REMARKS ON SOBOLEV REGULARITY IN ANOSOV SYSTEMS

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ABSTRACT. We consider the problem of expressing the regularity of different objects which appear in Anosov systems in terms of Sobolev spaces. We show in examples that one cannot expect any improvements in the regularity of Anosov splitting from the classical results that give regularity in $W^{r_0,\infty}$. We also develop a Sobolev regularity theory for solutions of cohomology equations, both commutative and non-commutative, and bootstrap of regularity arguments from Sobolev to other classes for a wide variety of functional equations involving Anosov systems.

1. INTRODUCTION

The goal of this paper is to give a public answer to a very natural question that has been asked me repeatedly, specially in recent times and has also appeared in papers (e.g. [DS].)

Roughly speaking, the question is whether it is possible to use Sobolev spaces in place of the more customary Hölder spaces for the study of the regularity of dynamical objects appearing in Anosov systems. This would be particularly useful in problems in dynamical systems or integral geometry which use both techniques from PDE's and from dyanical systems.

In this note, we will discuss the application of Sobolev spaces to several regularity issues appearing in the theory of Anosov systems. In particular, we will study Sobolev regularity for:

- The Anosov splitting and invariant section theorems. (In particular we will have to give special attention to the regularity of the Jacobians of the Anosov foliations.
- The solutions of Livsic cohomology equation, both commutative and non-commutative.
- The transfer operators of cohomologous cocycles and conjugacies among Anosov systems.

In all this cases, we will prove theorems and provide counterexamples that show that, at least in certain directions, the results we present are optimal.

We will also show that there is a critical regularity r_1 such that if the invariant objects are $W^{r_1,p}$ for any p, then the objects are essentially as smooth as the maps (one or two derivatives may be lost, depending on the objects we consider, to technical details). These bootstraps of regularity are Sobolev counterparts of the bootstraps that occurred

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in [Has92], [dlL92] in classical C^r spaces. The critical value r_1 in the Sobolev scale is the same as was found in the C^r scale.

One topic that we have not included is the study of the Mather spectrum of the push-forward acting on sections, when the spaces of sections are endowed with Sobolev norms. This will be covered in revised versions of [dlL98].

One insight that seems to come from all the answers is that the functions that are invariant under Anosov systems seem to be rather lacunary (in the sense of harmonic analysis) and that therefore, the spaces where they belong seem to be computable from just scaling arguments. If this were true, little would be gained from using Sobolev (or other scales of) spaces except the discussion of the borderline cases. Nevertheless, if confirmed, other developments could follow since the theory of lacunary functions is quite well developed in harmonic analysis. Of course, the borderline cases, besides their intrinsic interest, seem to appear sometimes in geometric problems.

We call attention to the fact that in this paper we have only considered the regularity of solutions of equations involving Anosov systems when the systems themselves are C^r. One series of questions we have not addressed is the study of the regularity when the Anosov systems themselves have regularities that are measured in Sobolev classes. We think that this further study could be rewarding. Already the theory of existence of solutions of differential equations with Sobolev coefficients has very interesting results and applications to fluid dynamics and conservation laws (See [DL89a],[DL89b]). For example, if we consider the Lagrangian transport in equations in hydrodynamics, the regularity of the vector field is best measured in Sobolev or Besov classes. Many of the problems considered in some modern papers in hydrodynamics would be quite easily recognized by practitioners in dynamical systems (See e.g. [Vis98].) The composition operator – crucial for dynamical systems and for cohomology equations – has been studied in Sobolev regularities in [LdC90, LdC97]. Unfortunately, at this moment, we have few results in the direction of dynamical systems which are in Sobolev class. (A modest result in this direction is part D) of Theorem 9.)

2. Preliminaries and notation

The facts that we have used about Sobolev spaces are not very sophisticated by the standards of practitioners of Partial Differential Equations and can be found in almost any advanced textbook in partial differential equations. General references that we have found useful are [Ada75, Nik61, Ste70, Tay96, Tay97]. Needless to say, there are many other places which also contain the facts that we will use and these are not the original references for the results.

For the sake of simplicity, in this note we will consider mainly Sobolev spaces of integer order. We recall that we define the Sobolev spaces in $\mathbf{R}^n$ by:

Definition 1. *When k is an integer, $1 \leq p \leq \infty$, we define*

$$\varphi \in W^{k,p}(\mathbf{R}^n) \Leftrightarrow \partial_x^\alpha \varphi \in L^p(\mathbf{R}^n) \qquad 0 \leq |\alpha| \leq k$$

where the derivatives are taken in the sense of distributions. We consider $W^{k,p}$ endowed with the norm,

$$\|\varphi\|_{W^{k,p}} = \sum_{|\alpha| \le k} \|\partial_x^\alpha \varphi\|_{L^p}$$

which makes it into a Banach space.

An immediate characterization of these spaces is that instead of partial derivatives along the coordinate axis, we can take derivatives along many finite sets of smooth vector fields which span the whole space. This definition generalizes to manifolds without trouble and is the one we will consider.

An further important characterization of Sobolev spaces is the following well known ([Ste70], p.303, Theorem 3).

Lemma 1. *Assume that $1 < p < \infty$, $k \in \mathbf{N}$. Then*

$$\varphi \in W^{k,p} \Leftrightarrow (-\Delta + \mathrm{Id})^{k/2} \varphi \in L^p \tag{2.1}$$

Moreover, for some $C > 0$

$$C^{-1} \|(-\Delta + \mathrm{Id})^{k/2} \varphi\|_{L^p} \le \|\varphi\|_{W^{k,p}} \le C \|(-\Delta + \mathrm{Id})^{k/2} \varphi\|_{L^p}$$

An extension of this characterization will play an important role in our work (see Lemma 3).

We call attention to the fact that this characterization cannot be extended to the cases $p = 1$, $p = \infty$ and that indeed there are well known counterexamples (See e.g. the discussion in [Ste70] p. 304)

When k is an even integer the meaning of $(-\Delta + \mathrm{Id})^{k/2}$ is clear as a differential operator. When k is an odd integer, it can be interpreted using the calculus of self-adjoint operators and, more importantly, there are integral kernel representations (called the Bessel potentials in [Ste70]). These integral representations are quite explicit in Euclidean spaces.

Using the operators $(-\Delta + \mathrm{Id})^{k/2}$ for non-integer values of k, one can define Sobolev spaces of fractional order. $W^{k,p} = \{u | (-\Delta + \mathrm{Id})^{k/2} u \in L^p\}$ with the obvious norm. (These spaces are called the "potential spaces" in [Ste70]) Indeed, k can be taken to be complex. (Another useful scale of spaces that interpolates the integer valued Sobolev spaces is the so-called Besov spaces.)

One useful fact about Sobolev spaces defined in this way is that they are interpolation spaces in the k variable (See [Tay96] p. 275 ff.) This has the consequence:

Lemma 2. *For every p, $1 \le p \le \infty$, every $k < l \in \mathbf{R}$, every $0 \le \theta \le 1$, every $u \in W^{l,p}$ we have.*

$$||u||_{W^{\theta k + (1-\theta) l, p}} \le K ||u||^{\theta}_{W^k} ||u||^{1-\theta}_{W^l} \tag{2.2}$$

Moreover, if a linear operator T maps $W^{k,p}$ into itself and $W^{l,p}$ into itself, we have that it also maps $W^{\theta k + (1-\theta) l, p}$ into itself and we have:

$$||T||_{W^{\theta k + (1-\theta) l, p}} \le K ||T||^{\theta}_{W^k} ||T||^{1-\theta}_{W^l} \tag{2.3}$$

Since we are going to be working with dynamical systems and consider compositions with highly iterated diffeomorphisms it will be significantly easier for us to use only differential operators and not integral kernels since the change of variable formulas are much easier to prove and to handle (they reduce to the chain rule). Hence our general regularity results will be restricted to Sobolev spaces of integer (and even) order. On the other hand, our counterexamples will be constructed in easy to understand manifolds such as tori and, hence, for the counterexamples, we will use fractional Sobolev spaces, which make the results sharper.

Restricting to integers k in the definition of Sobolev spaces and to even k in Lemma 1 and its analogues will lead to results which are sometimes less than what one can conjecture is optimal by one or two derivatives. (This lack of optimality in our results can be exhibited — and mitigated — by applying interpolation as we will detail later in Remark 21).

This loss of one derivative in our results on Sobolev regularity in Livsic theorem can presumably be remedied by developing more geometric version of the integral characterizations of Sobolev spaces and studying their behavior under iterates of the dynamical systems. This seems to us quite doable, but we decided not to pursue this research in this note.

We will also use the notation from [Ste70] of Λ_r for the Lipschitz spaces. These spaces agree with the usual C^r spaces when r is not an integer and C^r means the space of $[r]$-times differentiable functions whose $[r]$ derivatives are Hölder continuous of exponent $r - [r]$. For integer r, they are different both from the spaces of r times continuously differentiable functions and from the $r-1$ times differentiable functions with Lipschitz derivatives.

A very important fact about Sobolev spaces is the Sobolev imbedding theorem (see [Ste70], V 2.2 for the first part and [Tay97] p. 29 Proposition 8 for the second part). We only state it under the hypothesis that we will need it.

Lemma 3. *For compact manifolds of dimension n,*

$$W^{k,p} \subset L^q \tag{2.4}$$

when

$$\frac{1}{q} \le \left[\frac{1}{p} - \frac{k}{n}\right]_+ . \tag{2.5}$$

(where we denote by $t_+ = \max(t, 0)$) and the inclusion operator is continuous. (The case that the R.H.S. of (2.5) is zero, can be taken to mean that $q = \infty$ is allowed.)

When $r = k - \frac{k}{p} > 0$ we have

$$W^{k,p} \subset \Lambda_r$$

This allows us to show:

Lemma 4. *When $\frac{1}{p} - \frac{k}{n} = 0$, and the manifold is compact, the space $W^{k,p}$ is a Banach algebra under multiplication.*

Proof. Recall that, by Leibnitz rule

$$D^k(\varphi \cdot \psi) = \sum D^i\varphi D^{k-i}\psi \binom{k}{i} .$$

By Sobolev's theorem

$$\begin{aligned} D^i\varphi \in L^{q_\varphi} \quad &\text{with} \quad \frac{1}{q_\varphi} = \frac{1}{p} - \frac{k-i}{n} , \\ D^{k-i}\psi \in L^{q_\psi} \quad &\text{with} \quad \frac{1}{q_\psi} = \frac{1}{p} - \frac{k-(k-i)}{n} \end{aligned}$$

By Hölder's inequality

$$D^i\varphi D^{k-i}\psi \in L^q$$

with

$$\frac{1}{q} = \frac{1}{q_\varphi} + \frac{1}{q_\psi} = \frac{1}{p} - \frac{k}{n} + \frac{i}{n} + \frac{1}{p} - \frac{k}{n} + \frac{k-i}{n} = 2\left[\frac{1}{p} - \frac{k}{n}\right] + \frac{k}{n} = \frac{1}{p}$$

by our assumption.

The case when the number of derivatives is zero is special. We need that we are in a compact manifold to show that $\varphi \in L^\infty$, $\psi \in L^\infty$ and hence, the product is in $L^\infty \subset L^p$. ☐

3. The invariant section theorem

A natural place to start the investigation of Sobolev regularities in dynamical systems is the celebrated invariant section theorem from [HP70]. This theorem has become a basic tool in the study of invariant objects in dynamical systems.

We reproduce the statement of [HP70] supplementing it with one observation c).

Theorem 1. *Let $p : Y \to X$ be a vector bundle over a metric space X. Let $D \subset Y$ be the unit ball bundle and $F : D \to D$ a map covering a homeomorphism $f : X \to X$. Suppose $0 \le \kappa < 1$ and that, for each $x \in X$, the restriction $F_x : D_x \to D_{f(x)}$ has Lipschitz constant $\le \kappa$. Then:*

a) *There is a unique section $g_0 : X \to D$ whose image is invariant under F.*
b) *Let $\mathrm{Lip}(f^{-1}) = \lambda < \infty$. Let $0 < \alpha \le 1$ be such that $\kappa\lambda^\alpha < 1$.*

Then, g_0 satisfies a Hölder condition of order α. Moreover

c) *Suppose that X is a C^1 manifold, Y is an C^1 vector bundle, and F and f are C^1. If $\kappa\lambda < 1$, then g_0 is C^1.*
d) *Suppose that X is a C^{r+1} manifold, Y is an C^{r+1} vector bundle, and F and f are C^{r+1}. If $\kappa\lambda^{r+\alpha} < 1$, then g_0 is C^r and the r derivative satisfies a Hölder condition of order α.*

Proof. The proofs of a), b), c) is in [HP70]. The proof of point *d)* follows by applying the *tangent functor trick* [AR67]. ☐

In the following we would like to explore the possibility that there are some improvements of these results using Sobolev spaces. Note that if the conclusions are that a section is C^r then, we conclude automatically that it is in $W^{r,\infty}$ (this remains true even for fractional r). Hence, the only improvements that we could hope for is that we obtain that it is in a certain $W^{r',p}$ with an $r' > r$.

Remark 1. It is important to notice that the regularity of the invariant section is limited by $r_0 = \log(\lambda)/\log(\kappa^{-1})$. It does not improve assuming more regularity in F, f. We will see later in examples that, as it is well known, this limitation does belong. $\square$

Remark 2. Notice that the Lipschitz constants in the hypothesis depend on the distance that we are considering, but nevertheless, the regularity in the conclusions does not. $\square$

Remark 3. Notice that, since we have uniqueness, an invariant section of F^2 is an invariant section for F.

Hence, it suffices to take κ in such a way that $\kappa^2 \geq \mathrm{Lip}(F_{f(x)} \circ F)$ and λ such that $\lambda^2 \geq \mathrm{Lip}(f^{-2})$. More generally,

$$\kappa > \inf_i \left(\mathrm{Lip}(F_{f^{i-1}(x)} \circ \cdots F_x)\right)^{1/i} \qquad \lambda > \inf_i \left(\mathrm{Lip}(f^i)\right)^{1/i} \tag{3.1}$$

Note also that

$$\begin{aligned}
&\sup_x(\mathrm{Lip}\, F_{f^{i+k}(x)} \circ \cdots \circ F_x) \\
&\qquad \leq \sup_x(\mathrm{Lip}\, F_{f^{i+k}(x)} \circ \cdots \circ F_{f^i(x)}) \cdot \sup_x(\mathrm{Lip}\, F_{f^{i-1}(x)} \circ \cdots \circ F_x) \\
&\qquad = \sup_x(\mathrm{Lip}\, F_{f^k(x)} \circ \cdots \circ F_x) \cdot \sup_x(\mathrm{Lip}\, F_{f^{i-1}(x)} \circ \cdots \circ F_x)
\end{aligned}$$

Hence by a subadditivity argument for the logarithms, the infimum in (3.1) is actually the limit.

A similar argument works for λ.

Hence, if we denote

$$\begin{aligned}
\tilde\lambda(f^{-1}) &= \lim_{i\to\infty}(\mathrm{Lip}\, f^{-i}(x))^{1/i} \\
\tilde\kappa(F) &= \lim_{i\to\infty}(\sup_x \mathrm{Lip}\, F_{f^i(x)} \circ \cdots \circ F_x)^{1/i}
\end{aligned}$$

we have that $\tilde\kappa, \tilde\lambda$ can be used in place of κ, λ in Theorem 1.

Notice that, in contrast with κ, λ the numbers $\tilde\kappa, \tilde\lambda$ do not change if the distances in X, Y are changed by an equivalent metric.

One should also notice that the $\tilde\kappa$, $\tilde\lambda$ are the values of the spectral radius of push-forward operators in continuous vector fields. (See [Mat68].)

Hence, if we are going to produce examples showing that the results of Theorem 1 cannot be improved even after the refinements contained in this remark, we will need to make sure that we choose norms in such a way that the above limit is exactly the value after a few steps. $\square$

Our first example shows that one cannot get any improvements using Sobolev spaces. That is, we cannot get Sobolev spaces that contain a larger number of derivatives than those predicted by the classical result.

Example 1. *Consider the map of $\mathbf{T}^2$ to itself $A(x,y) = (2x+y, x+y)$. (As it is well known, A has eigenvalues λ, λ^{-1} with $\lambda > 1$, an irrational number. The eigenvectors of A are irrational and orthogonal, since A is symmetric.)*

Consider now the trivial bundle $\mathbf{T}^2 \times \mathbf{C}$ And the map defined on sections by

$$[\mathcal{L}v](x) = \mu v(Ax) + P(x,y) \tag{3.2}$$

for some $0 < \mu < 1$ and were $P(x,y)$ is $P(x,y) = \sigma \exp(2\pi i x)$, with σ any number different from zero.

Let $r_0 = -\log\mu/\log\lambda$.

As predicted by Theorem 1, the invariant section for this map is in $\Lambda_{r_0-\delta} \subset W^{r_0-\delta,\infty}$ for any $\delta > 0$. Nevertheless it is not in $W^{r_0,p}$ for any $1 \le p \le \infty$. (Also, we note it is not in $\Lambda_{r_0+\delta}$.)

Another sense in with the Sobolev regularity does not lead to improvements will be developed in Section 8. We will show that if an invariant section happens to be in $W^{r_1,p}$, where r_1 is a number similar to r_0 depending on expansion rates, etc. then, it is roughly smooth as the map. We postpone a detailed discussion since even a precise formulation will need the development of more notation.

Remark 4. It will be clear after the proof that the example can be considerably extended. One can use other trig polynomials, other Anosov systems. Indeed, the behavior we report should be quite typical. At least, it is C^ω generic in the spaces of skew products. Of course, one can obtain similar results for real bundles rather than complex just by taking real parts of our example. The complex example has been chosen to simplify the notation. A purely real example can be obtained taking real parts since our example is linear. □

Remark 5. Note that if we consider the map $F_0 : \mathbf{T}^2 \times \mathbf{C}$

$$F_0(x,y,z) = \left(A\begin{pmatrix} x \\ y \end{pmatrix}, \mu z \right)$$

then, the set $\{(x,y,0)\}$ is an invariant normally hyperbolic manifold in the sense of [Fen74], [HPS77].

The image of the section invariant by $\mathcal{L}$ in (3.2) is an invariant manifold for

$$F_\sigma(x,y,z) = A\begin{pmatrix} x \\ y \end{pmatrix},\ \ \mu z + \sigma P(x,y)$$

Hence, Example 1 also shows that the regularity of the normally hyperbolic manifolds is, in general, limited by the ratios of expansion rates irrespectively of the regularity of the map.

We also note that this lack of regularity happens even if μ is not resonant with the spectrum of A. (see the Remark 6 later). □

Proof. In this very simple case, sections are just complex valued functions on the torus. Note that a section Ψ has an invariant image if and only if it satisfies:

$$\Psi \circ A = \mu\Psi + P \tag{3.3}$$

We first claim that the invariant section of (3.2) is the graph of

$$\Psi(x, y) = \sum_{i=0}^{\infty} \mu^i P \circ A^i \tag{3.4}$$

Where the sum is understood in a weak enough sense. For example, in C^0.

Indeed, to establish the claim, note that the sum in (3.4) converges uniformly (by Weierstrass M-test) and that the uniform convergence allows one to rearrange terms and verify that indeed it satisfies (3.4) when substituted in.

Note that one of the consequences of the invariant section theorem is that it is the only bounded invariant section.

We can compute the Fourier coefficients of Ψ by:

$$\hat{\Psi}_k = \sum_{i=0}^{\infty} \mu^i \hat{P}_{A^{t-i}k}$$

(In the case that we have considered in the example, of course, $A = A^t$.)

Since the eigenvectors of A are irrational, any vector $k \in \mathbf{Z}^2$ has non-zero components both on the stable and unstable components, hence $|(A^t)^i k| \geq C_k \lambda^i$.

Note that in our example for $k^{(i)} = (A^t)^i \binom{1}{0}$ we have $\hat{\Psi}_{k^{(i)}} = \mu^i$.

If we consider $\Phi = (-\Delta + 1)^{r/2}\Psi$ – of course in the sense of distributions – we see that $\hat{\Phi}_{k^{(i)}} = \mu^i(4\pi^2|k^{(i)}|^2 + 1)^{r/2}$.

We see that if $\lambda^r\mu \geq 1$, that is if $r \geq r_0$, then $\hat{\Phi}_{k^{(i)}}$ does not tend to zero as i tends to infinity. This shows, by Riemann–Lebesgue theorem that Φ cannot be in any L^p space and establishes the conclusions. □

Remark 6. Lead by the results in Sternberg theorem one could conjecture that, when the Mather spectrum of the map restricted to the manifold is non resonant with the Mather spectrum of the map in the normal direction, one could improve the persistence theorem of normally hyperbolic manifolds to produce more derivatives.

Indeed, if one assumes that k derivatives of v exist, one obtains that they should satisfy: (3.3), we obtain:

$$D^k v = \mu(D^k v) \circ A\, A^{\otimes k} + D^k P \tag{3.5}$$

which is equivalent to

$$\mu^{-1}(D^k v) \circ A^{-1}\, (A^{\otimes k})^{-1} - D^k P \circ A^{-1} = D^k v \tag{3.6}$$

We note that if we considers (3.5) as an equation for the D^k it is not difficult to arrange that it has solutions. (It is easy to arrange that for each partial derivative, either (3.5) or (3.6)) define a contraction operator and, therefore, have a continuous solution.

Nevertheless, these candidates for a derivative fail to be the true derivatives as it was established in the Example 1.

This is in sharp contrast to what happens in the Sternberg linearization theorem where the solutions of the equations satisfied by the derivatives – assuming that they exist – are indeed derivatives. In our case, the solutions of (3.5) and of (3.6) exist but are not derivatives.

This phenomenon can be understood by noting that the solution of one equation is obtained iterating the system in one direction while the other is obtained by iterating in the opposite. □

Remark 7. It is instructive to note that the invariant section that we have produced is rather *lacunary*. The coefficients tend to group in different scales. This makes it clear that the spaces to which it belongs are going to be determined by rather simple scaling properties. We believe that this is quite typical from the objects invariant under hyperbolic systems or otherwise related to them. Indeed, it seems that it is this lacunarity what makes it possible certain properties such as decays of correlations, central limit theorem. If this intuition could be made more substantial, it suggests that many of the methods of harmonic analysis developed for lacunary functions could be useful for hyperbolic systems.

Note that for these functions, the spaces in a certain scale that they belong to can be ascertained (except for the borderline cases) by simple scaling arguments. □

4. The Anosov splitting

One of the most common proofs of the regularity of the invariant Anosov splitting is by an appeal to the invariant section theorem [HP70]. Of course, one could have hopes that in this most restrictive situation, the examples for the invariant section theorem did not apply. Indeed, there are other proofs of the Hölder regularity of the Anosov splitting that follow another route [Fen74], [Fen77], which produce better regularity results than the invariant section theorem.

Nevertheless, it turns out that one can produce essentially the same phenomena as in Example 1.

We will denote by $E^{s,(f)}$, $E^{u,(f)}$ the stable and unstable bundles of the Anosov map f. When there is no danger of confusion with the system we are considering, we will suppress the (f) from the index, as it is customary.

Example 2. *Consider $\mathbf{T}^{k+l} = \mathbf{T}^k \times \mathbf{T}^l$ with $l, k \geq 2$ and write a point in $\mathbf{T}^{k+l}$ as (x, y) with $x \in \mathbf{T}^k$, $y \in \mathbf{T}^l$.*

Take f of the form

$$f(x, y) = \big(Ax, B(y) + e_u\varphi(x)\big) \tag{4.1}$$

where:

1) *A and B are Anosov linear automorphisms of $\mathbf{T}^k$ and $\mathbf{T}^l$ respectively.*
2) *e_u is an unstable eigenvector of B.*
3) *$\varphi : \mathbf{T}^k \to \mathbf{R}$ is an appropriate function. (We will show that choosing φ dense set of analytic functions works.)*

We can also arrange – to make the example more interesting in view of Remark 2 – that

4) *The matrices A, A_s, B are such that their norm agrees with their spectral radius. (e.g. A,B are symmetric)*

Denote by γ the eigenvalue of B corresponding to e_u, and by A_s, A_u, B_s, B_u the restriction of A, B to their stable and unstable subspaces.

Then, for every $\delta > 0$, the stable splitting of f is in $W^{r_0-\delta,\infty}$, where $r_0 = (-\log ||A_s|| + \log(\gamma))/\log ||A^{-1}||$ Nevertheless, it is not in $W^{r_0,p}$ for any $1 \le p \le \infty$.

Remark 8. Note that, as remarked in [dlL92], the eigenvalues of the derivatives at periodic points are just powers of those of A and B. If those are chosen to be nonresonant, the return map at every fixed point has a germ which is linearizable. (By Sternberg theorem.)

In particular, the obstructions for regularity of foliations found in [Ano69] p. 193 ff. vanish.

Indeed, any local obstruction to regularity of invariant foliations vanishes. (Note that the only local obstructions to regularity need to come from periodic orbits, since aperiodic orbits can be reduced to constants in a neighborhood.)

We call attention that, in lower dimensional systems, the vanishing of the Anosov obstructions indeed implies that the foliations are regular (See [HK90].) □

Remark 9. This example originated in [dlL92] where it was used as a counterexample to some problems in rigidity theory. In [JPdlL95] it was also used as a counterexample to integrability with smooth leaves of distributions. In [NT98], following a suggestion of A. Katok, it is shown that the derivatives of this example also provide counterexamples to improvements of their results in regularity of transfer operators. We will consider similar remarks in Section 9. □

Remark 10. One could hope that by restricting the class of Anosov systems one considers one could find a class of Anosov systems for which the Sobolev regularity indeed leads to better results.

We also note that this example starts to work only in dimension 4 or higher. We do not know of similar examples in dimensions 2 and 3.

We have not investigated these possibilities in detail, for symplectic and other geometric structures. Note however that Example 2 is volume preserving.

We conjecture, however, that the conclusions of optimality of Sobolev regularity will remain to valid if we consider more restrictive classes of Anosov systems such as symplectic or even geodesic flows or if we consider two or three dimensional maps.

One particularly interesting place to study these questions is in the examples constructed in [HW97], [HW99]. We conjecture that these examples also have the same lacunary effect and that the regularity cannot be improved in the Sobolev classes. □

Remark 11. We note that, even if in Example 2 we showed that the results of applying Theorem 1 are sharp, there are many interesting situations where the results of [Fen74],[Fen77] provide better regularity results than the invariant section theorem. The main reason is because the regularity produced in Theorem 1 is the ratio of the supremum of logarithm of contraction and the infimum of the expansion in some bundles. The method of [Fen74], [Fen77] considers these ratios over individual orbits and then takes the supremum. Of course, taking the supremum after taking the ratios is going to lead to sharper results. Moreover, it happens sometimes in geometrical problems that the presence of a geometric structure forces cancellations along the orbits. These cancellations do not show up when we take suprema over the expansions on the bundle.

We should also note that one could also improve further the results in [Fen74],[Fen77] by noting that one could take the asymptotic ratios [HK90].

Of course, for examples as simple as the ones we have been using, these refinements lead to no improvement. □

Proof. Denote by f_0 the linear Anosov map:

$$f_0(x,y) = (Ax, By)$$

Following standard techniques in the perturbation theory for Anosov splittings, we will find the invariant splitting for f as the graph of a linear bundle map from $E^{s,(f_0)}$ to $E^{u,(f_0)}$.

Note that, since f_0 is a product, we have $E^{s,(f_0)}_{x,y} = E^{s,(A)}_x \oplus E^{s,(B)}_y$ (with the obvious identification of considering $E^{s,(A)}_x$ as a subspace of $T_{x,y}\mathbf{T}^{k+l}$.)

Note moreover, that, since the unperturbed bundles are analytic, the regularity of the splitting is exactly the same as the regularity of the function whose graph gives the splitting.

We will search for the invariant graph only among linear bundle maps

$$W_{(x,y)} : E^{s,(f_0)}_{x,y} \mapsto E^{u,(f_0)}_{x,y}$$

of the form

$$W_{(x,y)}(a,b) = (0, e_u < \Psi(x), a >) \tag{4.2}$$

where (a,b) correspond to the components of $E^{s,(f_0)}_{(x,y)}$ along the direction of the first factor and the second factor respectively, $\Psi : \mathbf{T}^k \mapsto \mathbf{R}^k$ and $< \cdot, \cdot >$ denotes the scalar product.

Since the fixed point of this graph transform is unique, finding a fixed point of the form (4.2) gives us the Anosov splitting.

Note that a point in the graph of a function of the form (4.2) is transformed to:

$$DF(x,y)(a,b,0,e_u < \Psi(x), b >) = (A_s a, B_s b, 0, \gamma < \Psi(x), a > + e_u \nabla\phi(x)a) \tag{4.3}$$

In order that the point in the right hand side of (4.3) is in the graph of the the function given by (4.2), it is necessary and sufficient that:

$$< \Psi(Ax), A_s a > = \gamma < \Psi(x), a > + \nabla\varphi(x)a \tag{4.4}$$

Equivalently:

$$\Psi(Ax)A_s^t = \gamma\Psi(x) + \nabla\varphi(x) \tag{4.5}$$

or:

$$\gamma^{-1}\Psi(Ax)A_s^t - \nabla\varphi(x) = \Psi(x) \tag{4.6}$$

We can see by a reasoning very similar to the one we used in Example 1, that

$$\Psi(x) = \sum_{i=0}^{\infty} \gamma^{-i}(\nabla\varphi)(A^i x)(A_s^t)^i \tag{4.7}$$

is the only continuous solution of (4.5). Namely, we observe that the sum in (4.7) converges uniformly on C^0 by Weierstrass M-test and that substituting the sum in (4.6), and rearranging terms, which is allowed by the uniform convergence, we obtain that indeed the Ψ given in (4.7) is a solution – the only continuous one – of (4.6).

Note that the methods used in the study of the invariant section theorem give us that the Ψ given by (4.7) is in Λ_r when $||A||^r||A_s^t||\gamma^{-1} < 1$. That is, when $r \leq r_0$.

Now, to prove the negative results, we note that the Fourier coefficients of Ψ can be computed.

$$\hat{\Psi}_k = \sum_{i=0}^{\infty} \gamma^{-i} \widehat{(\nabla\varphi)}_{(A^t)^{-i}k} (A_s^t)^i \tag{4.8}$$

If we choose to consider a k such that it has a component over the fastest growing eigendirection of $(A^t)^{-i}k$, we have that

$$|(A^t)^{-i}k| \geq ||A^{-1}||^i C$$

If we pick a φ in such a way that $\widehat{(\nabla\varphi)}_k$ has a component on the left eigendirection of A_s^t, we will have:

$$|\widehat{(\nabla\varphi)}_{(A^t)^{-i}k}(A_s^t)^i| \geq C||A^{-1}||^i||A_s^t||^i$$

If we pick φ in such that $\widehat{(\nabla\varphi)}_{(A^t)^{-i}k} = 0$ when $i \neq 0$, and we consider Ψ as in (4.7), we see that $(-\Delta + 1)^{r_0/2}\Psi$ has coefficients that do not go to zero.

By Riemann-Lebesgue theorem, this contradicts that it is in any L^p.

Given the formula (4.8), we can see that the fact that

$$|\hat{\Psi}_k| \geq C|k|^{-r_0} \tag{4.9}$$

holds in an open and dense set of polynomials. Indeed, given (4.8) then, we see that we have (4.9) unless some relations – a linear manifold of positive codimension – holds among the coefficients. Since, we have several choices for k – infinitely many in case that the leading eigenvalue is irrational – we see that we have (4.9) except in a set of large codimension. □

5. Absolute continuity of the Anosov foliations

One of the most important properties of Anosov foliations from the point of view of ergodic theory is that they are absolutely continuous.

Moreover, from the point of view of regularity theory, we need to prove some extra properties of the Jacobian and discuss how are they related to the regularity of the leaves.

In this section, we will study the properties that we will use later. Since these properties are of a geometric nature, and related to the properties of the map – which we always measure in C^r spaces –, the regularity properties we discuss are better expressed in C^r spaces. Of course, if we started to consider the results for Anosov systems whose regularity is measured in Sobolev spaces, we would need to express these results in Sobolev classes.

We recall that the Anosov foliation theorem (see e.g [Ano69]) states that we can find invariant foliations tangent to the stable and unstable spaces. If the map is C^r, the foliations consists of C^r leaves, the r-jet of the leaves is continuous, the $r-1$ jet is Hölder, and the foliations are absolutely continuous. Moreover, the Jacobian of the foliation is differentiable along the leaves of the foliation.

The following result is obtained combining results of [Ano69] and extensions of them found in [dlLMM86].

Theorem 2. *Let f be a C^r $r = 1, 2, \ldots, \infty, \omega$ Anosov diffeomorphism on a compact manifold M of dimension n. Denote by s the dimension of the stable bundle.*

Then, given any point $p \in M$, we can find a sufficiently small open neighborhood $\mathcal{U}$, $p \in \mathcal{U}$ and a mapping (called a local parameterization of the stable foliation) from

$$\Gamma : U \times V \mapsto \mathcal{U}$$

(where $U \subset \mathbf{R}^s$, $V \subset \mathbf{R}^{n-s}$ are open sets)

The map Γ satisfies that:

1) *For every x in U, $\{\Gamma(x, y) | y \in V\}$ is the connected component of a leaf of the stable foliation restricted to $\mathcal{U}$.*
2) *$\partial_y^\alpha \Gamma$ is Hölder in $U \times V$ for $|\alpha| < r$. It is continuous for $\alpha = r$.*

More geometrically:

2') *The $r - 1$ jets of the stable foliation are Hölder, the r-jet is continuous.*

If, moreover, $r \geq 2$ the local parameterization Γ can also be chosen to satisfy

3) *The mapping Γ is absolutely continuous. That is*
$$|\Gamma(E \times E')| = \int_E dx \int_{E'} dy\, J(x, y)$$
where $|\ \ |$ denotes Lebesgue measure.
4) *$\partial_y^\alpha J(x, y)$ is Hölder for $|\alpha| \leq r - 2$ and continuous for $|\alpha| = r - 1$.*

Analogous results hold for unstable manifolds and for flows. In the later case, we can also consider the center-stable and center-unstable foliations.

Remark 12. The results 1), 2), 2') are quite standard in stable manifold theory. They can be found in [Ano69], [HP70], [HPS77].

Result 3) is the fundamental theorem 3) in [Ano69]. Alternative proofs of related results in terms of holonomy maps appeared in [PS72],[Mn87]. For us, the formulation in terms of parameterizations is more useful since we need to use also 4), which is hard to even formulate in the holonomy maps language.

Result 4) is stated for the C^∞ case in [dlLMM86]. The proof presented there for the C^∞ case is to establish 4) for every r. (Hence the proof presented there establishes the result we are stating here.) The idea of the proof is to examine in more detail the proof of 1) in [Ano69]. Observe that the forms considered in [Ano69] converge not only in the C^0 norm, but also with the derivatives along the foliation.

A very transparent reason why 4) is true is that if we had just one parameterization (as we can do in perturbations of linear flows in the torus), we would have

$$\log J \circ \Gamma^{-1} - \log J \circ \Gamma^{-1} \circ f = \log \det Df|_{E^s} \tag{5.1}$$

Note that the right hand side of (5.1) admits $r-1$ derivatives along the stable directions. The fact that all continuous solutions of an equation of the form (5.1) when f is an Anosov system and the right hand side admits derivatives along the stable directions is one of the parts of the smooth Livsic theorem in [dlLMM86]. See also [NT98], where an argument similar to the one in this paragraph is used to study the regularity of the Jacobian starting with the method of [PS72]. □

6. Regularity classes associated with Anosov foliations

Since the regularity of the foliations is quite asymmetric (quite differentiable along leaves, but only Hölder when we move across leaves) it is clear that it will be useful to define regularity classes that incorporate these regularities.

Along with these regularity classes we can define differential operators adapted to the geometry of the Anosov foliations. The fact that the leaves of the stable a and unstable foliation are differentiable allows us to define differential operators which are tangent to the foliation and which have smooth coefficients along the manifold and continuous over the whole manifold.

The derivatives are to be understood in the usual sense of calculus of differential geometry. Following standard practice, we will refer to this sense of derivatives as the "strong' sense, even if, as we will see, the use of this word is somewhat misleading. Since we are dealing with operators with continuous coefficients, it could well happen that strong derivatives are not weak derivatives. We will show, that, for our case, indeed, these strong derivatives are weak derivatives, except for those of order exactly r.

These operators can be composed (provided that the total order is smaller than r) even if the coefficients are only continuous. This gives the class of operators tangent to the foliation some algebraic structure. Later we will see that one can also define adjoints when the order is up to $r-1$.

Definition 2. *Given a C^r Anosov diffeomorphism f, we say that a function $\Phi : M \mapsto \mathbf{R}$ is in $C_s^{l,(f)}$, $(r \le l)$ when the restrictions of Φ to all the leaves of the stable foliation are uniformly C^l and moreover the l-jets are continuous on the manifold.*

We denote by $\mathcal{D}_s^{k,l,(f)}$ the space of differential operators of order k tangent to the stable foliation with coefficients in $C_s^{l,(f)}$. This makes classical sense when $k, l \le r$. As we will see later (see Remark 13), we can make sense of these operators in the sense of distributions when $k \le l \le r-1$.

When the Anosov system we are referring to is clear from the context — as it will be most of the time in this paper — we will suppress the (f) from the notation.

Analogous definitions hold for the unstable foliation or for flows rather than diffeomorphisms. In the later case, we also consider the center-stable direction, the center-unstable and the derivatives along the flow.

Remark 13. One of the consequences of the regularity of the Jacobian is that it is possible to define the adjoint of an operator in $\mathcal{D}^{k,l}$. Indeed, using a partition of unity, we can reduce to the case when the test functions have support in one of the patches covered by the parameterization

$$
\begin{aligned}
\int d\lambda \Psi(L\varphi) &= \int\int dx\, dy J(x,y)\ \Psi\circ\Gamma(x,y) \sum_{|\alpha|\le k} a_\alpha(x,y)\partial_y^\alpha \varphi\circ\Gamma(x,y) \\
&= \int\int dx\, dy\, J(x,y)\ \varphi\circ\Gamma(x,y) \\
&\qquad \sum_{|\alpha|\le k} (-1)^{|\alpha|}\partial_y^\alpha \left[a_\alpha(x,y)J^{-1}(x,y)\Psi\circ\Gamma(x,y)J(x,y)\right]\, dx\, dy\ .
\end{aligned}
\tag{6.1}
$$

Expanding the derivative we obtain a differential operator in the y variables acting on Ψ. Making back the change of variables, we can see that the adjoint is a differential operator. The coefficients of the highest order terms are as differentiable as those of the original operator. The lower order terms are less differentiable since they involve derivatives with respect to the y of the $a_\alpha(x,y)$. This allows us to define the operators directly in the weak sense.

This is more satisfactory than to define them as combinations of the partial derivatives with respect to the coordinates (understood, of course in the weak sense) multiplied by continuous coefficients.

Note however that since the Jacobian is only $r-1$ times continuously differentiable, this weak definition only makes sense for operators of order $r-1$.

We call attention to the fact that, in our case, even if it is true that it is possible to define derivatives in the strong sense of order up to r, we can define weak derivatives only up to order $r-1$. (Hence, for this case, it is easier to have strong derivatives than weak derivatives!)

Fortunately, as the above argument shows, in the common case, both definitions agree and, more importantly for us, they satisfy the same formulas under changes of variables. □

Definition 3. *Given a real valued function, we denote by $D_s\varphi$ the derivative restricted to the tangent bundle. That is: $D_s\varphi_{(x)}E^s_x \to \mathbf{R}$ is defined by $(Z\varphi)(x) = D_s\varphi(x)Z(x)$ for every vector field tangent to the stable foliation.*

Similar definitions hold for functions taking values in manifolds. In particular, it is possible to define D^i_s for $i \le r$.

Using Remark 13, it is possible to define the derivatives in the weak sense and D^i_s when $i \le r-1$. Since we will not consider the case $i = r$, we will not introduce a different notation for the two different concepts. We will try to explain which one we are using. (Except in some very specific places, it will be the weak derivative.)

Of course, analogous definitions hold for unstable foliations or indeed for any foliation with C^r leaves and continuous jets.

Of fundamental importance from what follows is the next result.

Theorem 3. *Let $\mathcal{F}^1, \cdots, \mathcal{F}^N$ be a finite number of foliations in a compact manifold M. Assume:*

i) *The leaves of the manifold are C^r, r even. The r-jet of the leaves are continuous across M.*

ii) *The foliations are transverse, that is, denoting by* $\mathcal{F}^i_x$ *the leaf of* $\mathcal{F}^i$ *passing through* x*, we have*

$$T_x M = \bigoplus_{i=1}^{N} T_x \mathcal{F}^i_x$$

iii) *The foliations are absolutely continuous and it is possible to find local parameterizations as in 3) of Theorem 2 so that*

$$\partial_y^\alpha J(x, y) \text{ is continuous for } |\alpha| \le k$$

Let φ *be any Schwartz distribution such that*

iv) $D^j_{\mathcal{F}^i}\varphi \in L^p \quad i = 1, \dots, N,\ j = 0, \dots, k,\ 1 < p < \infty$

We conclude

$$\varphi \in W^{k,p}$$

Similarly, if a set of foliations satisfies i) and ii) (not necessarily iii)), given a function φ *satisfying*

$$D^j_{\mathcal{F}^i}\varphi \in \Lambda_0 \subset C^0, \quad 0 \le j \le r$$

where the derivative is understood in the strong sense (since the introduction of weak derivatives required iii)), then

$$\varphi \in \Lambda_k \ .$$

Remark 14. For us, the most important sets of foliations are the stable/unstable foliations in the case of Anosov diffeomorphisms, or for the case of flows the stable, unstable foliations and the foliation consisting of flow lines.

In applications occurring in rigidity of actions, there are situations when one has to consider other sets of foliations. □

Remark 15. The reason why we need that the Jacobian is k times continuously differentiable is so that we can indeed make sense of the notion of weak derivatives along the stable manifold as indicated in the previous remark.

Note that the theories based on strong derivatives do not have this restriction. Strong derivatives can be defined without using the regularity of the Jacobian, just using the smoothness of the leaves. □

Proof. The proof of this theorem is very similar to the proof of Lemma 2.5 in [dlLMM86]. Indeed, the use of Sobolev spaces was already suggested there.

One of the important differences between the line of argument developed here and that in [dlLMM86] is that, in that paper, the derivatives along the foliation were understood in the strong sense whereas in this paper we directly consider weak derivatives throughout.

One advantage of this procedure is that we can obtain uniqueness from the usual elliptic theory. Of course, when we want to apply the result, we need to verify the existence of derivatives in the sense of distributions.

For $k = 1$ the proof is elementary. Note that any first order operator can be expressed as a combination with continuous — hence bounded, since M is compact — coefficients of operators tangent to the foliations. Since the operators tangent to the foliations are bounded in L^p we are done.

The proof for $k > 1$, k as in the hypothesis is significantly more subtle. First we show that using the properties of the Jacobian we can construct elliptic operators $\Box_i$ of order k, which are tangent to the foliation $\mathcal{F}_i$ (here is where we use the fact that k is even). Then $\Box = \sum_i \Box_i$ will be elliptic (by the transversality of the foliations).

Then, we need to apply the following well known result from elliptic regularity theory.

Theorem 4. *Let $\Box$ be an elliptic operator of degree k with continuous coefficients. Assume that $u \in \mathcal{D}'(M)$ — the space of distributions — and that*

$$\Box u = \eta \in L^p \ .$$

Then,

$$u \in W^{k,p} \ .$$

In case that $\eta \in C^0$, then

$$u \in \Lambda_k$$

Proof. The proof of this theorem can be found in [ADN59], Chapter V. but the proof for our case is significantly easier since we do not have to discuss the boundary.

A pedagogical exposition of the argument can be found in [Tay97], p.379 ff. In this book, however, one can only find the result for $H^k \equiv W^{2,p}$ spaces. The only difference in the proof of the result claimed here with the proof in [Tay97] is that the estimates (11.9), (11.25) have to be replaced by the $W^{k,p}$ estimates (see [Ste70], p.77) or the Λ_k estimates, which follow from the theory of Riesz potentials (see [Ste70], p.143). ∎

Applying Theorem 4 to the operators $\Box$, we obtain the desired result in Theorem 3 in the $W^{k,p}$ scale.

For the Λ_r scales, this is weaker that the result claimed in Theorem 3 since the elliptic regularity approach needs assumption iii).

For the Λ_r regularity, there is another approach which does not use elliptic theory and which, therefore, does not require *iii*). It is based on approximation and interpolation [Jou88]. (An earlier version of this approach, which included extra hypothesis is [Jou86].) ∎

Remark 16. In several respects, the approach of [Jou88] is superior to the elliptic approach. It can work with any number of derivatives while the elliptic regularity approach — as it is written here — needs an even integer number. Also, it can deal with derivatives up to order r, while the elliptic regularity approach can deal only with derivatives up to order $r - 1$. (We need to be able to integrate by parts to make sure that we can define weak derivatives.)

More importantly, since the approach of [Jou88] does not use the Jacobian, it does not need that the foliations are absolutely continuous. This allows to consider several situations in dynamics where the foliations are not absolutely continuous. In [dlL92], there are extensions of the approach to situations where the functions are not defined on manifolds, but rather in Cantor sets enjoying certain geometric properties.

On the other hand, the elliptic regularity approach can deal with any number of foliations, while the approach of [Jou88] is restricted to two foliations. Also, the approach in [Jou88] cannot deal with Sobolev spaces.

A third approach to regularity lemmas is the approach of [HK90]. This approach is based on estimating very simply the Fourier coefficients just by expressing the integrals defining the Fourier coefficients in the coordinates giving the trivialization of the foliations.

In order to be able to carry out this step, one also needs the smoothness of the Jacobian.

Since the characterization of regularity in C^r by the size of the Fourier coefficients is not very sharp, this method leads to a loss of the number of derivatives that is roughly the same as the dimension of the manifold. The situation for Sobolev spaces is significantly worse since there are no characterizations of L^p spaces (except $p = 2$) by sizes of Fourier coefficients.

On the other hand, being simple, the approach in [HK90], can work in a variety of circumstances where elliptic theory is somewhat cumbersome. In [dlL97] it was shown how to use it to establish analytic regularity. In that case, losing a finite number of derivatives is of no consequence, and elliptic theory is somewhat difficult for analytic regularity since some basic tools such as cut-offs, stop working. Nevertheless, there are textbook studies of elliptic equations and analytic regularity, which we think can be extended to cover the problems considered here (See [Fri69] p. 199 ff.)

Some extensions of the above methods that I believe would be welcome are:

- We think that it would be quite interesting if the approach in [HK90] could be supplemented by some more sophisticated harmonic analysis (e.g., instead of estimating just the sizes of Fourier coefficients estimating the sizes of Littlewood-Paley decompositions or, perhaps wavelets). Of course, the Littlewood-Paley decomposition is closely related to the elliptic estimates.
- Develop the analytic regularity results using the analytic elliptic regularity.
- Extend the method of [Jou88] to a larger number of foliations.
- Extend the method of [Jou88] to the Cantor sets that appear as basic sets in Axiom A systems.
- Extend the method of [Jou88] to analytic regularity.

Of course, introducing another method would be even more welcome. $\square$

7. The smooth Livsic theorem

In this section we discuss the regularity of solutions of cohomology equations These are equations where the unknown is the function $\varphi : M \to \mathbf{R}$ and take the form:

$$\varphi \circ f - \varphi = \eta \tag{7.1}$$

$$X\varphi = \eta \tag{7.2}$$

where f, (resp . X) is a given Anosov diffeomorphism (resp. vector field) in a compact manifold M, $\eta : M \mapsto \mathbf{R}$ is a given function.

Similarly, the non-commutative analogues (again the unknown is $\varphi : M \to G$, G a Lie group with whose operation we will denote by just juxtaposing the elements and we will denote the inverse Lie operation by a superindex of -1. We will denote by φ^{-1} the function that to a point x associates $(\varphi(x))^{-1}$.

$$\varphi \circ f \varphi^{-1} = \eta \tag{7.3}$$

$$X\varphi = \eta \tag{7.4}$$

where in the case (7.3), η, φ are functions on M taking values on a Lie group G. In the case of (7.3), φ takes values in a Lie group G and and η takes values in its Lie algebra $\mathcal{G}$.

We want to establish that if f – or X – and η are regular, then any continuous φ solving (7.1), (7.2), (7.3) or (7.4), has to be smooth.

The main result of this section will be:

Theorem 5. *Let f be a C^r Anosov diffeomorphism of the compact manifold M, $r \in 2, \cdots$. (resp. X a C^r Anosov vector field on M). Let η be a function in $W^{l,p}$, $l \in \mathbf{N}$ $(0 < l \leq r)$. $1 \leq p \leq \infty$. Assume that f preserves Lebesgue measure or, more generally that there is a $0 < \lambda < 1$ such that:*

$$||J_f||_{L^\infty}^{1/p} ||D_s f||_{L^\infty} \leq \lambda$$
$$||J_{f^{-1}}||_{L^\infty}^{1/p} ||(D_u f)^{-1}||_{L^\infty} \leq \lambda$$

where J_f denotes the Jacobian of f with respect to the Lebesgue measure. (In the case of flows, we need the same hypothesis when f the time one map of the flow.)

Assume that φ is a continuous function that satisfies (7.1) *(resp.* (7.2), (7.3) (7.2)*). Then, $\varphi \in W^{l',p}$, where $l' = l$ when l is an even integer or $l = 1$, $l' = l - 1$ when l is odd.*

Remark 17. The result is not optimal in the case of odd l. One can certainly take this result and improve it using interpolation. (see Remark 21.)

Nevertheless, it seems quite possible that one can construct a direct proof of these improved results. This seems, however to require the development of certain techniques such as fractional derivatives and Bessel potentials associated to the Anosov foliations. Even if it looks to us that this is reasonable straightforward to do, it would be somewhat lengthy. □

Remark 18. Notice that, in the same way as in the more familiar C^r cases, these regularity results do not require that the system is transitive or, in the non-commutative case that η is close to the identity. Such assumptions are needed for the well known Livsic theorem that states that conditions on periodic orbits are necessary and sufficient for the existence of a solution. □

The bootstrap of regularity will be done in different stages. Each stage will require a different argument. Furthermore, we will need different arguments for the commutative and non-commutative case. We develop these arguments and the corresponding theorems in the rest of this section.

We will study first the (somewhat simpler) commutative case and start by proving some theorems that accomplish different stages of the bootstrap of regularity.

Lemma 5. *Let f be a transitive C^r Anosov diffeomorphism in a compact manifold M, $r \geq 2$. Denote by μ_+, μ_- the forward and backwards SRB measure for f. Let $\varphi : M \to \mathbf{R}$, $\varphi \in L^1(\mu^+) \cap L^1(\mu^-)$ be such that*

$$\varphi - \varphi \circ f = \eta \tag{7.5}$$

where $\eta(x)$ is a continuous function with a modulus of continuity ω (i.e., $|\eta(x) - \eta(y)| \leq \omega(d(x, y))$, $\omega : \mathbf{R}^+ \to \mathbf{R}^+$ non-decreasing $\omega(0) = 0$) satisfying for some $\lambda < 1$ (and hence *for all $\lambda < 1$)*

$$\sum_{i=0}^{\infty} \omega(\lambda^i t) = \Gamma(t) < \infty \tag{7.6}$$
$$\lim_{t \to 0} \Gamma(t) = 0$$

Then φ agrees Lebesgue a.e. with a continuous function of modulus of continuity Γ.

If $\omega(t) = Ct^\alpha$ for some $0 < \alpha \leq 1$, (i.e., η is Hölder) of then, φ agrees a.e. with a Hölder function of the same exponent as η.

Remark 19. Notice that a consequence of φ agreeing a.e. with a continuous function is that then we can conclude that η satisfies the usual condition on periodic orbits $0 = \sum_{i=0}^{T-1} \eta \circ f^i(x)$ when $f^T(x) = x$. □

Proof. If φ satisfies (7.5), then

$$\varphi(x) = \eta(x) + \cdots + \eta \circ f^i(x) + \varphi \circ f^{i+1}(x)$$

Summing the above equations for $i = 0$ to $N - 1$ and dividing by N, we obtain

$$\varphi(x) = \frac{1}{N} \sum_{i=0}^{N-1} (N - i) \eta \circ f^i(x) + \frac{1}{N} \sum_{i=1}^{N} \varphi \circ f^i(x)$$

Substituting the above expression evaluated for x and y, taking limits as $N \to \infty$ and recalling that for Lebesgue almost all points x, $\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \varphi \circ f^i(x) = \int \varphi \, d\mu^+$ we have:

$$\varphi(x) - \varphi(y) = \lim_{N\to\infty} \left[\sum_{i=0}^{N-1} \eta \circ f^i(x) - \eta \circ f^i(y) - \sum_{i=0}^{N-1} \frac{i}{N} [\eta \circ f^i(x) - \eta \circ f^i(y)] \right] \tag{7.7}$$

for Lebesgue almost all x, y.

We recall that, by the absolute continuity of the stable foliation there is a dense set of stable leaves in which the formula (7.7) for x, y a.e. with respect to the Riemannian measure on that leaf.

If x, y are on the same stable leaf and (7.7) is valid for them we note that

$$\begin{aligned}
\Big| \sum_{i=0}^{N-1} \frac{i}{N} (\eta \circ f^i(x) - \eta \circ f^i(y)) \Big| &\leq \sum_{i=0}^{N-1} \frac{i}{N} \omega(\lambda^i d_s(x, y)) \\
&\leq \sum_{i=0}^{\sqrt{N}} \frac{i}{N} \omega(\lambda^i d_s(x, y)) + \sum_{\sqrt{N}+1}^{N-1} \frac{i}{N} \omega(\lambda^i d_s(x, y)) \\
&\leq \frac{1}{\sqrt{N}} \sum_{i=0}^{\infty} \omega(\lambda^i d_s(x, y)) + \sum_{\sqrt{N}+1}^{N} \omega(\lambda^i d_s(x, y))
\end{aligned} \tag{7.8}$$

where $d_s(x, y)$ denotes the distance of x, y on the stable leaf on which they lie.

Both terms of the R.H.S. of (7.8) tend to zero as $N \to \infty$ provided $\sum_i^\infty \omega(\lambda^i d_s(x, y)) < \infty$.

Similarly, we can bound

$$\Big|\sum_{i=0}^{N-1} \eta\circ f^i(x) - \eta\circ f^i(y)\Big| \le \sum_i^N \omega(\lambda^i d_s(x,y)) = \Gamma(d(x,y))$$

Hence, we have established that for a dense set of stable leaves

$$|\varphi(x)-\varphi(y)| \le \Gamma(d_s(x,y)) \tag{7.9}$$

almost everywhere on the leaf. Hence, we conclude that the function φ restricted to this leaf, agrees almost everywhere on the leaf with a continuous function of modulus of continuity Γ. Since we have that Γ is independent of the leaf, then, the result can be improved from a dense set of leaves to all the leaves.

An identical argument with f^{-1} in place of f and μ^- in place of μ^+, we get that for a dense set of unstable leaves, for almost all points (and, hence, for all leaves and all points) we have:

$$|\varphi(x)-\varphi(y)| \le \Gamma(d_u(x,y)) \tag{7.10}$$

when x,y are in the same unstable leaf and d_u is the distance along this leaf.

Using the uniform transversality of the Anosov foliations, we have

$$C^{-1}\max(d_s(x,y),d_u(x,y)) \le d(x,y) \le C\max(d_s(x,y),d_u(x,y))$$

when $d(x,y)\le\delta$ for some conveniently chosen $\delta>0$. From this and (7.9), (7.10), the desired result follows.

The result about Hölder continuity follows because when $\omega(t)=ct^\alpha$

$$\sum \omega(\lambda^i(t) = \left(\frac{C}{1-\lambda^\alpha}\right) t^\alpha$$

□

Now we state the main results of this section. They allow us to bootstrap the regularity in Sobolev classes for solutions of the cohomology equations. We call attention to the fact that there is a difference between the commutative and the non commutative cases.

Theorem 6. *Let f be a C^r Anosov diffeomorphism (resp. X a C^r Anosov vector field) $r\ge 2$. Assume that φ,η satisfy (7.1) (resp. (7.2)) and that:*

i) *$\varphi\in C^0$ and $\eta\in W^{k,p}$, $1\le k\le r$, $1<p<\infty$*

ii) *f, (resp. X) preserve a measure equivalent to Lebesgue*

or, more generally

ii′) *f (resp. X_1) satisfy*

$$\begin{aligned} \|J_f\|_{L^\infty}^{1/p}\|D_sf\|_{L^\infty} &\le \lambda < 1 \\ \|J_{f^{-1}}\|_{L^\infty}^{1/p}\|(D_uf)^{-1}\|_{L^\infty} &\le \lambda<1 \end{aligned} \tag{7.11}$$

Then $\varphi\in W^{k',p}$ where $k'=k$ when k is even or 1, $k'=k-1$ when k is odd and different from 1. In case that $k=1$, the cases $p=1$, $p=\infty$ are allowed, also $r=1$ suffices.

If we assume that $\varphi \in W^{l,p} \cap C^0$, $1 \le \lambda \le r$ it suffices to assume, instead of ii) or ii′) that

$$\begin{aligned} \|J_f\|_{L^\infty}^{1/p} \|D_s f\|_{L^\infty}^{l} \le \lambda < 1 \\ \|J_{f^{-1}}\|_{L^\infty}^{1/p} \|(D_u f)^{-1}\|_{L^\infty}^{l} \le \lambda < 1 \end{aligned} \tag{7.12}$$

to reach the same conclusions.

In case f is transitive and preserves a measure equivalent to Lebesgue, using Lemma 5, the assumption $\varphi \in C^0$ can be relaxed to $\varphi \in L^1$, still reaching the same conclusions.

Remark 20. Note that, since $f(M) = M \int J_f = |M|$ and, therefore, $\|J_f\|_{L^\infty} \ge 1$. Hence, we will use $\|J_f\|_{L^\infty}^{\alpha} \le \|J_f\|_{L^\infty}^{\beta}$ when $\alpha \le \beta$. □

Theorem 7. *Let f be a C^r Anosov system, $r \ge 2$ (resp. X a C^r Anosov vector field). Assume that φ, η satisfy* (7.3) *(resp.* (7.4)
 i) *$\varphi \in C^0$, $\eta \in W^{k,p}$, $1 \le k \le r$, $\infty > p \ge \frac{n}{k}$*
 ii) *or ii) of Theorem 6 hold.*

Then $\varphi \in W^{k',p}$ where $k' = k$ if k even or 1, $k' = k - 1$ when k is odd and not 1. In case that $k = 1$, we can allow $1 \le p \le \infty$, $r \ge 1$ in the other hypotheses.

Remark 21. We note that the fact that we can only deal with even k presumably does not belong. This is due to the fact that we want to use positive operators elliptic operators in Theorem3 and, at the same time, we want to use differential operators in our work so as to have easy change of variable formulas.

As we conjectured before, it seems doable to develop a theory of pseudo-differential operators and their changes of variables. (Perhaps it is even done in the literature but we are not aware of it.)

Nevertheless, it is easy to use interpolation to deduce from Theorem 6 results that have less restrictions in the k.

In the commutative case, when f is transitive, the fundamental Livsic theorem [Liv71] tells us that a necessary and sufficient condition for a Hölder function η to be a coboundary (i.e. to be such that (7.1) has a solution) is that the sum over all periodic orbits is zero.

Then, when k, p are such that the Sobolev imbedding theorem implies that the $W^{k,p}$ norm dominates a Hölder norm, the set of coboundaries is closed in $W^{k,p}$. We denote this space by $B^{k,p}$

If k is even, $1 < p < \infty$ we can use Theorem 6 to produce a bounded linear operator T that maps $B^{k,p}$ into $W^{k,p}$. The operator T is defined as the operator that to a function $\eta \in B^{k,p}$ associates the function φ solving (7.1) and $\int \varphi d\mu = 0$.

Note also that, since the solutions of (7.1) and the normalization condition are unique, we can consider that the operator T is the same in all the $B^{k,p}$ spaces.

If we can define this operator and apply theorem 6 for two different values of k (say k_1, k_2 such that the Sobolev imbedding gives that the spaces are Hölder), then, applying Lemma 2, the operator T will also be bounded in all the Sobolev spaces with intermediate k.

Similar considerations apply for flows. □

7.1. **Proof of Theorem 6.** Now, we start the proof of Theorem 6. This will require to study different bootstrap arguments that we will state as separate lemmas since they will appear again in the proof of other theorems.

We first study the case of (7.1) in the non-transitive case and $r = 1$.

The first stage of bootstrap is to obtain the existence of one derivative.

Lemma 6. *Assume that f is a C^1 mapping preserving an stable bundle — and hence, having an stable foliation. Assume that φ and η are as in (7.1) and that $\eta \in W^{1,p}$. Then φ is differentiable in the sense of distributions along the stable direction and $D_s\varphi \in L^p$.*

Moreover, the formula

$$D_s\varphi = \sum_{i=0}^{\infty} D_s\eta \circ f^i D_s f^i \tag{7.13}$$

holds. The sum on the R.H.S. converges in the L^p sense.

Proof. Given a vector field Z tangent to the stable foliation and with coefficients in $C^1_s(f)$ we can define a flow Z_t by integrating the differential equations restricted to the stable leaf.

If $\eta \in W^{1,p}$, we know that $Z\eta$ can be defined in the sense of distributions on each leaf and $Z\eta \in L^p(M)$ and (appealing to the absolute continuity of the Anosov foliations) for almost all leaves.

As in one of the standard proofs of Sobolev imbedding ([Ste70], V 2.2) we have:

$$\eta(Z_t(x)) - \eta(x) = \int_0^t (Z\eta) \circ Z_s(x)\, ds = \int_0^t [(D_s\eta)Z] \circ Z_s(x)\, ds$$

where the equality should be understood as equality between L^p functions. (Note that Z_a is absolutely continuous with respect to Lebesgue since it is Lipschitz on each leaf and the foliation is absolutely continuous.)

Similarly:

$$\begin{aligned} \eta \circ f^i(Z_t(x)) - \eta \circ f^i(x) &= \int_0^t Z(\eta \circ f^i) \circ Z_a(x)\, da \\ &= \int_0^t \big[(D_s\eta) \circ f^i(D_s f^i Z)\big] \circ Z_a(x)\, da \end{aligned} \tag{7.14}$$

We note that if (7.1) holds, then

$$\varphi(x) = \eta(x) + \cdots + \eta \circ f^i(x) + \varphi \circ f^{i+1}(x) \tag{7.15}$$

Subtracting (7.15) for x and y and applying (7.14), we have:

$$\begin{aligned} \varphi \circ Z_t(x) - \varphi(x) = &\int_0^t \sum_{i=0}^{N} \big[(D_s\eta) \circ f^i(D_s f^i Z) \circ f^i\big] \circ Z_a(x)\, da \\ &+ \varphi \circ Z_t \circ f^{N+1}(x) - \varphi \circ f^{N+1}(s) \end{aligned} \tag{7.16}$$

We note that, since x, $Z_t(x)$ are in the same stable manifold and φ is uniformly continuous,

$$\varphi \circ f^{N+1} \circ Z_t(x) - \varphi \circ f^{N+1}(x)$$

converges to zero as $N \to \infty$ uniformly in x.

We also note that

$$\|\Gamma \circ f\|_{L^p} \leq \|J_f\|_{L^\infty}^{1/p} \|\Gamma\|_{L^p}$$

Hence, the general term in the series (7.16) can be bounded by

$$\begin{aligned} \|(D_s\eta D_s f^i Z) \circ f^i\|_{L^p} &\leq \|J_f\|_{L^\infty}^{i/p} \|D_s\eta\|_{L^p} \|D_s f^i Z\|_{L^\infty} \\ &\leq (\|J_f\|_{L^\infty}^{1/p} \|D_s f\|_{L^\infty})^i \|D_s\eta\|_{L^p} \end{aligned} \tag{7.17}$$

We have, therefore, established that the series in the R.H.S. of (7.13) converges in L^p.

To show that it is indeed a derivative, we note that, taking limits $N \to \infty$ in (7.16) we have:

$$\varphi(Z_t(x)) - \varphi(x) = \int_0^t \Gamma \circ Z_a(x)\, da$$

where Γ is the series in the R.H.S. of (7.13).

Again, using the absolute continuity of the foliation, we obtain that $\Gamma = Z\varphi$ in the sense of distributions.

Since Z is arbitrary, we have established (7.13)

This finishes the proof of Lemma 6. □

Higher stable derivatives are much easier (at least conceptually). We will just show that the series obtained taking derivatives along the stable direction term by term in (7.13) converges in L^p. This implies that indeed this is the distributional derivative and that the distributional derivative is in L^p.

This is the content of the following lemma. We call attention that the bootstrap of regularity can be done independently for stable derivatives and unstable derivatives. This will be relevant later.

Lemma 7. *Let f be a C^r map with a stable direction. Assume that φ, η satisfy* (7.1), *$D_s^j\eta \in L^p$ for $j \leq \ell$. (This is clearly implied by $\eta \in W^{\ell,p}$.) If either*

- *$\varphi \in C^0$ and f satisfies the first of* (7.11)
- *$\varphi \in W^{k,p}$ and f satisfies the first of* (7.12)

Then,

$$D_s^\ell \varphi \in L^p$$

Proof. Note that, by Faa-di-Bruno formula

$$D_s^{k-1}(D_s\eta \circ f^i D_s f^i) = \sum_{j=1}^{k} \sum_{i_1+\cdots+i_j=j} \sigma_{j;i_i,\ldots,i_j}\ D_s^j\eta \circ f^i D_s^{i_1} f^i \cdots D_s^{i_j} f^i \tag{7.18}$$

where $\sigma_{j;i_1,\ldots,i_j}$ are combinatorial coefficients whose explicit expression (which we will not need in these notes) can be found in many places, including [AR67].

Note that, by the bounds in [dlLMM86], p.547, we have

$$\|D_s^i f^l\|_{L^\infty} \le \|D_s f\|_{L^\infty}^{l\,i} C_i \tag{7.19}$$

Using that

$$\|D^j \eta \circ f^l\|_{L^p} \le \|D_s^j \eta\|_{L^p} \|J_f\|_{L^\infty}^{1/p} \tag{7.20}$$

we obtain:

$$\|D_s^\ell (\eta \circ f^j)\|_{L^p} \le \|D^\ell \eta\| (\|J_f\|^{1/p} \|D_s f\|^\ell)^j \tag{7.21}$$

The reason why we can use ii′) instead of ii) is the following: Again if we start from (7.14) and take ℓ derivatives we obtain

$$D_s^\ell \varphi = \sum_{i=0}^{N} D_s^\ell (\eta \circ f^i)(x) + D_s^\ell \varphi \circ f^{N+1}(x)$$

Using the same argument as before (i.e., use (7.18), (7.19), (7.20)) we can bound

$$\|D_s^\ell \varphi \circ f^{N+1}\|_{L^p} \le \|J_f\|^{N\frac{1}{p}} \|D_s^\ell \varphi\|_{L^p} \|D_S f\|_{L^\infty}^{Ni}$$

which under assumption ii) converges to zero in L^p.

We are then left with the formula

$$D_s^\ell \varphi = \sum_{i=0}^{\infty} D_s^\ell (\eta \circ f^i)$$

where the sum converges in L^p sense by (7.21).

We have, therefore, established Lemma 7.

□

Note that if f is an Anosov system and it satisfies (7.11) and (7.12) then we can apply Lemma 7 to both f and f^{-1} (the stable direction for f^{-1} is the unstable for f). Then, we can apply Lemma 3 and establish Theorem 6 for the case of commutative cohomology equations over Anosov diffeomorphisms.

The result for commutative equations for flows can be established using the results we have already established.

Note that (7.2) implies

$$\varphi \circ X_t(x) - \varphi(x) = \int_0^t \eta \circ X_a(x)\, da \tag{7.22}$$

If we fix t (to, for the sake of simplicity, 1), we see that the mapping X_1 has stable directions, which are integrable and absolutely continuous.

If we apply Lemma 7 to X_1 and to $X_1^{-1} = X_{-1}$ we obtain the existence of $D_s^k \varphi$, $D_u^k \varphi$. The existence of derivatives along the flow follows from the equation.

Then, again, applying Lemma 3 we establish Theorem 6 for the case of commutative cohomology equation over flows.

Remark 22. Note that the bootstrap of regularity along the stable directions is valid for $p = 1, \infty$. Nevertheless, when we need to apply Theorem 3 we need to exclude these values for p. □

7.2. **Proof of Theorem 7.** The proof proceeds, roughly, along the same lines as the proof of Theorem 6. We will bootstrap the derivatives along the stable and unstable directions separately and then apply Theorem 3 to conclude regularity. Nevertheless, the actual bootstraps will require different arguments since the non-commutativity will prevent some of the rearrangements that we did in the commutative case. Indeed, we do not have any analogue of Lemma 5.

We will assume, for the sake of simplifying the notation that the groups we consider are groups of matrices. This will allow us to use norms in the estimates of products and additive notation. The use of norms in products can be avoided using the local Lipschitz nature of the product in Lie groups and the additive notation can be avoided using the exponential mapping, but this is only a typographical matter.

As in the commutative case, we start by studying the case $r = 1$, establish that under the assumptions there is a derivative and that this derivative is given by an explicit formula.

Lemma 8. *Let $f \in C^r$, $r \geq 2$, and preserves an stable direction. Assume φ, η satisfy* (7.3)

$$\varphi \in C^0 \ , \quad D_s\eta \in L^p \qquad 1 \leq p \leq \infty$$

and that ii′) of Theorem 5 holds. Then

a) $D_s\varphi \in L^p$.

and we have

b) $D_s\varphi(x) = \sum_{i=0}^{\infty} \varphi\ \varphi^{-1} \circ f^i(x) D_s\eta \circ f^i(x) \varphi \circ f^{i+1}(x)$

Of course, similar results hold for the unstable derivatives.

Remark 23. Note that since φ takes values in a compact set and $g \to g^{-1}$ is uniformly differentiable in a compact set, the same conclusions of regularity are reached for φ^{-1} in place of φ. This remark also applies to higher derivatives. □

Remark 24. The only reason we need to assume $r = 2$ in this Lemma and not just $r = 1$ is that we need to ensure that the stable manifold is absolutely continuous and that we can define the first derivatives in the weak sense. □

Proof of Lemma 8. We start by observing that, if (7.3) holds then,

$$\varphi(x) = \eta(x) \cdots \eta \circ f^N(x) \varphi \circ f^{N+1}(x) \tag{7.23}$$

Hence, we can write

$$
\begin{aligned}
\varphi(x) - \varphi(y) &= [\eta(x) - \eta(y)]\eta \circ f(x) \cdots \eta \circ f^N(x)\varphi \circ f^{N+1}(x) + \\
&\quad + \eta(y)[\eta \circ f(x) - \eta \circ f(y)]\eta \circ f^2(x) \cdots \eta \circ f^N(x)\varphi \circ f^{N+1}(x) + \\
&\quad + \cdots + \\
&\quad + \eta(y) \cdots \eta \circ f^{N-1}(y)[\eta \circ f^N(x) - \eta \circ f^N(y)]\varphi \circ f^{N+1}(x) \\
&\quad + \eta(y) \cdots \eta \circ f^N(y)[\varphi \circ f^{N+1}(x) - \varphi \circ f^{N+1}(y)] \\
&= [\eta(x) - \eta(y)]\varphi \circ f(x) \\
&\quad + \varphi(y)\varphi^{-1} \circ f(y)[\eta \circ f(x) - \eta \circ f(y)]\varphi \circ f^2(x) \\
&\quad + \varphi(y)\varphi^{-1} \circ f^2(y)[\eta \circ f^2(x) - \eta \circ f^2(y)]\varphi \circ f^3(x) \\
&\quad + \cdots + \\
&\quad + \varphi(y)\varphi^{-1} \circ f^{N-1}(y)[\eta \circ f^N(x) - \eta \circ f^N(y)]\varphi \circ f^{N+1}(x) \\
&\quad + \varphi(y)\varphi^{-1} \circ f^N(y)[\varphi \circ f^{N+1}(x) - \varphi \circ f^{N+1}y]
\end{aligned}
\tag{7.24}
$$

This remarkable cancellation was used in [NT96].

It will be important to observe that the terms in this sum except for the last one are independent of N. That is, if we take the limit as $N \to \infty$ of (7.24) except for the last term, we obtain a series.

If Z is a vector field tangent to the stable foliation, we can write

$$
\begin{aligned}
&\varphi \circ Z_t(x) - \varphi(x) = \\
&\quad + \left[\int_0^1 [(D_s\eta)Z] \circ Z_a(x)\, da\right]\varphi \circ f(x) \\
&\quad + \varphi(y)\varphi^{-1} \circ f(y)\left[\int_0^1 [D_s(\eta \circ f)Z]Z_a(x)\, da\ \right]\varphi \circ f^2(x) \\
&\quad + \cdots + \varphi(y)\varphi^{-1} \circ f^{N-1}(y)\left[\int_0^1 (D_s\eta \circ f)Z] \circ Z_a(x)\, dz\right]\varphi \circ f^N(x) \\
&\quad + \varphi(y)\varphi^{-1} \circ f^N(y)[\varphi \circ f^{N+1}(x) - \varphi^{N+1}(y)]
\end{aligned}
\tag{7.25}
$$

Note also that, because $\varphi(y)\varphi^{-1} \circ f^i(y)$ are matrices, using the linearity, we can change the order of the integration with the first factors in the term, which do not depend on the integration variable. Of course, in more general groups similar formulas exist, but we have to introduce the derivative of the group operation.

We note that, since φ,φ^{-1} are continuous are bounded and x, y are in the same stable manifold we have that the last term in (7.25) converges to zero uniformly.

We also can estimate the general term of (7.25). The term in braces already estimated in (7.17) and the factors involving φ, φ^{-1} are uniformly bounded.

Hence, we have established that

$$
\varphi \circ Z_t(x) - \varphi(x) = \int_0^t \Gamma \circ Z_a(x)Z
$$

with Γ the L^p function defined by

$$\Gamma(x) = \sum_{i=0}^{\infty} \varphi(x)\varphi^{-1} \circ f^i(x),\ D_s\eta \circ f^i(x),\ \varphi \circ f^{i+1}(x) \tag{7.26}$$

Hence, this is the stable derivative and this establishes Lemma 8. $\square$

From Lemma 8 the results that we claimed for $k = 1$ in Theorem 7 follow immediately. When f is a diffeomorphism apply Lemma 8 to f^{-1} to obtain the derivative along the unstable foliation and apply Theorem 3. For flows, applying Lemma 8 to X_1, X_{-1} we obtain existence of the stable and unstable derivatives. The derivative along the flow follows from the equation and, then, we can apply Theorem 3.

To finish the proof of Theorem 7 we just argue by induction in the number of derivatives that the series in (7.2) can be differentiated repeatedly term by term and that the result converges in an appropriate L^q space. The induction stops when the term by term derivative would require derivatives of η not included in the assumptions. Eventually, we will show that $D_s^k\varphi \in L^p$. This shows that the derivative in the sense of distributions exists and that it belongs to the appropriate space.

The argument that we have to use is significantly more complicated than in the commutative case, since in this case, we use just all the intermediate derivatives are in L^p. Note that using the Sobolev imbedding (Lemma 3) we expect that if the function is indeed in $W^{k,p}$ derivatives of lower order are in a L^q space with a $q > p$.

The reason why we have to pay attention to the exponent L^q depending on the order is that products of derivatives belong to an L^q spaces with a lower exponent than each of the factors. We need to make the induction argument without loosing too much in each step in order to avoid ending up with higher loses than necessary. As it turns out, the right hypothesis to carry out the induction is that the derivatives are in the L^q space predicted by the Sobolev imbedding theorem from the fact that $\varphi \in W^{k,p}$.

We will formulate our induction step after we explain some of the more technical results, which will motivate the definitions.

The following elementary result is the key to the induction procedure alluded to above. We express the derivatives of the general term of b) in Lemma 8.

Proposition 1. *The $k-1$ stable derivative of the general term of* (7.2)

$$D_s^{i_0-1}[\varphi\varphi^{-1} \circ f^i D_s\eta \circ f^i \varphi \circ f^{i+1}] \tag{7.27}$$

can be expressed, applying Leibnitz rule, as a sum of terms satisfying the following properties.

a) *The term is a product of four factors of the form*

$$D_s^{i_1}\varphi\ , \quad D_s^{i_2}(\varphi^{-1}f^i)\ , \quad D_s^{i_3}(\eta \circ f^i)\ , \quad D_s^{i_4}(\varphi \circ f^{i+1})$$

b) *All the derivatives of φ, φ^{-1} appearing in the terms above are of order smaller or equal than $i_0 - 1$.*

c) *The sum of the orders of the derivatives of each factor is precisely i_0.*

The proof of this proposition is quite well known. It is just the Leibnitz theorem for derivatives of the product. One just needs to verify that it holds in the weak sense that we are using. But this is quite well known from the theory of distributions.

The induction lemma is

Lemma 9. *Assume that we are in the conditions of Theorem 7. Assume inductively that for $i < i_0$ we have*

$$D_s^i \varphi \in L^{q_i}$$

(and, hence, by Remark 23, $D_s^i \varphi^{-1} \in L^{q_i}$) where

$$\frac{1}{q_i} = \frac{1}{p} - \frac{k}{n} + \frac{i}{n} = \frac{i}{n} \ .$$

(Recall that we are assuming in Theorem 7 that p is such $\frac{1}{p} - \frac{k}{n} = 0$.) Then:

a) *All the terms in the derivative* (7.27) *are in $L^{q_{i_0}}$.*
b) *The series obtained summing the derivatives of the general terms* (7.27) *converge in $L^{q_{i_0}}$.*

Proof. Part a) follows by observing that, by Hölder's inequality, we have that the inverse of Lebesgue exponents in the L^p spaces are additive in products. (That is, if $\Psi_i \in L^{p_i}$, then $\Psi_1 \Psi_2 \cdots \Psi_N \in L^q$ with $(1/q) = (1/p_1) + \cdots + (1/p_N)$.)

Using the inductive hypothesis,

$$\begin{gathered} D_s^{i_1} \varphi \in L^{q_1} \ , \quad D_s^{i_2} \varphi^{-1} \circ f^i \in L^{q_2} \ , \quad D_s^{i_3} (\eta \circ f^i) \in L^{q_3} \\ D_s^{i_4} \varphi \circ f^{i+1} \in L^{q_4} \ , \text{ with } \ \frac{1}{q_1} = \frac{i_1}{n} \ , \quad \frac{1}{q_2} = \frac{i_2}{n} \ , \quad \frac{1}{q_3} = \frac{i_3}{n} \ , \quad \frac{1}{q_4} = \frac{i_4}{n} \ . \end{gathered}$$

Hence, the whole term belongs to L^q, with

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4} = \frac{i_1 + i_2 + i_3 + i_4}{n} = \frac{i_0}{n} = \frac{1}{q_{i_0}}$$

where the last equalities follow from c) of Proposition 1 and the definition of q_{i_0}.

Moreover, we can bound the $L^{q_{i_0}}$ norm of the terms by the products of the L^{q_j} norms $j = 1, \dots, 4$ of the corresponding factors.

We note that, by the same arguments we used before,

$$\begin{aligned} \|D_s^{i_2} \varphi^{-1} \circ f^l\|_{L^{q_2}} &\le K \|J_f\|_{L^\infty}^{1/q_2} \|D_s f\|_{L^\infty}^{l i_2} \\ \|D_s^{i_3} \eta \circ f^l\|_{L^{q_3}} &\le K \|J_f\|_{L^\infty}^{1/q_3} \|D_s f\|_{L^\infty}^{l i_3} \end{aligned}$$

and the other terms are independent from i.

Now, we note that $i_2 \ge 0$, $i_3 \ge 1$, $1/q_3 \ge 1/p$. Hence all the terms in the derivative can be bounded by an exponentially decreasing bound. ☐

From Lemmas 8 and 9, we obtain easily Theorem 7.

Lemma 8 allows us to obtain the existence of the first stable derivatives (and applying it to the inverse) of the unstable derivatives. From that, we can apply repeatedly, Lemma 9 to obtain the existence of stable and unstable derivatives of order up to k and that they are in L^p. Once we have that, we apply Theorem 3 to obtain the desired result.

8. Bootstraping Regularity of Invariant Sections

In [Has92] and (resp. [dlL92]) it was discovered that, given smooth Anosov systems, there are critical regularities r_1 — depending only on expansion rates — such that if the Anosov splitting (resp. any conjugacy with the given system) is $C^{r_1+\delta}$, $\delta > 0$, then it is essentially as smooth as the map (a finite number of derivatives, independent of the dimension may be lost to different issues of a technical nature).

The main goal of this section is to prove a similar result for Sobolev regularities. We will show that if the objects we consider are in $W^{r_1+\delta,p}$ — with r_0 precisely the same as in the previous papers — then, they are C^{r-A} where r is the regularity of the map and A is a small number which is independent of the system considered.

More precisely, we will prove the following result for invariant sections of bundle maps (not necessarily contractions) that shows that any sufficiently regular section is as regular as the map.

This result, as we will note later implies results in Sobolev classes analogous to those of [dlL92] for regularities of conjugacies and to those of [Has92] for Anosov splittings.

Theorem 8. *Let f be a C^r Anosov mapping of a compact manifold M. Let Y be a bundle over M and F a C^r map covering f. Let σ be a section of Y invariant under F. Assume that $\sigma \in W^{\ell,p}$ where*

a) $1 \le p \le \infty, \quad \ell < r-1$
b) $\frac{1}{p} - \frac{\ell}{n} - \frac{1}{n} \ge 0$
c) *ℓ and p are such that*

$$\begin{aligned} |DF_{f^{-1}(x)}|\,|D_u f^{-1}(x)|^{\ell} |J_{f^{-1}(x)}|^{1/p} &< 1 \\ |DF^{-1}_{f(x)}|\,|D_s f(x)|^{\ell} |J_{f(x)}|^{1/p} &< 1 \end{aligned}$$

Then $\sigma \in C^{r-\varepsilon}$.

Remark 25. It will be important to remark that, if assumption c) holds for some ℓ, p, then it continues to hold if we increase ℓ or increase p. This will allow us to repeat the bootstrap of regularity argument. □

Remark 26. We note that, in analogy with Remark 3, condition c) in Theorem 8 can be replaced by, for some N

$$\begin{aligned} |DF_{f^{-1}(x)} \cdots DF_{f^{-N}(x)}|\,|D_u f^{-N}(x)|^{\ell} |J_{f^{-N}(x)}|^{1/p} &\le K < 1 \\ |DF^{-1}_{f(x)} \cdots DF^{-1}_{f^N(x)}|\,|D_s f^{N}(x)|^{\ell} |J_{f^{N}(x)}|^{1/p} &\le K < 1\ . \end{aligned}$$

□

Remark 27. If one assumes that the map satisfies some extra property such as preserving a symplectic structure, or that it is lower dimensional, it is possible that one can get a bootstrap of regularity starting even in lower regularities. One result of this kind is [FK91], where it is shown that for symplectic Anosov maps of $\mathbf{T}^2$, if the foliations are C^2, then they are C^∞. □

8.1. **Proof of Theorem8.** Note that the invariant section satisfies

$$\begin{aligned} \sigma(x) &= F_{f^{-1}(x)}\sigma(f^{-1}(x)) \\ \sigma(x) &= F^{-1}_{f(x)}\sigma(f(x)) \end{aligned} \tag{8.1}$$

Under the assumption that σ is ℓ times differentiable in the weak sense, we can take $D_u^{\ell,(f)}$ of the first of (8.1) and $D_s^{\ell,(f)}$ of the second and obtain:

$$\begin{aligned} D_u^\ell\sigma(x) &= DF_{f^{-1}(x)} \circ \sigma \circ f^{-1}(x) D_u^\ell \sigma \circ f^{-1}(x)[D_u f^{-1}(x)]^{\otimes r} + R_u(x) \\ D_s^\ell\sigma(x) &= DF^{-1}_{f(x)} \circ \sigma \circ f^{-1}(x) D_s^\ell \sigma \circ f(x)[D_u f(x)]^{\otimes r} + R_s(x) \end{aligned} \tag{8.2}$$

where R_s, R_u are multilinear expressions in $D_s^j\sigma \circ f$, $D_u^j\sigma \circ f^{-1}$ for $j \le \ell$ affected by coefficients which are derivatives of F, f, F^{-1}, f of order not more than ℓ.

Note that by Sobolev's imbedding theorem $\sigma \in L^\infty$ and, since DF, DF^{-1} are Lipschitz, we have

$$DF_{f^{-1}} \circ \sigma \circ f^{-1} \in L^\infty \qquad DF_{f^{-1}} \circ \sigma \circ f \in L^\infty$$

and similarly for all the other derivatives of F evaluated at σ.

Also note that R_s, R_u only involve derivatives of order up to $\ell-1$ multiplied by functions in L^∞. It is also clear that $R_s, R_u \in L^p$ because they are the difference of two functions which are in L^p. (Recall that we are assuming that $D_u^\ell F_{f^{-1}} \sigma \circ f^{-1} \in L^p$, $D_s^\ell F^{-1}_{f(x)} \circ \sigma \circ f \in L^p$ and the first term in the right hand side of (8.2) is also in L^p.)

We will, however, need somewhat more detailed information. This is provided by the following proposition

Proposition 2. *Using Faa-di-Bruno formula, we can express R_s, R_u (introduced in (8.2)) as a sum of terms in such a way that*

a) *Each term is the product of derivatives of F, f, F^{-1}, f^{-1} and derivatives of σ along $\mathcal{F}_s$ or $\mathcal{F}_u$.*
b) *The total order of the derivatives of σ which appear in one term is smaller or equal than ℓ.*
c) *There is no term involving a derivative of σ of order ℓ.*
d) *At least one of the factors is a first derivative.*
e) *Each of the terms is in $L^{\tilde{p}}$, with $1/\tilde{p} = 1/p - 1/n$.*

Proof of Proposition 2. Items a), b), c), d) are just consequences of Faa-di-Bruno formula. Item e) follows from an argument which we will carry out in detail and which has very similar flavor to the argument we used in the proof of Lemma 9.

We note that, in a product of terms the inverse of the Lebesgue exponent is additive. Moreover, by Sobolev's theorem the inverse Lebesgue exponent of $D_s^i\sigma$ is $[\frac{1}{p} - \frac{\ell}{n} - \frac{i}{n}]_+$ where $[t] = \max(t, 0)$. Note that, with our assumptions, this inverse exponent is 0 when $i = 1$ and is $\frac{1}{p} - \frac{\ell}{n} + \frac{i}{n}$ when $i \ge 2$.

Therefore, the term which includes factors $D^{i_1}\sigma \cdots D^{i_k}\sigma$ has an inverse Lebesgue exponent

$$\sum_{i_k \ge 2} \Big(\frac{1}{p} - \frac{\ell}{n}\Big) + \sum_{i_k \ge 2} \frac{i_k}{n} = N\Big(\frac{1}{p} - \frac{\ell}{n}\Big) + \frac{1}{n} \sum_{i_k \ge 2} i_k \tag{8.3}$$

where N is the number of k with $i_k \geq 2$. We note that, obviously $N \geq 0$ and, under our assumptions, $\frac{1}{p} - \frac{\ell}{n} = \frac{-1}{n} < 0$.

Moreover $\sum_{i_k \geq 2} i_k \leq \ell - 1$ because the sum of all i_k is ℓ and there is at least one which is one. Hence, we obtain that the inverse Lebesgue exponent is smaller than $\frac{\ell-1}{n}$.

This result can be improved by observing that the only case when $N = 0$ is when $k = \ell$, $i_1 = i_2 = \cdots = i_\ell = 1$. In such a case, all the factors are in L^∞.

In all the other cases $N \geq 1$ and hence we can bound (8.3) by

$$-\frac{1}{n} + \frac{\ell - 1}{n} = \frac{\ell - 2}{n} = \frac{1}{p} - \frac{1}{n}$$

This establishes the claim. □

We denote by $\mathcal{A}^s_x$, $\mathcal{A}^u_x$ the operators acting on linear maps given by the first terms in (8.2) so that we can write (8.2) as:

$$\begin{aligned} (D^\ell_u \sigma)(x) &= \mathcal{A}^u_x (D^\ell_u \sigma) \circ f^{-1}(x) + R_u(x) \\ (D^\ell_s \sigma)(x) &= \mathcal{A}^s_x (D^\ell_s \sigma) \circ f(x) + R_s(x) \end{aligned} \tag{8.4}$$

Note that, by applying (8.4) repeatedly we obtain

$$\begin{aligned} (D^\ell_u \sigma)(x) = R_u(x) + \mathcal{A}^u_x R_u \circ f^{-1} + \cdots + A^u_x \mathcal{A}^u_{f^{-1}(x)} \cdots \mathcal{A}^u_{f^{-N}(x)} R_u \circ f^{-N-1}(x) \\ + A^u_x \cdots \mathcal{A}^u_{f^{-N}(x)} (D^\ell_u \sigma) \circ f^{-N-1}(x) \\ (D^\ell_s \sigma(x) = R_s(x) + \mathcal{A}^s_x R_s \circ f(x) + \cdots + A^s_x \cdots \mathcal{A}_{f^N(x)} R_s \circ f^{N+1}(x) \\ + \mathcal{A}^s_x \cdots \mathcal{A}^s_{f^N(x)} (D^\ell_s \sigma) \circ f^{N+1}(x) \end{aligned} \tag{8.5}$$

Assumption c) of Theorem 8 implies that the last term of (8.5) go to zero in L^p. Hence, we derive formulas for the stable and unstable derivatives

$$\begin{aligned} (D^\ell_u \sigma)(x) &= \sum_{N=0}^{\infty} \mathcal{A}^u_x \cdots \mathcal{A}^u_{f^{-N}(x)} R_u \circ f^{-N-1}(x) \\ (D^\ell_s \sigma)(x) &= \sum_{N=0}^{\infty} \mathcal{A}^s_x \cdots \mathcal{A}^s_{f^N} R_s \circ f^{N+1}(x) \end{aligned} \tag{8.6}$$

where the sums of the right hand side converge in L^p if we assume that $D^\ell_u \sigma \in L^\infty$.

We also note that exactly the same argument shows that if $D^\ell_u \sigma$, $D^\ell_u \sigma$ are continuous, the series (8.6) converges uniformly.

Indeed, this follows from our previous result by taking $p = \infty$. Observe also that if $\sigma \in C^{\ell,(f)}_s$ (or $C^{\ell,(f)}_s$) then the terms in the corresponding series are continuous.

Once we have this representation, we argue that the series obtained by differentiating term by term along the unstable (or stable) directions the first and the second series in (8.6) we obtain a series that converges uniformly in L^p. (We will show this in Lemma 10.)

Once we have that, we can finish the result very quickly in several different ways.

Note that we can use the regularity theorem to conclude that $\sigma \in W^{\ell+1,p} \subset W^{\ell,\tilde{p}}$ with $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{n}$. We can keep on repeating the argument to obtain better ℓ and better p. The argument stops when $\ell = r - 1$ but we can take p finite arbitrarily large.

In particular, we can arrange (using Sobolev imbedding theorem for large enough p) that $D_s^\ell\sigma$, $D_s^\ell\sigma$ are continuous functions.

Once we have that, we can bootstrap the existence of stable derivatives in the strong sense. This allows us to get the existence of up to r stable and unstable derivatives of φ.

Then, we can apply the result in [Jou88] and obtain that $\sigma \in \Lambda_{r-\varepsilon}$.

Hence, to prove Theorem 8 the only thing that remains to do is to prove

Lemma 10. *Under the conditions of Theorem 8,*

1) *If either of the series in* (8.6) *converges in L^p, then if $\ell < r-1$*
   a.1) *Each of the terms of the series can be differentiated along the corresponding foliation in the weak sense.*
   a.2) *The series of the derivatives of the terms constructed in the point above converges in L^p.*
2) *If the series converge uniformly, then if $\ell < r$*
   b.1) *Each term can be differentiated along the corresponding foliation.*
   b.2) *The series of the differentiated terms converges uniformly.*

Proof. We will work the details only for the D_s case. The D_u case follows from it by taking inverse mappings (or a direct proof can be done exactly in the same way).

The fact that the derivatives exist follows from the fact that each of the terms in the series, only involves derivatives of σ of order up to $\ell-1$.

The fact that $D_sR_s \in L^p$ follows by exactly the same argument that we used to show $R_s \in L^p$.

Note that, by Faa-di-Bruno formula D_sR_s is a sum of terms which satisfy properties very similar (and which follow from) to those in Proposition 2. Namely, a), d) hold as well as b) and c) with ℓ replaced by $\ell+1$.

Proceeding as in the proof of e) in Proposition 2 we obtain that we can bound the inverse Lebesgue exponent of the term that contains $D_s^{i_1}\sigma\cdots D^{i_k}\sigma$ by

$$N\Big(\frac{1}{p}-\frac{\ell}{n}\Big)+\sum_{i_k\ge 2} i_k$$

where N is the number of $i_k \ge 2$.

The only case when $N=0$ is when $k=\ell+1$ and $i_1=i_2=\cdots=i_{\ell+1}=1$. In such a case, all the terms are in L^∞ (indeed they are Hölder).

In all the other cases, $N \ge 1$, $\sum_{i_k\ge 2} i_k \ge \ell$ since the sum is $\ell+1$ and there is at least one index which is equal to 1. Hence, we conclude that the term has an inverse Lebesgue exponent smaller or equal than $-1/n+\ell/n$.

Moreover,

$$(8.7)\quad \begin{aligned} &D_s \left[DF^{-1}_{f^N(x)} \cdots DF^{-1}_{(x)} R_s \circ f^N(x)(D_s f^N(x))^{\otimes \ell} \right] \\ &= \left[\sum_{i=0}^{N} DF^{-1}_{f^N(x)} \cdots DF^{-1}_{f^{i+1}(x)} D_s(DF^{-1}_{f^i(x)}) DF^{-1}_{f^{i-1}(x)} \cdots DF^{-1}_x \right] \\ &\quad + DF^{-1}_{f^N(x)} \cdots DF^{-1}_x D_s(R_s \circ f^N(x))[D_s f^N(x)]^{\otimes \ell} \\ &\quad + DF^{-1}_{f^N(x)} \cdots DF^{-1}_x R_s \circ f^N(x) D^2_s [f^N(x)]^{\otimes \ell} \end{aligned}$$

The bounds of the third term are quite similar to bounds we have done before. Note that $\|R_s \circ f^N\|_{L^p} \leq \|J_f\|^{N\frac{1}{p}}$ and the results in [dlLMM86] show that $\|D^2_s f^N\|_{L^\infty} \leq \|D_s f\|^N_{L^\infty} K$. Obviously $\|DF^{-1}_{f^N} \cdots DF^{-1}_x\|_{L^\infty} \leq \|DF^{-1}_x\|^N_{L^\infty}$.

Hence, using the assumption c) of Theorem 8 we can bound the third term by a converging exponential.

The third term can be similarly bounded using that $D_s(R_s \circ f^N) = (D_s R_s) \circ f^N D_s f^N$. We have already shown that $D_s R_s \in L^p$. Hence we can bound the L^p norm of the composition.

Using again the same hypothesis we obtain a geometric convergence (indeed we obtain an extra factor $D_s f^N$, which makes the convergence even faster.

The first term is only slightly more complicated to bound. Note that

$$\|D_s(DF^{-1}_{f^i(x)})\|_{L^\infty} = \|D_s DF^{-1}_{f^i(x)} D_s f^i(x)\|_{L^\infty} \leq K \|D_s f\|^i_{L^\infty} \ .$$

Hence, we can bound

$$\begin{aligned} \left\| \sum_{i=1}^{N} DF^{-1}_{f^N} \cdots D_s(DF^{-1}_{f^i(x)}) \cdots DF^{-1}_x \right\| &\leq \sum_{i=1}^{N} \|DF^{-1}\|^{N-1}_{L^\infty} K \|D_s f\|^i_{L^\infty} \\ &\leq \|DF^{-1}\|^{N-1}_{L^\infty} \left(K \sum_{i=1}^{\infty} \|D_s f\|^i_\infty \right) \\ &\leq \|DF^{-1}\|^{N-1}_{L^\infty} K \ . \end{aligned}$$

Then, if we bound the other factors in this term as we have done for the other cases, using again assumption c) of Theorem 8, we obtain that the first term also is bounded by a convergent geometric series.

This finishes the proof of Lemma 10 and, therefore of Theorem 8. □

9. Some remarks about transfer equations and conjugacy equations

The equation that describe that two dynamical systems f, g are equivalent under the change of variables h is

$$(9.1)\quad f \circ h = h \circ g$$

If we take derivatives of (9.1) (assuming that h is differentiable), we obtain

$$(9.2)\quad Df \circ h Dh = Dh \circ g Dg$$

Equation (9.2) can be described by saying that the cocycles Dg and $Df \circ h$ (note that both are cocycles over g) are cohomologous with a transfer map Dh.

It is interesting to know what the equation (9.2) implies for the regularity of Dh given that we have information on the regularity of Dg and $Df \circ h$ (and, of course on regularity of g) and some geometric information about rates of growth of the cocycles Dg and $Df \circ h$ (and also properties of g such as it being an Anosov system and bounds on its expansion rates under positive and negative iteration).

With this motivation (it is not the only possible motivation since similar equations appear often in rigidity theory), we will be studing the regularity of matrix valued functions (or more generally sections of a bundle of linear maps) ϕ which satisfy an equation of the form

$$\alpha(x)\phi(x) = \phi \circ g(x)\beta(x) \tag{9.3}$$

given that we know information about the regularities of α, β, g and information on rates of growth of the cocycles α, β and the derivatives of g. In the paper [NT98] one can find a rather complete theory in the case that we consider α, β, ϕ to be taking values in diffeomorphism groups and we consider regularity measured in the classical C^r spaces. As it is remarked in there, since a linear map can also be considered as a diffeomorphism, their results also imply information about the more mundane case of bundle maps. In this section, we want to consider a Sobolev theory for the above equations (9.3). We will discuss what the theory developed implies for our main motivation (9.1),(9.2). We also note that rather than considering that α, β taking values on matrices, we can consider them taking values in Lie groups.

The first result we present on transfer equations is an extension of Theorem 8. We give several variants of the result depending on the exact hypothesis on differentiability. As we will see, the proof of all these results requires similar (but not identical) ideas. We have striven to present a variety of arguments. Even more variants could be obtained by combining the arguments in several of them.

We call special attention to variant D). Note that it assumes Sobolev hypothesis on what – in the applications to conjugacy equations – are the derivatives of the dynamical system. (At this stage of our knowledge, we still assume C^r regularity in the dynamical system)

Theorem 9. *Let g be a C^r, $1 \leq r$ Anosov diffeomorphism over a compact manifold M of dimension n. Let E be a bundle over M and $\alpha(x) : E_{h(x)} \to E_{g(x)}$, $\beta(x) : E_x \to E_{g(x)}$, $\phi(x) : E_x \to E_{h(x)}$ be linear maps on the fibers of the bundle.*

Denote by Γ_x the linear operator on $\mathcal{L}(E_x, E_{h(x)})$ to $\mathcal{L}(E_{g(x)}, E_{h\circ g(x)})$ defined by:

$$\Gamma_x \phi = \alpha(x)\phi\beta^{-1}(x)$$

Set $\gamma^+ = \sup_x ||\Gamma_x||$, $\gamma^- = \sup_x ||(\Gamma_x)^{-1}||$,

A) Assume that

1): *α, β are C^r_s (resp. C^r_u)*

2): *$D^l_s\phi \in L^p$*

3): *$l \in \mathbf{N}$, $l \leq r-1$, $1 \leq p \leq \infty$ are such that*

$$
\begin{aligned}
&\gamma^- \|J_g\|_{L^\infty}^{1/p} ||D_s g||_{L^\infty}^l \leq \lambda < 1 \\
&\textit{resp. } \gamma^+ \|J_{g^{-1}}\|_{L^\infty}^{1/p} ||(D_u g)||_{L^\infty}^l \leq \lambda < 1
\end{aligned} \tag{9.4}
$$

Then $D_s^k \phi \in L^p$, $1 \leq k \leq r$. (resp. $D_u^k \phi \in L^p$, $1 \leq k \leq r$).

B) In the case that we take $p = \infty$, we can replace in the above the hypothesis 2) and 3) by

2'): *$\phi \in C_s^l$. (resp. C_u^l.)*
3'): *We have:*

$$
\begin{aligned}
&\gamma^- ||D_s g||_{L^\infty}^l \leq \lambda < 1 \\
&\textit{resp. } \gamma^+ ||(D_u g)||_{L^\infty}^l \leq \lambda < 1
\end{aligned} \tag{9.5}
$$

Then, we obtain:

$$\phi \in C_s^r$$

(resp. $\phi \in C_u^r$.)

C) Assume 2), 3)

4): *$l \leq r - 3$*

and, in place of 1)

1'): *$\alpha, \beta \in C_s^r \cap C_u^r$ (In particular, if $\alpha, \beta \in C^r$)*

Then, we have:

$$\phi \in C_s^r \cap C_u^r$$

(In particular $\phi \in \Lambda_{r-\varepsilon}$ for all $\varepsilon > 0$.)

D) Assume:

5): *For some $m \in \mathbf{N}$, $m \leq r-1$, for all $0 < j \leq m$ and some $1 \leq p \leq \infty$ we have:*

$$D_s^j \alpha, D_s^j \beta \in L^p$$

6): *$\alpha, \beta, D_s\alpha, D_s\beta \in L^\infty$.*
7): *For some $l \in \mathbf{N}$ $l < m$, for all $k \leq l$ we have: $D_s^k \phi \in L^p$,*
8): *l, p satisfy assuption 3).*

Then, $D_s^j \phi \in L^p$ $\forall 0 \leq j \leq m$.

Remark 28. We believe that hypothesis 4) does not belong in the results for part C), (indeed it can be weakened a bit by distinguishing subcases on whether l is odd or even) but we certainly use a version of it in the present proof. □

Remark 29. The classical statement in terms of C_s^r, C_u^r spaces is very similar to the method of proof of Theorem 6.1 in [dlL92] and the statement that one obtains applying for linear diffeomorphisms the results of [NT98]. □

Remark 30. Notice that the result in C) is somewhat stronger than the one would have obtained applying B) to the stable and unstable directions and then, applying Theorem 3.

The reason is that when we have regularity both in the stable and unstable directions, we can use this to improve the p. (Note that this part of the bootstrap argument is not available when one is bootstrapping only the stable or the unstable directions.)

Eventually, for high enough p, the week derivatives are strong derivatives and we can use the more customary bootstrap of strong derivatives. The bootstrap of strong derivatives is more efficient because it can improve to the same differentiability of the map (the weak derivative bootstrap can only bootstrap to the number of derivatives of the Jacobian) Moreover, we can obtain global regularity using the method of [Jou88] rather than the elliptic regularity, which obviates the restriction of an even number of derivatives.

One could also apply the same type of reasoning with the regularity assumed in variant D) assuming L^p bounds

□

Before embarking in the proof of Theorem 9, we would like to describe and prove a corollary of version B) which extends to Sobolev regularity of Theorem 6.1 of [dlL92]. We will also discuss that we have some a-priori regularity for these equations and that, the gap between the a-priori regularity and the regularity that leads to bootstraps really does belong.

Corollary 1. *Assume that f and g are C^r Anosov systems $1 \leq r$ and that they are conjugated by a homeomorphism h such that h in* (9.1) *satisfies:*

1): *$h \in C^1$, $Dh \in W^{l,p}$.*

2): *l, p g satisfy condition 3) 4) of Theorem 9*

Then, $h \in C_s^{r,(g)} \cap C_u^{r,(g)}$. (Of course, applying the result in [Jou88]*, this implies that h in $\Lambda_{r-\varepsilon}$ for all $\varepsilon > 0$.)*

Proof of Corollary 1. We note that, as pointed out in (9.2), Dh satisfies a transfer equation with $\alpha(x) = Df \circ h(x)$, $\beta(x) = Dg(x)$.

Since we assumed that $h \in C^1$, we conclude that $\alpha \in C_s^1$, β, of course is in $C_s^{r-1} \subset C_s^1$.

Applying Theorem 9, we conclude that $h \in C_s^1$.

Then, we conclude that $\alpha \in C_s^1$ and applying the Theorem 9 again, we conclude that $h \in C_s^2$.

We can keep on repeating the procedure and stop at $h \in C_s^r$.

A similar argument applies to the unstable derivatives. Then, applying the regularity result, we obtain the desired result. □

It is important to note that the regularity hypothesis of Theorem 9 involve that the solutions we consider are sufficiently smooth to start with. If we start with situations where the regularity we start with is not high enough, one can get that that the original regularity does not bootstrap.

Examples of this kind can be obtained taking derivatives of Example 2.

We note that, the fact that the unstable foliation of f is not smooth implies that there is no smooth conjugacy between $f_0(x, y) = (Ax, By)$, which indeed has smooth invariant foliations and f. (Recall that any smooth conjugacy has to preserve the invariant foliations.)

A direct proof that the conjugacy provided by structural stability between f_0 and f in Example 2 can be found in [dlL92], where rather explicit formulas for the conjugacy are written down. One can see that the formulas involve lacunary functions and, hence, that they do not belong to the Sobolev spaces with a number of derivatives higher than the regularity measured in C^r classes.

We note that one can get the conjugacies in Example 2 to be as smooth as desired to the critical value of bootstrap. Hence, this regularity is optimal.

Remark 31. We also note that it is possible to choose A, B so that f_0 has a non-resonant spectrum. In this case, if $f \in C^\infty$, the germs of the return at periodic orbits are C^∞ conjugate to those of f_0. Hence, we see that the obstructions to conjugacy cannot be local obstructions related to periodic orbits. We still do not understand very well what are these global obstructions. One interesting paper studying global obstructions in a simpler case is [PY90]. □

9.1. **Proof of Theorem 9.** Now, we turn to the proof of Theorem 9. We remark that it is very similar to the proof of of Theorem 8.

We will give first a proof of version A) in full detail and, then, call attention to the variations and the new ideas that are required to establish the other versions.

Indeed note that Γ is a bundle map (the bundle is a bundle of linear maps). Nevertheless, there are differences in the proof since in the proof of Theorem 9 we have the extra structure that the bundle map is linear in the fibers. We will also be able to bootstrap the stable and unstable directions separately. This leads to the fact that the conditions in the l and p are sharper than the conditions that we had in Theorem 8.

We note that, using the operator Γ that was introduced in the statement of Theorem 9, we have that the transfer equation can be written more compactly as:

$$\Gamma_x \phi(x) = \phi \circ g(x) \tag{9.6}$$

If we take l weak derivatives, we obtain

$$\Gamma_x D_s^l \phi(x) + R_l(x) = (D_s^l \phi) \circ g(x) D_s^l g(x) \tag{9.7}$$

where R_l is an expression which is the sum of terms. Each term is the product of three factors. Each of these factors in a term is a derivative of α, β,g ϕ. The sum of the orders the derivatives of the factors of a term is l. (This can be obtained easily applying Leibnitz rule to the L.H.S and Faa-di-Bruno to the R.H.S. of (9.6).)

More precisely, we see that all the terms in R_l are of the forms

$$(D_s^{l-k}\Gamma_x) D_s^k \phi(x); \quad 0 \le k \le l-1 \tag{9.8}$$

$$\begin{aligned} &D_s^k \phi \circ g(x) D_s^{i_1} g(x) \cdots D_s^{i_k} g(x);\\ &\qquad 0 \le k \le l-1; \quad 0 \le k \le l-1; i_1 + \cdots + i_k = l-1 \end{aligned} \tag{9.9}$$

depending on whether the terms have been generated by the application of the Leibnitz formula or of the Faa-di-Bruno formula.

We also note that from (9.7) and our assumptions it follows that $R_l \in L^p$.

Applying repeatedly (9.7), we obtain:

$$
\begin{aligned}
D^l_s\phi(x) &= -\Gamma_x^{-1}R_l(x) - \Gamma_x^{-1}\Gamma_{g(x)}R_l\circ g(x)D^l_s g(x)\\
&\quad - \cdots -\\
&\quad - \Gamma_x^{-1}\cdots\Gamma^{-1}_{g^N(x)}R_l\circ g^N(D^l_s g)\circ g^{N-1}(x)\cdots D^l_s g(x) +\\
&\quad + \Gamma_x^{-1}\cdots\Gamma^{-1}_{g^N(x)}(D^l_s\phi)\circ g^N(x)(D^l_s g)\circ g^{N-1}(x)(D^l_s g)\circ g^{N-1}(x)\cdots D^l_s g(x)
\end{aligned}
\tag{9.10}
$$

We note that, under the hypothesis of Theorem 9, the last term in (9.10) converges to zero in L^p and that the general term of the sum converges in L^p.

The bounds to establish the L^p convergence of (9.10) are simpler than those that we established in the proof of Theorem 8. Notice that, in our present case, in each term, only one of the factors requires L^p bounds. All the others require only L^∞ bounds.

Hence, we obtain a representation

$$
D^l_s\phi(x) = \sum_{j=0}^{\infty}\Gamma_x^{-1}\cdots\Gamma^{-1}_{g^N(x)}R_l\circ g^N(D^l_s g)\circ g^{N-1}(x)\cdots D^l_s g(x)
\tag{9.11}
$$

As before, we will argue that the series (9.11) can be differentiated term by term and that the series converges.

Again, this is significantly easier than the corresponding argument in Theorem 8, since the linearity reduces the number of terms that we have to consider and, more importantly, we do not need to consider products of the derivatives we are bootstrapping.

We will study separately the terms (9.8) and (9.9).

For the terms coming from (9.8), we note that, applying the rule for the derivative of the product

$$
D_s\left[\Gamma_x^{-1}\cdots\Gamma^{-1}_{g^N(x)}[(D^{l-k}_s\Gamma^{-1})D^k_s\phi]\circ g^N(D^l_s g)\circ g^{N-1}(x)\cdots D^l_s g(x)\right]
\tag{9.12}
$$

can be expressed as a sum of not more than $2N+2$ terms each of which contains:

- At most N factors Γ^{-1}
- At most 1 factor involving $D_s\Gamma^{-1}_{g^i(x)} = (D_s\Gamma^{-1})_{g^i(x)}D_s g^i(x)$,
- Exactly $N-1$ factors of the form $(D^l_s g)\circ g^i(x)$
- Exactly one factor of the form $D^{l+1}_s g^i(x)$

Using the bounds (7.19) for the last two factors in the above list and the bounds in hypothesis 3) of Theorem 9, as well as the obvious bound

$$
\|D_s\Gamma^{-1}_{g^i(x)}\|_{L^\infty} \le \|D_s(\Gamma^{-1})_{g^i(x)}\|_{L^\infty}\|D_s g^i\|_{L^\infty} \le K
$$

we obtain that each term is bounded in L^p norm by a geometric series in N.

For the terms of the form (9.9), we note that

$$
D_s\left[\Gamma_x^{-1}\cdots\Gamma^{-1}_{g^N(x)}[D^k_s\phi(x)D^{i_1}_s g\cdots D^{i_k}_s g]\circ g^N(D^l_s g)\circ g^{N-1}(x)\cdots D^l_s g(x)\right]
\tag{9.13}
$$

can also be bounded because the terms that we obtain taking derivatives have a very similar structure to those obtained expanding (9.12).

Indeed, expanding the derivative in (9.13), we obtain $3N + 2$ terms. We note that all of these terms contain at least as many factors which are stable derivatives of g as the terms that we studied before and they do not contain any more terms of the form Γ.

Hence, we have shown that if we are under the hypothesis of part A) of the theorem, we can conclude that there exists one derivative more. This establishes the results in part A) proceeding by induction. We note that if the hypothesis hold for one l, they hold for higher l.

The arguments for the other variants are very similar. We observe that, under the hypothesis, we obtain (9.11) and that we can take derivatives. In part C) we need an extra argument to crossover from weak derivatives to strong derivatives.

The results in part B) are obtained by observing that spaces of continuous functions are closed under L^∞ norms. If we assume inductively that all the terms that appear in (9.11) are continuous, we will get that the derivative converges uniformly and that therefore $\phi \in C_s^{l+1}$. Hence we can proceed by induction.

The proof of part C) involves an interesting twist. If we perform the Sobolev bootstrap in part A), once or twice, (so that we can apply Theorem 3, here is where hypothesis 4) enters), we can conclude that $\phi \in W^{l+\sigma,p}$ (where σ is either 1 or 2). Applying Sobolev imbedding (Lemma 3), we can conclude that $\phi \in W^{l,p'}$ with $p' > p$. We can repeat this step often enough to conclude $\phi \in W^{l,\tilde{p}}$ for any $\tilde{p} < \infty$. Hence, applying our argument again, we obtain $\phi \in W^{l+1,\tilde{p}}$. Hence, if we apply the smooth Sobolev imbedding theorem (Lemma 2.4) we obtain, taking $\tilde{p}$ large enough that $\phi \in C^l$. Then, we can switch to the Hölder space bootstrap that we established in part B) of the theorem and we conclude that $\phi \in C_s^r \cap C_u^r$.

The proof of part D) follows along the same lines as the proof of part A). Note that the argument to derive (9.11) only needs the L^p bounds that show that the last term of (9.10) tends to zero and that the general term is bounded by a geometric series. This remains exactly the same. Similarly, given that we assumed L^∞ bounds on the derivatives of α and β, the arguments to show that the derivative converges remain the same.

This finishes the proof of Theorem 9.

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