Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models

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Abstract

We consider spread-out models of self-avoiding walk, bond percolation, lattice trees and bond lattice animals on $\mathbb{Z}^d$, having long finite-range connections, above their upper critical dimensions $d = 4$ (self-avoiding walk), $d = 6$ (percolation) and $d = 8$ (trees and animals). The two-point functions for these models are respectively the generating function for self-avoiding walks from the origin to $x \in \mathbb{Z}^d$, the probability of a connection from 0 to x, and the generating function for lattice trees or lattice animals containing 0 and x. We use the lace expansion to prove that for sufficiently spread-out models above the upper critical dimension, the two-point function of each model decays, at the critical point, as a multiple of $|x|^{2-d}$ as $x \to \infty$. We use a new unified method to prove convergence of the lace expansion. The method is based on x-space methods rather than the Fourier transform. Our results also yield unified and simplified proofs of the bubble condition for self-avoiding walk, the triangle condition for percolation, and the square condition for lattice trees and lattice animals, for sufficiently spread-out models above the upper critical dimension.

Contents

1 Introduction

1.1 Critical two-point functions

In equilibrium statistical mechanical models at criticality, correlations typically decay according to a power law, rather than exponentially as is the case away from the critical point. We consider models of self-avoiding walks, bond percolation, lattice trees and bond lattice animals on the lattice $\mathbb{Z}^d$. Let $|x|$ denote the Euclidean norm of $x \in \mathbb{Z}^d$. Assuming translation invariance, and denoting the critical two-point function for any one of these models by $U_{p_c}(x, y) = U_{p_c}(y - x)$, the power-law decay is traditionally written as

$$U_{p_c}(x) \sim \text{const.} \frac{1}{|x|^{d-2+\eta}}, \quad \text{as } |x| \to \infty. \tag{1.1}$$

The critical exponent η is known as the *anomalous dimension*, and depends on the model under consideration. Its value is believed to depend on d but otherwise to be *universal*, which means insensitive to many details of the model's definition.

The above models have *upper critical dimensions*

$$d_c = \begin{cases} 4 & \text{for self-avoiding walk} \\ 6 & \text{for percolation} \\ 8 & \text{for lattice trees and lattice animals} \end{cases} \tag{1.2}$$

above which critical exponents cease to depend on the dimension. Our purpose in this paper is to prove (1.1) for $d > d_c$, with $\eta = 0$, for certain long-range models having a small parameter. The small parameter is used to ensure convergence of the *lace expansion*. There is now a large literature on the lace expansion, but proving (1.1) for $d > d_c$ remained an open question. To make this paper more self-contained, a review of the basic steps in the derivation of the lace expansion will be included.

All past approaches to the lace expansion have relied heavily on the Fourier transform of the two-point function. We present a new approach to the lace expansion, based directly in x-space. Our approach provides a unified proof of convergence of the expansion, with most of the analysis applying simultaneously to all the models under consideration. There is one model-dependent step in the convergence proof, involving estimation of certain Feynman diagrams. The Feynman diagrams are model-specific, and converge when $d > d_c$. This is the key place where the assumption $d > d_c$ enters the analysis. We use a new method to estimate the relevant Feynman diagrams, based in x-space rather than using the Fourier transform.

As we will explain in more detail below, weaker versions of (1.1) have been obtained previously, for the Fourier transform of the two-point function. These statements for the Fourier transform follow as corollaries from our x-space results. In addition, our results immediately imply the bubble, square and triangle conditions for sufficiently spread-out models of self-avoiding walks, lattice trees and lattice animals, and percolation, for $d > d_c$. These diagrammatic conditions, which had been obtained previously using Fourier methods, are known to imply existence (with mean-field values) of various critical exponents.

For $d \leq d_c$, it remains an open question to prove the existence of η. In fact, it has not been proved for self-avoiding walk nor for lattice trees or animals that $U_{p_c}(x)$ is even finite for $2 \leq d \leq d_c$. For percolation, the two-point function is a probability, so it is certainly finite. However, it has not been proved for $2 \leq d \leq 6$ that it approaches zero as $|x| \to \infty$, except for $d = 2$ [24]. Such a result is known to imply absence of percolation at the critical point [4], which, for general dimensions, is an outstanding open problem in percolation theory.

For self-avoiding walks, partial results suggesting that $\eta = 0$ for $d = d_c = 4$ have been obtained in [8] for a hierarchical lattice and in [23] for a variant of the Edwards model. Contrary to other critical exponents at the upper critical dimension, no logarithmic factors appear to leading order. It is believed that $\eta > 0$ for self-avoiding walk for $2 \leq d < 4$ [28]. Interestingly, there is numerical evidence that $\eta < 0$ for percolation when $3 \leq d < 6$ [1], and it has been conjectured that $\eta < 0$

also for lattice trees and lattice animals when $2 \leq d < 8$ [7] (see also [31] for $d = 3$ and [27] for $d = 8 - \epsilon$). The exponent η is believed to be related to the exponents γ for the susceptibility and ν for the correlation length by the scaling relation $\gamma = (2 - \eta)\nu$. Some exact but nonrigorous values of γ and ν have been predicted (see [11, 22, 28, 31]), which lead to the exact predictions $\eta = \frac{5}{24}$ for 2-dimensional self-avoiding walk and percolation, and $\eta = -1$ for 3-dimensional lattice trees and animals.

1.2 Main results

The spread-out models are defined in terms of a function $D : \mathbb{Z}^d \to [0, \infty)$, which depends on a positive parameter L. We will take L to be large, providing a small parameter L^{-1}. We will consider only those D which obey the conditions imposed in the following definition.

Definition 1.1. Let h be a non-negative bounded function on $\mathbb{R}^d$ which is piecewise continuous, symmetric under the lattice symmetries, supported in $[-1, 1]^d$, and normalised so that $\int_{[-1,1]^d} h(x) d^d x = 1$. Then for large L we define

$$D(x) = \frac{h(x/L)}{\sum_{x \in \mathbb{Z}^d} h(x/L)}. \tag{1.3}$$

Since $\sum_{x \in \mathbb{Z}^d} h(x/L) \sim L^d$ (using a Riemann sum approximation to $\int_{[-1,1]^d} h(x) d^d x$), the assumption that L is large ensures that the denominator of (1.3) is nonzero. We also define $\sigma^2 = \sum_x |x|^2 D(x)$.

The sum $\sum_x |x|^p D(x)$ can be regarded as a Riemann sum, and is asymptotic to a multiple of L^p for $p > 0$. In particular, σ and L are comparable. A basic example obeying the conditions of Definition 1.1 is given by the function $h(x) = 2^{-d}$ for $x \in [-1, 1]^d$, $h(x) = 0$ otherwise, for which $D(x) = (2L + 1)^{-d}$ for $x \in [-L, L]^d \cap \mathbb{Z}^d$, $D(x) = 0$ otherwise.

Next, we define the models we consider. Let $\Omega_D = \{x \in \mathbb{Z}^d : D(x) > 0\}$. By Definition 1.1, Ω_D is finite and $\mathbb{Z}^d$-symmetric. A *bond* is a pair of sites $\{x, y\} \subset \mathbb{Z}^d$ with $y - x \in \Omega_D$. For $n \geq 0$, an n-step *walk* from x to y is a mapping $\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ such that $\omega(i+1) - \omega(i) \in \Omega_D$ for $i = 1, \ldots, n-1$. We sometimes consider a walk to be a set of bonds, rather than a set of sites. Let $\mathcal{W}(x, y)$ denote the set of walks from x to y, taking any number of steps. An n-step *self-avoiding walk* is an n-step walk ω such that $\omega(i) \neq \omega(j)$ for each pair $i \neq j$. Let $\mathcal{S}(x, y)$ denote the set of self-avoiding walks from x to y, taking any number of steps. A *lattice tree* is a finite connected set of bonds which has no cycles. A *lattice animal* is a finite connected set of bonds which may contain cycles. Although a tree T is defined as a set of bonds, we write $x \in T$ if x is an endpoint of some bond of T, and similarly for lattice animals. Let $\mathcal{T}(x, y)$ denote the set of lattice trees containing x and y, and let $\mathcal{A}(x, y)$ denote the set of lattice animals containing x and y.

Given a finite set B of bonds and a nonnegative parameter p, we define its *weight* to be

$$W_{p,D}(B) = \prod_{\{x,y\} \in B} pD(y - x). \tag{1.4}$$

If B is empty, we set $W_{p,D}(\varnothing) = 1$. The random walk and self-avoiding walk two-point functions are defined respectively by

$$S_p(x) = \sum_{\omega \in \mathcal{W}(0,x)} W_{p,D}(\omega), \qquad \sigma_p(x) = \sum_{\omega \in \mathcal{S}(0,x)} W_{p,D}(\omega). \tag{1.5}$$

For any $d > 0$, $\sum_x S_p(x)$ converges for $p < 1$ and diverges for $p > 1$, and $p = 1$ plays the role of a critical point. It is well-known [35] that, for $d > 2$,

$$S_1(x) \sim \text{const.} \frac{1}{|x|^{d-2}}, \quad \text{as } |x| \to \infty, \tag{1.6}$$

so that $\eta = 0$. A standard subadditivity argument [12, 21, 28] implies that $\sum_x \sigma_p(x)$ converges for $p < p_c$ and diverges for $p > p_c$, for some finite positive critical value p_c.

The lattice tree and lattice animal two-point functions are defined by

$$\rho_p(x) = \sum_{T \in \mathcal{T}(0,x)} W_{p,D}(T), \qquad \rho_p^a(x) = \sum_{A \in \mathcal{A}(0,x)} W_{p,D}(A). \tag{1.7}$$

A standard subadditivity argument implies that there are positive finite p_c and p_c^a such that $\sum_x \rho_p^{(a)}(x)$ converges for $p < p_c^{(a)}$ and diverges for $p > p_c^{(a)}$ [25, 26].

Turning now to bond percolation, we associate independent Bernoulli random variables $n_{\{x,y\}}$ to each bond $\{x, y\}$, with

$$\mathbb{P}(n_{\{x,y\}} = 1) = pD(x - y) \qquad \mathbb{P}(n_{\{x,y\}} = 0) = 1 - pD(x - y), \tag{1.8}$$

where $p \in [0, (\max_x D(x))^{-1}]$. (Note that p is not a probability.) A configuration is a realisation of the bond variables. Given a configuration, a bond $\{x, y\}$ is called *occupied* if $n_{\{x,y\}} = 1$ and otherwise is called *vacant*. Let $C(x)$ denote the random set of sites y such that there is a path from x to y consisting of occupied bonds. The percolation *two-point function* is defined by

$$\tau_p(x) = \mathbb{P}_p(x \in C(0)), \tag{1.9}$$

where $\mathbb{P}_p$ is the probability measure on configurations induced by the bond variables. There is a critical value $p_c \in (0, 1)$ such that $\sum_x \tau_p(x) < \infty$ for $p \in [0, p_c)$ and $\sum_x \tau_p(x) = \infty$ for $p \geq p_c$. This critical point can also be characterised by the fact that the probability of existence of an infinite cluster of occupied bonds is 1 for $p > p_c$ and 0 for $p < p_c$ [3, 29].

We use $U_p(x)$ to refer to the two-point function of all models simultaneously. We use p_c to denote the critical points for the different models, although they are, of course, model-dependent. In what follows, it will be clear from the context which model is intended.

Let

$$a_d = \frac{d\Gamma(\frac{d}{2} - 1)}{2\pi^{d/2}}. \tag{1.10}$$

We write $O(f(x, L))$ to denote a quantity bounded by const.$f(x, L)$, with a constant that is independent of x and L but may depend on d. We define ϵ by

$$\epsilon = \begin{cases} 2(d-4) & \text{(self avoiding walk)} \\ d-6 & \text{(percolation)} \\ d-8 & \text{(lattice trees and animals)} \end{cases} \tag{1.11}$$

and write

$$\epsilon_2 = \epsilon \wedge 2. \tag{1.12}$$

Our main result is the following theorem.

Theorem 1.2. *Let $U_{p_c}(x)$ denote the critical two-point function for self-avoiding walk, percolation, lattice trees or lattice animals. Let $d > d_c$, and fix any $\alpha > 0$. There is a constant $A = 1+O(L^{-2+\alpha})$ depending on d, L and the model, and an L_0 depending on d, α and the model, such that for $L \geq L_0$*

$$U_{p_c}(x) = \frac{a_d A}{\sigma^2 |x|^{d-2}} \left[1 + O\Big(\frac{L^{\epsilon_2+\alpha}}{|x|^{\epsilon_2-\alpha}}\Big) + O\Big(\frac{L^2}{|x|^{2-\alpha}}\Big)\right] \tag{1.13}$$

as $|x| \to \infty$. Constants in the error terms for (1.13) and $A-1$ depend on α.

We expect that Threorem 1.2 remains true with $\alpha = 0$, but it is convenient in our analysis to allow a small power of L or $|x|$ to enter into error estimates. Results closely related to Theorem 1.2, for nearest-neighbour models in very high dimensions, are proved in [13] using a different method.

The leading asymptotics of the critical random walk two-point function $S_1(x)$ are also given by (1.13), with $A = 1$. This will be discussed in detail, in Proposition 1.6 below. The second error term in (1.13) represents an error term in the expansion for random walk, while the first error term represents the difference between random walk and the other models. The fact that the power $|x|^{2-d}$ appears as the leading power in (1.13), independent of the precise form of D or the value of large L, is an illustration of universality.

As was pointed out in Section 1.1, it is a consequence of (1.13) for percolation that there is no percolation at the critical point. In other words, for $d > 6$ and for L large, with probability 1 there is no infinite cluster of occupied bonds when $p = p_c$. There are, however, large emerging structures present at $p = p_c$ that are loosely referred to as the incipient infinite cluster. The result of Theorem 1.2 for percolation provides a necessary ingredient for work of Aizenman [2] in this regard. Roughly speaking, Aizenman showed that if a (then unproved) weaker statement than (1.13) holds for $d > 6$, then at p_c the largest percolation clusters present within a box of side length M are of size approximately M^4 and are approximately M^{d-6} in number. Details can be found in [2]. Equation (1.13) now implies that Aizenman's conclusions do hold for sufficiently spread-out models with $d > 6$.

The following corollary will follow immediately from Theorem 1.2. The conclusion of the corollary was proved previously in [28] for self-avoiding walk, in [14] for percolation, and in [15] for lattice trees and lattice animals. The corollary is known to imply existence (with mean-field values) of various critical exponents [5, 6, 28, 34].

Corollary 1.3. *For $d > d_c$ and $L \geq L_0$, the self-avoiding walk bubble condition, the percolation triangle condition, and the lattice tree and lattice animal square conditions all hold. These diagrammatic conditions are respectively the statements that the following sums are finite:*

$$\sum_{x\in\mathbb{Z}^d} \sigma_{p_c}(x)^2, \qquad \sum_{x,y\in\mathbb{Z}^d} \tau_{p_c}(0,x)\tau_{p_c}(x,y)\tau_{p_c}(y,0), \qquad \sum_{w,x,y\in\mathbb{Z}^d} \rho^{(a)}_{p_c^{(a)}}(w)\rho^{(a)}_{p_c^{(a)}}(x-w)\rho^{(a)}_{p_c^{(a)}}(y-x)\rho^{(a)}_{p_c^{(a)}}(y).$$

Theorem 1.2 implies a related result for the Fourier transform of the critical two-point function. Given an absolutely summable function f on $\mathbb{Z}^d$, we denote its Fourier transform by

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} f(x)e^{ik\cdot x}, \qquad k \in [-\pi,\pi]^d. \tag{1.14}$$

In general, (1.1) can be expected to correspond to

$$\hat{U}_{p_c}(k) \sim \text{const.}\frac{1}{|k|^{2-\eta}}, \quad \text{as } k \to 0. \tag{1.15}$$

However, some care is required with this correspondence. In particular, if $\eta = 0$ then $U_{p_c}(x)$ is not summable, and hence its Fourier transform is not well-defined. Moreover, the inverse Fourier transform of a function asymptotic to a multiple of $|k|^{-2}$, which does exist for $d > 2$, is not necessarily asymptotic to a multiple of $|x|^{2-d}$ without further assumptions. A counterexample is given in [28, page 32].

The situation is well-understood for random walk [33]. For $d > 2$, it is the case that $S_1(x)$ is given by the inverse Fourier transform

$$S_1(x) = \int_{[-\pi,\pi]^d} \frac{e^{-ik\cdot x}}{1-\hat{D}(k)} \frac{d^d k}{(2\pi)^d}. \tag{1.16}$$

Therefore, it is reasonable to assert that

$$\hat{S}_1(k) = \frac{1}{1-\hat{D}(k)}, \tag{1.17}$$

even though $S_1(x)$ is not summable. Our assumptions on D imply that $1 - \hat{D}(k) \sim \sigma^2|k|^2/(2d)$ as $k \to 0$. Comparing with (1.15) and (1.17), this gives the k-space version of the statement that $\eta = 0$ for random walk.

For the models of Theorem 1.2, we have the following corollary. A proof of the corollary will be given in Section 2. The quantity $\hat{U}_{p_c}(k)$ appearing in the corollary represents the Fourier transform of the corresponding x-space two-point functions $U_{p_c}(x)$, in the sense that the x-space two-point functions are given by the inverse Fourier transform of the k-space quantities. It will be part of the proof to demonstrate this correspondence. Recall that $\epsilon_2 = \epsilon \wedge 2$.

Corollary 1.4. *For $d > d_c$ and $L \geq L_0$, the Fourier transforms of the critical two-point functions of the models of Theorem 1.2 obey*

$$\hat{U}_{p_c}(k) = \frac{2dA}{\sigma^2|k|^2}\left[1 + \Delta_L(k)\right], \qquad |\Delta_L(k)| \leq \begin{cases} const.\, |k|^{\epsilon_2} & (\epsilon \neq 2) \\ const.\, |k|^2 \log|k|^{-1} & (\epsilon = 2) \end{cases} \tag{1.18}$$

as $k \to 0$, with an L-dependent constant in the error term Δ_L. The constant A is the same as the constant of Theorem 1.2.

The conclusion of Corollary 1.4 for self-avoiding walk was established in [28, Theorem 6.1.6], with $|\Delta_L(k)| \leq \text{const.}|k|^a$ for any $a < \frac{d-4}{2} \wedge 1$. For percolation, it was proved in [18, Theorem 1.1] that, under the hypotheses of Corollary 1.4, $\lim_{k\to 0} |k|^2 \hat{\tau}_{p_c}(k) = A$, with no error estimate but with joint control in the limit $(k, h) \to (0, 0)$, where h is a magnetic field. For lattice trees, the conclusion of Corollary 1.4 was implicitly proved in [10], with $|\Delta_L(k)| \leq \text{const.}|k|^a$ for some unspecified $a > 0$.

The proof of Theorem 1.2 also yields the following result for the asymptotic behaviour of the critical points of self-avoiding walk and percolation. We do not obtain such a result for lattice trees and lattice animals. Much stronger results have been obtained for nearest-neighbour self-avoiding walk and percolation by pushing lace expansion methods further [17]. See [32] for related results obtained without using the lace expansion, including for lattice trees.

Corollary 1.5. *Let $\alpha > 0$. For self-avoiding walk and percolation with $d > d_c$, as $L \to \infty$*

$$1 \leq p_c \leq 1 + O(L^{-2+\alpha}). \tag{1.19}$$

In [19, 20], (1.19) is improved to $1 \leq p_c \leq 1 + O(L^{-d})$ for self-avoiding walk.

1.3 Overview of the proof

In this section, we isolate four propositions which will be combined in Section 2 to prove Theorem 1.2.

We define I by $I(x) = \delta_{0,x}$, and denote the convolution of two functions f, g on $\mathbb{Z}^d$ by

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(x-y)g(y). \tag{1.20}$$

Consider the random walk two-point function $S_z(x)$. By separating out the contribution from the zero-step walk, and extracting the contribution from the first step in the other walks, S_z can be seen to obey the convolution equation

$$S_z = I + (zD * S_z). \tag{1.21}$$

The lace expansion is a modification of this convolution equation, for the models we are considering, that takes interactions into account via a kind of inclusion-exclusion.

To state the lace expansion in a unified fashion, a change of variables is required. This change of variables is explained in Section 3. To each $p \le p_c$, we associate

$$z = \begin{cases} p & \text{(self-avoiding walk and percolation)} \\ p\rho_p^{(a)}(0) & \text{(lattice trees and animals).} \end{cases} \tag{1.22}$$

We denote by z_c the value which corresponds to p_c in the above definition. It is possible in principle that $z_c = \infty$ for lattice trees and animals, but we will rule out this possibility in Section 2, and we proceed in this section under the assumption that $\rho_{p_c}^{(a)}(0) < \infty$. Since the right hand side of (1.22) is increasing in p, it defines a one-to-one mapping. For $p = p(z)$ given by (1.22), we also define

$$G_z(x) = \begin{cases} \sigma_p(x) & \text{(self-avoiding walk)} \\ \rho_p^{(a)}(x)/\rho_p^{(a)}(0) & \text{(lattice trees and animals)} \\ \tau_p(x) & \text{(percolation).} \end{cases} \tag{1.23}$$

We will explain in Section 3 how the lace expansion gives rise to a function Π_z on $\mathbb{Z}^d$ and to the convolution equation

$$G_z = I + \Pi_z + (zD * (I + \Pi_z) * G_z). \tag{1.24}$$

The function Π_z is symmetric under the symmetries of $\mathbb{Z}^d$. For self-avoiding walk, a small modification of the usual analysis [9, 28] has been made to write the lace expansion in this form. (A remainder term in the percolation expansion will be shown to vanish, in Section 2.)

The identity (1.24) reduces to (1.21) when $\Pi_z \equiv 0$. Our method involves treating each of the models as a small perturbation of random walk, and the function $\Pi_z(x)$ should be regarded as a small correction to $\delta_{0,x}$. As we will show in Section 2, $\Pi_z(x)$ is small uniformly in x and $z \le z_c$ for large L and decays at least as fast as $|x|^{-(d+2+\epsilon)}$, when $d > d_c$. In particular, $\sum_x |x|^{2+s}|\Pi_z(x)|$ converges for $z \le z_c$, for any $s < \epsilon$, so Π_z has a finite $(2+s)$ moment. We assume the above bounds on $\Pi_z(x)$ in the remainder of this section.

Equations (1.21) and (1.24) can be rewritten as

$$I = (I - \mu D) * S_\mu = G_z - \Pi_z - (zD * (I + \Pi_z) * G_z). \tag{1.25}$$

Let $\lambda \in \mathbb{R}$. Writing

$$G_z = \lambda S_\mu + (I * G_z) - \lambda(I * S_\mu) \tag{1.26}$$

and using the first representation of (1.25) for I in $I * G_z$ and the second in $I * S_\mu$, we obtain

$$G_z = \lambda((I + \Pi_z) * S_\mu) + (S_\mu * E_{z,\lambda,\mu} * G_z), \tag{1.27}$$

with

$$E_{z,\lambda,\mu} = [I - \mu D] - \lambda[I - zD * (I + \Pi_z)]. \tag{1.28}$$

By symmetry, odd moments of $E_{z,\mu,\lambda}(x)$ vanish. We fix λ and μ so that the zeroth and second moments also vanish, i.e.,

$$\sum_{x \in \mathbb{Z}^d} E_{z,\lambda,\mu}(x) = \sum_{x \in \mathbb{Z}^d} |x|^2 E_{z,\lambda,\mu}(x) = 0. \tag{1.29}$$

Here we are assuming, as discussed above, that Π_z has finite second moment. Thus we take

$$\lambda \;=\; \lambda_z = \frac{1}{1 + z\sigma^{-2} \sum_x |x|^2 \Pi_z(x)}, \tag{1.30}$$

$$\mu \;=\; \mu_z = 1 - \lambda_z[1 - z - z\sum_x \Pi_z(x)]. \tag{1.31}$$

For simplicity, we will write $E_z(x) = E_{z,\mu_z,\lambda_z}(x)$. Then (1.27) becomes

$$G_z(x) = \lambda_z((I + \Pi_z) * S_{\mu_z})(x) + (S_{\mu_z} * E_z * G_z)(x). \tag{1.32}$$

The critical point obeys the identity

$$1 - z_c - z_c \sum_x \Pi_{z_c}(x) = 0, \tag{1.33}$$

and hence $\mu_{z_c} = 1$. To see this, we sum (1.24) over x to obtain

$$\sum_x G_z(x) = \frac{1 + \sum_x \Pi_z(x)}{1 - z - z\sum_x \Pi_z(x)}. \tag{1.34}$$

The left side is finite below the critical point, but diverges as $z \uparrow z_c$ [5, 7, 28]. Under the assumption made above on Π_z, the critical point thus corresponds to the vanishing of the denominator of (1.34).

Using the decay of Π_z in x, we will argue that, at z_c, the first term of (1.32) gives $\lambda_{z_c}[1 + \sum_y \Pi_{z_c}(y)]S_1(x)$ as the leading behaviour of $G_{z_c}(x)$. The second term will be shown to be an error term which decays faster than $|x|^{-(d-2)}$. In terms of the Fourier transform, we understand this second term as follows. By our choice of the parameters λ_z and μ_z, $\hat{E}_{z_c}(k)$ should behave to leading order as k^{2+a} for some positive a. Assuming that $\hat{G}_{z_c}(k)$ behaves like k^{-2}, and since $\hat{S}_1(k)$ behaves like k^{-2} by (1.17), the second term of (1.32) would be of the form k^{-2+a}, which should correspond to x-space decay of the form $|x|^{-(d-2+a)}$. Our proof will be based on this insight.

The proof will require:

i. information about the asymptotics of $S_\mu(x)$ (model-independent),

ii. an estimate providing bounds on the decay rate of a convolution in terms of the decay of the functions being convolved (model-independent),

iii. a mechanism for proving that $\Pi_z(x)$ decays faster than $|x|^{-(d+2)}$ (model-dependent), and

iv. given this decay of $\Pi_z(x)$, an upper bound guaranteeing adequate decay of $(S_{\mu_z} * E_z)(x)$ (model-independent).

The third item is the part of the argument that is model-dependent. The restriction $d > d_c$ enters here, in the bounding of certain Feynman diagrams that are specific to the model under consideration.

The first ingredient in the above list, namely asymptotics for the random walk generating function, is provided by the following proposition. The proof of the proposition is deferred to Section 6. More general results for the critical generating function $S_1(x)$ can be found in work of Uchiyama [35]. However, we are unable to apply results of [35] directly, since we need control of the parameter L in our estimates that is not readily extractable from [35]. Our proof of Proposition 1.6 will also make use of an analysis of $S_\mu(x)$ for $\mu < 1$, which we will need in proving Proposition 1.9 below.

Proposition 1.6. *Let $d > 2$, and suppose D satisfies the conditions of Definition 1.1. Then, for L sufficiently large, $\alpha > 0$, $\mu \le 1$ and $x \in \mathbb{Z}^d$,*

$$S_\mu(x) \le \delta_{0,x} + O\Big(\frac{1}{L^{2-\alpha}\,(|x|+1)^{d-2}}\Big), \tag{1.35}$$

$$S_1(x) = \frac{a_d}{\sigma^2}\frac{1}{(|x|+1)^{d-2}} + O\Big(\frac{1}{(|x|+1)^{d-\alpha}}\Big). \tag{1.36}$$

In (1.35)–(1.36), *constants in error terms may depend on α.*

For the second ingredient in the list above, we will use the following proposition, whose estimates show that the decay rate of functions implies a corresponding decay for their convolution. The elementary proof of the proposition will be given in Section 5.

Proposition 1.7.
(i) If functions f, g on $\mathbb{Z}^d$ satisfy $|f(x)| \le (|x|+1)^{-a}$ and $|g(x)| \le (|x|+1)^{-b}$ with $a \ge b > 0$, then there exists a constant C depending on a, b, d such that

$$\Big|(f*g)(x)\Big| \le \begin{cases} C(|x|+1)^{-(a\wedge b)} & (a > d) \\ C(|x|+1)^{d-(a+b)} & (a < d \text{ and } a+b > d). \end{cases} \tag{1.37}$$

(ii) Let $d > 2$, and let f, g be functions on $\mathbb{Z}^d$, where g is $\mathbb{Z}^d$-symmetric. Suppose that there are $A, B, C > 0$ and $s > 0$ such that

$$f(x) = \frac{A}{(|x|+1)^{d-2}} + O\Big(\frac{B}{(|x|+1)^{d-2+s}}\Big) \tag{1.38}$$

$$|g(x)| \le \frac{C}{(|x|+1)^{d+s}}. \tag{1.39}$$

Let $s_2 = s \wedge 2$. Then

$$(f * g)(x) = \frac{A \sum_y g(y)}{(|x|+1)^{d-2}} + e(x) \tag{1.40}$$

with

$$e(x) = \begin{cases} O(C(A+B)(|x|+1)^{-(d-2+s_2)}) & (s \neq 2) \\ O(C(A+B)\log(|x|+2)(|x|+1)^{-d}) & (s = 2), \end{cases} \tag{1.41}$$

where the constant in the error term depends on d and s.

For the third ingredient, we will use the following proposition. The proof of the proposition involves model-dependent diagrammatic estimates, and is given in Section 4.

Proposition 1.8. *Let $q < d$, and suppose that*

$$G_z(x) \leq \beta(|x|+1)^{-q} \quad (x \neq 0) \tag{1.42}$$

with β/L^{q-d} bounded away from zero. Then for sufficiently small β (which requires L to be large) the following statements hold.
(a) Let $z \leq 2$, and assume $\frac{1}{2}d < q < d$. For self-avoiding walk, there is a c depending on d and q such that

$$|\Pi_z(x)| \leq c\beta\, \delta_{0,x} + \frac{c\beta^3}{(|x|+1)^{3q}}. \tag{1.43}$$

(b) Define $p = p(z)$ implicitly by (1.22), and fix a positive constant R. Let z be such that $\rho^{(a)}_{p(z)}(0) \leq R$, and assume $\frac{3}{4}d < q < d$. For lattice trees or lattice animals, there is a c depending on d, q and R such that

$$|\Pi_z(x)| \leq c\beta\delta_{0,x} + \frac{c\beta^2}{(|x|+1)^{3q-d}}. \tag{1.44}$$

(c) Let $z \leq 2$, and assume $\frac{2}{3}d < q < d$. For percolation, there is a c depending on d and q such that

$$|\Pi_z(x)| \leq c\beta\delta_{0,x} + \frac{c\beta^2}{(|x|+1)^{2q}}. \tag{1.45}$$

The main hypothesis in Proposition 1.8 is an assumed bound on the decay of the two-point function. To motivate the form of the assumption, we first note that $G_z(x)$ cannot be expected to decay faster than $D(x)$. Let χ_L denote the indicator function of the cube $[-L, L]^d$. By Definition 1.1,

$$D(x) \leq O(L^{-d})\chi_L(x) \leq O(L^{q-d}(|x|+1)^{-q}), \tag{1.46}$$

and the upper bound is achieved when $|x|$ and L are comparable. This helps explain the assumption that β/L^{q-d} is bounded away from zero in the proposition. Note that $G_z(0) = 1$ for all $z \leq z_c$.

We will apply Proposition 1.8 with $q = d - 2$. However, to do so, we will have to deal with the fact that *a priori* we do not know that (1.42) holds for z near z_c with $q = d - 2$. Note that, for $q = d - 2$, the conditions on d in the above proposition correspond to $d > d_c$, with d_c given by (1.2). Also, using ϵ defined in (1.11), all three bounds of the lemma can be unified (after weakening the self-avoiding walk bound) in the form

$$|\Pi_z(x)| \leq c\beta\, \delta_{0,x} + \frac{c\beta^2}{(|x|+1)^{d+2+\epsilon}}. \tag{1.47}$$

Note that $\epsilon > 0$ if and only if $d > d_c$. It is at this stage of the analysis, and *only* here, that the upper critical dimension enters our analysis.

Finally, the fourth ingredient is the following proposition. Its proof is model-independent and is given in Section 7.

Proposition 1.9. *Fix $z \leq z_c$, $0 < \gamma < 1$, $\alpha > 0$ and $\kappa > 0$. Let $\kappa_2 = \kappa \wedge 2$. Assume that $z \leq C$ and that $|\Pi_z(x)| \leq \gamma(|x|+1)^{-(d+2+\kappa)}$. Then there is a c depending on C, κ, α but independent of z, γ, L such that for L sufficiently large*

$$|(E_z * S_{\mu_z})(x)| \leq \begin{cases} c\gamma L^{-d} & (x \neq 0) \\ c\gamma L^{\kappa_2}(|x|+1)^{-(d+\kappa_2-\alpha)} & (\text{all } x). \end{cases} \tag{1.48}$$

In (1.48), we are interested in the case where α is close to zero (and small compared to κ_2), so that the upper bound decays faster in $|x|$ than $|x|^{-d}$. It will be crucial in the proof of Proposition 1.9 that E_z is $\mathbb{Z}^d$-symmetric, and that we have chosen λ_z and μ_z such that the zeroth and second moments of E_z vanish. The coefficients of terms of order $|x|^{2-d}$ and $|x|^{-d}$, which would typically be present in the convolution of a function decaying like $|x|^{-(d+2+\kappa)}$ with S_{μ_z}, then vanish and hence are absent in the upper bound of (1.48). This can partially be seen from the first term of (1.40), where the leading term vanishes if and only if $\sum_y g(y) = 0$.

2 Proof of the main results

In this section, we prove Theorem 1.2 and Corollaries 1.3–1.5, assuming Propositions 1.6–1.9. The proof will be based on the following elementary lemma. The lemma states that under an appropriate continuity assumption, if an inequality implies a stronger inequality, then in fact the stronger inequality must hold.

Lemma 2.1. *Let f be a nonnegative function defined on an interval $[z_1, z_c)$, and let $a \in (0,1)$ be given. Suppose that*

i. f is continuous on the interval $[z_1, z_c)$.

ii. $f(z_1) \leq a$.

iii. for each $z \in (z_1, z_c)$, if $f(z) \leq 1$ then in fact $f(z) \leq a$. (In other words, one inequality implies a stronger inequality.)

Then $f(z) \leq a$ for all $z \in [z_1, z_c)$.

Proof. By the third assumption, $f(z) \notin (a,1]$ for all $z \in (z_1, z_c)$. By the first assumption, $f(z)$ is continuous in $z \in [z_1, z_c)$. Since $f(z_1) \le a$ by the second assumption, the above two facts imply that $f(z)$ cannot enter the forbidden interval $(a,1]$ when $z \in (z_1, z_c)$, and hence $f(z) \le a$ for all $z \in [z_1, z_c)$. □

We will employ Lemma 2.1 to prove the following proposition, which lies at the heart of our method. The proposition provides a good upper bound on the critical two-point function for nonzero x. There is an additional detail required in the proof for lattice trees and lattice animals, and we therefore treat these models separately from self-avoiding walk and percolation. The relevant difference between the models is connected with the fact that $\sigma_z(0) = \tau_z(0) = 1$, whereas $\rho_p^{(a)}(0) > 1$ and we do not know *a priori* that $\rho_{p_c}^{(a)}(0) < \infty$. In the proof, we establish the finiteness of $\rho_{p_c}^{(a)}(0)$. As usual, α should be regarded as almost zero in Proposition 2.2.

Proposition 2.2. *Fix $d > d_c$ and $\alpha > 0$. For L sufficiently large depending on d and α,*

$$G_{z_c}(x) \le \frac{C}{L^{2-\alpha}(|x|+1)^{d-2}} \qquad (x \ne 0). \tag{2.1}$$

In addition, $z_c \le 1 + O(L^{-2+\alpha})$, and, for lattice trees and lattice animals, $\rho_{p_c}^{(a)}(0) < O(1)$. The constants in all the above statements depend only on d and α, and not on L.

Proof. We prove the desired bound for $\alpha < \frac{\epsilon \wedge 1}{2}$, because the bound for large α follows from that for small α. In the following, let K denote the smallest constant that can be used in the error bound of (1.35), i.e. $K = \sup_{L \ge 1, x \ne 0} L^{2-\alpha}(|x|+1)^{d-2} S_1(x) \in (0,\infty)$.

Self-avoiding walk and percolation. We will prove that $G_z(x)$ obeys the upper bound of (2.1) uniformly in $z < z_c$. This is sufficient, by the monotone convergence theorem.

Let

$$g_x(z) = (2K)^{-1} L^{2-\alpha}(|x|+1)^{d-2} G_z(x), \qquad g(z) = \sup_{x \ne 0} g_x(z). \tag{2.2}$$

For self-avoiding walk and percolation, we will employ Lemma 2.1 with $z_1 = 1$,

$$f(z) = \max\{g(z), \tfrac{1}{2} z\}, \tag{2.3}$$

and a chosen arbitrarily in $(\frac{1}{2}, 1)$. We verify the assumptions of Lemma 2.1 one by one, with the bound (2.1) then following immediately from Lemma 2.1. In the course of the proof, the desired upper bound on z_c will be shown to be a consequence of a weaker bound than (2.1), in (2.7). Since the proof actually establishes (2.1), (2.7) then follows.

(i) Continuity of each g_x on $[0, z_c)$ is immediate from the fact that $\sigma_z(x)$ is a power series with radius of convergence z_c, and from the continuity in z of $\tau_z(x)$ proved in [4]. We need to argue that the supremum of these continuous functions is also continuous. For this, it suffices to show that the supremum is continuous on $[0, z_c - t)$ for every small $t > 0$. It is a standard result that $\sigma_z(x)$ and $\tau_z(x)$ decay exponentially in $|x|$, with a decay rate that is uniform in $z \in [0, z_c - t)$ (though not in L) [11, 28]. Thus $g_x(z)$ can be made less than any $\delta > 0$, uniformly in $z \in [0, z_c - t)$, by taking $|x|$ larger than some $R = R(L, t, \delta)$. However, choosing x_0 such that $D(x_0) > 0$, we see that $g_{x_0}(z) \ge (2K)^{-1} L^{2-\alpha}(|x_0|+1)^{d-2} z D(x_0) \ge (2K)^{-1} L^{2-\alpha}(|x_0|+1)^{d-2} D(x_0) \equiv \delta_0$. Hence

the supremum is attained for $|x| \leq R(L,t,\delta_0)$, which is a finite set, and hence the supremum is continuous and the first assumption of Lemma 2.1 has been established.

(ii) For the second assumption of the lemma, we note that $\tau_1(x) \leq \sigma_1(x) \leq S_1(x)$ and apply the uniform bound of (1.35) to conclude that $g(1) \leq 1/2$. Since we have restricted a to be larger than $\frac{1}{2}$, this implies $f(1) < a$.

(iii) Fix $z \in (1, z_c)$. We assume that $f(z) \leq 1$, which implies

$$G_z(x) \leq \frac{2KL^{-2+\alpha}}{(|x|+1)^{d-2}} \qquad (x \neq 0). \tag{2.4}$$

We will apply Proposition 1.8 with $q = d-2$ and $\beta = KL^{-2+\alpha}$. Since we have taken $\alpha < \frac{1}{2}$, we have $\beta \ll 1$ and $\beta / L^{q-d} = KL^{\alpha} \gg 1$ for sufficiently large L depending on α. Proposition 1.8 then implies that

$$|\Pi_z(x)| \leq cKL^{-2+\alpha}\delta_{0,x} + \frac{c\,K^2\,L^{-4+2\alpha}}{(|x|+1)^{d+2+\epsilon}} \leq \frac{c\,KL^{-2+\alpha}}{(|x|+1)^{d+2+\epsilon}}, \tag{2.5}$$

where $\epsilon > 0$ was defined in (1.11). It addition, for percolation, as argued at the end of Section 4.5, the remainder term $R_z^{(N)}(x)$ vanishes in the limit $N \to \infty$ under the assumption (2.4), yielding the form (1.24) of the expansion.

Summing (1.24) over $x \in \mathbb{Z}^d$ gives

$$\sum_x G_z(x) = \frac{1 + \sum_x \Pi_z(x)}{1 - z - z\sum_x \Pi_z(x)} > 0, \tag{2.6}$$

which is finite for $z < z_c$. The numerator is positive by (2.5), and hence the denominator is also positive. Therefore, since $z \leq 2$ by our assumption that $f(z) \leq 1$, (2.5) implies that

$$z < 1 - z\sum_x \Pi_z(x) \leq 1 + O(L^{-2+\alpha}). \tag{2.7}$$

Since $a \in (\frac{1}{2}, 1)$, this implies that $z < 2a$ for all $z < z_c$, when L is large. Thus, to prove that $f(z) \leq a$, it suffices to show that $g(z) \leq a$.

The bound (2.5) also implies that λ_z and μ_z are well-defined by (1.30)–(1.31), and that $\lambda_z \to 1$ uniformly in $z \leq z_c$. Using the convolution bound of Proposition 1.7(i), (1.35), and the first bound of (2.5), it then follows that

$$|(\Pi_z * S_{\mu_z})(x)| \leq \frac{O(L^{-4+2\alpha})}{(|x|+1)^{d-2}} = \frac{o(L^{-2+\alpha})}{(|x|+1)^{d-2}} \qquad (x \neq 0). \tag{2.8}$$

By Proposition 1.9 with $\kappa = 2\alpha < \epsilon$ and $\gamma = cKL^{-2+\alpha}$, for L large we have

$$|(E_z * S_{\mu_z})(x)| \leq \begin{cases} O(L^{-2+\alpha-d}) & (x \neq 0) \\ O(L^{-2+3\alpha})(|x|+1)^{-(d+\alpha)} & (\text{all } x). \end{cases} \tag{2.9}$$

Using the first bound for $0 < |x| \leq L$ and the second bound for $|x| \geq L$, we conclude from this that

$$|(E_z * S_{\mu_z})(x)| \leq O(L^{-4+2\alpha})(|x|+1)^{-(d-2)} \qquad (x \neq 0). \tag{2.10}$$

By Proposition 1.7(i), (2.4) and (2.9), it then follows that

$$\begin{aligned}|(E_z * S_{\mu_z} * G_z)(x)| &\leq |(E_z * S_{\mu_z})(x)| + \sum_{y\neq 0} |(E_z * S_{\mu_z})(x-y)|\,|G_z(y)| \\ &\leq \frac{O(L^{-4+4\alpha})}{(|x|+1)^{d-2}} = \frac{o(L^{-2+\alpha})}{(|x|+1)^{d-2}} \qquad (x \neq 0),\end{aligned} \tag{2.11}$$

where we have used (2.10) to bound the first term in the first inequality. Using the fact that $\lambda_z = 1 + o(1)$ as $L \to \infty$, and the definition of K, it then follows from the identity (1.32) that for L sufficiently large we have

$$G_z(x) \leq (1+o(1))S_1(x) + \frac{o(L^{-2+\alpha})}{(|x|+1)^{d-2}} \leq \frac{2a\,K}{L^{2-\alpha}\,(|x|+1)^{d-2}} \qquad (x \neq 0). \tag{2.12}$$

This yields $g(z) \leq a$, and completes the proof for self-avoiding walk and percolation.

Lattice trees and lattice animals. We will first prove that $G_z(x)$ obeys the upper bound of (2.1) uniformly in $z < z_c$.

By (1.3) and the fact that h is bounded, there is a $\delta_1 \geq 1$ such that $D(x) \leq \delta_1 |\Omega_D|^{-1}$ for all x. The number of n-bond lattice trees or lattice animals containing the origin is less than the number $b_n(L)$ of n-bond lattice trees on the Bethe lattice of coordination number $|\Omega_D|$ (the uniform tree of degree $|\Omega_D|$), which contain the origin. A standard subadditivity argument, together with the fact that, as $L \to \infty$, $\lim_{n\to\infty} b_n(L)^{1/n} \sim e|\Omega_D| \leq 3|\Omega_D|$ (see, e.g., [32]), implies that $b_n(L) \leq (n+1)(3|\Omega_D|)^n$. Therefore, for lattice trees or lattice animals,

$$\rho_p^{(a)}(0) \leq \sum_{n=0}^{\infty} (n+1)(3\delta_1 p)^n = \frac{1}{(1-3\delta_1 p)^2}. \tag{2.13}$$

Let $p_1 = \frac{1}{6\delta_1}$. We use $z_1 = p_1 \rho_{p_1}^{(a)}(0)$ in Lemma 2.1. Note that z_1 is well-defined, since (2.13) gives $\rho_p^{(a)}(0) \leq 4$ for $p \leq p_1$. In addition, (2.13) implies that $p_c \geq (3\delta_1)^{-1} > p_1$, so $z_c > z_1$. We again fix $a \in (\frac{1}{2}, 1)$, and we use the function $f(z)$ of (2.3) in Lemma 2.1, taking now

$$g(z) = \sup_{x\neq 0} g_x(z) \quad \text{with} \quad g_x(z) = \frac{1}{8K} L^{2-\alpha}(|x|+1)^{d-2} G_z(x), \tag{2.14}$$

The desired bound on $G_z(x)$, for $z < z_c$, together with the desired bound on z_c, will follow once we verify the three conditions of Lemma 2.1. We verify these conditions now.

(i) Continuity of $f(z)$ follows from the exponential decay of $\rho_p^{(a)}(x)$ for $p < p_c$, as in the previous discussion, together with the continuity of $\rho_p^{(a)}(0)$ for $p < p_c$.

(ii) By the remarks surrounding the definition of z_1, we have $\frac{z_1}{2} \leq \frac{4}{2\cdot 6\delta_1} \leq \frac{1}{3} < a$. Moreover, this implies $z_1 < \frac{2}{3} < 1$. It remains to show that

$$G_{z_1}(x) \leq \frac{8aK}{L^{2-\alpha}(|x|+1)^{d-2}} \qquad (x \neq 0). \tag{2.15}$$

Since $\rho_{p_1}^{(a)} \geq 1$, we have $G_{z_1}(x) \leq \rho_{p_1}^{(a)}(x)$. Each lattice tree or lattice animal containing 0 and x can be decomposed in a non-unique way into a walk from 0 to x with a lattice tree or lattice animal attached at each site along the walk. Therefore $\rho_{p_1}^{(a)}(x) \le \rho_{p_1}^{(a)}(0) S_{z_1}(x)$. Using $\rho_{p_1}^{(a)}(0) \le 4$, it follows from Proposition 1.6 that $G_{z_1}(x) \le 4KL^{-2+\alpha}(|x|+1)^{-(d-2)}$, which implies (2.15).

(iii) Fix $z \in (z_1, z_c)$. The assumption that $f(z) \le 1$ implies the bound $\rho_p^{(a)}(0) \le z/p_1 \le 12\delta_1$, and we take $R = 12\delta_1$ in Proposition 1.8(b). We then proceed as in the discussion for self-avoiding walk and percolation. We obtain (2.7) as before, so that $z < 2a$ as required. The proof of (2.12) also proceeds as before.

The above discussion proves that $G_z(x)$ is bounded by the right side of (2.1) and that $\rho_p^{(a)}(0) \le 4$, uniformly in $z < z_c$, and that $z_c \le 1 + O(L^{-2+\alpha})$. The proof is then completed by observing that $\lim_{p \uparrow p_c} \rho_p^{(a)}(x) = \rho_{p_c}^{(a)}(x)$, by monotone convergence. □

Proposition 2.2 establishes the hypotheses of Proposition 1.8, with β proportional to $L^{-2+\alpha}$, $z = z_c$ and $q = d-2$. Hence the hypotheses of Proposition 1.9 are also now established, with $z = z_c$, $\kappa = \epsilon$, and $\gamma = O(L^{-2+\alpha})$. The conclusion of Proposition 1.9 has therefore also been established. Moreover, since Proposition 2.2 gives a bound on $G_z(x)$ uniformly in $z \le z_c$, the bounds of Proposition 2.2 and Proposition 1.9 hold uniformly in $z \le z_c$. We will use this in the following.

Proof of Theorem 1.2. Fix $z = z_c$, and recall the observation below (1.33) that $\mu_{z_c} = 1$. Define

$$H(x) = \lambda_{z_c} \sum_{n=0}^{\infty} \left((I + \Pi_{z_c}) * (E_{z_c} * S_1)^{*n} \right)(x), \tag{2.16}$$

where the superscript $*n$ denotes an n-fold convolution and $(E_{z_c} * S_1)^{*0} = I$. Propositions 1.7(i), 1.8 and 1.9 guarantee that the series in (2.16) converges absolutely, and that

$$H(x) = \lambda_{z_c} \delta_{0,x} + O\Big(\frac{L^{-2+\epsilon_2+\alpha}}{(|x|+1)^{d+\epsilon_2-\alpha}} \Big) \tag{2.17}$$

with $\epsilon_2 = \epsilon \wedge 2$. Iteration of (1.32) then gives

$$G_{z_c}(x) = (S_1 * H)(x). \tag{2.18}$$

By Proposition 1.7(ii) and the asymptotic formula of (1.36), this yields

$$G_{z_c}(x) = \frac{a_d A}{\sigma^2 (|x|+1)^{d-2}} + O\Big(\frac{L^{-2+\epsilon_2+\alpha}}{(|x|+1)^{d-2+\epsilon_2-\alpha}} \Big) + O\Big(\frac{1}{(|x|+1)^{d-\alpha}} \Big), \tag{2.19}$$

with $A = \hat{H}(0)$. This proves Theorem 1.2, apart from the assertion that $A = 1 + O(L^{-2+\alpha})$.

The constant $A = \hat{H}(0)$ can be evaluated as follows. Since (2.16) is absolutely summable over x, the Fourier transform

$$\hat{H}(k) = \lambda_{z_c} (1 + \hat{\Pi}_{z_c}(k)) \sum_{n=0}^{\infty} [\hat{E}_{z_c}(k) \hat{S}_1(k)]^n \tag{2.20}$$

is continuous in k. Using (1.29), the fact that $E_{z_c}(x)$ decays like $|x|^{-(d+2+\epsilon)}$, and dominated convergence, we have

$$\lim_{k \to 0} |k|^{-2} \hat{E}_{z_c}(k) = \lim_{k \to 0} \sum_x E_{z_c}(x) |k|^{-2} \Big(\cos(k \cdot x) - 1 - \frac{|k|^2 |x|^2}{2d} \Big) = 0. \tag{2.21}$$

Since $\hat{S}_1(k)$ diverges like a multiple of $|k|^{-2}$ by (1.17), we conclude from (1.30) and the conclusion of Proposition 1.8 that

$$A = \hat{H}(0) = \lim_{k\to 0} \hat{H}(k) = \lambda_{z_c}(1+\hat{\Pi}_{z_c}(0)) = \frac{1+\hat{\Pi}_{z_c}(0)}{1+z_c\,\sigma^{-2}\sum_x |x|^2\Pi_{z_c}(x)} = 1+O(L^{-2+\alpha}). \tag{2.22}$$

$\square$

Proof of Corollary 1.3. The corollary follows immediately from Theorem 1.2 and the convolution bound of Proposition 1.7(i). $\square$

To prove Corollary 1.4, the following lemma will be useful.

Lemma 2.3. *Let $f(x)$ be a $\mathbb{Z}^d$-symmetric function which obeys the bound $|f(x)| \leq (|x|+1)^{-(d+2+\kappa)}$ with $\kappa > 0$. Then*

$$\hat{f}(k) = \hat{f}(0) + \frac{|k|^2}{2d}\nabla^2\hat{f}(0) + e(k) \tag{2.23}$$

with

$$|e(k)| \leq \begin{cases} const.\ |k|^{2+(\kappa\wedge 2)} & (\kappa \neq 2) \\ const.\ |k|^4 \log|k|^{-1} & (\kappa = 2). \end{cases} \tag{2.24}$$

Proof. By the $\mathbb{Z}^d$-symmetry of $f(x)$,

$$\hat{f}(k) = \sum_x f(x)\cos(k\cdot x) = \hat{f}(0) + \frac{|k|^2}{2d}\nabla^2\hat{f}(0) + \sum_x \left(\cos(k\cdot x) - 1 + \frac{(k\cdot x)^2}{2}\right) f(x). \tag{2.25}$$

The expression in brackets of the third term is bounded in absolute value both by $|k|^4|x|^4/24$ and $2+|k|^2|x|^2/2$. The third term of (2.25) is therefore bounded by

$$\frac{|k|^4}{24}\sum_{x:|x|\leq|k|^{-1}} |x|^4\,|f(x)| + \sum_{x:|x|>|k|^{-1}} \left(2+\frac{|k|^2|x|^2}{2}\right)|f(x)|. \tag{2.26}$$

Using the assumed upper bound on $f(x)$ then gives (2.24) and completes the proof. $\square$

Proof of Corollary 1.4. We first assume $\epsilon \neq 2$, and comment on the minor modifications required for $\epsilon = 2$ at the end of the proof.

Let $\hat{F}_z(k) = 1 - z\hat{D}(k)(1+\hat{\Pi}_z(k))$. For $z < z_c$, as in (2.6) we have

$$\hat{G}_z(k) = \frac{1+\hat{\Pi}_z(k)}{\hat{F}_z(k)}. \tag{2.27}$$

As we have noted above, the bounds of Proposition 1.8 have been established with $q = d-2$, uniformly in $z \leq z_c$. Therefore by Lemma 2.3, we have for $z \leq z_c$

$$1+\hat{\Pi}_z(k) = 1+\hat{\Pi}_z(0) + O_L(|k|^2), \tag{2.28}$$

$$\hat{F}_z(k) = \hat{F}_z(0) + \frac{|k|^2}{2d}\nabla^2\hat{F}_z(0) + O_L(|k|^{2+(\epsilon\wedge 2)}), \tag{2.29}$$

with L-dependent error estimates. Also, as observed in (2.7), $\hat{F}_z(0) > 0$ for $z < z_c$. Thus we have the infra-red bound

$$0 < \hat{G}_z(k) \leq O_L(|k|^{-2}) \tag{2.30}$$

uniformly in $z < z_c$.

Since $G_{z_c}(x)$ behaves like $|x|^{-(d-2)}$, it is not summable over x and hence the summation defining $\hat{G}_{z_c}(k)$ is not well-defined. We define

$$\hat{G}_{z_c}(k) = \lim_{z \uparrow z_c} \hat{G}_z(k) = \frac{1 + \hat{\Pi}_{z_c}(k)}{\hat{F}_{z_c}(k)}. \tag{2.31}$$

This is a sensible definition, because $G_{z_c}(x)$ is then given by the inverse Fourier transform of $\hat{G}_{z_c}(k)$. In fact, using monotone convergence in the first step, and (2.30) and the dominated convergence theorem in the last step (since $d \geq d_c + 1 > 2$), we have

$$G_{z_c}(x) = \lim_{z \uparrow z_c} G_z(x) = \lim_{z \uparrow z_c} \int_{[-\pi,\pi]^d} \hat{G}_z(k) e^{-ik \cdot x} \frac{d^d k}{(2\pi)^d} = \int_{[-\pi,\pi]^d} \hat{G}_{z_c}(k) e^{-ik \cdot x} \frac{d^d k}{(2\pi)^d}. \tag{2.32}$$

Since $\hat{F}_{z_c}(0) = 0$ by (1.33), (2.28)–(2.29) then imply

$$\hat{G}_{z_c}(k) = \frac{2d(1 + \hat{\Pi}_{z_c}(0))}{\nabla^2 \hat{F}_{z_c}(0)|k|^2} \left[1 + O_L(|k|^{\epsilon \wedge 2})\right] = \frac{2dA}{\sigma^2 |k|^2} \left[1 + O_L(|k|^{\epsilon \wedge 2})\right]. \tag{2.33}$$

In the last equality, we used (1.33) and (2.22).

The case $\epsilon = 2$ can be treated by adding an extra factor $\log |k|^{-1}$ to (2.29) and (2.33). $\square$

Proof of Corollary 1.5. Recall the elementary fact that for self-avoiding walk and percolation, $p_c = z_c \geq 1$. The corollary then follows immediately from Proposition 2.2. (The bound $z_c \leq 1 + O(L^{-2+\alpha})$ is uninformative concerning p_c for lattice trees and lattice animals, since we have proved only that $\rho_{p_c}^{(a)}(0) \in [1,4]$.) $\square$

It remains to prove Propositions 1.6–1.9. After reviewing the lace expansion in Section 3, these four propositions will be proved in Sections 6, 5, 4 and 7 respectively.

3 The lace expansion

In this section, we review the key steps in the derivation of the lace expansion. In particular, we will describe how for each of our models the lace expansion gives rise to the convolution equation (1.24), which can be written as

$$G_z(x) = \delta_{0,x} + \Pi_z(x) + (zD * G_z)(x) + (\Pi_z * zD * G_z)(x). \tag{3.1}$$

For self-avoiding walk, the lace expansion was introduced by Brydges and Spencer in [9]. Our treatment of the expansion for self-avoiding walk differs slightly from the usual treatment, to allow for a simultaneous treatment of lattice trees and animals. For percolation, and for lattice trees and lattice animals, the expansions were introduced by Hara and Slade in [14, 15]. For overviews, see [16, 28]. Proofs and further details can be found in the above references.

This section, together with Section 4, contains the model-dependent part of our analysis.

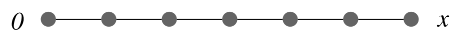

Figure 1: A string of beads.

3.1 Inclusion-exclusion

The expansion can be understood intuitively as arising from repeated use of the inclusion-exclusion relation. We describe this now in general terms, postponing a more precise (but more technical) description to Sections 3.2–3.3.

The two-point function for each of the models under consideration is a sum, over geometrical objects, of weights associated with these objects. The geometrical objects are self-avoiding walks, lattice trees, or lattice animals containing the two points 0 and x. This is the case also for percolation when $p < p_c$. For example, for the nearest-neighbour model $\tau_p(x) = \sum_{A \in \mathcal{A}(0,x)} p^{|A|}(1-p)^{|\partial A|}$ for $p < p_c$, where ∂A represents the boundary bonds of A and $|A|$ is the number of bonds in A. We view these geometrical objects as a string of mutually-avoiding beads, as depicted in Figure 1. For self-avoiding walk, the beads are simply lattice sites, whose mutual avoidance keeps the walk self-avoiding. For lattice trees, the string represents the unique path, or *backbone*, in the tree from 0 to x, and the beads represent lattice trees corresponding to branches along the backbone. These branches are mutually-avoiding, to preserve the overall tree structure.

For lattice animals and percolation, we need to introduce the notion of a pivotal bond. A bond $\{a, b\}$ in $A \in \mathcal{A}(x, y)$ is called *pivotal* for the connection from x to y if its removal would disconnect the animal into two connected components, with x in one component and y in the other. A lattice animal A containing x and y is said to have a *double connection* from x to y if there are two bond-disjoint paths in A between x and y, or if $x = y$. For lattice animals, the string in the string of beads represents the pivotal bonds for the connection of 0 and x. The beads correspond to the portions of the animal doubly-connected between pivotal bonds. The mutual avoidance of the beads is required for consistency with the pivotal nature of the pivotal bonds. This picture is the same both for lattice animals and for percolation.

The basic idea of the lace expansion is the same in all four models. It consists in approximating the two-point function by a sum of weights of geometrical objects represented by a string of beads, with the interaction between the first bead and all subsequent beads neglected. This treats the model as if it were a Markov process. The approximation causes configurations which do not contribute to the two-point function to be included, and these undesired contributions are then excluded in a correction term. The correction term is then subjected to repeated and systematic further application of inclusion-exclusion.

Let $\mathcal{D}(x, y)$ denote the set of all animals having a double connection between x and y, and, given a lattice animal, let $\Delta(x)$ denote the set of sites that are doubly-connected to x. We define

$$\psi_p^{(0)}(x) = \begin{cases} 0 & \text{(self-avoiding walk and lattice trees)} \\ (1-\delta_{0,x}) \sum_{A \in \mathcal{D}(0,x)} W_{p,D}(A) & \text{(lattice animals)} \\ (1-\delta_{0,x}) \mathbb{P}_p(x \in \Delta(0)) & \text{(percolation)} \end{cases} \tag{3.2}$$

and

$$a_p = \begin{cases} 1 & \text{(self-avoiding walk and percolation)} \\ \rho_p^{(a)}(0) & \text{(lattice trees and lattice animals).} \end{cases} \tag{3.3}$$

The procedure described in the preceding paragraph is implemented by writing

$$U_p(x) = a_p\delta_{0,x} + \psi_p^{(0)}(x) + a_p(pD * U_p)(x) + (\psi_p^{(0)} * pD * U_p)(x) + R_p^{(0)}(x). \tag{3.4}$$

The terms on the right side can be understood as follows. The first term is the contribution due to the case when the string of beads consists of a single bead and $x = 0$. The term $\psi_p^{(0)}(x)$ is the contribution due to the case when the string of beads consists of a single bead and $x \neq 0$. The convolutions correspond to the case where the string of beads consists of more than a single bead. The factors a_p and $\psi_p^{(0)}$ together give the contribution from the first bead, the factor pD is the contribution from the first piece of string, and the factor U_p is the contribution of the remaining portion of the string of beads. These two terms neglect the interaction between the first bead and the subsequent beads. This is corrected by the correction term $R_p^{(0)}(x)$, which is negative.

To understand the correction term, we first restrict attention to the combinatorial models, which excludes percolation. In this case, the correction term simply involves the contributions from configurations in which the first bead intersects some subsequent bead. The contribution due the case where the first such bead is actually the last bead is denoted $-\psi_p^{(1)}(x)$. If the first such bead is not the last bead, then suppose it is the j^{th} bead. The second through j^{th} beads are mutually avoiding, and the $(j+1)^{\text{st}}$ through last bead are mutually avoiding, and these two sets of beads avoid each other. We neglect the mutual avoidance between these two sets of beads, making them independent of each other, and add a correction term to exclude the undesired configurations included through this neglect. This leads to the identity

$$R_p^{(0)}(x) = -\psi_p^{(1)}(x) - (\psi_p^{(1)} * pD * U_p)(x) + R_p^{(1)}(x). \tag{3.5}$$

The inclusion-exclusion can then be applied to $R_p^{(1)}(x)$, and so on. For percolation, the above procedure can also be applied, but more care is needed in dealing with the probabilistic nature of the weights involved. The form of the terms arising in the expansion for percolation is, however, the same as the above. When the process is continued indefinitely, the result is

$$U_p(x) = a_p\delta_{0,x} + \psi_p(x) + a_p(pD * U_p)(x) + (\psi_p * pD * U_p)(x), \tag{3.6}$$

with

$$\psi_p(x) = \sum_{N=0}^{\infty} (-1)^N \psi_p^{(N)}(x). \tag{3.7}$$

The change of variables defined by (1.22)–(1.23) then gives our basic identity (3.1), once we define

$$\Pi_z(x) = a_p^{-1}\psi_p(x). \tag{3.8}$$

Care is needed for convergence of (3.7). We require convergence at $p = p_c$, which demands in particular that the individual terms in the sum over N are finite when $p = p_c$. This will be achieved by taking d greater than the critical dimension d_c. The role of large L is to ensure that the terms $\psi_{p_c}^{(N)}(x)$ are not only finite, but grow small with N sufficiently rapidly to be summable. These issues are addressed in detail in Section 4.

The above discussion has been at an informal level, to establish intuition for the expansion. Our next goal is to make this more precise.

3.2 Self-avoiding walk, lattice trees and lattice animals

For the combinatorial models, an elegant formalism introduced by Brydges and Spencer [9] can be used to make the discussion more precise, using the notion of *lace*. We discuss this now.

Let R be an ordered set $R_0, R_1, \ldots, R_l$ of lattice animals, with l arbitrary. In particular, each R_j may be simply a lattice tree or a single site. Given R, we define

$$\mathcal{U}_{st}(R) \;= \begin{cases} -1 & \text{if } R_s \cap R_t \neq \varnothing \\ \;\;\,0 & \text{if } R_s \cap R_t = \varnothing. \end{cases} \tag{3.9}$$

In (3.9), the intersection is to be interpreted as the intersection of sets of *sites* rather than of bonds. For $0 \leq a \leq b$, we also define

$$K_R[a,b] = \prod_{a \leq s < t \leq b} (1 + \mathcal{U}_{st}(R)). \tag{3.10}$$

The two-point functions for self-avoiding walk, lattice trees, and lattice animals can be rewritten in terms of K_R.

Given a finite set B of bonds, we let $|B|$ denote its cardinality. For self-avoiding walk, we let R consist of the sites along the walk (the 'beads'), so that each R_i is the single site $\omega(i)$. Then the two-point function can be written

$$\sigma_p(x) = \sum_{\omega \in \mathcal{W}(0,x)} W_{p,D}(\omega) K_R[0, |\omega|]. \tag{3.11}$$

The sum is over all walks, with or without self-intersections, but K_R is nonzero only for self-avoiding walks, for which $K_R = 1$. Thus K_R provides the avoidance interaction.

For a lattice tree $T \ni 0, x$, we let the R_i denote the branches (the 'beads') along the backbone of T joining 0 to x. The two-point function for lattice trees can be written

$$\rho_p(x) = \sum_{\omega \in \mathcal{W}(0,x)} W_{p,D}(\omega) \left[\prod_{i=0}^{|\omega|} \sum_{R_i \in \mathcal{T}(\omega(i),\omega(i))} W_{p,D}(R_i) \right] K_R[0, |\omega|]. \tag{3.12}$$

The additional sums and product in (3.12), compared with (3.11), generate the branches attached along the backbone ω, and the factor K_R ensures that the branches do not intersect.

For a lattice animal $A \in \mathcal{A}(x,y)$, there is a natural order to the set of pivotal bonds for the connection from x to y, and each pivotal bond is directed in a natural way, as in the left to right order in Figure 1. Given two sites x, y and an animal A containing x and y, the *backbone* of A is defined to be the ordered set of directed pivotal bonds for the connection from x to y. In general this backbone is not connected. Let R denote the set $R_0, R_1, \ldots$ of connected components which remain after the removal of the backbone from A (the 'beads'). Let $B = ((u_1, v_1), \ldots, (u_{|B|}, v_{|B|}))$ be an arbitrary finite ordered set of directed bonds. Let $v_0 = 0$ and $u_{|B|+1} = x$. Then the two-point function for lattice animals can be written as

$$\rho_p^a(x) = \sum_{B : |B| \geq 0} W_{p,D}(B) \left[\prod_{i=0}^{|\omega|} \sum_{R_i \in \mathcal{D}(u_i, v_i)} W_{p,D}(R_i) \right] K_R[0, |B|]. \tag{3.13}$$

The lace expansion proceeds by expanding out the product defining K_R, in each of (3.11)–(3.13). An elementary but careful partial resummation is then performed, which leads to a result equivalent to that of the inclusion-exclusion procedure described in Section 3.1. We will review this procedure now, leading to precise definitions for $\psi_p(x)$ and hence, recalling (3.8), also for $\Pi_z(x)$.

An essential ingredient is the following definition, in which the notion of lace is defined. It involves a definition of graph connectivity, which for self-avoiding walk has been relaxed in the following compared to the usual definition [9, 28], to give a unified form of the expansion for all the models.

Definition 3.1. Given an interval $I = [a, b]$ of positive integers, we refer to a pair $\{s, t\}$ of elements of I as an *edge*. For $s < t$, we write simply st for $\{s, t\}$. A set of edges is called a *graph*. The set of graphs on $[a, b]$ is denoted $\mathcal{G}[a, b]$. A graph Γ is said to be *connected* if, as intervals, $\cup_{st\in\Gamma}[s, t] = [a, b]$. A *lace* is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a, b]$ is denoted by $\mathcal{L}[a, b]$. Given a connected graph Γ, the following prescription associates to Γ a unique lace $\mathsf{L}_\Gamma \subset \Gamma$: The lace L_Γ consists of edges $s_1t_1, s_2t_2, \ldots$ where, for $i \geq 2$,

$$s_1 = a, \quad t_1 = \max\{t : at \in \Gamma\} \quad t_i = \max\{t : \exists st \in \Gamma, s \leq t_{i-1}\} \quad s_i = \min\{s : st_i \in \Gamma\}. \tag{3.14}$$

The procedure terminates as soon as $t_N = b$. Given a lace L, the set of all edges $st \notin L$ such that $\mathsf{L}_{L\cup\{st\}} = L$ is called the set of edges *compatible* with L and is denoted $\mathcal{C}(L)$.

For $0 \leq a < b$ we define

$$J_R[a, b] = \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L} \mathcal{U}_{st}(R) \prod_{s't'\in\mathcal{C}(L)} (1 + \mathcal{U}_{s't'}(R)). \tag{3.15}$$

This has a nice interpretation in terms of the beads of Section 3.1. In that language, the product over $\mathcal{C}(L)$ in (3.15) is nonzero precisely when pairs of beads compatible with the lace L avoid each other, as in the product defining K_R. On the other hand, the product over L is nonzero precisely when the pairs of beads corresponding to lace edges do intersect each other. The number $N = N(L)$ of edges in L corresponds to the superscript in $\psi_p^{(N)}(x)$ in (3.7).

The function $\psi_p(x)$ is defined, for the different models, by

$$\begin{aligned}
\psi_p^{\rm saw}(x) &= \sum_{\substack{\omega:0\to x\\ |\omega|\geq 2}} W_{p,D}(\omega) J_\omega[0, |\omega|], &(3.16)\\
\psi_p^{\rm lt}(x) &= \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} W_{p,D}(\omega) \left[\prod_{i=0}^{|\omega|} \sum_{R_i\in\mathcal{T}(\omega(i),\omega(i))} W_{p,D}(R_i)\right] J_R[0, |\omega|], &(3.17)\\
\psi_p^{\rm la}(x) &= (1 - \delta_{0,x}) \sum_{R\in\mathcal{D}(0,x)} W_{p,D}(R)\\
&\quad + \sum_{B:|B|\geq 1} W_{p,D}(B) \left[\prod_{i=0}^{|\omega|} \sum_{R_i\in\mathcal{D}(u_i,v_i)} W_{p,D}(R_i)\right] J_R[0, |B|], &(3.18)
\end{aligned}$$

for any p for which the right side converges. We then define z in terms of p as in (1.22), and introduce $G_z(x)$ and $\Pi_z(x)$ as in (1.23) and (3.8). The following theorem gives the basic convolution equation (3.1) for the combinatorial models. For self-avoiding walk, the proof involves a minor modification of the standard proof given in [9, 28], to account for the relaxed definition of connectivity. For lattice trees and lattice animals, the proof is given in [15].

Theorem 3.2. *For any $p < p_c$ for which the series defining $\psi_p(x)$ is absolutely summable over x (with absolute values taken inside the sums in* (3.16)–(3.18)*), the convolution equations* (3.6) *and* (3.1) *hold.*

Sketch of proof. The proof relies on the elementary identity

$$K_R[0,b] = K_R[1,b] + J_R[0,b] + \sum_{a=1}^{b-1} J_R[0,a]K_R[a+1,b], \quad (b \geq 1). \tag{3.19}$$

To prove (3.19), we first expand the product in (3.10) to obtain $K_R[0,b] = \sum_{\Gamma \in \mathcal{G}[0,b]} \prod_{st \in \Gamma} U_{st}(R)$. Graphs with no edge containing 0 contribute $K_R[1,b]$. Graphs with an edge containing 0 are then partitioned according to the interval supporting the connected component containing 0, and give rise to the remaining two terms in the identity.

In the first term on the right side of (3.19), interactions between the first and subsequent beads do not occur, corresponding to the term $a_p(pD * U_p)(x)$ of (3.6). The second term gives rise to the term $\psi_p(x)$ of (3.6). (For lattice animals, the first term of (3.18) arises from the case of just one bead, which does not appear in (3.19).) The last term represents an effective decoupling of the interaction between beads 0 to a and beads $a+1$ to b, and gives rise to the final term of (3.6). $\square$

For $N \geq 1$, let $\mathcal{L}^{(N)}[a,b]$ denote the set of laces in $\mathcal{L}[a,b]$ consisting of exactly N edges. We define

$$J_R^{(N)}[a,b] = \sum_{L \in \mathcal{L}^{(N)}[a,b]} \prod_{st \in L} \mathcal{U}_{st}(R) \prod_{s't' \in \mathcal{C}(L)} (1 + \mathcal{U}_{s't'}(R)). \tag{3.20}$$

For $N \geq 1$, the quantity $\psi_p^{(N)}(x)$ discussed in Section 3.1 then corresponds to $(-1)^N$ times the contribution to (3.16)–(3.18) arising from the replacement of J by $J^{(N)}$ in those formulas. This representation of $\psi_p^{(N)}(x)$ leads to the formula $\psi_p(x) = \sum_{N=0}^{\infty} (-1)^N \psi_p^{(N)}(x)$ (with the $N = 0$ term arising only for lattice animals and given by the first term of (3.18)). This formula was discussed via the inclusion-exclusion approach in the discussion leading to (3.7).

3.3 Percolation

The lace expansion discussed in Section 3.2 is combinatorial in nature, but the expansion for percolation is inherently probabilistic. It relies entirely on inclusion-exclusion and does not make use of an interaction term $\mathcal{U}_{st}$. It is interesting that an expansion based on such an interaction can be carried out for oriented percolation [30], which has an additional Markovian structure not present in ordinary percolation. However, this has not been done outside the oriented setting. The expansion we present here, based on inclusion-exclusion, applies to quite general percolation models, including oriented percolation. Before giving a precise statement of the expansion, we first revisit the discussion of Section 3.1.

The discussion of the first application of inclusion-exclusion can be recast as follows, in the context of percolation. Let $g_p^{(0)}(x) = \mathbb{P}_p(x \in \Delta(0))$ denote the probability that 0 and x are doubly connected. If these two sites are not doubly connected, then there is a first pivotal bond (u,v)

for the connection. As in the discussion of lattice animals in Section 3.2, we may regard this bond as being directed. Let $F(0,u,v,x)$ denote the event that 0 and x are connected, but not doubly connected, and that (u,v) is the first pivotal bond for the connection. We would like to approximate $\mathbb{P}_p(F(0,u,v,x))$ by $(g_p^{(0)} * pD * \tau_p)(x)$, which treats the first bead in the string of beads as independent of the beads that follow. To discuss the error in this approximation, we will use the following definitions.

Definition 3.3. (a) Given a set of sites $A \subset \mathbb{Z}^d$ and a bond configuration, two sites x and y are *connected in* A if there is an occupied path from x to y having all of its sites in A, or if $x = y \in A$.
(b) The *restricted two-point function* is defined by

$$\tau_p^A(x,y) = \mathbb{P}_p(x \text{ and } y \text{ are connected in } \mathbb{Z}^d \backslash A).$$

(c) Given a bond $\{u,v\}$ and a bond configuration, we define $\tilde{C}^{\{u,v\}}(x)$ to be the set of sites which remain connected to x in the new configuration obtained by setting $\{u,v\}$ to be vacant.

It can be shown [28, Lemma 5.5.4] that

$$\mathbb{P}_p(F(0,u,v,x)) = pD(v-u)\mathbb{E}\Big[I[u \in \Delta(0)]\tau_p^{\tilde{C}^{\{u,v\}}(0)}(v,x)\Big], \tag{3.21}$$

where $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}_p$. The restricted two-point function in the above identity is a random variable, since the set $\tilde{C}^{\{u,v\}}(0)$ is random. The approximation discussed above amounts to replacing the restricted two-point function simply by $\tau_p(x-v)$, and gives

$$\begin{aligned}\mathbb{P}_p(F(0,u,v,x)) &= g_p^{(0)}(u)pD(v-u)\tau_p(x-v) \\ &\quad -pD(v-u)\mathbb{E}\Big[I[x \in \Delta(0)]\Big(\tau_p(x-v) - \tau_p^{\tilde{C}^{\{u,v\}}(0)}(v,x)\Big)\Big]. \end{aligned}\tag{3.22}$$

To understand the correction term in (3.22), we introduce the following definition. Two sites x and y are *connected through* A if they are connected in such a way that every occupied path from x to y has at least one bond with an endpoint in A, or if $x = y \in A$. Then, by definition,

$$\tau_p(x-v) - \tau_p^A(v,x) = \mathbb{P}_p(v \text{ is connected to } x \text{ through } A). \tag{3.23}$$

Therefore

$$\begin{aligned}\mathbb{P}_p(F(0,u,v,x)) &= g_p^{(0)}(u)pD(v-u)\tau_p(x-v) \\ &\quad -pD(v-u)\mathbb{E}\Big[I[x \in \Delta(0)]\Big(\mathbb{E}I[v \text{ is connected to } x \text{ through } \tilde{C}^{\{u,v\}}(0)]\Big)\Big]. \end{aligned}\tag{3.24}$$

In (3.24), we encounter the occurrence of a nested expectation, corresponding to a pair of distinct percolation configurations. This is the analogue for percolation of the occurrence of independent strings of beads in the combinatorial models. The two percolation configurations interact with each other via the event in the inner expectation, which requires a specific kind of intersection between them.

An example of a pair of configurations contributing to this nested expectation is depicted in Figure 2. In the figure, (u',v') is the first pivotal bond for the connection from v to x such that v is connected to u' through $\tilde{C}^{\{u,v\}}(0)$. It is possible that there is no such pivotal bond, corresponding

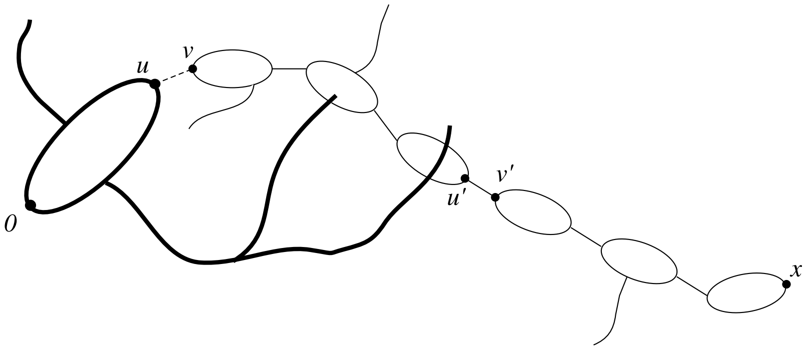

Figure 2: Schematic representation of the two configurations in a contribution to the nested expectation of (3.24). Bold lines represent occupied bonds in the outer expectation, while thin lines represent occupied bonds in the inner expectation.

to a picture in which $u' = x$, and in that case no further expansion is performed. In the case where there is such a pivotal bond, we perform the expansion again by treating the portion of the cluster of x following u' as independent of the portion preceding u', in a manner similar to the first application of inclusion-exclusion performed above. This is discussed in detail in [14, 28], and we now just state the conclusion.

In doing so, we will use subscripts to coordinate random sets with the corresponding expectations. For example, we write the subtracted term in (3.24) as

$$pD(v-u)\mathbb{E}_0\Big[I[u \in \Delta(0)]\Big(\mathbb{E}_1 I[v \text{ is connected to } x \text{ through } \tilde{C}_0^{\{u,v\}}(0)]\Big)\Big] \qquad (3.25)$$

to emphasise that the set occurring in the inner expectation is a random set with respect to the outer expectation. We will also make use of the following definition. Given sites x, y and a set of sites A, let $E(x,y;A)$ be the event that x is connected to y through A and there is no directed pivotal bond for the connection from x to y whose first endpoint is connected to x through A. We make the abbreviation $I_j = I[E(y_j', y_{j+1}; \tilde{C}_{j-1})]$ with $\tilde{C}_{j-1} = \tilde{C}_{j-1}^{\{y_j, y_j'\}}(y_{j-1}')$ and $y_0' = 0$, and we write $p_{u,v} = pD(v-u)$. In this notation, the situation with $u' = x$ discussed in the previous paragraph makes a contribution to (3.25) equal to

$$p_{u,v}\mathbb{E}_0\Big[I[u \in \Delta(0)]\Big(\mathbb{E}_1 I[E(v,x;\tilde{C}_0)]\Big)\Big]. \qquad (3.26)$$

Let

$$\psi_p^{(0)}(x) = (1-\delta_{0,x})\mathbb{P}_p(x \in \Delta(0)). \qquad (3.27)$$

For $n \geq 1$, we define

$$\begin{aligned}\psi_p^{(n)}(x) \;=\; & \sum_{(y_1,y_1')} p_{y_1,y_1'} \cdots \sum_{(y_n,y_n')} p_{y_n,y_n'} \mathbb{E}_0\Big(I[y_1 \in \Delta(0)] \\ & \times \mathbb{E}_1 I_1 \mathbb{E}_2 I_2 \mathbb{E}_3 I_3 \ldots \mathbb{E}_{n-1} I_{n-1} \mathbb{E}_n I[E(y_n', x; \tilde{C}_{n-1})]\Big), \end{aligned} \qquad (3.28)$$

where the sums are over directed bonds and *all* the expectations are nested. Define

$$\Psi_p^{(n)}(x) = \sum_{j=0}^{n} (-1)^j \psi_p^{(j)}(x) \qquad (3.29)$$

and

$$R_p^{(n)}(x) = \sum_{(y_1,y_1')} p_{y_1,y_1'} \cdots \sum_{(y_{n+1},y_{n+1}')} p_{y_{n+1},y_{n+1}'} \mathbb{E}_0 \Big(I[y_1 \in \Delta(0)]$$
$$\times \mathbb{E}_1 I_1 \mathbb{E}_2 I_2 \ldots \mathbb{E}_n \Big(I_n(\tau_p(x - y_{n+1}') - \tau_p^{\tilde{C}_n}(y_{n+1}', x)) \Big) \Big). \tag{3.30}$$

The following theorem is proved in [14]; see also [16, 28].

Theorem 3.4. *For $p < p_c$ and $N \geq 0$,*

$$\tau_p(x) = \delta_{0,x} + \Psi_p^{(N)}(x) + (pD * \tau_p)(x) + (\Psi_p^{(N)} * pD * \tau_p)(x) + (-1)^{N+1} R_p^{(N)}(x). \tag{3.31}$$

As we will show in Section 4.5, the limit $N \to \infty$ can be taken in (3.31) under the hypotheses of Proposition 1.8(c), with the remainder term vanishing in the limit. Defining $\Pi_p(x) = \psi_p(x) = \sum_{j=0}^{\infty} (-1)^j \psi_p^{(j)}(x)$, (3.31) then becomes

$$\tau_p(x) = \delta_{0,x} + \Pi_p(x) + (pD * \tau_p)(x) + (\Pi_p * pD * \tau_p)(x). \tag{3.32}$$

This is equivalent to (3.1), with $z = p$ and $G_z(x) = \tau_p(x)$.

4 Lace expansion diagrams

We begin in Section 4.1 by recalling the well-established procedure by which the lace expansion for self-avoiding walk gives rise to diagrammatic upper bounds for $\psi_p^{(N)}(x)$ [9, 28]. We then bound these diagrams to prove Proposition 1.8(a). For the other models, there are also diagrammatic upper bounds. These bounds can be expressed in the form $\psi_p^{(N)}(x) \leq M^{(N)}(x,x)$, where $M^{(N)}(x,y)$ is a recursively defined function having a diagrammatic interpretation. In Section 4.2, we prove Lemma 4.1, a key lemma that will be used to bound $M^{(N)}(x,y)$. In Sections 4.3–4.5, we recall the well-established procedure by which the expansions of Section 3 give rise to diagrams [14, 15] for lattice trees, lattice animals and percolation. We will not provide complete proofs here but attempt only to motivate the diagrams. Once the diagrams have been identified, we estimate them using Lemma 4.1. This will provide a proof of Proposition 1.8(b-c). In addition, in Section 4.5, we will argue that, for percolation, $\lim_{N\to\infty} R_p^{(N)}(x) = 0$ under the hypotheses of Proposition 1.8(c).

Our bounds here are in contrast to all previous diagrammatic estimates in lace expansion analyses, which have been for $\sum_x \psi_p(x)$ rather than for fixed-x quantities [9, 14, 15].

4.1 Self-avoiding walk diagrams

For self-avoiding walk, (4.33) simplifies to

$$\psi_p^{(N)}(x) = (-1)^N \sum_{\substack{\omega:0\to x \\ |\omega|\geq 2}} W_{p,D}(\omega) J_R^{(N)}[0, |\omega|]. \tag{4.1}$$

The diagrammatic representation of an expression of the form (4.1) has been discussed many times in the literature, for example in [9, 28]. Here we focus on the differences that arise because of the weakened definition of connectivity used in Definition 3.1.

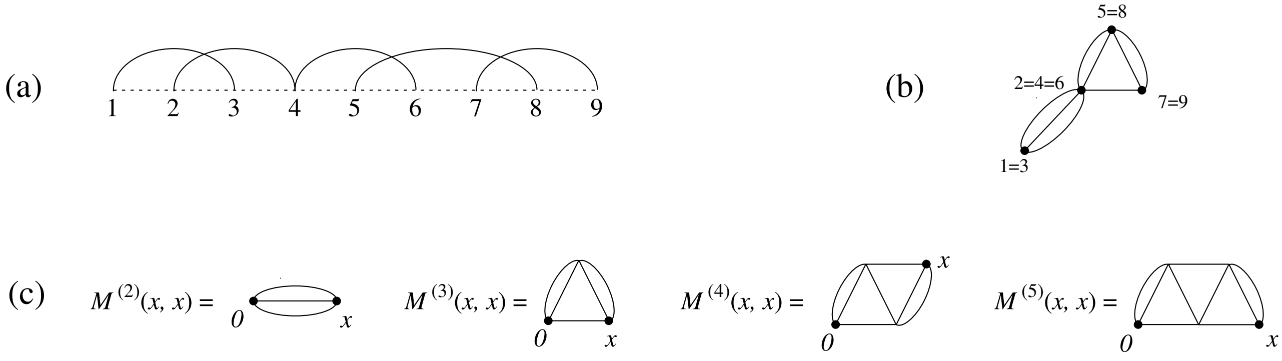

Figure 3: (a) A lace with five edges, dividing the time interval into eight subintervals. (b) The self-avoiding walk diagram corresponding to this lace. (c) Diagrams for $\pi_p^{(n)}(x)$.

The factor $\prod_{st\in L}\mathcal{U}_{st}$ in $J^{(N)}$ imposes N bead intersections, which are self-intersections of the random walk. These self-intersections divide the underlying time interval into subintervals, as illustrated in Figure 3(a). The factor $\prod_{st\in\mathcal{C}(L)}(1+\mathcal{U}_{st})$ in $J^{(N)}$ is then bounded by replacing each factor $1+\mathcal{U}_{st}$ for which s and t lie in distinct subwalks by the factor 1. This produces a bound that can be interpreted as involving a self-avoiding walk on each time subinterval, with no interaction between the walks corresponding to distinct time intervals. For example, the lace of Figure 3(a) gives rise to the diagram of Figure 3(b).

A simplification for these diagrams occurs in the case where two lace edges abut and do not overlap. In this case, after discarding the interaction between distinct subwalks, the interaction decouples across the time coordinate where an abuttal occurs. If we define $\pi_p^{(N)}(x)$ to be the contribution to the summation in (4.1) only from laces with no abuttal, then we are led to

$$0 \le \psi_p^{(N)}(x) \le \sum_{m=1}^{N} \sum_{\substack{(n_1,\dots,n_m):\\ n_1+\cdots+n_m=N}} (\pi_p^{(n_1)} * \cdots * \pi_p^{(n_m)})(x). \tag{4.2}$$

The quantity $\pi_p^{(n)}(x)$ is the quantity that has appeared in previous lace expansion analyses [9, 28]. We encounter ψ_p instead, because we have used a definition of graph connectivity in Definition 3.1 that is relaxed compared to previous analyses, to achieve a unified treatment with lattice trees and lattice animals.

We will bound $\psi_p^{(N)}(x)$ by combining a bound on $\pi_p^{(n)}(x)$ with Proposition 1.7(i). To bound $\pi_p^{(n)}(x)$, we define

$$\sigma_p'(x) = \sigma_p(x) - \delta_{0,x}, \tag{4.3}$$

$$A(u,v,x,y) = \sigma_p'(v-u)\sigma_p(y-u)\delta_{v,x}, \tag{4.4}$$

$$M^{(2)}(x,y) = \sigma_p'(x)^2\sigma_p(y), \tag{4.5}$$

$$M^{(n)}(x,y) = \sum_{u,v\in\mathbb{Z}^d} M^{(n-1)}(u,v)A(u,v,x,y) \qquad (n\ge 3). \tag{4.6}$$

The standard bounds of [9] can then be written as

$$0 \le \pi_p^{(1)}(x) \le \delta_{0,x} \sum_{v \in \Omega_D} pD(v)\sigma_p'(v), \tag{4.7}$$

$$0 \le \pi_p^{(n)}(x) \le M^{(n)}(x,x) \qquad (n \ge 2). \tag{4.8}$$

The power $3q$ in the desired decay $(|x|+1)^{-3q}$ can be understood from the fact that there are three distinct routes from 0 to x in the diagrams for $M^{(n)}(x,x)$; see Figure 3(c).

Proof of Proposition 1.8(a). For self-avoiding walk, we have $z = p$, $G_z(x) = \sigma_p(x)$ and $\Pi_z(x) = \psi_p(x)$. The hypotheses of the proposition are that $\sigma_p'(x) \le \beta(|x|+1)^{-q}$, with $2q > d$, and that $p \le 2$. Since $\sigma_p(0) = 1$, it follows that $\sigma_p(x) \le (|x|+1)^{-q}$. (The lower bound on βL^{q-d} assumed in Proposition 1.8 is not needed for self-avoiding walk.) We must show that

$$\psi_p(x) \le c\beta\delta_{0,x} + c\beta^3(|x|+1)^{-3q}. \tag{4.9}$$

By definition of σ_p' and the hypotheses, it follows from (4.7) that

$$0 \le \pi_p^{(1)}(x) \le 2^{1-q}\beta\delta_{0,x}. \tag{4.10}$$

By (4.4) and hypothesis,

$$A(u,v,x,y) \le \frac{\beta}{(|v-u|+1)^q(|y-u|+1)^q}\delta_{v,x}. \tag{4.11}$$

Let

$$S(x) = \sum_{y \in \mathbb{Z}^d} \frac{1}{(|y|+1)^q(|x-y|+1)^q}, \qquad \bar{S} = \sup_{x \in \mathbb{Z}^d} S(x). \tag{4.12}$$

Note that $\bar{S} < \infty$ if $2q > d$, by Proposition 1.7(i). Diagrammatically, $S(x)$ corresponds to an open bubble, and the condition $\bar{S} < \infty$ is closely related to the bubble condition [28, Section 1.5].

We will show that (4.11) implies there is a constant C such that

$$M^{(n)}(x,y) \le \beta^n (C\bar{S})^{n-2} \frac{1}{(|x|+1)^{2q}(|y|+1)^q} \qquad (n \ge 2). \tag{4.13}$$

This gives the conclusion of Proposition 1.8(a), apart from the fact that the $n = 2$ term here has a factor β^2 rather than the required β^3. The missing factor of β can be recovered by noting that, by definition, $M^{(2)}(x,x)$ can be written as $\sigma_p'(x)^3$, and we obtain one factor of β for each factor of σ_p'.

To prove (4.13), we use induction on n. The case $n = 2$ follows immediately from (4.5) and the assumed bound on σ_p'. To advance the induction, we assume (4.13) for $n-1$ and show that it holds also for n. The inductive hypothesis and (4.11) then give

$$M^{(n)}(x,y) \le \sum_{u \in \mathbb{Z}^d} \frac{\beta^{n-1}(C\bar{S})^{n-3}}{(|u|+1)^{2q}(|x|+1)^q} \frac{\beta}{(|x-u|+1)^q(|y-u|+1)^q} \qquad (n \ge 3). \tag{4.14}$$

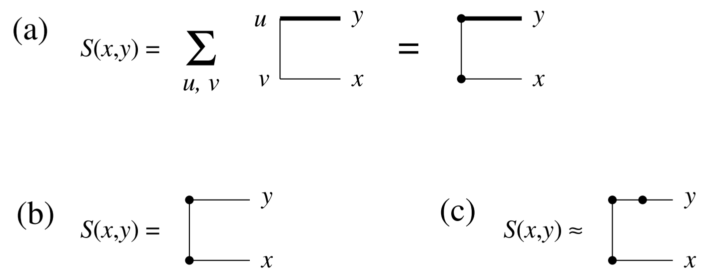

Figure 4: (a) The diagram for $S(x, y)$. Thin lines decay with power q_1 and the thick line decays with power q_2. Unlabelled vertices are summed over. (b) The open triangle diagram for percolation ($q_1 = q_2 = q$). (c) The open square diagram for lattice trees and lattice animals ($q_1 = q, q_2 = 2q-d$).

It therefore suffices to show that there is a C for which

$$\sum_{u \in \mathbb{Z}^d} \frac{1}{(|u|+1)^{2q}} \frac{1}{(|x-u|+1)^q (|y-u|+1)^q} \le \frac{C\bar{S}}{(|x|+1)^q (|y|+1)^q} \quad (n \ge 3). \tag{4.15}$$

To prove (4.15), we consider four cases.
Case 1: $|u| \ge |x|/2$ and $|u| \ge |y|/2$. In this contribution to (4.15), we may bound the factor $(|u|+1)^{-2q}$ above by $2^{2q}(|x|+1)^{-q}(|y|+1)^{-q}$. The remaining summation over u is then bounded above by $\bar{S}$, as required.
Case 2: $|u| \ge |x|/2$ and $|u| \le |y|/2$. The second inequality implies that $|y-u| \ge |y|/2$. We then argue as in Case 1.
Case 3: $|u| \le |x|/2$ and $|u| \ge |y|/2$. This is the same as Case 2, by symmetry.
Case 4: $|u| \le |x|/2$ and $|u| \le |y|/2$. This follows as above, using $|y-u| \ge |y|/2$ and $|x-u| \ge |x|/2$. □

4.2 The diagram lemma

In this section, we present a lemma that will be useful for the diagrammatic estimates for lattice trees, lattice animals and percolation. It involves a constant $\bar{S}$, which is defined for $q_1, q_2 > 0$ by

$$S(x,y) = \sum_{u,v \in \mathbb{Z}^d} \frac{1}{(|u-v|+1)^{q_1}(|y-u|+1)^{q_2}(|x-v|+1)^{q_1}}, \qquad \bar{S} = \sup_{x,y \in \mathbb{Z}^d} S(x,y). \tag{4.16}$$

It is possible that $\bar{S} = \infty$, depending on the values of d and the q_i. However, if

$$2q_1 + d > 2q_1 + q_2 > 2d \tag{4.17}$$

then $\bar{S} < \infty$. In fact, given (4.17), it follows from Proposition 1.7(i) that

$$S(x,y) \le \frac{C}{(|x-y|+1)^{2q_1+q_2-2d}}. \tag{4.18}$$

Finiteness of $\bar{S}$ is related to the triangle condition for percolation [5] and to the square condition for lattice trees and lattice animals [34]. To see this, we first note the diagrammatic representation of $S(x,y)$ in figure 4(a). When $q_1 = q_2 = q$, which is the relevant case for percolation, this corresponds to the open triangle diagram depicted in Figure 4(b). When $q_1 = q$ and $q_2 = 2q - d$, which is the relevant case for lattice trees and lattice animals, S corresponds to the open square diagram depicted in Figure 4(c). To understand this for the square diagram, we interpret the line decaying with power q_2 as arising from a convolution of two two-point functions decaying with power q, in accordance with Proposition 1.7(i).

The following lemma is the key lemma that will be used in bounding diagrams for lattice trees, lattice animals and percolation. Its statement involves functions $A^{(0)} : \mathbb{Z}^{2d} \to [0,\infty)$, $A^{(i)} : \mathbb{Z}^{4d} \to [0,\infty)$ for $i \geq 1$, $A^{(\mathrm{end})} : \mathbb{Z}^{4d} \to [0,\infty)$, and functions $M^{(N)} : \mathbb{Z}^{2d} \to [0,\infty)$ defined for $N \geq 1$ by

$$M^{(N)}(x,y) = \sum_{u_1,v_1,\dots,u_N,v_N \in \mathbb{Z}^d} A^{(0)}(u_1,v_1) \prod_{i=1}^{N-1} A^{(i)}(u_i,v_i,u_{i+1},v_{i+1}) A^{(\mathrm{end})}(u_N,v_N,x,y). \tag{4.19}$$

(For $N = 1$, the empty product over i is interpreted as 1.) The proof of Lemma 4.1 can be extended to $q_1 < d$ and q_2 obeying (4.17), but since $q_2 \leq q_1$ in our applications, we add this assumption to simplify the proof.

Lemma 4.1. *Fix $q_2 \leq q_1 < d$ obeying (4.17), so that $\bar{S} < \infty$. Let $K_0 > 0$. Suppose that*

$$A^{(0)}(x,y) \leq K_0 \left\{ \frac{1}{(|x|+1)^{q_1}(|y|+1)^{q_2}} + \frac{1}{(|x|+1)^{q_2}(|y|+1)^{q_1}} \right\}, \tag{4.20}$$

and suppose that $A^{(i)}$ for $i \geq 1$ and $A^{(end)}$ satisfy

$$A^{(*)}(u,v,x,y) \leq \frac{K_*}{(|u-v|+1)^{q_1}} \left\{ \frac{1}{(|y-u|+1)^{q_2}(|x-v|+1)^{q_1}} + \frac{1}{(|y-u|+1)^{q_1}(|x-v|+1)^{q_2}} \right\} \tag{4.21}$$

with $K_* > 0$. Then there is a C depending on d, q_1, q_2 such that for $N \geq 1$

$$M^{(N)}(x,y) \leq (C\bar{S})^{N-1} \Big(\prod_{i=0}^{N-1} K_i \Big) K_{\mathrm{end}} \left\{ \frac{1}{(|x|+1)^{q_1}(|y|+1)^{q_2}} + \frac{1}{(|x|+1)^{q_2}(|y|+1)^{q_1}} \right\}. \tag{4.22}$$

Proof. The proof is by induction on N. To deal with the fact that $M^{(N)}$ is not defined literally by a convolution of $M^{(N-1)}$ with $A^{(\mathrm{end})}$, we proceed as follows. Let $\tilde{M}^{(N)}$ be the quantity defined by replacing $A^{(\mathrm{end})}$ by $A^{(N)}$ in the definition of $M^{(N)}$. Because all the constituent factors in the definitions of $M^{(N)}$ and $\tilde{M}^{(N)}$ obey the same bounds, it suffices to prove that $\tilde{M}^{(N)}$ obeys (4.22) with K_{end} replaced by K_N. We prove this by induction, with the inductive hypothesis that $\tilde{M}^{(N-1)}$ obeys (4.22) with K_{end} replaced by K_{N-1} and N replaced by $N-1$ on the right side.

For $x, y \in \mathbb{Z}^d$, let

$$T(x,y) = \sum_{u,v \in \mathbb{Z}^d} \left\{ \frac{1}{(|u|+1)^{q_1}(|v|+1)^{q_2}} + \frac{1}{(|u|+1)^{q_2}(|v|+1)^{q_1}} \right\} \times \frac{1}{(|u-v|+1)^{q_1}(|y-u|+1)^{q_2}(|x-v|+1)^{q_1}}. \tag{4.23}$$

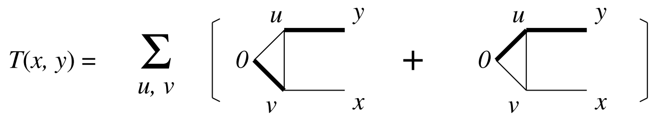

Figure 5: Diagram for $T(x, y)$. Thin lines decay with power q_1, while thick lines decay with power q_2.

This quantity is depicted in Figure 5. By definition, and using (4.20)–(4.21),

$$\tilde{M}^{(1)}(x, y) \leq K_0 K_1 \Big[T(x, y) + T(y, x)\Big]. \tag{4.24}$$

By the induction hypothesis, (4.19) and (4.21),

$$\tilde{M}^{(N)}(x, y) \leq (C\bar{S})^{N-2} \Big(\prod_{i=0}^{N} K_i\Big) \Big[T(x, y) + T(y, x)\Big]. \tag{4.25}$$

It therefore suffices to show that

$$T(x, y) \leq \frac{1}{2} C\bar{S} \left\{ \frac{1}{(|x| + 1)^{q_1}(|y| + 1)^{q_2}} + \frac{1}{(|x| + 1)^{q_2}(|y| + 1)^{q_1}} \right\}. \tag{4.26}$$

To prove (4.26), we write $T(x, y) \leq \sum_{i=1}^{4} T_i(x, y)$, with $T_i(x, y)$ defined to be the contribution to $T(x, y)$ arising from each of the following four cases. In the discussion of these four cases, C denotes a generic constant whose value may change from line to line.
Case 1. $|v| \geq |x - v|$ and $|u| \geq |u - y|$. This implies $|v| \geq |x|/2$ and $|u| \geq |y|/2$, so that

$$T_1(x, y) \leq C\bar{S} \left\{ \frac{1}{(|x| + 1)^{q_1}(|y| + 1)^{q_2}} + \frac{1}{(|x| + 1)^{q_2}(|y| + 1)^{q_1}} \right\}. \tag{4.27}$$

Case 2. $|v| \geq |x - v|$ and $|u| \leq |u - y|$. This implies $|v| \geq |x|/2$ and $|u - y| \geq |y|/2$. Then

$$\begin{aligned} T_2(x, y) \leq \frac{C}{(|y| + 1)^{q_2}} \sum_{u,v} &\left\{ \frac{1}{(|u| + 1)^{q_1}(|x| + 1)^{q_2}} + \frac{1}{(|u| + 1)^{q_2}(|x| + 1)^{q_1}} \right\} \\ &\times \frac{1}{(|u - v| + 1)^{q_1}(|x - v| + 1)^{q_1}}. \end{aligned} \tag{4.28}$$

The second term of (4.28) is bounded above by $C\bar{S}(|x| + 1)^{-q_1}(|y| + 1)^{-q_2}$, as required. We bound the first term using Proposition 1.7(i) to estimate the two convolutions, obtaining a bound $C(|x|+1)^{-(3q_1+q_2-2d)}(|y|+1)^{-q_2}$. (Here we used the assumption $q_2 \leq q_1$ to ensure that $3q_1 - 2d > 0$, as required to apply Lemma 1.7(i).) It follows from (4.17) that $3q_1 + q_2 - 2d > q_1$, which gives the desired result.
Case 3. $|v| \leq |x - v|$ and $|u| \geq |u - y|$. This implies $|v - x| \geq |x|/2$ and $|u| \geq |y|/2$, and hence

$$\begin{aligned} T_3(x, y) \leq \frac{C}{(|x| + 1)^{q_1}(|y| + 1)^{q_2}} \sum_{u,v} &\left\{ \frac{1}{(|y - u| + 1)^{q_1 - q_2}(|v| + 1)^{q_2}} + \frac{1}{(|v| + 1)^{q_1}} \right\} \\ \times &\frac{1}{(|u - v| + 1)^{q_1}(|y - u| + 1)^{q_2}}. \end{aligned} \tag{4.29}$$

Each term is bounded by $C\bar{S}(|x|+1)^{-q_1}(|y|+1)^{-q_2}$, as required.
Case 4. $|v| \le |x-v|$ and $|u| \le |u-y|$. This implies $|v-x| \ge |x|/2$ and $|u-y| \ge |y|/2$, and hence

$$T_4(x,y) \le 2C\bar{S}\frac{1}{(|x|+1)^{q_1}(|y|+1)^{q_2}}. \tag{4.30}$$

Adding the contributions in the four cases yields (4.26) and completes the proof. □

Remark 4.2. Let

$$H(z,w,x,y) = \sum_{u,v} \frac{1}{(|z-u|+1)^q}\frac{1}{(|y-u|+1)^q}\frac{1}{(|w-v|+1)^q}\frac{1}{(|x-v|+1)^q}\frac{1}{(|u-v|+1)^q}. \tag{4.31}$$

By dividing into four cases according to whether $|z-u|$ is greater than or less than $|y-u|$ and whether $|w-v|$ is greater than or less than $|x-v|$, the above proof can be easily adapted to show that

$$H(w,z,x,y) \le \frac{C\bar{S}}{(|y-z|+1)^q(|x-w|+1)^q}, \tag{4.32}$$

where $\bar{S}$ is defined in (4.16) with now $q_1 = q_2 = q$. This will be used in Section 4.5 to analyse percolation.

4.3 Lattice tree diagrams

For lattice trees, the quantity $\psi_p^{(N)}(x)$ $(N \ge 1)$ can be understood either as arising from N applications of inclusion-exclusion, along the lines discussed in Section 3.1, or from the contribution to (3.17) from laces having N edges, as explained around (3.20). Explicitly,

$$\psi_p^{(N)}(x) = (-1)^N \sum_{\substack{\omega:0\to x\\ |\omega|\ge 1}} W_{p,D}(\omega)\left[\prod_{i=0}^{|\omega|}\sum_{R_i\in\mathcal{T}(\omega(i),\omega(i))} W_{p,D}(R_i)\right] J_R^{(N)}[0,|\omega|]. \tag{4.33}$$

For a nonzero contribution to $\psi_p^{(N)}(x)$, the factor $\prod_{st\in L}\mathcal{U}_{st}$ in $J^{(N)}$ enforces intersections between the beads R_s and R_t, for each $st \in L$. This leads to bounds in which the contribution to $\psi_p^{(N)}(x)$ from the N-edge laces can be bounded above by N-loop diagrams. We illustrate this in detail only for the simplest case $N=1$.

To bound $\psi_p^{(1)}(x)$, we proceed as follows. There is a unique lace $0|\omega|$ consisting of a single edge, and all other edges on $[0,|\omega|]$ are compatible with it. Therefore

$$\psi_p^{(1)}(x) = -\sum_{\substack{\omega:0\to x\\ |\omega|\ge 1}} W_{p,D}(\omega)\left[\prod_{i=0}^{|\omega|}\sum_{R_i\in\mathcal{T}(\omega(i),\omega(i))} W_{p,D}(R_i)\right]\mathcal{U}_{0|\omega|}\prod_{\substack{0\le s<t\le|\omega|\\ (s,t)\ne(0,|\omega|)}}(1+\mathcal{U}_{st}). \tag{4.34}$$

After relaxing the last product to $\prod_{1\le s<t\le|\omega|}(1+\mathcal{U}_{st})$, the trees $R_1,\ldots,R_l$, together with the bonds of ω connecting them, can be considered as a single lattice tree connecting $\omega(1)$ and x. Writing

this tree as T_1, writing $v = \omega(1)$, and stating the constraint imposed by $\mathcal{U}_{0|\omega|}$ in words, we obtain

$$
\begin{aligned}
0 \le \psi_p^{(1)}(x) &\le \sum_{v\in\Omega_D} pD(v) \sum_{R_0\in\mathcal{T}(0,0)} W_{p,D}(R_0) \sum_{T_1\in\mathcal{T}(v,x)} W_{p,D}(T_1) \\
&\qquad\qquad \times I[R_0 \text{ and the bead at } x \text{ of } T_1 \text{ share a common site}] \\
&\le \sum_{y\in\mathbb{Z}^d}\sum_{v\in\Omega_D} pD(v) \sum_{R_0\in\mathcal{T}(0,y)} W_{p,D}(R_0) \sum_{T_1\in\mathcal{T}(v,x)} W_{p,D}(T_1)\, I[(\text{bead at } x \text{ of } T_1) \ni y].
\end{aligned}
\tag{4.35}
$$

In (4.35), the summations over R_0 and T_1 can be performed independently. The summation over R_0 simply gives $\rho_p(y)$. For the summation over T_1, we note that there must be disjoint connections from v to x and from x to y, because y is in the last bead of T_1. Therefore the sum over T_1 is bounded above by $\rho_p(x-v)\rho_p(y-x)$. Define

$$\tilde{\rho}_p(x) = (pD * \rho_p)(x) = \sum_{v\in\Omega_D} pD(v)\rho_p(x-v). \tag{4.36}$$

Then the above bound gives

$$0 \le \psi_p^{(1)}(x) \le \sum_{y\in\mathbb{Z}^d} \tilde{\rho}_p(x)\rho_p(y-x)\rho_p(y). \tag{4.37}$$

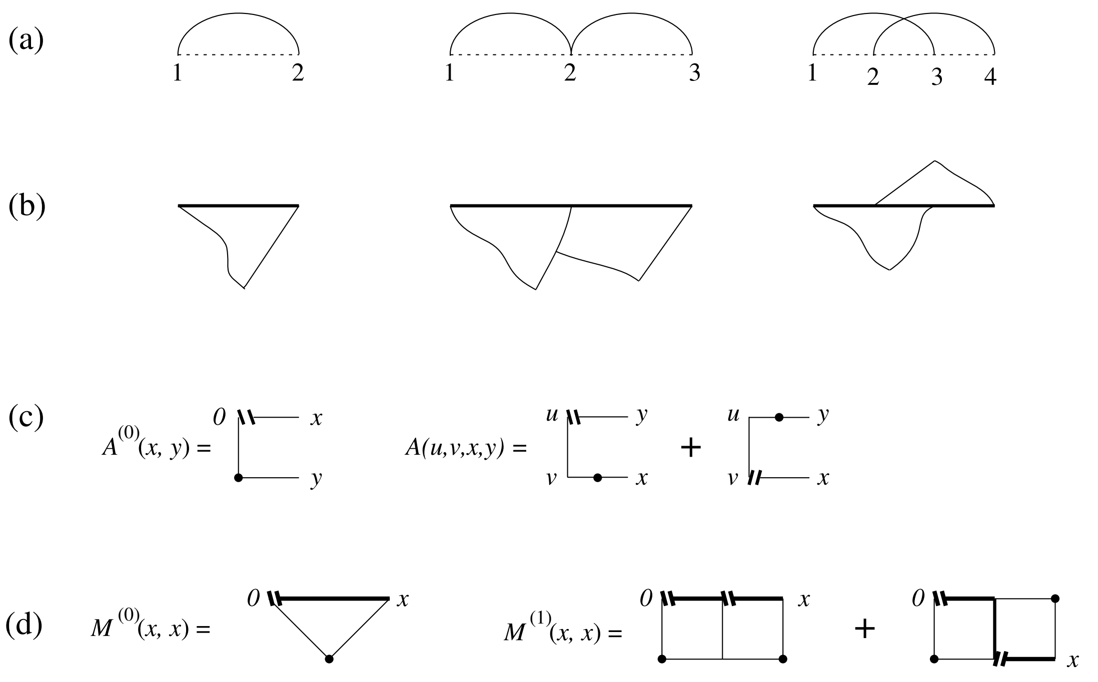

Figure 6: (a) The laces with one and two edges. (b) Bead intersections imposed by the laces. (c) Constituents for constructing $M^{(N)}$, where A stands for both $A^{(i)}$ and $A^{(\mathrm{end})}$. Lines ending with double bars represent $\tilde{\rho}$-lines. (d) The one-loop and two-loop lattice tree diagrams, with lines corresponding to the backbone drawn in bold.

For $N \geq 2$, a similar analysis can be performed, along the lines discussed in [15]. To state the resulting bound, we define

$$M^{(0)}(x,y) = A^{(0)}(x,y) = \tilde{\rho}_p(x) \sum_v \rho_p(y-v)\rho_p(v) \tag{4.38}$$

and

$$A^{(i)}(u,v,x,y) = \rho_p(v-u)\Big[\tilde{\rho}_p(y-u) \sum_{a\in\mathbb{Z}^d} \rho_p(a-v)\rho_p(x-a) + \tilde{\rho}_p(x-v) \sum_{a\in\mathbb{Z}^d} \rho_p(a-u)\rho_p(y-a)\Big], \tag{4.39}$$

with $A^{(\mathrm{end})} = A^{(i)}$. We define $M^{(N)}(x,y)$ $(N \geq 1)$ recursively by (4.19). Then, for $N \geq 1$, the resulting bound is

$$0 \leq \psi_p^{(N)}(x) \leq M^{(N-1)}(x,x). \tag{4.40}$$

The first few diagrams are depicted in Figure 6. The upper bound (4.40) differs from the bound of [15], which uses ρ_p in place of $\tilde{\rho}_p$ in (4.39). We could also use the bounds of [15] here, but the bounds with $\tilde{\rho}_p$ are easier to derive and lead ultimately to the same conclusion.

Proof of Proposition 1.8(b) for lattice trees. For lattice trees, we have $z = p\rho_p(0)$, $G_z(x) = \rho_p(x)/\rho_p(0)$ and $\Pi_z(x) = \psi_p(x)/\rho_p(0)$. The hypotheses of the proposition are that $G_z(x) \leq \beta(|x|+1)^{-q}$ for $x \neq 0$, with $\frac{3}{4}d < q < d$, that there is a constant R such that $\rho_p^{(a)}(0) \leq R$, and that βL^{q-d} is bounded away from zero. It follows that $\rho_p(x) \leq R\beta(|x|+1)^{-q}$ for $x \neq 0$. Since $\rho_p(0) \geq 1$, it is sufficient to conclude that

$$\psi_p(x) \leq c\beta\delta_{0,x} + c\beta^2(|x|+1)^{d-3q}, \tag{4.41}$$

where c may depend on R.

By definition

$$\tilde{\rho}_p^{(a)}(x) = pD(x)\rho_p^{(a)}(0) + \sum_{v\in\Omega_D : v\neq x} pD(v)\rho_p^{(a)}(x-v). \tag{4.42}$$

Note that $p = \sum_{v\in\Omega_D} pD(v) < \rho_p^{(a)}(0) \leq R$. The first term on the right side can be bounded as in (1.46), while the second term can be estimated by considering separately the contributions due to $|x| \geq 2L$ and $|x| < 2L$. The result is

$$\tilde{\rho}_p(x) \leq \frac{C}{L^{d-q}(|x|+1)^q} + \frac{C\beta}{(|x|+1)^q} \leq \frac{C\beta}{(|x|+1)^q}, \tag{4.43}$$

where we have invoked the hypothesis that βL^{q-d} is bounded away from zero.

Therefore, by definition and by Proposition 1.7(i),

$$M^{(0)}(x,x) \leq c\beta(|x|+1)^{d-3q}. \tag{4.44}$$

Similarly, $A^{(0)}(x,y)$ of (4.38) obeys the bound of (4.20) with $q_1 = q$ and $q_2 = 2q-d$. Moreover, the factor β on the right side of (4.44) can be replaced by β^2 when $x \neq 0$, since at least one of the two lower lines in the first diagram of Figure 6(b) must make a nonzero displacement when $x \neq 0$.

For $N \geq 1$, we will show that the hypotheses imply

$$M^{(N)}(x,x) \leq \frac{\beta^{N+1} C_1^N}{(|x|+1)^{3q-d}}, \tag{4.45}$$

where C_1 is a constant. By (4.40) and (4.44), this will complete the proof. The remainder of the proof is devoted to proving (4.45).

By Proposition 1.7(i) and the above remarks, the function $A^{(i)}$ defined in (4.39) obeys

$$A^{(i)}(u,v,x,y) \leq \frac{C\beta}{(|u-v|+1)^q} \left[\frac{1}{(|y-u|+1)^q (|x-v|+1)^{2q-d}} + \frac{1}{(|y-u|+1)^{2q-d}(|x-v|+1)^q} \right]. \tag{4.46}$$

Hence, (4.21) applies with $q_1 = q$, $q_2 = 2q - d$. By our assumption on q, it follows that $q_2 \leq q_1 < d$ and (4.17) is satisfied. Therefore, by Lemma 4.1, there is a constant C_1 such that

$$M^{(N)}(x,y) \leq \beta^{N+1} C_1^N \left\{ \frac{1}{(|x|+1)^q (|y|+1)^{2q-d}} + \frac{1}{(|x|+1)^{2q-d}(|y|+1)^q} \right\}. \tag{4.47}$$

This implies (4.45) and completes the proof for lattice trees. □

4.4 Lattice animal diagrams

The determination of the lattice animal diagrams is similar to that for lattice trees. It makes use of [15, Lemma 2.1], which can be rephrased in our present context as follows.

Lemma 4.3. *Given sets of lattice paths E_i $(i = 1, \ldots, n)$, let $\mathcal{A}_i$ denote the set of lattice animals which contain a path in E_i, and let $\mathcal{A}$ denote the set of lattice animals which contain disjoint paths in each of $E_1, \ldots, E_n$. Then*

$$\sum_{A \in \mathcal{A}} W_{p,D}(A) \leq \prod_{i=1}^{n} \left[\sum_{A_i \in \mathcal{A}_i} W_{p,D}(A_i). \right] \tag{4.48}$$

We denote the first term on the right side of (3.18) by $\psi_p^{(0)}(x)$ and denote the contribution to the second term due to $J_R^{(N)}[0,|B|]$ by $\psi_p^{(N)}(x)$. By Lemma 4.3,

$$\psi_p^{(0)}(x) = (1-\delta_{0,x}) \sum_{A \in \mathcal{D}(0,x)} W_{p,D}(A) \leq (1-\delta_{0,x}) \rho_p^a(x)^2. \tag{4.49}$$

By definition,

$$\psi_p^{(1)}(x) = -\sum_{|B|:|B|\geq 1} W_{p,D}(B) \left[\prod_{i=0}^{|B|} \sum_{R_i \in \mathcal{D}(v_i,u_{i+1})} W_{p,D}(R_i) \right] \mathcal{U}_{0|B|} \prod_{\substack{0 \leq s < t \leq |B| \\ st \neq 0|B|}} (1+\mathcal{U}_{st}), \tag{4.50}$$

where the sum over B is a sum over $|B|$ bonds (u_i, v_i) with $v_i - u_i \in \Omega_D$, where $v_0 = 0$ and $u_{|B|+1} = x$.

After relaxing the avoidance constraint in (4.50) to $\prod_{1\leq s<t\leq |B|}(1+\mathcal{U}_{st})$, the beads $R_1, \ldots, R_{|B|}$, together with the pivotal bonds connecting them, can be considered as a single lattice animal connecting v_1 and x. Writing this animal as A_1, and stating the constraint imposed by $\mathcal{U}_{0|B|}$ in words, we obtain

$$
\begin{aligned}
0 \leq \psi_p^{(1)}(x) &\leq \sum_{(u,v)} pD(v-u) \sum_{R_0\in\mathcal{D}(0,u)} W_{p,D}(R_0) \sum_{A_1\in\mathcal{A}(v,x)} W_{p,D}(A_1) \\
&\qquad \times I[R_0 \text{ and the last bead of } A_1 \text{ share a common site}] \\
&\leq \sum_y \sum_{(u,v)} pD(v-u) \sum_{R_0\in\mathcal{D}(0,u):R_0\ni y} W_{p,D}(R_0) \sum_{A_1\in\mathcal{A}(v,x)} W_{p,D}(A_1) I[(\text{last bead of } A_1)\ni y].
\end{aligned}
\tag{4.51}
$$

In (4.51), the summations over R_0 and A_1 can be performed independently. For the summation over R_0, we note that there must be a site w, and four disjoint connections joining 0 to w, w to u, u to 0, and w to y. For the summation over A_1, there must be disjoint connections joining x to v and x to y, because y is in the last bead of A_1. This is illustrated in Figure 7, where on the left we show a typical contribution to the one-loop diagram, and on the right we show the connections used to bound it. Therefore, using Lemma 4.3 we obtain

$$
0 \leq \psi_p^{(1)}(x) \leq \sum_{u,w,y\in\mathbb{Z}^d} \rho_p^a(u)\rho_p^a(w)\rho_p^a(u-w)\rho_p^a(y-w)\rho_p^a(x-y)\tilde{\rho}_p^a(x-u), \tag{4.52}
$$

where $\tilde{\rho}_p^a(x) = (pD * \rho_p^a)(x)$ as in (4.36). This diagram is depicted in Figure 8. The contribution arising from the term with $u = w = 0$ equals $\rho_p^a(0)^3$ times the triangle diagram of (4.37). Taking the full sum into account, the right side of (4.52) corresponds diagrammatically to the triangle diagram (4.37) with its vertex at the origin replaced by a triangle.

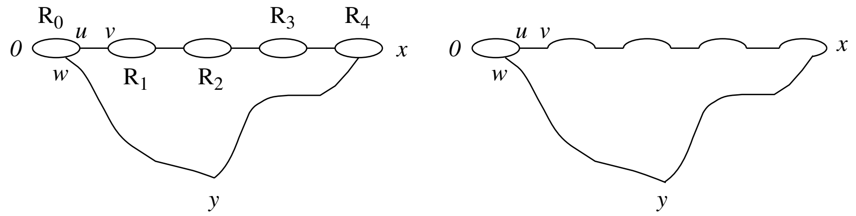

Figure 7: Configuration for the lattice animal one-loop diagram.

The above procedure can be extended to bound the higher-order terms. The resulting diagrams are the lattice tree diagrams, with an extra initial triangle as observed for $\psi_p^{(1)}(x)$. Now we define

$$
A^{(0)}(x,y) = \rho_p^a(x)\rho_p^a(y) \tag{4.53}
$$

and use the $A^{(i)} = A^{\text{end}}$ of (4.39) (with ρ replaced by ρ^a) to define $M^{(N)}$ recursively by (4.19) for $N \geq 1$. Then for $N \geq 1$ we have

$$
0 \leq \psi_p^{(N)}(x) \leq M^{(N)}(x,x). \tag{4.54}
$$

The cases $N = 1, 2$ are depicted in Figure 8.

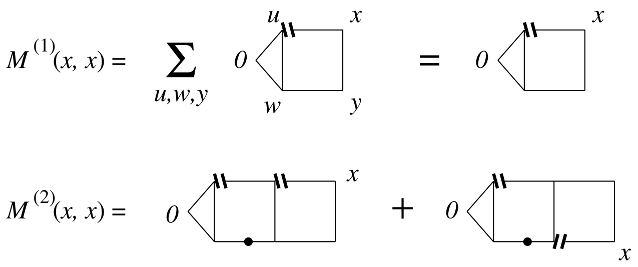

Figure 8: The one-loop and two-loop diagrams for lattice animals. Lines ending with double bars represent $\tilde{\rho}^a$-lines.

The bounds described above for lattice animals differ from those of [15] in two respects. One difference is that the diagrams of [15] involve additional small triangles that make no significant difference and need not be included. A second difference is that here we are using $\tilde{\rho}^a$ whereas only ρ^a was used in [15]. It is in fact possible to avoid the use of $\tilde{\rho}^a$, but a more involved argument than the one provided in [15] is necessary for this. However, the use of $\tilde{\rho}^a$ poses no difficulties and is simpler, so we will use it here.

Proof of Proposition 1.8(b) for lattice animals. The proof proceeds in the same way as for lattice trees. One minor difference for lattice animals is the presence of the term $\psi_p^{(0)}$, for which (4.49) implies

$$0 \le \psi_p^{(0)}(x) \le \frac{R^2\beta^2}{(|x|+1)^{2q}}(1-\delta_{0,x}). \tag{4.55}$$

Since $2q > 3q - d$ by assumption, this is smaller than what is required (second term of (4.41)). A second minor change is that the extraction of the extra factor β from the bound on $\psi_p^{(1)}$ is slightly different. □

4.5 Percolation diagrams

For percolation, the BK inequality [11] plays the role that Lemma 4.3 played for lattice animals. In particular, application of the BK inequality to (3.27) immediately gives

$$\psi_p^{(0)}(x) \le (1-\delta_{0,x})\tau_p(x)^2. \tag{4.56}$$

Higher order contributions can also be bounded using the BK inequality. For example, application of BK to the contribution to Figure 2 when $u' = x$ leads to the bound

$$\psi_p^{(1)}(x) \le \sum_{u,w,y,z\in\mathbb{Z}^d} \tau_p(u)\tau_p(w)\tau_p(u-w)\tilde{\tau}_p(y-u)\tau_p(y-z)\tau_p(z-w)\tau_p(x-y)\tau_p(x-z), \tag{4.57}$$

where

$$\tilde{\tau}_p(x) = \sum_{v\in\Omega_D} pD(v)\tau_p(x-v). \tag{4.58}$$

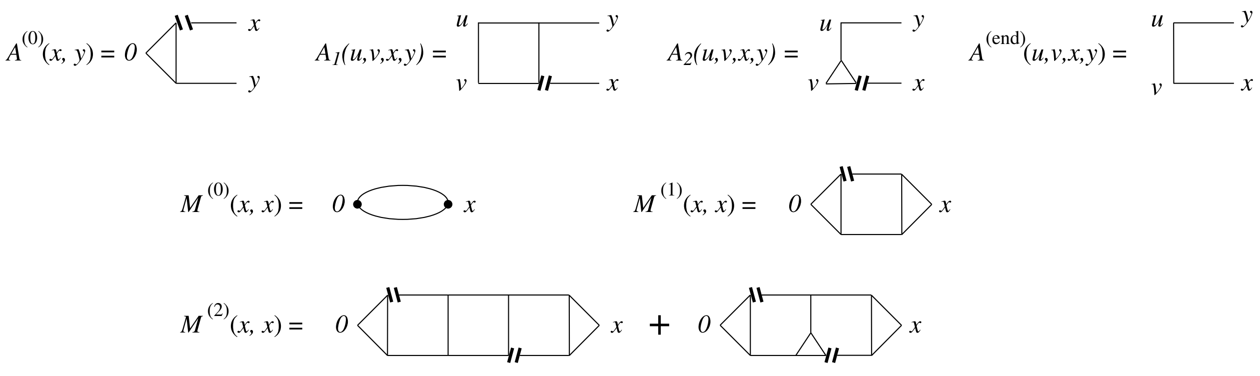

Figure 9: The diagrams for percolation. Lines ending with double bars represent $\tilde{\tau}^a$-lines.

The right side of (4.57) is depicted in Figure 9. It involves the two distinct routes $0 \to u \to y \to x$ and $0 \to w \to z \to x$ from 0 to x, which is suggestive of the fact that $\psi_p^{(1)}(x)$ could decay, like (4.56), twice as rapidly as the two-point function.

To state bounds on $\psi_p^{(N)}(x)$ for general N, we define

$$A^{(0)}(x, y) = \sum_{a,b \in \mathbb{Z}^d} \tau_p(a)\tau_p(b)\tau_p(a-b)\tilde{\tau}_p(x-a)\tau_p(y-b), \tag{4.59}$$

$$A_1(u, v, x, y) = \tau_p(u-v) \sum_{a,b \in \mathbb{Z}^d} \tau_p(u-a)\tau_p(v-b)\tau_p(a-b)\tau_p(y-a)\tilde{\tau}_p(x-b), \tag{4.60}$$

$$A_2(u, v, x, y) = \tau_p(y-u) \sum_{a,b \in \mathbb{Z}^d} \tau_p(u-a)\tau_p(v-a)\tau_p(a-b)\tau_p(v-b)\tilde{\tau}_p(x-b), \tag{4.61}$$

$$A^{(i)}(u, v, x, y) = A_1(u, v, x, y) + A_2(u, v, x, y) \quad (i \geq 1), \tag{4.62}$$

$$A^{(\mathrm{end})}(u, v, x, y) = \tau_p(u-v)\tau_p(u-x)\tau_p(v-y). \tag{4.63}$$

The above quantities are depicted in Figure 9.

We define $M^{(N)}$ for $N \geq 1$ by (4.19). It then follows from [14, Proposition 2.4] that, for $N \geq 1$,

$$0 \leq \psi_p^{(N)}(x) \leq M^{(N)}(x, x). \tag{4.64}$$

Consistent with (4.56), we define $M^{(0)}(x, x) = (1 - \delta_{0,x})\tau_p(x)^2$. We also recall from [14, Proposition 2.4] that for $N \geq 1$ the expansion remainder term $R_p^{(N)}(x)$ of (3.30) obeys

$$0 \leq R_p^{(N)}(x) \leq \sum_{u \in \mathbb{Z}^d} M^{(N)}(u, u)\tilde{\tau}_p(x-u). \tag{4.65}$$

We will use this below to conclude that $\lim_{N\to\infty} R_p^{(N)}(x) = 0$, assuming the hypotheses of Proposition 1.8(c). The vanishing of this limit was claimed below Theorem 3.4 and used under (2.5).

Proof of Proposition 1.8(c). For percolation, we have $z = p$, $G_z(x) = \tau_p(x)$ and $\Pi_z(x) = \psi_p(x)$. The hypotheses of the proposition are that $\tau_p(x) \leq \beta(|x|+1)^{-q}$ for $x \neq 0$ with $\frac{2}{3}d < q < d$, that βL^{q-d} is bounded away from zero, and that $p \leq 2$. It suffices to show that

$$\psi_p(x) \leq c\beta\delta_{0,x} + c\beta^2(|x|+1)^{-2q}. \tag{4.66}$$

It follows immediately from (4.56) that the contribution to ψ_p from $\psi_p^{(0)}$ does obey (4.66), and we concentrate now on $N \geq 1$.

By the assumed bound on τ, we conclude as in (4.43) that

$$\tilde{\tau}_p(x) \leq \frac{C\beta}{(|x|+1)^q}. \tag{4.67}$$

We will apply Lemma 4.1 with $q_1 = q_2 = q$. Our assumption on q implies that (4.17) is satisfied. We also need to verify that $A^{(0)}$, $A^{(i)}$, $A^{(\mathrm{end})}$ obey the assumptions of Lemma 4.1.

It is clear that $A^{(\mathrm{end})}$ obeys (4.21) with $q_1 = q_2 = q$ and $K_{\mathrm{end}} = O(1)$. For $A^{(0)}$, we note the decomposition

$$A^{(0)}(x,y) = \sum_{u,v} \Big[\tau_p(u)\tau_p(v) \Big] \Big[\tau_p(u-v)\tau_p(y-v)\tilde{\tau}_p(x-u) \Big]. \tag{4.68}$$

We can then apply Lemma 4.1, considering the first factor as $A^{(0)}$ and the second factor as $A^{(\mathrm{end})}$, to conclude that $A^{(0)}$ obeys (4.20) with $K_0 = C\beta$.

To check that $A^{(i)} = A_1 + A_2$ obeys (4.21) with $K = C\beta$, we begin with A_2. Define $a(u,v,x) = A_2(u,v,x,y)/\tau_p(y-u)$. This quantity is nothing but $A^{(0)}(x-v, u-v)$ of (4.68). Therefore,

$$a(u,v,x) \leq \frac{C\beta}{(|v-u|+1)^q(|x-v|+1)^q}, \tag{4.69}$$

and A_2 obeys (4.21) with $q_1 = q_2 = q$ and $K = C\beta$. For A_1, recalling the definition of H in (4.31), we see that $A_1(u,v,x,y)$ obeys the same bound as $C\beta\tau_p(u-v)H(u,v,x,y)$. By (4.32), A_1 obeys (4.21) with $K = C\beta$.

It then follows from Lemma 4.1 that for $N \geq 1$

$$0 \leq \psi_p^{(N)}(x) \leq M^{(N)}(x,x) \leq \frac{(\beta C)^N}{(|x|+1)^{2q}}. \tag{4.70}$$

The factor β^N here arises from the β's present in $A^{(0)}$ and in each of the $N-1$ factors of $A^{(i)}$. This gives an adequate bound for $N \geq 2$. To complete the proof, it suffices to argue that for $N=1$ the power of β in (4.70) can be replaced by β^2 when $x \neq 0$. This follows from the observation that for $N=1$ and $x \neq 0$, at least two diagram lines in $M^{(1)}(x,x)$ must undergo a nontrivial displacement, and each of these lines contributes a factor β. ∎

Proof that $\lim_{N\to\infty} R_p^{(N)}(x) = 0$ *under hypotheses of Proposition 1.8(c).* This is an immediate consequence of (4.65), (4.67), (4.70), and Proposition 1.7(i). ∎

5 Convolution bounds

In this section, we prove Proposition 1.7.

Proof of Proposition 1.7. (i) By definition,

$$|(f*g)(x)| \leq \sum_{y:|x-y|\leq|y|} \frac{1}{(|x-y|+1)^a}\frac{1}{(|y|+1)^b} + \sum_{y:|x-y|>|y|} \frac{1}{(|x-y|+1)^a}\frac{1}{(|y|+1)^b}. \tag{5.1}$$

Using $a \geq b$ and the change of variables $z = x - y$ in the second term, we see that

$$|(f * g)(x)| \leq 2 \sum_{y:|x-y|\leq|y|} \frac{1}{(|x-y|+1)^a} \frac{1}{(|y|+1)^b}. \tag{5.2}$$

In the above summation, $|y| \geq \frac{1}{2}|x|$. Therefore, for $a > d$ we have

$$|(f * g)(x)| \leq \frac{2^{b+1}}{(|x|+1)^b} \sum_{y:|x-y|\leq|y|} \frac{1}{(|x-y|+1)^a} \leq C(|x|+1)^{-b}. \tag{5.3}$$

Suppose now that $a < d$ and $a + b > d$. In this case, we divide the sum in (5.2) according to whether $\frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x|$ or $|y| \geq \frac{3}{2}|x|$. The contribution to (5.2) due to the first range of y is bounded above, as in (5.3), by

$$\frac{2^{b+1}}{(|x|+1)^b} \sum_{y:|x-y|\leq 3|x|/2} \frac{1}{(|x-y|+1)^a} \leq \frac{C}{(|x|+1)^b}(|x|+1)^{d-a}, \tag{5.4}$$

as required. When $|y| \geq \frac{3}{2}|x|$, we have $|y - x| \geq |y| - |x| \geq |y|/3$. Therefore, the contribution to (5.2) due to the second range of y is bounded above by

$$3^b \cdot 2 \sum_{y:|y|\geq 3|x|/2} \frac{1}{(|y|+1)^{a+b}} \leq \frac{C}{(|x|+1)^{a+b-d}}. \tag{5.5}$$

This completes the proof.

(ii) By (i), the convolution of g with the error term of f gives a result that is $O(BC(|x|+1)^{-(d-2+s)})$. This leaves us with the convolution of the main term with g, which is given by

$$\sum_{y\in\mathbb{Z}^d} g(y) \frac{A}{(|x-y|+1)^{d-2}} = \frac{A\sum_y g(y)}{(|x|+1)^{d-2}} + \sum_{y\in\mathbb{Z}^d} g(y) \left[\frac{A}{(|x-y|+1)^{d-2}} - \frac{A}{(|x|+1)^{d-2}}\right]. \tag{5.6}$$

The first term is the desired main term, so it remains to prove that the second term is an error term. We denote the second term by X.

We consider separately the contributions to X due to $|y| > \frac{1}{2}|x|$ and $|y| \leq \frac{1}{2}|x|$, beginning with the former. This contribution to X, which we call X_1, is bounded above by

$$|X_1| \leq AC \sum_{y:|y|>|x|/2} \frac{1}{(|y|+1)^{d+s}} \left[\frac{1}{(|x-y|+1)^{d-2}} + \frac{1}{(|x|+1)^{d-2}}\right]. \tag{5.7}$$

Using part (i) of the proposition, the first term is bounded above by

$$\frac{2^{2+s}AC}{(|x|+1)^{2+s}} \sum_{y\in\mathbb{Z}^d} \frac{1}{(|y|+1)^{d-2}} \frac{1}{(|x-y|+1)^{d-2}} \leq O\Big(\frac{AC}{(|x|+1)^{d-2+s}}\Big). \tag{5.8}$$

The second term of (5.7) obeys

$$\frac{AC}{(|x|+1)^{d-2}} \sum_{y:|y|>|x|/2} \frac{1}{(|y|+1)^{d+s}} = O\Big(\frac{AC}{(|x|+1)^{d-2+s}}\Big). \tag{5.9}$$

Combining these gives $X_1 = O(AC(|x|+1)^{-(d-2+s)})$, so X_1 is an error term.

Next, we consider the contribution to X due to $|y| \leq \frac{1}{2}|x|$, which we denote by X_2. We estimate this term by expanding the difference $\frac{1}{(|x-y|+1)^{d-2}} - \frac{1}{(|x|+1)^{d-2}}$ into powers of y. Because of the $\mathbb{Z}^d$-symmetry of $g(y)$, odd powers of y in the expansion give no contribution. Let $h(t) = [|x|(1+t)+1]^{2-d}$, for $|t| < 1$. Using the Fundamental Theorem of Calculus, it can be seen that $|h(t) - h(0) - h'(0)t| \leq c|t|^2|x|^2(|x|+1)^{-d}$ when $|t| \leq \frac{1}{2}$. Applying this with $t = |x|^{-1}|x-y| - 1$ (the case $x = 0$ does not contribute), we conclude that

$$|X_2| \leq \sum_{y:|y|\leq|x|/2} |g(y)| \frac{cA|y|^2}{(|x|+1)^d}. \tag{5.10}$$

Therefore, recalling that $s_2 = s \wedge 2$, we have

$$|X_2| \leq \frac{cAC}{(|x|+1)^d} \sum_{y:|y|\leq|x|/2} \frac{|y|^2}{(|y|+1)^{d+s}} \leq \begin{cases} cAC(|x|+1)^{-d-2+s_2} & (s \neq 2) \\ cAC(|x|+1)^{-d}\log(|x|+2) & (s = 2). \end{cases} \tag{5.11}$$

This completes the proof. $\square$

6 The random walk two-point function

In this section, we prove Proposition 1.6. We begin in Section 6.1 with an elementary proof of the bound

$$\delta_{0,x} \leq S_\mu(x) \leq \delta_{0,x} + O(L^{-d}), \tag{6.1}$$

which is uniform in $\mu \leq 1$ and $x \in \mathbb{Z}^d$. In Section 6.2, an integral representation for $S_\mu(x)$ is introduced, which is analysed in Sections 6.3–6.4 for $|x|$ large compared with L. The proof of the asymptotic formula (1.36) is then given in Section 6.5.

Once (1.36) and (6.1) have been proved, the bound (1.35) then follows easily. In fact, it suffices to prove (1.35) for $\mu = 1$, since $S_\mu(x)$ is increasing in μ. However, (1.35) for $\mu = 1$ follows immediately, by using (6.1) for $|x| \leq L$ and (1.36) for $|x| > L$.

6.1 Proof of the uniform bound

In this section, we prove the uniform bound (6.1). The lower bound of (6.1) follows immediately from the facts that $S_\mu(x)$ is positive for all x by definition and that $S_\mu(0)$ receives a contribution 1 from the zero-step walk. So it remains to prove the upper bound. For this, it suffices to consider $\mu = 1$, because $S_\mu(x)$ is increasing in μ.

In preparation, and for later use, we first note some properties of D. By Definition 1.1, $D(x) \leq O(L^{-d})$ and $\sigma \sim \text{const.}L$. In addition, it is proved in [19, Appendix A] that there are constants δ_2 and δ_3, such that for L sufficiently large,

$$1 - \hat{D}(k) \geq \delta_2 L^2 |k|^2 \qquad \text{for } |k| \leq L^{-1}, \tag{6.2}$$

$$1 - \hat{D}(k) \geq \delta_3 \qquad \text{for } k \in [-\pi,\pi]^d \text{ with } |k| \geq L^{-1}. \tag{6.3}$$

To prove the upper bound of (6.1), we rewrite $[1-\hat{D}(k)]^{-1}$ as $1+\hat{D}(k)+\hat{D}(k)^2[1-\hat{D}(k)]^{-1}$, to obtain

$$S_1(x) = \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1-\hat{D}(k)} = \delta_{0,x} + D(x) + \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} \frac{\hat{D}(k)^2 e^{-ik\cdot x}}{1-\hat{D}(k)}. \tag{6.4}$$

The second term is $O(L^{-d})$, so it remains to prove that the last term is also $O(L^{-d})$. We first estimate the absolute value of the last term by taking absolute values inside the integral. We then divide the integral over k into two parts, according to whether $|k|$ is greater than or less that L^{-1}.

For the integral over small k, we note that in general $|\hat{D}(k)| \leq 1$. Using (6.2) yields

$$\int_{|k|<L^{-1}} \frac{d^dk}{(2\pi)^d} \frac{\hat{D}(k)^2}{1-\hat{D}(k)} \leq \int_{|k|<L^{-1}} \frac{d^dk}{(2\pi)^d} \frac{1}{\delta_2 L^2 |k|^2} = O(L^{-d}). \tag{6.5}$$

Also, using (6.3), the integral over large k is bounded by

$$\int_{\{k\in[-\pi,\pi]^d : |k|\geq L^{-1}\}} \frac{d^dk}{(2\pi)^d} \frac{\hat{D}(k)^2}{1-\hat{D}(k)} \leq \frac{1}{\delta_3} \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} \hat{D}(k)^2 = \frac{1}{\delta_3} \sum_y D(y)^2 = O(L^{-d}). \tag{6.6}$$

This proves (6.1).

6.2 The integral representation

To prove the asymptotic formula (1.36), which states that

$$S_1(x) = \frac{a_d}{\sigma^2} \frac{1}{(|x|+1)^{d-2}} + O\Big(\frac{1}{(|x|+1)^{d-\alpha}}\Big), \tag{6.7}$$

we will use an integral representation for $S_\mu(x)$. By (6.1), $S_1(x) \leq O(L^{-d})$ for $x \neq 0$. This immediately implies (6.7) for $|x| \leq L^{1+\alpha/d}$. It therefore suffices, in what follows, to restrict attention to $|x| \geq L^{1+\alpha/d}$.

Although it is sufficient to consider only $\mu = 1$ to prove (6.7), we consider also $0 \leq \mu < 1$, as this will be used in the proof of the main error estimate in Section 7. Let

$$I_{t,\mu}(x) = \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} e^{-ik\cdot x}\, e^{-t[1-\mu\hat{D}(k)]}. \tag{6.8}$$

It follows from (1.21) that $\hat{S}_\mu(k) = [1-\mu\hat{D}(k)]^{-1}$. Thus, for $0 \leq \mu \leq 1$ we have the integral representation

$$S_\mu(x) = \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1-\mu\hat{D}(k)} = \int\limits_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} e^{-ik\cdot x} \int_0^\infty dt\, e^{-t[1-\mu\hat{D}(k)]} = \int_0^\infty dt\, I_{t,\mu}(x). \tag{6.9}$$

The integration variable t plays the role of a time variable, with the dominant contribution to $S_1(x)$ due to $t \approx |x|^2/\sigma^2$. With this in mind, we write $S_\mu(x) = S_\mu^<(x;T) + S_\mu^>(x;T)$ with

$$S_\mu^<(x;T) = \int_0^T dt\, I_{t,\mu}(x), \qquad S_\mu^>(x;T) = \int_T^\infty dt\, I_{t,\mu}(x), \tag{6.10}$$

and choose T to be equal to

$$T_x = \left(\frac{|x|}{\sigma}\right)^{2-2\alpha/d}, \tag{6.11}$$

where α is the small parameter of Proposition 1.6. With this choice, it will turn out that $S_1^>(x)$ contains the leading term of (6.7), whereas $S_1^<(x)$ is an error term. Our analysis will make use of different estimates on $I_{t,1}(x)$ for each of these two terms.

6.3 Integration over $[T, \infty]$

In this section, we prove that for $|x| \geq L^{1+\alpha/d}$ and L sufficiently large depending on α, we have

$$S_1^>(x; T_x) = \int_{T_x}^{\infty} I_{t,1}(x)\, dt = \int_{T_x}^{\infty} p_t(x)\, dt + O\Big(\frac{L^{-\alpha}}{(|x|+1)^{d-\alpha}}\Big), \tag{6.12}$$

where

$$p_t(x) = \left(\frac{d}{2\pi\sigma^2 t}\right)^{d/2} \exp\left(-\frac{d\,|x|^2}{2t\sigma^2}\right). \tag{6.13}$$

The proof will make use of the following lemma, which extracts the leading term from $I_{t,1}(x)$.

Lemma 6.1. *Let $d > 2$, and suppose D obeys Definition 1.1. Then there are finite L-independent constants τ and c_1 such that for $t \geq \tau$*

$$I_{t,1}(x) = p_t(x) + r_t(x) \tag{6.14}$$

with

$$|r_t(x)| \leq c_1\, L^{-d} t^{-d/2-1} + e^{-t\delta_3}. \tag{6.15}$$

Before proving Lemma 6.1, we show how integration of its bound leads to a proof of (6.12). It suffices to show that the integral of the error term $r_t(x)$ in (6.14) is bounded by the error term of (6.12). By (6.15) we have

$$\left|\int_{T_x}^{\infty} dt\, r_t(x)\right| \leq cL^{-d}\, T_x^{-d/2} + \delta_3^{-1}\, e^{-\delta_3 T_x}. \tag{6.16}$$

The second term can be absorbed into the first term. In fact, since $T_x \geq cL^{2(\alpha/d)(1-\alpha/d)}$, for any positive N we have

$$e^{-\delta_3 T_x} \leq \frac{c_N}{T_x^N} = c_N \frac{L^{-d}}{T_x^{(d+2)/2}} \frac{L^d}{T_x^{N-(d+2)/2}}, \tag{6.17}$$

with the last factor less than 1 for L and N sufficiently large depending on α. In addition, since $|x| > L$ and $d > \alpha$, the first term of (6.16) is equal to

$$cL^{-d}\, T_x^{-d/2} = c\frac{L^{-\alpha}}{(|x|+1)^{d-\alpha}}. \tag{6.18}$$

This proves (6.12).

Proof of Lemma 6.1. By Taylor's theorem and symmetry, for $k \in [-\pi, \pi]^d$ we have

$$1 - \hat{D}(k) = \frac{\sigma^2 |k|^2}{2d} + R(k) \tag{6.19}$$

with

$$|R(k)| \leq \frac{1}{4!} \sum_x D(x)(x \cdot k)^4 \leq \text{const.} L^4 |k|^4. \tag{6.20}$$

Let $k_t^2 = 4d\sigma^{-2} t^{-1} \log t$. We write $I_{t,1}(x) = \sum_{j=1}^4 I_t^{(j)}(x)$ with

$$I_t^{(1)}(x) = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x - t\sigma^2 |k|^2/(2d)}, \tag{6.21}$$

$$I_t^{(2)}(x) = -\int_{k_t < |k| < \infty} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x - t\sigma^2 |k|^2/(2d)}, \tag{6.22}$$

$$I_t^{(3)}(x) = \int\limits_{|k| < k_t} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x - t\sigma^2 |k|^2/(2d)} \left(e^{-tR(k)} - 1 \right), \tag{6.23}$$

$$I_t^{(4)}(x) = \int\limits_{k \in [-\pi,\pi]^d :\, |k| > k_t} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x}\, e^{-t[1-\hat{D}(k)]}. \tag{6.24}$$

The integrals $I_t^{(1)}(x)$ through $I_t^{(3)}(x)$ combine to give the contribution to $I_{t,1}(x)$ due to $|k| \leq k_t$, while $I_t^{(4)}(x)$ represents the contribution from $|k| > k_t$.

The first integral can be evaluated exactly to give

$$I_t^{(1)}(x) = p_t(x). \tag{6.25}$$

We therefore set $r_t(x) = \sum_{j=2}^4 I_t^{(j)}(x)$ and show that $r_t(x)$ obeys (6.15). By definition,

$$|I_t^{(2)}(x)| \leq \int\limits_{k_t < |k| < \infty} \frac{d^d k}{(2\pi)^d} e^{-t\sigma^2 |k|^2/(2d)} \leq c(t\sigma^2)^{-d/2} e^{-t\sigma^2 k_t^2/(4d)} \leq cL^{-d} t^{-d/2-1}. \tag{6.26}$$

The integral $I_t^{(3)}(x)$ is bounded as follows. First, we note that for $|k| < k_t$ it follows from (6.20) and the definition of k_t that $|tR(k)| \leq c(\log t)^2/t$, which is less than 1 for sufficiently large t. Using the bound $|e^x - 1| \leq |x|$ for $|x| \leq 1$, and increasing the integration domain to $\mathbb{R}^d$ in the last step, we have

$$|I_t^{(3)}(x)| \leq c \int\limits_{|k| < k_t} \frac{d^d k}{(2\pi)^d}\, e^{-t\sigma^2 |k|^2/(2d)}\, |tR(k)| \leq ctL^4 \int\limits_{|k| < k_t} \frac{d^d k}{(2\pi)^d}\, e^{-t\sigma^2 |k|^2/(2d)}\, |k|^4 \leq cL^{-d} t^{-d/2-1}. \tag{6.27}$$

Finally we estimate $I_t^{(4)}(x)$. We divide the integration domain according to whether $|k|$ is greater than or less than L^{-1}. By (6.2), as in (6.26) the contribution due to $|k| \le L^{-1}$ is at most

$$\int_{k_t<|k|<\infty} \frac{d^dk}{(2\pi)^d}\, e^{-t\delta_2 L^2|k|^2} \le cL^{-d}t^{-d/2-1}. \tag{6.28}$$

By (6.3), the contribution due to $|k| > L^{-1}$ is at most

$$\int_{k\in[-\pi,\pi]^d:\,|k|>L^{-1}} \frac{d^dk}{(2\pi)^d}\, e^{-t\delta_3} \le e^{-t\delta_3}. \tag{6.29}$$

Combining (6.26)–(6.29) then gives the desired bound

$$|r_t(x)| \le \sum_{j=2}^{4} |I_t^{(j)}(x)| \le c\,L^{-d}t^{-d/2-1} + e^{-t\delta_3}. \tag{6.30}$$

□

6.4 Integration over $[0, T]$

In this section, we prove the following lemma, which will also be used in the main error estimate of Section 7.

Lemma 6.2. *Let $|x| \ge L^{1+\alpha/d}$ and $T \le T_x$. Then for $0 \le \mu \le 1$ and sufficiently large L depending on α*

$$S_\mu^<(x;T) = \int_0^T I_{t,\mu}(x)\,dt \le \frac{1}{(|x|+1)^{d+2}}. \tag{6.31}$$

We will prove this using the following lemma, whose proof involves a standard large deviations argument.

Lemma 6.3. *For $x \in \mathbb{Z}^d$, $t \ge 0$ and $t_0 = dL\|x\|_\infty/(2\sigma^2)$,*

$$0 \le I_{t,1}(x) \le \begin{cases} \exp[-\|x\|_\infty/L + \sigma^2 t/(dL^2)] & (0 \le t < \infty) \\ \exp[-d\|x\|_\infty^2/(4\sigma^2 t)] & (t \ge t_0). \end{cases} \tag{6.32}$$

Proof of Lemma 6.2 assuming Lemma 6.3. We first note that for fixed $x \in \mathbb{Z}^d$, $I_{t,\mu}(x)$ is nonnegative and is monotone increasing in μ. In fact, expanding the exponential $e^{t\mu\hat{D}(k)}$ in (6.8) and interchanging the integral and the sum (justified by absolute convergence), gives

$$I_{t,\mu}(x) = e^{-t} \sum_{n=0}^{\infty} \frac{(t\mu)^n}{n!} \int_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d}\, e^{-ik\cdot x}\hat{D}(k)^n = e^{-t} \sum_{n=0}^{\infty} \frac{(t\mu)^n}{n!} D^{*n}(x), \tag{6.33}$$

where D^{*n} denotes the n-fold x-space convolution. Because $D^{*n}(x)$ is nonnegative, this representation immediately implies the non-negativity of $I_{t,\mu}(x)$, together with its monotonicity in μ. Therefore $S_\mu^<(x;T)$ is increasing in μ and in T, and it suffices to prove (6.31) for $\mu = 1$ and $T = T_x$.

In this case, (6.32) gives

$$\int_0^{T_x} dt\, I_{t,1}(x) \le \int_0^{t_0} dt\, \exp\Big[-\frac{\|x\|_\infty}{L} + \frac{\sigma^2 t}{dL^2}\Big] + \int_{t_0}^{T_x} dt\, \exp\Big[-\frac{d\|x\|_\infty^2}{4\sigma^2 t}\Big]. \tag{6.34}$$

The first integral can be performed exactly. For the second, we use the fact that for $a \ge T$,

$$\int_0^T dt\, e^{-a/t} = a \int_{a/T}^\infty du\, u^{-2} e^{-u} \le \frac{T^2}{a} e^{-a/T}. \tag{6.35}$$

Now choose $T = T_x$ and $a = d\|x\|_\infty^2/(4\sigma^2) \ge T_x$, for $|x| \ge L^{1+\alpha/d}$. This gives

$$\begin{aligned}\int_0^{T_x} dt\, I_{t,1}(x) &\le c\exp\Big[-\frac{\|x\|_\infty}{2L}\Big] + \frac{4\sigma^2}{d\|x\|_\infty^2} T_x^2 \exp\Big[-\frac{d\|x\|_\infty^2}{4\sigma^2 T_x}\Big] \\ &\le c\,\exp\Big(-c\frac{|x|}{L}\Big) + c\Big(\frac{|x|}{L}\Big)^{2-4\alpha/d} \exp\Big(-c\Big(\frac{|x|}{L}\Big)^{2\alpha/d}\Big). \end{aligned}\tag{6.36}$$

For $|x| \ge L^{1+\alpha/d}$, we have $|x|/L \ge |x|^{(\alpha/d)/(1+\alpha/d)}$. The integral $\int_0^{T_x} dt\, I_{t,1}(x)$ therefore decays at least as fast as a constant multiple of an exponential of a power of $|x|$, and hence eventually decays faster than $|x|^{-(d+2)}$. This completes the proof of (6.31). □

Proof of Lemma 6.3. Since D is supported only on $\|x\|_\infty \le L$, we have $D^{*n}(x) = 0$ for $\|x\|_\infty > nL$. Since $0 \le D^{*n}(x) \le 1$ for all n, we can therefore bound (6.33) using the inequality

$$\sum_{n=N}^\infty \frac{t^n}{n!} \le e^t \frac{t^N}{N!} \le e^t \Big(\frac{et}{N}\Big)^N \tag{6.37}$$

as

$$I_{t,1}(x) = e^{-t} \sum_{n \ge \|x\|_\infty/L} \frac{t^n}{n!} D^{*n}(x) \le e^{-t} \sum_{n \ge \|x\|_\infty/L} \frac{t^n}{n!} \le \Big(\frac{et}{\|x\|_\infty/L}\Big)^{\|x\|_\infty/L}. \tag{6.38}$$

Thus, $I_{t,1}(x)$ decays in $|x|$ more rapidly than any exponential, and we may define the quantity

$$\phi_t(s) = \sum_{x\in\mathbb{Z}^d} e^{sx_1}\, I_{t,1}(x) \qquad (s \in \mathbb{R}). \tag{6.39}$$

We claim that

$$\phi_t(s) = \exp\Big[t \sum_x D(x)\big[\cosh(sx_1) - 1\big]\Big]. \tag{6.40}$$

This follows by interchanging sums in (6.33) to obtain

$$\phi_t(s) = e^{-t} \sum_{n=0}^\infty \frac{t^n}{n!} \sum_x e^{sx_1} D^{*n}(x) = e^{-t} \sum_{n=0}^\infty \frac{t^n}{n!} \Big[\sum_x e^{sx_1} D(x)\Big]^n = \exp\Big[-t + t\sum_x e^{sx_1} D(x)\Big], \tag{6.41}$$

and then using $D(x) = D(-x)$ and $\sum_x D(x) = 1$.

By definition, $\phi_t(s) = \sum_y e^{s|y_1|} I_{t,1}(y) \geq e^{s|x_1|} I_{t,1}(x)$ for any $x \in \mathbb{Z}^d$, and therefore $I_{t,1}(x) \leq e^{-s|x_1|}\phi_t(s)$. The $\mathbb{Z}^d$-symmetry and the formula (6.40) for $\phi_t(s)$ then give

$$I_{t,1}(x) \leq \exp\Big[-s\|x\|_\infty + t\sum_y D(y)\Big[\cosh(sy_1) - 1\Big]\Big]. \tag{6.42}$$

When $s \leq L^{-1}$, we have $s|y_1| \leq 1$ for any y contributing to $\sum_y D(y)[\cosh(sy_1) - 1]$. Since $\cosh x \leq 1 + x^2$ for $|x| \leq 1$, we obtain

$$0 \leq \sum_y D(y)\Big[\cosh(sy_1) - 1\Big] \leq s^2 \sum_y D(y) y_1^2 = s^2 \frac{\sigma^2}{d}. \tag{6.43}$$

Thus, for $s \leq L^{-1}$ we have

$$I_{t,1}(x) \leq \exp\Big[-s\|x\|_\infty + \sigma^2 t d^{-1} s^2\Big]. \tag{6.44}$$

Putting $s = L^{-1}$ in (6.44) gives the first bound of (6.32).

The minimum of the right side of (6.44) is attained at $s = d\|x\|_\infty/(2\sigma^2 t)$, but we may use (6.44) only for $s \leq 1/L$. This condition will be valid provided $t \geq t_0$. Using the minimal value of s in (6.44) gives the second bound of (6.32). □

6.5 Proof of the asymptotics

We now prove (6.7). As discussed below (6.7), it suffices to consider $|x| \geq L^{1+\alpha/d}$. For $|x| \geq L^{1+\alpha/d}$, we have already proved a version of (6.7) having lower limit of integration T_x rather than 0, since the combination of (6.12) and (6.31) gives

$$S_1(x) = S_1^{<}(x;T_x) + S_1^{>}(x;T_x) = \int_{T_x}^\infty p_t(x)\,dt + O\Big(\frac{L^{-\alpha}}{(|x|+1)^{d-\alpha}}\Big) \qquad (|x| \geq L^{1+\alpha/d}). \tag{6.45}$$

We will show that (6.7) holds as stated with lower limit of integration 0, if $|x| \geq L^{1+\alpha/d}$. For this, it suffices to show that the quantity $R(x) = \int_0^{T_x} p_t(x)dt$ obeys the bound

$$R(x) \leq O\Big(\frac{1}{(|x|+1)^d}\Big) \qquad (|x| \geq L^{1+\alpha/d}). \tag{6.46}$$

By definition,

$$|R(x)| \leq cL^{-d}\int_0^{T_x} dt\, t^{-d/2} e^{-c'|x|^2/(tL^2)}. \tag{6.47}$$

To estimate the right side, we use the fact that

$$\int_0^T dt\, t^{-b}\, e^{-a/t} = a^{1-b}\int_{a/T}^\infty u^{b-2} e^{-u} du \leq \begin{cases} a^{-1}T^{2-b}e^{-a/T} & (1 < b \leq 2) \\ C(b)a^{1-b}e^{-a/2T} & (b > 2) \end{cases} \tag{6.48}$$

if $a \geq T > 0$ and $b > 1$, where $C(b)$ is a b-dependent constant. This inequality can be proved for $1 < b \leq 2$ using $u^{b-2} \leq (a/T)^{b-2}$. For $b > 2$, it can be proved using $e^{-u} \leq e^{-u/2}e^{-a/2T}$.

We apply (6.48) with $a = c|x|^2/L^2$, which is greater than T_x when $|x| \geq L^{1+\alpha/d}$ and L is large. The exponent b equals $d/2$, which is greater than 1 for $d > 2$. The result is that

$$|R(x)| \leq cL^{-d} e^{-c''(|x|/L)^{2\alpha/d}} \left(\frac{|x|}{L}\right)^{q(d)}, \tag{6.49}$$

for some power $q(d)$. For $|x| \geq L^{1+\alpha/d}$ sufficiently large, we therefore have

$$|R(x)| \leq cL^{-d} \left(\frac{L}{|x|}\right)^d \leq \frac{c}{(|x|+1)^d}. \tag{6.50}$$

This completes the proof of (6.7) if $|x| \geq L^{1+\alpha/d}$, and hence for all x. □

7 The main error estimate

In this section, we prove Proposition 1.9. We first obtain bounds on $E_z(x)$ and $\hat{E}_z(k)$ in Section 7.1, and then complete the proof of Proposition 1.9 in Section 7.2.

7.1 Bounds on E_z

First we derive the following bound on $E_z(x)$ from the assumed bound on $\Pi_z(x)$.

Lemma 7.1. *Under the assumptions of Proposition 1.9,*

$$|E_z(x)| \leq \begin{cases} c\gamma & (x=0) \\ c\gamma L^{-d} & (0 < |x| < 2L) \\ c\gamma |x|^{-(d+2+\kappa)} & (|x| \geq 2L). \end{cases} \tag{7.1}$$

Proof. By virtue of its definition in (1.28), we can write

$$E_z(x) = (1-\lambda_z)\delta_{0,x} - (D * N_z)(x). \tag{7.2}$$

with

$$N_z(x) = [(1-\lambda_z) + \lambda_z z\hat{\Pi}_z(0)]\delta_{0,x} - \lambda_z z \Pi_z(x). \tag{7.3}$$

To derive bounds on N_z and thus on E_z, we first derive bounds on Π_z and λ_z. Assuming $|\Pi_z(x)| \leq \gamma(|x|+1)^{-(d+2+\kappa)}$, we have

$$\sum_y |\Pi_z(y)| \leq c\gamma, \qquad \sum_y |y|^2\, |\Pi_z(y)| \leq c\gamma. \tag{7.4}$$

Also, by the formula for λ_z of (1.30) and our assumption that $z \leq C$, it follows that

$$\lambda_z = O(1), \qquad \lambda_z - 1 = O(\gamma). \tag{7.5}$$

The bounds (7.4) and (7.5) imply

$$N_z(x) = O(\gamma)\delta_{0,x} + O\Big(\frac{\gamma}{(|x|+1)^{d+2+\kappa}}\Big), \qquad \sum_x N_z(x) = O(\gamma), \tag{7.6}$$

and hence

$$(D * N_z)(x) = \sum_{|y|\le L} N_z(x-y)D(y) = \sum_{|y|\le L} |N_z(x-y)|\, O(L^{-d}) = O(\gamma L^{-d}). \tag{7.7}$$

By (7.2), this proves (7.1) for $0 \le |x| < 2L$. For $|x| \ge 2L$, we note that $|x-y| \ge |x|/2$ when $|y| \le L$. For such y, (7.6) implies $|N_z(x-y)| = O(\gamma|x|^{-(d+2+\kappa)})$, and therefore

$$(D * N_z)(x) = \sum_{|y|\le L} N_z(x-y)D(y) = O\Big(\frac{\gamma}{|x|^{d+2+\kappa}}\Big)\sum_{|y|\le L} D(y) = O\Big(\frac{\gamma}{|x|^{d+2+\kappa}}\Big). \tag{7.8}$$

□

Next, we use the above bound on $E_z(x)$ to derive a bound on $\hat{E}_z(k)$.

Lemma 7.2. *Let $\kappa_2 = \kappa \wedge 2$. As $k \to 0$, under the assumptions of Lemma 1.9,*

$$|\hat{E}_z(k)| \le \begin{cases} c\gamma L^{2+\kappa_2}|k|^{2+\kappa_2} & (\kappa \ne 2) \\ c\gamma |k|^4\,(L^4 + \log|k|^{-1}) & (\kappa = 2). \end{cases} \tag{7.9}$$

Proof. The proof proceeds as in the proof of Lemma 2.3. Since $\hat{E}_z(0) = \nabla^2\hat{E}_z(0) = 0$, as in (2.25)–(2.26) we have

$$|\hat{E}_z(k)| \le c|k|^4 \sum_{x:|x|\le|k|^{-1}} |x|^4\,|E_z(x)| + c\sum_{x:|x|>|k|^{-1}} (1+|k|^2|x|^2)\,|E_z(x)|. \tag{7.10}$$

A calculation using Lemma 7.1 then implies that for $|k| \le (2L)^{-1}$,

$$|\hat{E}_z(k)| \le \begin{cases} c\gamma[L^4|k|^4 + |k|^{2+\kappa_2}] & (\kappa \ne 2) \\ c\gamma[L^4|k|^4 + |k|^4\log|k|^{-1}] & (\kappa = 2). \end{cases} \tag{7.11}$$

The above bounds imply (7.9) for $|k| \le (2L)^{-1}$. The case $|k| > (2L)^{-1}$ is bounded simply as

$$|\hat{E}_z(k)| \le \sum_x |E_z(x)| = O(\gamma), \tag{7.12}$$

which satisfies (7.9) for $|k| > (2L)^{-1}$. □

7.2 Proof of Proposition 1.9

In this section, we prove Proposition 1.9. It suffices to consider the case of small α. We begin by proving the x-dependent bound, which is valid for all x. The uniform bound, valid for $x \ne 0$, will follow immediately from this proof. The proof is divided into three cases, according to the value of x. We prove the x-dependent bound first assuming $\kappa \neq 2$, and comment on the minor modifications for $\kappa = 2$ at the end.

Case 1: $x = 0$. The uniform bound (6.1) on $S_{\mu_z}(x)$ implies that

$$|(E_z * S_{\mu_z})(0)| \leq |E_z(0)| + O(L^{-d}) \sum_y |E_z(y)|. \tag{7.13}$$

Lemma 7.1 implies $\sum_y |E_z(y)| = O(\gamma)$, and hence (7.13) is $O(\gamma)$ and satisfies (1.48).

Case 2: $0 < |x| \leq L^{1+\alpha/(d+\kappa_2)}$. For arbitrary $x \neq 0$, it follows from Lemma 7.1 and (6.1) that

$$\begin{aligned}(E_z * S_{\mu_z})(x) &= E_z(x) S_{\mu_z}(0) + \sum_{y:y\neq x} E_z(y) S_{\mu_z}(x-y) \\ &= O(\gamma L^{-d}) + \sum_y |E_z(y)|\, O(L^{-d}) = O(\gamma L^{-d}).\end{aligned} \tag{7.14}$$

This proves the first bound of (1.48). Also, when $0 < |x| \leq L^{1+\alpha/(d+\kappa_2)}$, (7.14) implies

$$|(E_z * S_{\mu_z})(x)| = O(\gamma L^{-d}) = \frac{|x|^{d+\kappa_2-\alpha}}{L^d} O\Big(\frac{\gamma}{|x|^{d+\kappa_2-\alpha}}\Big) \leq O\Big(\frac{\gamma L^{\kappa_2}}{|x|^{d+\kappa_2-\alpha}}\Big). \tag{7.15}$$

Case 3: $|x| > L^{1+\alpha/(d+\kappa_2)}$. We fix $T = (\frac{|x|}{2\sigma})^{2-2\alpha/(d+\kappa_2)}$, which is equal to $T_{x/2}$ of (6.11) with a smaller α. We then define X_1 and X_2 by

$$(E_z * S_{\mu_z})(x) = \sum_y E_z(x-y) S^{<}_{\mu_z}(y;T) + \sum_y E_z(x-y) S^{>}_{\mu_z}(y;T) = X_1 + X_2. \tag{7.16}$$

The contribution X_1 is further divided as

$$X_1 = \sum_{y:|y|\leq|x|/2} E_z(x-y) S^{<}_{\mu_z}(y;T) + \sum_{y:|y|>|x|/2} E_z(x-y) S^{<}_{\mu_z}(y;T) = X_{11} + X_{12}. \tag{7.17}$$

It remains to estimate X_{11}, X_{12} and X_2.

For X_{12}, by our choice of T we can use (6.31). Since $\sum_y |E_z(y)| = O(\gamma)$, we obtain

$$|X_{12}| \leq \sum_{y:|y|>|x|/2} |E_z(x-y)| \frac{1}{(|y|+1)^{d+2}} \leq \sum_y |E_z(x-y)| O\Big(\frac{1}{|x|^{d+2}}\Big) = O\Big(\frac{\gamma}{|x|^{d+2}}\Big). \tag{7.18}$$

For X_{11}, we use (1.35) to obtain

$$S^{<}_{\mu_z}(y;T) \leq S_{\mu_z}(y) \leq \delta_{0,y} + O\Big(\frac{1}{L^{2-\alpha}(|y|+1)^{d-2}}\Big). \tag{7.19}$$

Since $|x-y| \geq |x|/2 \geq 2L$ (for large L), the third bound of Lemma 7.1 gives

$$|X_{11}| \leq \Big[1 + \sum_{y:|y|\leq|x|/2} O\Big(\frac{1}{L^{2-\alpha}(|y|+1)^{d-2}}\Big)\Big] O\Big(\frac{\gamma}{|x|^{d+2+\kappa}}\Big) \leq O\Big(\frac{\gamma}{|x|^{d+\kappa}}\Big). \tag{7.20}$$

To control X_2, we use the integral representation (6.10) for $S^{>}_{\mu_z}$ to write

$$X_2 = (E_z * S^{>}_{\mu_z})(x) = \int_T^\infty dt (I_{t,\mu_z} * E_z)(x) = \int_T^\infty dt \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \hat{I}_{t,\mu_z}(k) \hat{E}_z(k). \qquad (7.21)$$

By (6.8), $\hat{I}_{t,\mu_z}(k) = e^{-t[1-\mu_z \hat{D}(k)]}$. Since $1 - \mu_z \hat{D} \geq [1-\hat{D}]/2$, it follows from Lemma 7.2 that

$$|X_2| \leq \int_T^\infty dt \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{-t[1-\hat{D}(k)]/2}\, c\gamma L^{2+\kappa_2} |k|^{2+\kappa_2}. \qquad (7.22)$$

We divide the k-integral according to whether $|k|$ is greater or less than L^{-1}, as in the analysis of $I_t^{(4)}(x)$ in (6.28)–(6.29). This gives

$$\int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} e^{-t[1-\hat{D}(k)]/2}\, |k|^{2+\kappa_2} \leq O(L^{-(d+2+\kappa_2)}) t^{-(d+2+\kappa_2)/2} + O(e^{-\delta_3 t}). \qquad (7.23)$$

The second error term can be absorbed into the first for $t \geq T$ and sufficiently large L, by arguing exactly as was done for (6.16). Performing the t-integral then gives

$$|X_2| \leq O(\gamma L^{-d}) \int_T^\infty t^{-(d+2+\kappa_2)/2}\, dt \leq \frac{O(\gamma L^{\kappa_2 - \alpha})}{|x|^{d+\kappa_2-\alpha}}. \qquad (7.24)$$

Combining (7.18), (7.20) and (7.24) gives the desired estimate

$$\Big|(E_z * S_{\mu_z})(x)\Big| \leq O\Big(\frac{\gamma L^{\kappa_2-\alpha}}{|x|^{d+\kappa_2-\alpha}}\Big). \qquad (7.25)$$

for $|x| > L^{1+\alpha/(d+\kappa_2)}$.

The case $\kappa = 2$ adds extra factors $\log |k|^{-1}$, $|\log t|$, and $\log(|x|/L)$ in (7.22), (7.23), (7.24), and (7.25). However, the extra logarithm of (7.25) can be absorbed in $|x|^{d+\kappa_2-\alpha}$, by slightly increasing the exponent α of (1.48). □

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