estimates for most of the relevant terms. By Lemma 3.1, $|\zeta_{n+1}| \le CK_1\beta(n+1)^{-(d-2)/2}$. By (3.41), $|X_1| \le CK_4'\beta a(k)^{1+\epsilon'}$, for any $\epsilon' \in (\delta, \epsilon \wedge 1 \wedge \frac{d-4}{2})$. By Lemma 2.5(v), $|e_{n+1}(k)| \le K_4'\beta(n+1)^{-d/2}$. It remains to estimate W. We will show below that W obeys the bound

$$|W| \le \frac{CK_4'\beta}{a(k)^{1+\rho}(n+1)^{d/2}}(1 + K_3\beta + K_5). \tag{3.62}$$

Before proving (3.62), we will first show that it is sufficient for the advancement of (H4).

In preparation for this, we first note that it suffices to consider only large n. In fact, since $|f_n(k;z)|$ is bounded uniformly in k and in z in a compact set by Assumption S, and since $a(k) \le 2$, it is clear that both inequalities of (H4) hold for all $n \le N$, if we choose K_4 and K_5 large enough (depending on N). We therefore assume in the following that $n \ge N$ with N large.

Also, care is required to invoke (H3) or (H4), as applicable, in estimating the factor $f_n(k)$ of (3.60). Given k, (H3) should be used for the value n for which $\gamma(n+1)^{-1}\log(n+1) < a(k) \le \gamma n^{-1}\log n$ ((H4) should be used for larger n). We will now show that, as anticipated in the discussion of (H4) in Section 2.3, the bound of (H3) actually implies the bound of (H4) in this case. To see this, we use Lemma 2.2 to see that there are q, q' arbitrarily close to 1 such that

$$|f_n(k)| \le Ce^{-qa(k)n} \le \frac{C}{(n+1)^{q\gamma n/(n+1)}} \le \frac{C}{n^{q'\gamma}} \le \frac{C}{n^{d/2}}\frac{n^{2+\rho}}{n^{q'\gamma+2+\rho-d/2}} \le \frac{C}{n^{d/2}a(k)^{2+\rho}}, \tag{3.63}$$

where we used the fact that $\gamma + 2 + \rho - d/2 > 0$ by (2.2). Thus, taking $K_4 \gg 1$, we may use the bound of (H4) also for the value of n to which (H3) nominally applies. We will do so in what follows, without further comment.

Advancement of the second bound of (H4) assuming (3.62). To advance the second estimate in (H4), we use (3.60), (H4), and the bounds found above, to obtain

$$\begin{aligned}\Big|f_{n+1}(k) - f_n(k)\Big| \ &\le \ \Big|f_n(k)\Big| \ \Big| - a(k)b_{n+1} + X_1 + \zeta_{n+1}\Big| + |W| + |e_{n+1}(k)| \\ &\le \ \frac{K_4}{n^{d/2}a(k)^{2+\rho}}\left(a(k)b_{n+1} + CK_4'\beta a(k)^{1+\epsilon'} + \frac{CK_1\beta}{(n+1)^{(d-2)/2}}\right) \\ &\quad + \frac{CK_4'\beta(1 + K_3\beta + K_5)}{(n+1)^{d/2}a(k)^{1+\rho}} + \frac{K_4'\beta}{(n+1)^{d/2}}.\end{aligned} \tag{3.64}$$

Since $b_{n+1} = 1 + \mathcal{O}(\beta)$ by (3.9), and since $(n+1)^{-(d-2)/2} < [a(k)/\gamma \log(n+1)]^{(d-2)/2} \leq Ca(k)$, the second estimate in (H4) follows for $n+1$ provided $K_5 \gg K_4$ and β is sufficiently small.

Advancement of the first bound of (H4) assuming (3.62). To advance the first estimate of (H4), we argue as in (3.64) to obtain

$$\begin{aligned} \left|f_{n+1}(k)\right| &\leq \left|f_n(k)\right| \left|1 - a(k)b_{n+1} + X_1 + \zeta_{n+1}\right| + |W| + |e_{n+1}(k)| \\ &\leq \frac{K_4}{n^{d/2}a(k)^{2+\rho}} \left(|1 - a(k)b_{n+1}| + CK_4'\beta a(k)^{1+\epsilon'} + \frac{CK_1\beta}{(n+1)^{(d-2)/2}} \right) \\ &\quad + \frac{CK_4'\beta(1+K_3\beta+K_5)}{(n+1)^{d/2}a(k)^{1+\rho}} + \frac{K_4'\beta}{(n+1)^{d/2}}. \end{aligned} \tag{3.65}$$

We need to argue that the right-hand side is no larger than $K_4(n+1)^{-d/2}a(k)^{-2-\rho}$. To achieve this, we will use separate arguments for $a(k) \leq \frac{1}{2}$ and $a(k) > \frac{1}{2}$. These arguments will be valid only when n is large enough.

Suppose that $a(k) \leq \frac{1}{2}$. Since $b_{n+1} = 1 + \mathcal{O}(\beta)$ by (3.9), for β sufficiently small we have

$$1 - b_{n+1}a(k) \geq 0. \tag{3.66}$$

Hence, the absolute value signs on the right side of (3.65) may be removed. Therefore, to obtain the first estimate of (H4) for $n+1$, it now suffices to show that

$$1 - ca(k) + \frac{CK_1\beta}{(n+1)^{(d-2)/2}} \leq \frac{n^{d/2}}{(n+1)^{d/2}}, \tag{3.67}$$

for c within order β of 1. The term $ca(k)$ has been introduced to absorb $b_{n+1}a(k)$, the order β term in (3.65) involving $a(k)^{1+\epsilon'}$, and the last two terms of (3.65). However, $a(k) > \gamma(n+1)^{-1}\log(n+1)$. From this, it can be seen that (3.67) holds for n sufficiently large and β sufficiently small.

Suppose, on the other hand, that $a(k) > \frac{1}{2}$. By (1.13), there is a positive η, which we may assume lies in $(0, \frac{1}{2})$, such that $-1 + \eta < 1 - a(k) < \frac{1}{2}$. Therefore $|1 - a(k)| \leq 1 - \eta$ and

$$|1 - b_{n+1}a(k)| \leq |1 - a(k)| + |b_{n+1} - 1|\,|a(k)| \leq 1 - \eta + 2|b_{n+1} - 1|. \tag{3.68}$$

Hence

$$|1 - a(k)b_{n+1}| + CK_4'\beta a(k)^{1+\epsilon'} + \frac{CK_1\beta}{(n+1)^{(d-2)/2}} \leq 1 - \eta + C(K_1 + K_4')\beta, \tag{3.69}$$

and the right side of (3.65) is at most

$$\begin{aligned} &\frac{K_4}{n^{d/2}a(k)^{2+\rho}} \left[1 - \eta + C(K_1 + K_4')\beta\right] + \frac{CK_4'(1+K_3\beta+K_5)\beta}{(n+1)^{d/2}a(k)^{2+\rho}} \\ &\leq \frac{K_4}{n^{d/2}a(k)^{2+\rho}} \left[1 - \eta + C(K_5K_4' + K_1)\beta\right]. \end{aligned} \tag{3.70}$$

This is less than $K_4(n+1)^{-d/2}a(k)^{-2-\rho}$ if n is large and β is sufficiently small.

This advances the first bound in (H4), assuming (3.62).

Bound on W. We now obtain the bound (3.62) on W. As a first step, we rewrite W as

$$W = \sum_{j=0}^{n-1} g_{n+1-j}(k) \sum_{l=j+1}^{n} [f_{l-1}(k) - f_l(k)]. \tag{3.71}$$

Let

$$m(k) = \max\{l \in \{2, \dots, n\} : a(k) \leq \gamma l^{-1} \log l\}. \tag{3.72}$$

For $l \leq m(k)$, f_l is in the domain of (H3), while for $l > m(k)$, f_l is in the domain of (H4). By hypothesis, $a(k) > \gamma(n+1)^{-1}\log(n+1)$. We divide the sum over l into two parts, corresponding respectively to $l \leq m(k)$ and $l > m(k)$, yielding $W = W_1 + W_2$, By Lemma 2.5(i),

$$\begin{aligned} |W_1| &\leq \sum_{j=0}^{m(k)} \frac{K_4'\beta}{(n+1-j)^{d/2}} \sum_{l=j+1}^{m(k)} |f_{l-1}(k) - f_l(k)| & (3.73)\\ |W_2| &\leq \sum_{j=0}^{n-1} \frac{K_4'\beta}{(n+1-j)^{d/2}} \sum_{l=(m(k)\vee j)+1}^{n} |f_{l-1}(k) - f_l(k)|. & (3.74) \end{aligned}$$

The term W_2 is easy, since by (H4) and Lemma 3.2 we have

$$|W_2| \leq \sum_{j=0}^{n-1} \frac{K_4'\beta}{(n+1-j)^{d/2}} \sum_{l=j+1}^{n} \frac{K_5}{a(k)^{1+\rho}\, l^{d/2}} \leq \frac{CK_5K_4'\beta}{a(k)^{1+\rho}(n+1)^{d/2}}. \tag{3.75}$$

For W_1, we have the estimate

$$|W_1| \leq \sum_{j=0}^{m(k)} \frac{K_4'\beta}{(n+1-j)^{d/2}} \sum_{l=j+1}^{m(k)} |f_{l-1}(k) - f_l(k)|. \tag{3.76}$$

For $1 \leq l \leq m(k)$, it follows from Lemma 2.2 and (3.34) that

$$|f_{l-1}(k) - f_l(k)| \leq Ce^{-qa(k)l}\left(a(k) + \frac{K_3\beta}{l^{(d-2)/2}}\right), \tag{3.77}$$

with $q = 1 - \mathcal{O}(\beta)$. We fix a small $r > 0$, and bound the summation over j in (3.76) by summing separately over j in the ranges $1 \leq j \leq (1-r)n$ and $(1-r)n \leq j \leq m(k)$ (the latter range may be empty). We denote the contributions from these two sums by $W_{1,1}$ and $W_{1,2}$ respectively.

To estimate $W_{1,1}$, we will make use of the bound

$$\sum_{l=j+1}^{\infty} e^{-qa(k)l} l^{-p} \leq \begin{cases} Ce^{-qa(k)j}a(k)^{p-1} & (0 \leq p < 1) \\ Ce^{-qa(k)j} & (p > 1), \end{cases} \tag{3.78}$$

which can be proved by approximating the sum by an integral. With (3.76) and (3.77), this gives

$$\begin{aligned} |W_{1,1}| &\leq \frac{CK_4'\beta}{(n+1)^{d/2}} \sum_{j=0}^{(1-r)n} e^{-qa(k)j}\,(1+K_3\beta) \\ &\leq \frac{CK_4'\beta}{(n+1)^{d/2}} \frac{1+K_3\beta}{a(k)} \leq \frac{CK_4'\beta}{(n+1)^{d/2}} \frac{1+K_3\beta}{a(k)^{1+\rho}}. & (3.79) \end{aligned}$$

For $W_{1,2}$, we have

$$|W_{1,2}| \leq \sum_{j=(1-r)n}^{m(k)} \frac{CK_4'\beta}{(n+1-j)^{d/2}} \sum_{l=j+1}^{m(k)} e^{-qa(k)l} \left(a(k) + \frac{K_3\beta}{l^{(d-2)/2}}\right). \tag{3.80}$$

Since l and $m(k)$ are comparable and large, it follows as in (3.63) that

$$e^{-qa(k)l} \left(a(k) + \frac{K_3\beta}{l^{(d-2)/2}}\right) \leq \frac{C}{a(k)^{2+\rho} l^{d/2}} \left(a(k) + \frac{K_3\beta}{l^{(d-2)/2}}\right) \leq \frac{C(1+K_3\beta)}{a(k)^{1+\rho} l^{d/2}}. \tag{3.81}$$

Hence, by Lemma 3.2,

$$|W_{1,2}| \leq \frac{C(1+K_3\beta)K_4'\beta}{a(k)^{1+\rho}} \sum_{j=(1-r)n}^{m(k)} \frac{1}{(n+1-j)^{d/2}} \sum_{l=j+1}^{m(k)} \frac{1}{l^{d/2}} \leq \frac{C(1+K_3\beta)K_4'\beta}{a(k)^{1+\rho}(n+1)^{d/2}}. \tag{3.82}$$

Summarising, by (3.79), (3.82), and (3.75), we have

$$|W| \leq |W_{1,1}| + |W_{1,2}| + |W_2| \leq \frac{CK_4'\beta}{a(k)^{1+\rho}(n+1)^{d/2}}(1 + K_3\beta + K_5), \tag{3.83}$$

which proves (3.62).

4 Proof of the main results

As a consequence of the completed induction, it follows from Lemma 2.1 that $I_1 \supset I_2 \supset I_3 \supset \cdots$, so $\cap_{n=1}^{\infty} I_n$ consists of a single point $z = z_c$. Since $z_0 = 1$, it follows from (H1) that $z_c = 1 + \mathcal{O}(\beta)$. We fix $z = z_c$ throughout this section. The constant A is defined by $A = \prod_{i=1}^{\infty}[1+r_i(0)] = 1+\mathcal{O}(\beta)$. To define v, we note that it follows from (H2) that there is a constant $v = 1 + \mathcal{O}(\beta)$ such that

$$|v_n(z_c) - v| \leq \mathcal{O}(\beta n^{-(d-4)/2}). \tag{4.1}$$

4.1 Proof of Theorem 1.1

Proof of Theorem 1.1(a). By (H3),

$$|f_n(0; z_c) - A| = \prod_{i=1}^{n}[1 + r_i(0)]\big|1 - \prod_{i=n+1}^{\infty} [1 + r_i(0)]\big| \leq \mathcal{O}(\beta n^{-(d-4)/2}). \tag{4.2}$$

Suppose k is such that $a(k/\sqrt{Dn}) \leq \gamma n^{-1} \log n$, so that (H3) applies. Here, we use the γ of (2.2). By (1.9), $a(k) = \sigma^2 k^2/2d + \mathcal{O}(k^{2+2\epsilon})$ with $\epsilon > \delta$, where we now allow constants in error terms to depend on L. Using this, together with (2.18)–(2.19), (4.1), and $\delta < 1 \wedge \frac{d-4}{2}$, we obtain

$$\begin{aligned} \frac{f_n(k/\sqrt{v\sigma^2 n}; z_c)}{f_n(0; z_c)} &= \prod_{i=1}^{n} \left[1 - v_i a(\frac{k}{\sqrt{v\sigma^2 n}}) + \mathcal{O}(\beta a(\frac{k}{\sqrt{v\sigma^2 n}}) i^{-\delta})\right] \\ &= e^{-k^2/2d}[1 + \mathcal{O}(k^{2+2\epsilon} n^{-\epsilon}) + \mathcal{O}(k^2 n^{-\delta})]. \end{aligned} \tag{4.3}$$

With (4.2), this gives the desired result.

Proof of Theorem 1.1(b). Since $\delta < 1 \wedge \frac{d-4}{4}$, it follows from (2.34)–(2.35) and (4.1)–(4.2) that

$$\frac{\nabla^2 f_n(0; z_c)}{f_n(0; z_c)} = -v\sigma^2 n[1 + \mathcal{O}(\beta n^{-\delta})]. \tag{4.4}$$

Proof of Theorem 1.1(c). The claim is immediate from Lemma 2.3, which is now known to hold for all n.

4.2 Proof of Theorem 1.2

By Theorem 1.1(a), $\chi(z_c) = \infty$. Therefore $z_c \geq p_c$. We need to rule out the possibility that $z_c > p_c$. Theorem 1.1 also gives (1.16) at $z = z_c$. By assumption, the series

$$G(z) = \sum_{m=2}^{\infty} g_m(0; z), \quad E(z) = \sum_{m=2}^{\infty} e_m(0; z) \tag{4.5}$$

therefore both converge and are $\mathcal{O}(\beta)$ uniformly in $z \leq z_c$. For $z < p_c$, the basic recursion relation (1.1) gives

$$\chi(z) = 1 + z\chi(z) + G(z)\chi(z) + E(z), \tag{4.6}$$

and hence

$$\chi(z) = \frac{1 + E(z)}{1 - z - G(z)}, \quad (z < p_c). \tag{4.7}$$

Since $E(z) = \mathcal{O}(\beta)$ and $\lim_{z \uparrow p_c} \chi(z) = \infty$, it follows from (4.7) that

$$1 - p_c - G(p_c) = 0. \tag{4.8}$$

By (1.20), (4.8) holds also when p_c is replaced by z_c. If $p_c \neq z_c$, then it follows from the mean-value theorem that

$$z_c - p_c = G(p_c) - G(z_c) = -(z_c - p_c) \sum_{m=2}^{\infty} \partial_z g_m(0; t) \tag{4.9}$$

for some $t \in (p_c, z_c)$. However, by a bound of Assumption G, the sum on the right side is $\mathcal{O}(\beta)$ uniformly in $t \leq z_c$. This is a contradiction, so we conclude that $z_c = p_c$.

4.3 Proof of Theorem 1.3

We begin by noting that

$$\frac{1}{(2R+1)^d} \sum_{y \in C_R(\lfloor x\sqrt{v\sigma^2 n} \rfloor)} p_n(y) = (Q_R * p_n)(\lfloor x\sqrt{n} \rfloor), \tag{4.10}$$

where $Q_R(x) = (2R+1)^{-d}$ for $x \in C_R(0)$, and otherwise equals zero. Therefore,

$$\frac{1}{(2R+1)^d} \sum_{y \in C_R(\lfloor x\sqrt{v\sigma^2 n}\rfloor)} p_n(y) = \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot\lfloor x\sqrt{v\sigma^2 n}\rfloor} \hat{Q}_R(k) f_n(k; z_c). \tag{4.11}$$

Before proceeding with the proof, we first derive some relevant properties of Q_R.

By definition, $|\hat{Q}_R(k)| \leq 1$. The Fourier transform $\hat{Q}_R(k)$ is given in terms of the Dirichlet kernel

$$M(t) = \sum_{j=-R}^{R} e^{ijt} = \frac{\sin[(2R+1)t/2]}{\sin(t/2)} \tag{4.12}$$

by

$$\hat{Q}_R(k) = \prod_{i=1}^{d} \frac{M(k_i)}{2R+1}. \tag{4.13}$$

Since $|M(t)|$ is bounded above by both $2R+1$ and $|\sin(t/2)|^{-1}$, we have

$$|\hat{Q}_R(k)| \leq \frac{1}{(2R+1)\sin(\|k\|_\infty/2)} \leq \frac{\pi}{2\|k\|_\infty R}. \tag{4.14}$$

Let $\kappa \in (0,1)$. When the right side of (4.14) is less than 1, the bound is degraded by raising the right side to the power κ. On the other hand, when the right side is greater than 1, we may still raise it to the power κ and have a correct bound, since the left side is at most 1. Therefore, for every $\kappa \in (0,1)$, we have

$$|\hat{Q}_R(k)| \leq \left[\frac{\pi}{2\|k\|_\infty R}\right]^\kappa. \tag{4.15}$$

We divide the domain of integration $[-\pi,\pi]^d$ of (4.11) into a small-k region $S_n = \{k \in [-\pi,\pi]^d : a(k) \leq \gamma n^{-1}\log n\}$ and a large-k region $L_n = \{k \in [-\pi,\pi]^d : a(k) > \gamma n^{-1}\log n\}$. By Theorem 1.1(a), the integral over the small-k region equals

$$A \int_{S_n} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot\lfloor x\sqrt{v\sigma^2 n}\rfloor} e^{-nDk^2/2d} [1 + \mathcal{O}(k^2 n^{1-\delta}) + \mathcal{O}(k^2 R^2)], \tag{4.16}$$

where we allow constants in error terms to depend on L. For the error terms, we bound the complex exponential by 1 and obtain a factor $n^{-d/2}$ from the integration. For the leading term, we extend the integration domain to $\mathbb{R}^d$ and perform the integral exactly. The error incurred by extending the integration domain is at most $O(n^{-d/2-\gamma^-})$, where γ^- is any positive number less than γ. Therefore, the small-k region gives

$$A\left(\frac{d}{2\pi v\sigma^2 n}\right)^{d/2} \left[e^{-dx^2/2} + \mathcal{O}(n^{-\gamma^-}) + \mathcal{O}(n^{-\delta}) + \mathcal{O}(R^2 n^{-1})\right]. \tag{4.17}$$

The integral over the large-k region can be bounded using (H4), (4.15) and (1.11) by

$$K_4 n^{-d/2} \int_{L_n} \frac{d^d k}{(2\pi)^d} |\hat{Q}_R(k)| a(k)^{-2-\rho} \leq \mathcal{O}(n^{-d/2} R^{-\kappa}) \int_{L_n} d^d k \, k^{-4-2\rho-\kappa}. \tag{4.18}$$

The integral is bounded provided $\kappa < d - 4 + 2\rho$. By (2.2), we may choose a positive κ obeying this bound.

Therefore, the left side of (4.11) is given by

$$A \left(\frac{d}{2\pi v \sigma^2 n} \right)^{d/2} \left[e^{-dx^2/2} + \mathcal{O}(R^{-\kappa}) + \mathcal{O}(n^{-\gamma^-}) + \mathcal{O}(n^{-\delta}) + \mathcal{O}(R^2 n^{-1}) \right]. \tag{4.19}$$

For x^2 less than a sufficiently small multiple of $\log R$, this has the desired asymptotic form, provided $R = R_n \to \infty$ as $n \to \infty$, with $R_n = o(n^{1/2})$. $\square$

4.4 Formulae for z_c, A, v

The identity (1.20) follows after we let $n \to \infty$ in (3.3), using Lemma 3.1.

To determine A, we use a summation argument. Let $\chi_n = \sum_{k=0}^{n} f_k(0)$. By (1.1),

$$\begin{aligned} \chi_n &= 1 + \sum_{j=1}^{n} f_j(0) = 1 + \sum_{j=1}^{n} \sum_{m=1}^{j} g_m(0) f_{j-m}(0) + \sum_{j=1}^{n} e_j(0) \\ &= 1 + z\chi_{n-1} + \sum_{m=2}^{n} g_m(0) \chi_{n-m} + \sum_{m=1}^{n} e_m(0). \end{aligned} \tag{4.20}$$

Using (3.3) to rewrite z, this gives

$$f_n(0) = \chi_n - \chi_{n-1} = 1 + \zeta_n \chi_{n-1} - \sum_{m=2}^{n} g_m(0)(\chi_{n-1} - \chi_{n-m}) + \sum_{m=1}^{n} e_m(0). \tag{4.21}$$

By Theorem 1.1(a), $\chi_n \sim nA$ as $n \to \infty$. Therefore, using Lemma 3.1 to bound the ζ_n term, taking the limit $n \to \infty$ in the above equation gives

$$A = 1 - A \sum_{m=2}^{\infty} (m-1) g_m(0) + \sum_{m=1}^{\infty} e_m(0). \tag{4.22}$$

With (1.20), this gives (1.21).

To prove (1.22), we use (4.1), (2.1) and Lemma 2.5 to obtain

$$v = \lim_{n \to \infty} v_n = \frac{-\sigma^{-2} \sum_{m=2}^{\infty} \nabla^2 g_m(0)}{1 + \sum_{m=2}^{\infty} (m-1) g_m(0)}. \tag{4.23}$$

Equation (1.22) then follows, once we rewrite the denominator using (1.20).

A Appendix: Example of D_L

In this Appendix, we verify that the function D_L defined in (1.14) obeys the requirements of Assumption D. The bounds of (1.9) and (1.10) follow easily, and clearly $\hat{D}_L(0) = 1$. We therefore concentrate on establishing (1.11) and (1.12)–(1.13). Before proceeding, we first note that it is sufficient to prove the lower bounds of (1.11) and (1.12) respectively for $\|k\|_\infty \leq bL^{-1}$ and $\|k\|_\infty \geq bL^{-1}$, for any small positive b, since together these bounds imply the corresponding statements with $b = 1$. We will prove this modified form of the bounds.

For the upper bound of (1.11), we use only the symmetry of D_L and the inequality $1-\cos t \leq \frac{1}{2}t^2$ to obtain

$$0 \leq a_L(k) = \sum_{x \in \mathbb{Z}^d} [1 - \cos(k \cdot x)] D_L(x) \leq \frac{1}{2} \sum_{x \in \mathbb{Z}^d} (k \cdot x)^2 D_L(x) = \frac{1}{2} k^2 \sigma^2 \quad (k \in [-\pi, \pi]^d). \tag{A.1}$$

For a lower bound, we use $1 - \cos t \geq \frac{1}{2}t^2 - \text{const.}t^{2+2\epsilon}$, (1.9) and (1.14) to conclude that there is a small positive b such that

$$a_L(k) \geq \frac{1}{2} k^2 \sigma^2 - \text{const.} k^{2+2\epsilon} L^{2+2\epsilon} \geq \text{const.} L^2 k^2 \quad (\|k\|_\infty \leq bL^{-1}). \tag{A.2}$$

For (1.12)–(1.13), it suffices to show that $\hat{D}_L(k)$ is bounded away from 1 for $\|k\|_\infty \geq bL^{-1}$ and bounded away from -1 for $k \in [-\pi, \pi]^d$. Let $\tilde{h}(k) = \int_{\mathbb{R}^d} h(x) e^{ik \cdot x} d^d x$ denote the Fourier integral transform of h. Then it suffices to show that

$$|\hat{D}_L(k/L) - \tilde{h}(k)| \to 0 \tag{A.3}$$

uniformly in $k \in [-L\pi, L\pi]^d$, since the characteristic function $\tilde{h}$ of a piecewise continuous probability density is bounded away from -1, and is bounded away from 1 on the complement of any compact neighbourhood of 0. In the remainder of the proof, we prove (A.3).

Let X_L be a discrete random variable with probability mass function

$$\mathbb{P}(X_L = z) = D_L(Lz) = \frac{h(z)}{\sum_{w \in L^{-1}\mathbb{Z}^d} h(w)} \quad (z \in L^{-1}\mathbb{Z}^d). \tag{A.4}$$

Then X_L has characteristic function $\hat{D}_L(k/L)$. By definition,

$$\hat{D}_L(k/L) = \frac{\sum_{z \in L^{-1}\mathbb{Z}^d} h(z) e^{ik \cdot z}}{\sum_{z \in L^{-1}\mathbb{Z}^d} h(z)}. \tag{A.5}$$

The right side involves Riemann sums, and hence

$$\lim_{L \to \infty} \hat{D}_L(k/L) = \tilde{h}(k) \quad \text{for } k = \mathcal{O}(1). \tag{A.6}$$

We want to extend (A.6) to uniform convergence for $k \in [-L\pi, L\pi]^d$.

Given $y \in \mathbb{R}^d$, let z_y denote the closest point in $L^{-1}\mathbb{Z}^d$ to y, with a fixed arbitrary rule used to choose from among the closest points when there is not a unique closest point. Let Y_L be a continuous random variable with density h_L given by

$$h_L(y) = \frac{L^d h(z_y)}{\sum_{z \in L^{-1}\mathbb{Z}^d} h(z)} \quad (y \in \mathbb{R}^d). \tag{A.7}$$

Thus Y_L has a density which is a piecewise constant analogue of the probability mass function of X_L. It follows that Y_L has the same distribution as $X_L + U_L$, where U_L is uniform on $[-\frac{1}{2L}, \frac{1}{2L}]^d$ and independent of X_L. Let $\tilde{h}_L$ and $\tilde{u}_L$ respectively denote the characteristic functions of Y_L and U_L. Then

$$\hat{D}(k/L) = \frac{\tilde{h}_L(k)}{\tilde{u}_L(k)} \quad (k \in \mathbb{R}^d). \tag{A.8}$$

We claim that

$$|\tilde{h}_L(k) - \tilde{h}(k)| \to 0, \qquad \text{uniformly in } k \in \mathbb{R}^d. \tag{A.9}$$

To see this, we fix $\epsilon > 0$ and choose R, independent of L, such that $\int_{\mathbb{R}^d \setminus [-R,R]^d} |h(y)| d^d y < \epsilon$ and $\int_{\mathbb{R}^d \setminus [-R,R]^d} |h_L(y)| d^d y < \epsilon$. This is possible since h is integrable, using the monotonicity assumption on h. Since h is almost everywhere continuous, for almost all $y \in \mathbb{R}^d$ we have

$$\lim_{L \to \infty} h_L(y) = h(y). \tag{A.10}$$

Therefore, by dominated convergence, $\int_{[-R,R]^d} |h(y) - h_L(y)| d^d y < \epsilon$ for L sufficiently large. The claim (A.9) then follows from the inequality $|\tilde{h}_L(k) - \tilde{h}(k)| \leq \|h_L - h\|_1$.

By definition,

$$\tilde{u}_L(k) = L^d \int_{[-\frac{1}{2L}, \frac{1}{2L}]^d} e^{ik\cdot x} d^d x = \prod_{i=1}^d \frac{2L \sin\left(\frac{k_i}{2L}\right)}{k_i}. \tag{A.11}$$

Therefore, for any fixed $\alpha > 0$, $\tilde{u}_L(k) \to 1$ uniformly in k such that $\|k\|_\infty \leq L^{1-\alpha}$. With (A.8) and the uniform convergence of (A.9), this proves that the convergence in (A.3) holds uniformly in k such that $\|k\|_\infty \leq L^{1-\alpha}$.

It remains to consider the case $\|k\|_\infty \geq L^{1-\alpha}$. By (A.11), $\tilde{u}_L(k) \geq (2\pi^{-1})^d$ uniformly in $k \in [-L\pi, L\pi]^d$. Also, by the Riemann–Lebesgue lemma, $\lim_{L\to\infty} \sup_{\{k : \|k\|_\infty \geq L^{1-\alpha}\}} \tilde{h}(k) = 0$. Therefore, by (A.8) and (A.9),

$$|\hat{D}(k/L) - \tilde{h}(k)| \leq \left(\frac{\pi}{2}\right)^d |\tilde{h}_L(k)| + |\tilde{h}(k)| \to 0 \tag{A.12}$$

uniformly in k such that $\|k\|_\infty \geq L^{1-\alpha}$. This completes the proof of (1.12)–(1.13).

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