# One-sided random walk with weak pinning: pathwise descriptions of the phase transition [1]

Preliminary Draft

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**Abstract**

We consider a one-dimensional random walk which is conditioned to stay non-negative and is "weakly pinned" to zero. This model is known to exhibit a phase transition as the strength of the weak pinning varies. We prove path space limit theorems which describe the macroscopic shape of the path for all values of the pinning strength. If the pinning is less than (resp. equal to) the critical strength, then the limit process is the Brownian meander (resp. reflecting Brownian motion). If the pinning strength is supercritical, then the limit process is a positively recurrent Markov chain with a strong mixing property.

## 1 Introduction

In this paper, we discuss a localization-delocalization phase transition of a discrete interface above the attractive wall. Large number of works have been done on this subject, especially for one-dimensional interfaces, e.g. [BDZ00, Bur81, Cha82, Fis84, Upt99, vLHi81] among all (See also [BDZ00, CaVe00, DeVe00, DMRR92, IoVe00] for extensions in higher dimensions). The model considered in this paper is one-dimensional and is based on a symmetric, nearest neighbor random walk $(\{S_n\}_{n\geq 0}, \{P^x\}_{x\in\mathbf{Z}})$ on $\mathbf{Z}$. To be precise, $\{S_n\}_{n\geq 0}$ are integer valued random variables and $\{P^x\}_{x\in\mathbf{Z}}$ are probability measures such that the following hold for all $n\geq 1$ and $x\in\mathbf{Z}$; $S_1-S_0,\dots,S_n-S_{n-1}$ are $P^x$-independent and

$$P^x\{S_0=x\}=1,\quad P^x\{S_n-S_{n-1}=0\}=r<1,\quad P^x\{S_n-S_{n-1}=\pm 1\}=v/2, \tag{1.1}$$

where $r+v=1$. For $n\geq 1$ and $x\in\mathbf{Z}$, we set

$$L_n^x = \text{the number of } 0\leq j\leq n-1 \text{ such that } S_j=x. \tag{1.2}$$

We define a probability measure $\mu_n^x$ for $x,n\in\mathbf{N}=\{0,1,\dots\}$ by $\mu_0^x=P^x$ and

$$\mu_n^x(d\omega)=\frac{1}{Z_n^x}P^x\left[\exp\left(\beta L_n^0\right): d\omega\cap\{S_1\geq 0,\dots,S_n\geq 0\}\right],\quad n\geq 1 \tag{1.3}$$

where $\beta\in\mathbf{R}$ and $Z_n^x$ is the normalizing constant.

In the case $\beta>0$ which we are mainly interested in, the basic feature of the measure $\mu_n^x$ can be explained as follows. We refer to the exponential factor in (1.3) as the *weak pinning* which tries to localize the path $(S_k)_{k=1}^n$ near the origin. The effect of the weak pinning, however,

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has to compete with the entropic fluctuation of the random walk, which together with the restriction to $\{S_1 \geq 0, \ldots, S_n \geq 0\}$, has a tendency to push the path away from the origin. We call the latter tendency *entropic repulsion*. The competetion between weak pinning and entropic repulsion has been well studied at the level of the free energy. We set

$$\beta_c = \ln \frac{2}{1+r}, \tag{1.4}$$

$$\lambda_{\pm} = e^{\beta} \frac{(e^{\beta}-1)r \pm \sqrt{(e^{\beta}-1)^2 r^2 + (e^{\beta}-1)v^2}}{2(e^{\beta}-1)}, \quad \text{for } \beta > \beta_c. \tag{1.5}$$

Note that

$$\lambda_- < 0 < 1 < \lambda_+, \quad \text{and } |\lambda_-| < |\lambda_+|. \tag{1.6}$$

It is known [Bol99, Cha82, Fis84] and will also be seen later in this paper that the *free energy* for this model can explicitly be given as follows;

$$\psi(\beta) \stackrel{\text{def.}}{=} \lim_{n \nearrow \infty} n^{-1} \ln Z_n^x = \begin{cases} 0 & \text{if } \beta \leq \beta_c, \\ \ln(\lambda_+) > 0 & \text{if } \beta > \beta_c. \end{cases} \tag{1.7}$$

for all $x \in \mathbf{N}$. This means in particular that $\psi(\beta) = \psi(0)$ for $\beta \leq \beta_c$ and $\psi(\beta) > \psi(0)$ for $\beta > \beta_c$. We can therefore expect the following. For $\beta < \beta_c$, the effect of the weak pinning is smaller than that of the entropic repulsion and the pinning effect does not change the diffusive nature of the path in a qualitative way. On the other hand, if $\beta > \beta_c$, the effect of the weak pinning overwhelms that of the entropic repulsion and it confines the path near the origin. The purpose of this paper is to justify the above expected phenomena as limit theorems.

Let $p \geq 0$, $m \geq 1$ and $f : I^m \to \mathbf{R}$, where $I$ is either $[0, \infty)$ or $\mathbf{N}$. We say $f$ is *p-bounded* if there is $p \geq 0$ such that

$$\sup_{(x_j)_{j=1}^m \in I^m} (1 + |x_1| + \ldots + |x_m|)^{-p} |f(x_1, \ldots, x_m)| < \infty. \tag{1.8}$$

We say $f$ is *polynomially bounded* if it is $p$-bounded for some $p \geq 0$. We introduce a bilinear form

$$\langle f_1, f_2 \rangle_{\beta} = e^{-\beta} f_1(0) f_2(0) + \sum_{x \geq 1} f_1(x) f_2(x) \tag{1.9}$$

for $f_i : \mathbf{N} \to \mathbf{R}$ $(i = 1, 2)$ when the summation on the right-hand-side converges absolutely. For $\beta > \beta_c$, we define a function $\rho : \mathbf{N} \to (0, 1]$ by $\rho(x) = \rho(1)^x$, where

$$\rho(1) = \frac{-(e^{\beta}-1)r + \sqrt{(e^{\beta}-1)^2 r^2 + (e^{\beta}-1)v^2}}{(e^{\beta}-1)v} \in (0, 1). \tag{1.10}$$

**Theorem 1.1 a)** *If $\beta \leq \beta_c$, then*

$$\lim_{n \nearrow \infty} \mu_n^x \left[ f(S_n / \sqrt{vn}) \right] = \int_{[0,\infty)} f d\mu \tag{1.11}$$

*for all* $x \in \mathbf{N}$ *and for any polynomially bounded continuous function* $f : [0, \infty) \to \mathbf{R}$, *where*

$$\mu(dt) = \begin{cases} e^{-t^2/2} t dt & \text{if } \beta < \beta_c, \\ \sqrt{2/\pi} e^{-t^2/2} dt & \text{if } \beta = \beta_c. \end{cases} \tag{1.12}$$

**b)** *For* $\beta > \beta_c$, *there are constants* $C_{1.13} = C_{1.13}(\beta, v) > 0$ *and* $\varepsilon_{1.13} = \varepsilon_{1.13}(\beta, v) \in (0, 1)$ *as follows; if* $n \geq (1 + x) C_{1.13}$ *and* $f : \mathbf{N} \to \mathbf{R}$ *is polynomially bounded, then*

$$\left| \mu_n^x[f(S_n)] - \frac{\langle f, \rho \rangle_\beta}{\langle 1, \rho \rangle_\beta} \right| \leq B_{1.13}(f) \varepsilon_{1.13}^n, \tag{1.13}$$

*where* $B_{1.13}(f)$ *is a constant which depends only on* $\beta$, $v$ *and* $f$.

We have the following process level limit theorem for $\beta \leq \beta_c$;

**Theorem 1.2** *For* $\beta \leq \beta_c$, *consider probability measures* $\widetilde{\mu}_n^x$, $n \geq 1$, $x \in \mathbf{N}$ *on* $C([0, 1])$ *defined by*

$$\widetilde{\mu}_n^x = \mu_n^x \left\{ \left( S_{nt}/\sqrt{vn} \right)_{0 \leq t \leq 1} \in \cdot \right\}, \tag{1.14}$$

*where* $(S_t)_{t \geq 0}$ *is the linear interpolation of the random walk* $(S_n)_{n \in \mathbf{N}}$. *Then, for any fixed* $x \in \mathbf{N}$, *as* $n \nearrow \infty$, $\widetilde{\mu}_n^x$ *converges weakly to the law of a process* $W = (W_t)_{0 \leq t \leq 1}$. *The process* $W$ *can be described as follows;*

$$W_t = \begin{cases} (1 - \tau)^{-1/2} B_{\tau + t(1 - \tau)} & \text{if } \beta < \beta_c, \\ |B_t| & \text{if } \beta = \beta_c, \end{cases} \tag{1.15}$$

*where* $B = (B_t)_{0 \leq t \leq 1}$ *is a one-dimensional Brownian motion starting from 0 and* $\tau = \sup\{t \in [0, 1] \,;\, B_t = 0\}$.

**Remark 1.1** Similar limit theorems are discussed in the litratures. Bolthausen *et al* consider the case of one-dimensional weakly pinned Gaussian field in the delocalized phase [BDZ00, Section4]. In [Upt99], Upton discusses one-dimensional SOS interfaces and the phase boundaries of two-dimensional Ising model, where the thermodynamic paramerters are assumed to be sufficiently inside delocalized region.

We have a process level limit theorem also for $\beta > \beta_c$. We will have a positively recurrent Markov chain as the limit process. To describe the Markov chain obtaind as the limit, we introduce matrices $Q = (Q_{xy})_{x, y \in \mathbf{N}}$ and $R = (R_{xy})_{x, y \in \mathbf{N}}$ respectively by

$$Q = \begin{pmatrix} e^\beta r & e^\beta \frac{v}{2} & 0 & 0 & \cdots \\ \frac{v}{2} & r & \frac{v}{2} & 0 & \cdots \\ 0 & \frac{v}{2} & r & \frac{v}{2} & \ddots \\ 0 & 0 & \frac{v}{2} & r & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.16}$$

and

$$R_{xy} = \lambda_+^{-1}\rho(x)^{-1}Q_{xy}\rho(y). \tag{1.17}$$

We see by direct computation that

$$\lambda_+ = e^{\beta}\left(r + \frac{v}{2}\rho(1)\right) = r + \frac{v}{2}\left(\rho(1) + \rho(1)^{-1}\right). \tag{1.18}$$

This implies that $\sum_{y\in\mathbf{N}} Q_{xy}\rho(y) = \lambda_+\rho(x)$ for all $x \in \mathbf{N}$ and hence that $R$ is a transition probability for an $\mathbf{N}$-valued Markov chain. We denote this Markov chain by $(\{S_n\}_{n\in\mathbf{N}}, \{\nu^x\}_{x\in\mathbf{N}})$. It is easy to check that the Markov chain $\{\nu^x\}_{x\in\mathbf{N}}$ is reversible with respect to the probability measure $m$ on $\mathbf{N}$ defined by

$$m[f] = \frac{\langle\, f, \rho^2 \,\rangle_\beta}{\langle\, 1, \rho^2 \,\rangle_\beta} \quad f : \mathbf{N} \to [0, \infty). \tag{1.19}$$

In particular, $\{\nu^x\}_{x\in\mathbf{N}}$ is positively recurrent.

**Theorem 1.3** *For $\beta > \beta_c$, there are constants $C_{1.20} = C_{1.20}(\beta, v) > 0$ and $\varepsilon_{1.20} = \varepsilon_{1.20}(\beta, v) \in (0, 1)$ as follows; if $n \geq k + (1+x)C_{1.20}$, $f : \mathbf{N}^k \to \mathbf{R}$ is polynomially bounded and $F = f(S_1, \ldots, S_k)$, then*

$$|\mu_n^x[F] - \nu^x[F]| \leq B_{1.20}(f)\varepsilon_{1.20}^{n-k}, \tag{1.20}$$

*where $B_{1.20}(f)$ is a constant which depends only on $\beta$, $v$ and $f$.*

By Theorem 1.3, we get the following strong mixing properties for both $\mu_n^x$ and $\nu^x$ for $\beta > \beta_c$;

**Theorem 1.4** *For $\beta > \beta_c$, there are constants $C_{1.21} = C_{1.21}(\beta, v) > 0$ and $\varepsilon_{1.21} = \varepsilon_{1.21}(\beta, v) \in (0, 1)$ as follows; if $n \geq l + m$, $l \geq k + (1+x)C_{1.21}$, $f : \mathbf{N}^k \to \mathbf{R}$ and $g : \mathbf{N}^m \to \mathbf{R}$ are polynomially bounded, $F = f(S_1, \ldots, S_k)$ and $G = g(S_{l+1}, \ldots, S_{l+m})$ then*

$$|\mu_n^x(F\,G) - \mu_n^x(F)\mu_n^x(G)| \;\leq\; B_{1.21}(f, g)\varepsilon_{1.21}^{l-k}, \tag{1.21}$$

$$|\nu^x(F\,G) - \nu^x(F)\nu^x(G)| \;\leq\; B_{1.21}(f, g)\varepsilon_{1.21}^{l-k}, \tag{1.22}$$

*where $B_{1.21}(f, g)$ is a constant which depends only on $\beta$, $v$, $f$ and $g$.*

**Remark 1.2** In [BDZ00, Section4], it is mentioned that a similar statement to (1.21) is true for the one-dimensional weakly pinned Gaussian field in the localized phase. Localization of two-dimensional weakly pinned surfaces are discussed in [DeVe00, DMRR92, IoVe00], however without the attractive wall.

## 2 Proof of Theorem 1.1 (a)

### 2.1 Preliminaries for the proof

For $x, n \in \mathbf{N}$, we introduce the following random variables;

$$h_n = 1\{S_1 \geq 0, \ldots, S_n \geq 0\}, \quad n \geq 1, \tag{2.1}$$

$$e_n = \exp\left(\beta L_n^0\right), \quad \beta \in \mathbf{R}, n \geq 1, \tag{2.2}$$

$$z_n = e_n h_n, n \geq 1, \tag{2.3}$$

$$z_0 = e_0 = h_0 = 1, \tag{2.4}$$

where $1\{\ldots\}$ denotes the indicator function. Then, the measure $\mu_n^x$ can be rewritten as the following concise form;

$$\mu_n^x(d\omega) = \frac{1}{Z_n^x} P^x[z_n : d\omega], \quad x, n \in \mathbf{N}, \tag{2.5}$$

We need to introduce some more notations. For a function $\varphi : \mathbf{N} \to \mathbf{R}$, a formal power series

$$\widehat{\varphi}(s) = \sum_{n \in \mathbf{N}} s^n \varphi(n), \quad -1 \le s \le 1, \tag{2.6}$$

is called the *generating function* of $\varphi$. If $\varphi$ is non-negative and the convergence radius is one, then the following important relation between $\varphi$ and $\widehat{\varphi}$ is known as a Tauberian theorem [Fel70, page 447]; for $\alpha \in [0, \infty)$ and $C > 0$,

$$\widehat{\varphi}(s) \overset{s \nearrow 1}{\sim} C(1-s)^{-\alpha} \quad \text{if and only if} \quad \varphi(0) + \cdots + \varphi(n-1) \overset{n \nearrow \infty}{\sim} \frac{Cn^\alpha}{\Gamma(\alpha+1)}. \tag{2.7}$$

Moreover, if $\alpha \neq 0$ and $\varphi$ is monotone, then (2.7) is equivalent to

$$\varphi(n) \overset{n \nearrow \infty}{\sim} \frac{Cn^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.8}$$

We set

$$f^x(n) = \begin{cases} 0 & \text{if } n = 0, \\ P^x\{T_1 = n\}, & \text{if } n \ge 1, \end{cases} \tag{2.9}$$

where $T_m$ denotes the $m^{\text{th}}$-hitting time to the origin;

$$T_m = \inf\left\{ n \ge 1 \mid \sum_{k=1}^{n} 1\{S_k = 0\} = m \right\}. \tag{2.10}$$

It is then well known that the generating function is given by

$$\widehat{f^x}(s) = \begin{cases} \left(\frac{1-rs-\gamma(s)^{1/2}}{(1-r)s}\right)^{|x|} & \text{if } x \neq 0,\ |s| < 1, \\ 1 - \gamma(s)^{1/2} & \text{if } x = 0,\ |s| < 1, \end{cases} \tag{2.11}$$

where $\gamma(s) = (1-rs)^2 - s^2 v^2$.

### 2.2 Lemmas

We define

$$\theta_\pm(s) = 1 - \frac{e^\beta}{2}\left(rs + 1 \pm \sqrt{\gamma(s)}\right) \quad \text{for } s \in (-\infty, 1]. \tag{2.12}$$

It is then easy to see that

$$\{s \in (-\infty, 1]\ ;\ \theta_-(s) = 0\} = \begin{cases} \emptyset & \text{if } \beta < \beta_c, \\ \{1\} & \text{if } \beta = \beta_c, \\ \{1/\lambda_-, 1/\lambda_+\} & \text{if } \beta > \beta_c, \end{cases} \tag{2.13}$$

and that $\theta_-(s)$ is strictly decreasing on $[0, 1]$.

**Lemma 2.1** *For $x, y, n \in \mathbf{N}$, define*

$$\begin{aligned} z_n^y &= z_n 1\{S_n = y\}, && (2.14)\\ Z_n^{x,y} &= P^x[z_n^y]. && (2.15) \end{aligned}$$

*Then, for $0 \le s < 1$,*

$$\widehat{Z}^{x,0}(s) = \frac{1}{\theta_-(s) \vee 0}\left(\delta^{x,0} + (1-\delta^{x,0})\widehat{f^x}(s)\right). \tag{2.16}$$

*In particular,*

$$\widehat{Z}^{x,0}(1) = 1/\theta_-(1) \quad \text{if } \beta < \beta_c, \tag{2.17}$$

*and*

$$\widehat{Z}^{x,0}(s) \overset{s\nearrow 1}{\sim} C_{2.18}/(1-s)^{1/2}, \quad \text{if } \beta = \beta_c. \tag{2.18}$$

*where $C_{2.18} = \frac{1+r}{\sqrt{2v}}$.*

Proof: The equality (2.16) is essentially the same as the representation of Green function of the random walk in terms of its hitting times. We follow [Bol99]. Note first that

$$\widehat{Z}^{x,0}(s) = \delta^{x,0} + \sum_{n \ge 1} s^n P^x[z_n^0]. \tag{2.19}$$

To compute the right-hand-side, let us introduce

$$g^x(n) = \begin{cases} 0 & \text{if } n = 0,\\ P^x\{h_n : T_1 = n\} & \text{if } n \ge 1. \end{cases} \tag{2.20}$$

It is then, easy to see that

$$g^x(n) = \begin{cases} f^x(n) & \text{if } x \ge 1,\\ f^x(n) = r & \text{if } x = 0 \text{ and n=1},\\ f^x(n)/2 & \text{if } x = 0 \text{ and } n \ge 2. \end{cases} \tag{2.21}$$

and consequently that for $0 \le s \le 1$,

$$\widehat{g^x}(s) = \begin{cases} (rs + \widehat{f^0}(s))/2 & \text{if } x = 0,\\ \widehat{f^x}(s) & \text{if } x \ge 1. \end{cases} \tag{2.22}$$

We compute the right-hand-side of (2.19), first for the case $x \ge 1$;

$$\begin{aligned} &\sum_{n\ge 1} s^n P^x[z_n^0] && (2.23)\\ &= \sum_{n\ge 1} s^n \sum_{m \ge 1} e^{(m-1)\beta} \sum_{0<i_1<\ldots<i_m=n} P^x[h_n : T_1 = i_1, \ldots, T_m = i_m]\\ &= \sum_{n\ge 1} s^n \sum_{m \ge 1} e^{(m-1)\beta} \sum_{\substack{j_1 \ge 1, \ldots, j_m \ge 1\\ j_1 + \ldots + j_m = n}} g^x(j_1) g^0(j_2) \cdots g^0(j_m) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\geq 1}\sum_{j_1\geq 1}\cdots\sum_{j_m\geq 1} s^{j_1} g^x(j_1)\left(e^\beta s^{j_2} g^0(j_2)\right)\cdots\left(e^\beta s^{j_m} g^0(j_m)\right)\\
&= \widehat{g^x}(s)\sum_{m\geq 1}\left(e^\beta \widehat{g^0}(s)\right)^{m-1}\\
&= \widehat{g^x}(s)\sum_{m\geq 1}\left(1-\theta_-(s)\right)^{m-1}, \qquad (2.24)
\end{aligned}
$$

which, implies (2.16) for the case $x \geq 1$. Equality (2.16) for the case $x = 0$ can be seen similarly as above just by replacing $e^{(m-1)\beta}$ and $g^x(j_1)$ respectively by $e^{m\beta}$ and $g^0(j_1)$. $\square$

**Lemma 2.2** *For $0 \leq s < 1$,*

$$
\widehat{Z}^x(s) = \left(1-\delta^{x,0}\right)\frac{1-\widehat{f^x}(s)}{1-s} + \left(\tfrac{1}{2}e^\beta \frac{1-\widehat{f^0}(s)}{1-s} - \tfrac{1}{2}e^\beta + 1\right)\widehat{Z}^{x,0}(s). \qquad (2.25)
$$

*In particular,*

$$
\widehat{Z}^x(s) \overset{s\nearrow 1}{\sim} C_{2.26}/(1-s)^{1/2} \quad \text{if } \beta < \beta_c, \qquad (2.26)
$$

*where* $C_{2.26} = \sqrt{\frac{2}{v}}\left(x + \frac{ve^\beta}{2\theta_-(1)}\right)$.

Proof: We begin by showing that

$$
Z_n^x = \tfrac{1}{2}e^\beta \sum_{m=0}^{n-1} Z_m^{x,0} F_{n-m}^0 + Z_n^{x,0} + (1-\delta^{x,0})F_n^x, \qquad (2.27)
$$

where $F_n^x = P^x\{T_1 > n\}$. To prove this, we set

$$
\begin{aligned}
k_{m,n} &= 1\{S_m > 0, \dots, S_n > 0\}, \quad 1 \leq m \leq n,\\
d_n &= \max D_n \quad \text{if } D_n \overset{\text{def.}}{=} \{0 \leq j \leq n\ ;\ S_j = 0\} \neq \emptyset.
\end{aligned}
$$

We have

$$
P^x[k_{1,n}] = \begin{cases} F_n^x/2 & \text{if } x = 0,\\ F_n^x & \text{if } x \geq 1.\end{cases} \qquad (2.28)
$$

Note also that $D_n = \emptyset$ if and only if $x \geq 1$ and $T_1 > n$. Since we have either $d_n \in \{0, \dots, n\}$ or $D_n = \emptyset$, we see that $z_n$ can be decomposed as follows;

$$
\begin{aligned}
z_n &= \sum_{m=0}^{n-1} z_n 1\{d_n = m\} + z_n 1\{d_n = n\} + (1-\delta^{0,x}) z_n 1\{T_1 > n\}\\
&= \sum_{m=0}^{n-1} e^\beta z_m^0 k_{m+1,n} + z_n^0 + (1-\delta^{0,x})k_{1,n}.
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
Z_n^x &= e^\beta \sum_{m=0}^{n-1} P^x[z_m^0 k_{m+1,n}] + Z_n^{x,0} + (1-\delta^{x,0})P^x[k_{1,n}]\\
&= \tfrac{1}{2}e^\beta \sum_{m=0}^{n-1} Z_m^{x,0} F_{n-m}^0 + Z_n^{x,0} + (1-\delta^{x,0})F_n^x,
\end{aligned}
$$

where we have used (2.28) in the last line. This proves (2.27).

It is easy to see that

$$\widehat{F^x}(s) = \frac{1 - \widehat{f^x}(s)}{1 - s}. \tag{2.29}$$

Therefore, (2.25) is an easy consequence of (2.16), (2.29) and (2.27). We see (2.26) from (2.25) and the following asymptotic property for $\widehat{f^x}(s)$;

$$1 - \widehat{f^x}(s) \sim \begin{cases} \sqrt{2v(1-s)} & \text{if } x = 0, \\ x\sqrt{2(1-s)/v} & \text{if } x \neq 0, \end{cases} \qquad \text{as } s \nearrow 1. \tag{2.30}$$

□

To state and prove the next lemma, we will use the following notation. For a sequence $(a_n)_{n\geq 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \geq 1$.

**Lemma 2.3** *Let $\Phi_0$ be a constant and $\Phi_n = \varphi_n(S_1, \dots, S_n)$ for $n \geq 1$, where each $\varphi_n$ is a function on $\mathbf{N}^n$. Then,*

$$\begin{aligned} P^x[\Phi_n z_n] &= \Phi_0 + \sum_{k=1}^n P^x[z_{k-1}\Delta\Phi_k] + \sum_{k=1}^n P^x[z_{k-1}^0(e^\beta 1\{\Delta S_k \geq 0\} - 1)\Delta\Phi_k] \\ &\quad -\theta_-(1)\sum_{k=1}^n P^x[z_{k-1}^0\Phi_{k-1}]. \end{aligned} \tag{2.31}$$

*In particular,*

$$Z_n^x = 1 - \theta_-(1)\sum_{k=0}^{n-1} Z_k^{x,0}, \tag{2.32}$$

$$\begin{aligned} P^x[S_n^p z_n] &= x^p + v\sum_{1\leq j\leq p/2}\frac{p!}{(2j)!(p-2j)!}\sum_{k=0}^{n-1} P^x[S_k^{p-2j} z_k] \\ &\quad +\frac{v}{2}\{e^\beta - 1 - (-1)^p\}\sum_{k=0}^{n-1} Z_k^{x,0}. \end{aligned} \tag{2.33}$$

Proof: We observe that

$$\Delta z_n = \left(e^\beta 1\{\Delta S_n \geq 0\} - 1\right) z_{n-1}^0.$$

We therefore have

$$\begin{aligned} \Delta(z_n\Phi_n) &= z_n\Delta\Phi_n + \Phi_{n-1}\Delta z_n \\ &= (z_{n-1} + \Delta z_n)\Delta\Phi_n + \Phi_{n-1}\Delta z_n \\ &= z_{n-1}\Delta\Phi_n + \left(e^\beta 1\{\Delta S_n \geq 0\} - 1\right) z_{n-1}^0\Delta\Phi_n + \left(e^\beta 1\{\Delta S_n \geq 0\} - 1\right) z_{n-1}^0\Phi_{n-1} \end{aligned}$$

This implies (2.31), since $P^x[\Phi_n z_n] = \Phi_0 + \sum_{k=1}^n P^x[\Delta(z_k\Phi_k)]$ and $P^x[e^\beta 1\{\Delta S_n \geq 0\} - 1] = -\theta_-(1)$. The other two equalities are obtained easily from (2.31). □

## 2.3 Proof of Theorem 1.1(a)

It is enough to prove that for all $x, p \in \mathbf{N}$,

$$\lim_{n\nearrow\infty} \mu_n^x \left[(S_n/\sqrt{vn})^p\right] = \int_0^\infty x^p \mu(dx) = \begin{cases} 2^{p/2}\Gamma(\frac{p}{2}+1) & \text{if } \beta < \beta_c, \\ \sqrt{2^p/\pi}\,\Gamma(\frac{p+1}{2}) & \text{if } \beta = \beta_c. \end{cases} \tag{2.34}$$

We begin with the case $\beta < \beta_c$. Since $Z_n^x$ is decreasing in $n$ by (2.32), we see from (2.26) and (2.8) that

$$Z_n^x \sim C_{2.26}/(\pi n)^{1/2} \quad \text{as } n \nearrow \infty. \tag{2.35}$$

Therefore, if we set $M_n(p) = P^x[S_n^p z_n]$, then, (2.34) for $\beta < \beta_c$ is equivalent to that;

$$\lim_{n\nearrow\infty} n^{-\frac{p-1}{2}} M_n(p) = (C_{2.26}/\sqrt{\pi})(2v)^{p/2}\Gamma(\tfrac{p}{2}+1). \tag{2.36}$$

We prove (2.36) by induction. For $p = 0$, (2.36) can immediately be seen from (2.35). For $p = 1$, (2.36) can be seen from (2.17), (2.33) and (2.35). Let us consider the case $p \geq 2$. We may now assume (2.36) with $p$ replaced by $p-2j$ $(1 \leq j \leq p/2)$. We denote the right-hand-side of (2.36) by $m(p)$. It is then not difficult to see that

$$\begin{aligned} & n^{-\frac{p-1}{2}} \sum_{k=0}^{n-1} M_k(p-2j) \\ = \;& n^{-\frac{p-1}{2}} \sum_{k=0}^{n-1} k^{-\frac{p-2j-1}{2}} M_k(p-2j) k^{\frac{p-2j-1}{2}} \overset{n\nearrow\infty}{\longrightarrow} \begin{cases} \frac{m(p-2)}{(p-1)/2}, & \text{if } j = 1, \\ 0 & \text{if otherwise.} \end{cases} \end{aligned} \tag{2.37}$$

We have

$$M_n(p) = v \sum_{1\leq j\leq p/2} \frac{p!}{(2j)!(p-2j)!} \sum_{k=0}^{n-1} M_k(p-2j) + \mathcal{O}(1) \quad \text{as } n \nearrow \infty$$

by (2.17) and (2.33). We see from this and (2.37) that

$$\begin{aligned} \lim_{n\nearrow\infty} n^{-\frac{p-1}{2}} M_n(p) &= v\frac{p!}{2!(p-2)!}\frac{m(p-2)}{(p-1)/2} \\ &= vpm(p-2) = m(p), \end{aligned}$$

which completes the proof of (2.36).

Proof of (2.34) for $\beta = \beta_c$ is similar as above. What we have to show in this case is that;

$$\lim_{n\nearrow\infty} n^{-\frac{p}{2}} M_n(p) = \sqrt{(2v)^p/\pi}\,\Gamma(\tfrac{p+1}{2}). \tag{2.38}$$

We prove (2.38) by induction. For $p = 0$, (2.38) can immediately be seen from (2.32). For $p = 1$, we see from (2.18) and (2.7) that

$$\sum_{k=0}^{n-1} Z_k^{x,0} \sim 2C_{2.18} n^{1/2}/\pi \quad \text{as } n \nearrow \infty. \tag{2.39}$$

This and (2.33) imply (2.38) for $p = 1$. Let us consider the case $p \geq 2$. We may now assume (2.38) with $p$ replaced by $p - 2j$ $(1 \leq j \leq p/2)$. We denote the right-hand-side of (2.38) by $m(p)$. It is then not difficult to see that

$$\begin{aligned} & n^{-\frac{p}{2}} \sum_{k=0}^{n-1} M_k(p-2j) \\ = \ & n^{-\frac{p}{2}} \sum_{k=0}^{n-1} k^{-\frac{p-2j}{2}} M_k(p-2j) k^{\frac{p-2j}{2}} \stackrel{n \nearrow \infty}{\longrightarrow} \begin{cases} \frac{m(p-2)}{p/2} & \text{if } j = 1, \\ 0 & \text{if otherwise.} \end{cases} \end{aligned} \tag{2.40}$$

We have

$$M_n(p) = v \sum_{1 \leq j \leq p/2} \frac{p!}{(2j)!(p-2j)!} \sum_{k=0}^{n-1} M_k(p-2j) + \mathcal{O}(n^{1/2}) \quad \text{as } n \nearrow \infty$$

by (2.33) and (2.39). We see from this and (2.40) that

$$\begin{aligned} \lim_{n \nearrow \infty} n^{-\frac{p}{2}} M_n(p) &= v \frac{p!}{2!(p-2)!} \frac{m(p-2)}{p/2} \\ &= v(p-1)m(p-2) = m(p), \end{aligned}$$

which completes the proof of (2.38). □

## 3 Proof of Theorem 1.2

We consider the case $x = 0$ for simplicity. The modifications if otherwise are straightforward.

### 3.1 The critical case

In this case, the paths of $\{S_n\}$ can be embedded in the paths of $W(t) \equiv |B(t)|$ without changing the law under $\mu_n$. Note first that for all $k < n$,

$$\text{If } S_{k-1} > 0, \qquad \mu_n\{\Delta S_k = 0 | S_{k-1}\} = r; \mu_n\{\Delta S_k = \pm 1 | S_{k-1}\} = v/2. \tag{3.1}$$

$$\text{If } S_{k-1} = 0, \qquad \mu_n\{\Delta S_k = 0 | S_{k-1}\} = \frac{2r}{1+r}; \mu_n\{\Delta S_k = 1 | S_{k-1}\} = \frac{v}{1+r}. \tag{3.2}$$

By modifying the Skhrohod embedding, we can construct a series of stopping times $\{\sigma_k\}_{k \geq 0}$ so that $\{W(\sigma_k)\}_{k=1}^n$ is a realization of $(\{S_k\}_{k=1}^n, \mu_n^0)$. In fact, for constants $K_0$ and $K_1$ we define $\{\sigma_k\}_{k \geq 0}$ as follows; $\sigma_0 := 0$ and if $W(\sigma_{k-1}) \in \mathbf{N} \setminus \{0\}$ then

$$\sigma_k := \inf\{t \geq \sigma_{k-1}; |W(t) - W(\sigma_{k-1})| = 1 \text{ or } (t > K_1 + \sigma_{k-1} \text{ and } W(t) = W(\sigma_{k-1}))\}$$

if $W(\sigma_{k-1}) = 0$ then

$$\sigma_k := \inf\{t \geq \sigma_{k-1}; W(t) = 1 \text{ or } (t > K_0 + \sigma_{k-1} \text{ and } W(t) = 0)\}.$$

We can then, choose $K_0$ and $K_1$ (depending upon $\beta$ and $v$) so that $\widetilde{S}_k \stackrel{\text{def.}}{=} W(\sigma_k)$ satisfies

$$\text{If } \widetilde{S}_{k-1} > 0, \qquad P^W\{\Delta\widetilde{S}_k = 0|\widetilde{S}_{k-1}\} = r; P^W\{\Delta\widetilde{S}_k = \pm 1|\widetilde{S}_{k-1}\} = v/2. \tag{3.3}$$

$$\text{If } \widetilde{S}_{k-1} = 0, \qquad P^W\{\Delta\widetilde{S}_k = 0|\widetilde{S}_{k-1}\} = \frac{2r}{1+r}; P^W\{\Delta\widetilde{S}_k = 1|\widetilde{S}_{k-1}\} = \frac{v}{1+r}. \tag{3.4}$$

By discarding $k$'s such that $W(\sigma_{k-1}) = 0$, we obtain a series of i.i.d. $\{\tau_k; k \geq 0\} := \{\sigma_k - \sigma_{k-1}; k \geq 0, W(\sigma_{k-1}) \neq 0\}$. It is elementary to prove $E^W[\tau_k] = v$. By well-known Kolmogorov's inequality, we have

$$P^W[\exists k \leq n, \ |\sum_{j=1}^{k}(\tau_j - v)| > \delta n] \leq Var(\tau_1)/\delta^2 n. \tag{3.5}$$

Since the number of downcrossings from $1/\sqrt{n}$ to 0 approximate the local time, it holds, after rescaled by $\sqrt{n}$, for any $\varepsilon > 0$, there is $K_2 > 0$ such that for all large $n$,

$$P^W[\#\{k \leq n; W(\sigma_{k-1}) = 0\} > K_2\sqrt{n}/v] < \varepsilon. \tag{3.6}$$

The following is its natural consequence: there is $K_3 > 0$ such that for all large $n$,

$$P^W[\sum_{k\leq n; W(\sigma_{k-1})=0} (\sigma_k - \sigma_{k-1}) > K_3\sqrt{n}] < \varepsilon. \tag{3.7}$$

Choose $n$ so large that $1/n < \delta$ and $(K_2 \vee K_3)\sqrt{n} < \delta n$, where $\delta$ if beforehand chosen so that

$$P^W[\sup_{0<t,s<1; |s-t|<\delta} |W(vns) - W(vnt)| > \varepsilon\sqrt{n}] < \varepsilon. \tag{3.8}$$

Now it remains to put these together:

$$\begin{aligned}
\sup_{0<t<1}\left|\frac{W(vnt)}{\sqrt{n}} - \frac{\tilde{S}_{nt}}{\sqrt{n}}\right| &< \varepsilon + \sup_{0\leq k\leq n}\left|\frac{W(vk)}{\sqrt{n}} - \frac{S_k}{\sqrt{n}}\right| \\
&\equiv \varepsilon + \sup_{0\leq k\leq n}\left|\frac{W(vk)}{\sqrt{n}} - \frac{W(\sigma_k)}{\sqrt{n}}\right| \\
&< 2\varepsilon + \sup_{0\leq k\leq n}\left|\frac{W(vk)}{\sqrt{n}} - \frac{W(\sum_{j\leq k; W(\sigma_{j-1})\neq 0}(\sigma_j - \sigma_{j-1}))}{\sqrt{n}}\right| \\
&< 3\varepsilon + \sup_{0\leq k\leq n}\left|\frac{W(vk)}{\sqrt{n}} - \frac{W(\sum_{j\leq k; W(\sigma_{j-1})\neq 0} v)}{\sqrt{n}}\right| \\
&< 4\varepsilon
\end{aligned}$$

on the intersection of the events that appeared in (3.5), (3.6), (3.7) and (3.8).

We have thus proven $\widetilde{\mu}_n$ converges weakly to the law of $\{W(vnt)/\sqrt{n}; 0 \leq t \leq 1\}$ which in turn is identical to that of $\{\sqrt{v}W(t); 0 \leq t \leq 1\}$. □

### 3.2 Convergence of finite dimensional distributions for $\beta < \beta_c$

The purpose of this subsection is to prove the following

**Lemma 3.1** *Let $\beta < \beta_c$, $k \geq 1$ and $0 < t_1 < \ldots, < t_k \leq 1$. Suppose that $\{t_{j,n} > 0\ ;\ 1 \leq j \leq k,\ n \geq 1\}$ are such that $0 < t_{1,n} < \ldots, < t_{k,n}$ for all $n \geq 1$ and that $\sup_{n\geq 1} |t_{j,n} - nt_j| < \infty$ for all $1 \leq j \leq k$. Then, probability measures on $\mathbf{R}^k$ defined by*

$$\mu_n^0 \left\{ \left(S_{t_{j,n}}/\sqrt{vn}\right)_{j=1}^k \in \cdot \right\}, \quad ; n = 1, 2, \ldots \tag{3.9}$$

*converge weakly to the law of $\left(W_{t_j}\right)_{j=1}^k$.*

We introduce the following functions;

$$\begin{aligned} g_t(x) &= (2\pi t)^{-1} \exp\left(-\frac{x^2}{2t}\right), \quad t > 0,\ x \in \mathbf{R}, & (3.10)\\ h(x) &= 2\int_0^x g_1(y)dy, \quad x \geq 0, & (3.11)\\ g_t(x, y) &= g_t(x - y) - g_t(x + y), \quad t > 0,\ x, y \in \mathbf{R}, & (3.12)\\ h(t, x) &= h\left(\frac{x}{\sqrt{1-t}}\right), \quad (t, x) \in [0, 1] \times [0, \infty)\backslash\{(1, 0)\} & (3.13) \end{aligned}$$

According to Belkin [Bel72], the process $W$ for $\beta < \beta_c$ is a (time-inhomogeneous) diffusion process on $[0, \infty)$ with the following transition probability; for $0 \leq s < t \leq 1$ and $x, y \geq 0$

$$p(s, x|t, y) = \begin{cases} t^{-3/2} y \exp\left(-\frac{y^2}{2t}\right) h(t, y) & \text{if } s = 0 \text{ and } x = 0 \\ h(s, x)^{-1} g_{t-s}(x, y) h(t, y) & \text{if } s > 0 \text{ and } x > 0, \\ 0 & \text{if otherwise} \end{cases} \tag{3.14}$$

We set

$$S_n^{\min} = \min_{0\leq j\leq n} S_j. \tag{3.15}$$

**Lemma 3.2** *Let $\beta < \beta_c$. Then, for $0 \leq s < t < 1$, $x, y \geq 0$,*

$$\begin{aligned} \lim_{n\nearrow\infty} P^{y\sqrt{vn}} \left[ z_{n-\lfloor ns\rfloor} : \frac{S_{\lfloor nt\rfloor - \lfloor ns\rfloor}}{\sqrt{vn}} \leq x \right] &= P\{\min_{0\leq u\leq 1-s} B_u \geq -y,\ B_{t-s} \geq x - y\} & (3.16)\\ &= h(s, y) \int_0^x p(s, y|t, z). \end{aligned}$$

*The convergence is uniform in $(x, y) \in [0, \infty) \times [0, b]$ for any $b > 0$. In particular,*

$$\lim_{n\nearrow\infty} Z_{n-\lfloor ns\rfloor}^{y\sqrt{vn}} = h(s, y). \tag{3.17}$$

*uniformly in $y \in [0, b]$ for any $b > 0$.*

Proof: We write $\langle u \rangle = \lfloor u \rfloor - \lfloor ns \rfloor$ $(u \geq 0)$ in this proof. We define

$$\tau = \tau(y,n) = \inf\{j \geq 1\ ;\ S_j \leq S_0 - y\sqrt{vn}\}$$

and divide the expectation on the left-hand-side of (3.16) as follows;

$$\begin{aligned}
& P^{y\sqrt{vn}}\left[z_{\langle n \rangle} : S_{\langle nt \rangle} \leq x\sqrt{vn}\right] \\
&= P^0\left[\exp\left(\beta L_{\langle n \rangle}^{-y\sqrt{vn}}\right) : \frac{S^{\min}_{\langle n \rangle}}{\sqrt{vn}} \geq -y,\ \frac{S_{\langle nt \rangle}}{\sqrt{vn}} \leq x - y\right] \\
&= P^0\left[\frac{S^{\min}_{\langle n \rangle}}{\sqrt{vn}} > -y,\ \frac{S_{\langle nt \rangle}}{\sqrt{vn}} \leq x - y\right] \qquad (3.18) \\
&\quad + \sum_{k=0}^{\langle n \rangle - 1} P^0\left[\exp\left(\beta L_{\langle n \rangle}^{-y\sqrt{vn}}\right) : \frac{S^{\min}_{\langle n \rangle}}{\sqrt{vn}} \geq -y,\ \frac{S_{\langle nt \rangle}}{\sqrt{vn}} \leq x - y,\ \tau = k\right] \qquad (3.19)
\end{aligned}$$

The first term (3.18) converges to the right-hand-side of (3.16) by the invariance principle of Donsker. Moreover, the convergence is uniform in $(x,y) \in [0,\infty)^2$ by a result of Billingsley and Tops$\phi$e [BiTo67, Theorem 2]. On the other hand, the second term (3.19) is bounded from above by

$$\sum_{k=0}^{\langle n \rangle - 1} P^0\{\tau = k\} Z^0_{\langle n \rangle - k}. \qquad (3.20)$$

By using (2.35) and the fact that $P^0\{\tau = k\}$ is not greater than a constant times $y\sqrt{n}k^{-3/2}$, it is not difficult to bound (3.20) by a constant times $yn^{-1/3}$. This completes the proof. □

Proof of Lemma 3.1: Note first that it is enough to prove (3.9) for a special case: $t_{j,n} = \lfloor nt_j \rfloor$ $(1 \leq j \leq k)$. To see this, let $\{t_{j,n}\}$ be an arbitrary sequence which satisfies the assumptions in Lemma 3.1. Since $S_{t_{j,n}}$ and $S_{\lfloor nt_j \rfloor}$ differ at most by $|t_{j,n} - \lfloor nt_j \rfloor|$, the $L^1(\mu_n^0)$-difference of two vectors $\left(S_{t_{j,n}}/\sqrt{vn}\right)_{j=1}^k$ and $\left(S_{\lfloor nt_j \rfloor}/\sqrt{vn}\right)_{j=1}^k$ vanishes as $n \nearrow \infty$. Therefore, their limits in law coincide.

Following Iglehart [Igl74, proof of (2.23)], we prove (3.9) by induction on $k$. We will prove that for all $x_1, \ldots, x_k > 0$,

$$\lim_{n \nearrow \infty} \mu_n^0(A_k) = \int_0^{x_1} dy_1 \ldots \int_0^{x_k} dy_k p(0,0|t_1,y_1)p(t_1,y_1|t_2,y_2)\cdots p(t_{k-1},y_{k-1}|t_k,y_k), \qquad (3.21)$$

where $A_k = \left\{\frac{S_{\lfloor nt_1 \rfloor}}{\sqrt{vn}} \leq x_1, \ldots, \frac{S_{\lfloor nt_k \rfloor}}{\sqrt{vn}} \leq x_k\right\}$. We set $(s,t,x) = (t_{k-1}, t_k, x_k)$ to simplify the expressions. We start with the case $k = 1$. If $t = 1$, then, (3.21) follows from (1.11). Let us therefore assume $t < 1$. We then have

$$\mu_n^0\left\{\frac{S_{\lfloor nt \rfloor}}{\sqrt{vn}} \leq x\right\} = \frac{Z^0_{\lfloor nt \rfloor}}{Z^0_n} \int_{y \leq x} \mu^0_{\lfloor nt \rfloor}\left\{\frac{S_{\lfloor nt \rfloor}}{\sqrt{vn}} \in dy\right\} Z^{y\sqrt{vn}}_{n - \lfloor nt \rfloor} \qquad (3.22)$$

We see from Lemma 3.2 that $Z^{y\sqrt{vn}}_{n-\lfloor nt \rfloor}$ converges to $h(t,y)$ locally uniformly in $y$. We therefore have by (1.11) and (2.35) that

$$\lim_{n \nearrow \infty} \mu_n^0\left\{\frac{S_{\lfloor nt \rfloor}}{\sqrt{vn}} \leq x\right\} = t^{-1/2} \int_0^x \mu(t^{-1/2}dy) h(t,y)\, dy$$

$$= \int_0^x p(0,0|t,y)dy$$

which proves (3.21) for $k=1$. We now consider the case $k \geq 2$. Since

$$\frac{S_{\lfloor nt_j \rfloor}}{\sqrt{vn}} = \sqrt{\frac{m}{n}} \frac{S_{m \cdot \lfloor nt_j \rfloor / m}}{\sqrt{vm}}, \quad m = \lfloor ns \rfloor,$$

we have by the induction hypothesis and the observation at the beginning of the proof that

$$\lim_{n \nearrow \infty} \mu^0_{\lfloor ns \rfloor} \left\{ \left( S_{\lfloor nt_j \rfloor} / \sqrt{vn} \right)_{j=1}^{k-1} \in \cdot \right\} = \text{the law of } \left( \sqrt{s} W_{t_j/s} \right)_{j=1}^{k-1}.$$

Note that the above limit has the density

$$p_s\left(0,0|t_1,y_1\right) p_s\left(t_1,y_1|t_2,y_2\right) \cdots p_s\left(t_{k-2},y_{k-2}|s,y_{k-1}\right), \tag{3.23}$$

where

$$p_s\left(t,x|t',y\right) = s^{-1/2} p\left( t/s, x/\sqrt{s} | t'/s, y/\sqrt{s} \right).$$

If we set $h_s(t,x) = h(t/s, x/\sqrt{s})$, then we have obviously that

$$p_s\left(t,x|t',y\right) = \begin{cases} s^{1/2} p\left(0,0|t'/s, y/\sqrt{s}\right) \frac{h_s(t',y)}{h(t',y)}, & \text{if } t=0 \text{ and } x=0 \\ \frac{h_s(t,x)}{h(t,x)} p\left(t/s, x/\sqrt{s}|t'/s, y/\sqrt{s}\right) \frac{h_s(t',y)}{h(t',y)}, & \text{if } t>0 \text{ and } x>0. \end{cases} \tag{3.24}$$

We rewrite the probability on the left-hand-side of (3.21) as follows;

$$\mu^0_n(A_k) = \frac{Z^0_{\lfloor ns \rfloor}}{Z^0_n} \int_0^{x_{k-1}} \mu^0_{\lfloor ns \rfloor} \left\{ A_{k-2}, \frac{S_{\lfloor ns \rfloor}}{\sqrt{vn}} \in dy \right\} P^{y\sqrt{vn}} \left[ z_{n-\lfloor ns \rfloor} : \frac{S_{\lfloor nt \rfloor - \lfloor ns \rfloor}}{\sqrt{vn}} \leq x \right] \tag{3.25}$$

By (2.35), (3.25), (3.16) and (3.24) we conclude that

$$\begin{aligned} &\lim_{n \nearrow \infty} \mu^0_n(A_k) \\ &= s^{-1/2} \int_0^{x_1} dy_1 \ldots \int_0^{x_{k-2}} dy_{k-2} \int_0^{x_{k-1}} dy p_s\left(0,0|t_1,y_1\right) p_s\left(t_1,y_1|t_2,y_2\right) \cdots p_s\left(t_{k-2},y_{k-2}|s,y\right) \\ &\quad \times h(s,y) \int_0^x p(s,y|t,z)dz \\ &= \int_0^{x_1} dy_1 \ldots \int_0^{x_k} dy_k p(0,0|t_1,y_1) p(t_1,y_1|t_2,y_2) \cdots p(t_{k-1},y_{k-1}|t_k,y_k). \end{aligned} \tag{3.26}$$

This completes the proof of Lemma 3.1. □

### 3.3 Tightness for $\beta < \beta_c$

We continue to consider the distribution of $(S_{nt}/\sqrt{vn})_{0 \leq t \leq 1}$ under $\mu^0_n$ with $\beta < \beta_c$. We have shown in Lemma 3.1 that any finite dimensional distribution of converges. The purpose of this subsection is to prove the tightness of the distribution, which, together with Lemma 3.1, implies Theorem 1.2 for $\beta < \beta_c$. By the well-known Kolmogorov's tightness criterion, it is enough to prove that for $\varepsilon > 0$,

$$\lim_{\delta \searrow 0} \sup_{n \geq 1} \mu_n^0 \left\{ \max_{\substack{t, u \leq 1 \\ |t-u| \leq \delta}} |S_{nt} - S_{nu}| \geq \varepsilon\sqrt{n} \right\} = 0, \tag{3.27}$$

It is easy to see that (3.27) follows from the following estimate.

**Lemma 3.3** *For $\beta < \beta_c$, there is a constant $C_{3.28} = C_{3.28}(\beta, v) \in (0, \infty)$ such that for all $l \geq 1$ and $\lambda \geq 1$;*

$$\mu_n^0 \left\{ \max_{\substack{0 \leq j \leq l \\ 0 \leq k \leq n-j}} |S_{k+j} - S_k| \geq \lambda\sqrt{vl} \right\} \leq C_{3.28} \, (n/l)^{3/2} \exp\left(-\lambda/C_{3.28}\right). \tag{3.28}$$

We first prove the following exponential integrability property.

**Lemma 3.4** *Let $\beta < \beta_c$ and $\varepsilon > 0$ be such that*

$$\frac{v}{2}(e^{\varepsilon} - 1)(e^{\beta} - 1) \leq \theta_-(1). \tag{3.29}$$

*Then,*

$$\sup_{n \geq \varepsilon^2} P^x \left[ z_n \exp\left( \varepsilon \left| \frac{S_n - x}{\sqrt{n}} \right| \right) \right] \leq 2 \exp\left( \frac{\cosh(1) v \varepsilon^2}{2} \right). \tag{3.30}$$

Proof: We will show that

$$P^x \left[ z_n \exp\left(\varepsilon S_n - \varepsilon x\right) \right] \leq \exp\left( (\cosh(\varepsilon) - 1) vn \right). \tag{3.31}$$

Since $e^{|t|} \leq e^t + e^{-t}$ ($t \in \mathbf{R}$) and $\cosh(t) - 1 \leq \frac{1}{2}\cosh(1)t^2$ ($|t| \geq 1$), the desired inequality (3.30) follows from (3.31) by plugging $\pm\varepsilon/\sqrt{n}$ in place of $\varepsilon$. To prove (3.31), we set $\Phi_n = \exp\left(\varepsilon S_n - \varepsilon x\right)$ and $a_n = P^x[z_n \Phi_n]$. Then, it is easy to see that

$$\begin{aligned} P^x \left[ z_{k-1} \Delta\Phi_k \right] &= (\cosh(\varepsilon) - 1) v a_{k-1}, \\ P^x [z_{k-1}^0 (e^{\beta} 1\{\Delta S_k \geq 0\} - 1) \Delta\Phi_k] &= e^{-\varepsilon x} \frac{v}{2} (e^{\varepsilon} - 1)(e^{\beta} - 1) Z_{k-1}^{x,0}, \\ P^x [z_{k-1}^0 \Phi_{k-1}] &= e^{-\varepsilon x} Z_{k-1}^{x,0}. \end{aligned}$$

Hence we see from (2.31) that

$$a_n \leq 1 + (\cosh(\varepsilon) - 1) v \sum_{k=1}^{n} a_{k-1},$$

which implies (3.31). □

Proof of Lemma 3.3: We introduce the following notation;

$$M_{k,n} = \max_{k \le j \le n} |S_j - S_k|. \tag{3.32}$$

We divide the proof into two steps;

Step1: We first prove that there is a constant $C_{3.33} = C_{3.33}(\beta, v) \in (0, \infty)$ such that for all $y > 0$ and $\lambda \ge 1$;

$$\sup_{n \in \mathbf{N}} P^{y\sqrt{vn}} \left[ z_n : \frac{M_{0,n}}{\sqrt{vn}} \ge \lambda \right] \le C_{3.33} \exp\left(-\lambda / C_{3.33}\right), \tag{3.33}$$

$$\sup_{n \in \mathbf{N}} \sqrt{n} P^{0} \left[ z_n : \frac{M_{0,n}}{\sqrt{vn}} \ge \lambda \right] \le C_{3.33} \exp\left(-\lambda / C_{3.33}\right). \tag{3.34}$$

We follow Belkin [Bel72], cf. proof of (2.9) in that paper. We set $A_n = \{\frac{M_n}{\sqrt{vn}} \ge \lambda\}$ and $\tau = \inf\left\{ j \ge 0 \; ; \; \frac{|S_j - S_0|}{\sqrt{vn}} \ge \lambda \right\}$. Then, we have for any $m \ge 1$ that

$$\begin{aligned} & P^{y\sqrt{vn}}\left[z_n : A_n\right] \\ = \; & P^{y\sqrt{vn}}\left[z_n : \tau \le n-m\right] + P^{y\sqrt{vn}}\left[z_n : A_n \cap \{\tau > n-m\}\right]. \end{aligned} \tag{3.35}$$

We divide the first term of (3.35) into

$$\begin{aligned} I_1 &= P^{y\sqrt{vn}} \left[ z_n : \tau \le n-m, \frac{S_\tau}{\sqrt{vn}} < y - \lambda \right], \\ I_2 &= P^{y\sqrt{vn}} \left[ z_n : \tau \le n-m, \frac{S_\tau}{\sqrt{vn}} \ge y + \lambda \right]. \end{aligned}$$

Note that $I_1 \neq 0$ only if $y > \lambda$, in which case we have $z_\tau = 1$. Hence

$$\begin{aligned} I_1 &= P^{y\sqrt{vn}} \left[ Z^{S_\tau}_{n-\tau} : \tau \le n-m, \frac{S_\tau}{\sqrt{vn}} < y - \lambda \right] \\ &\le P^{y\sqrt{vn}} \left\{ \tau \le n-m, \frac{S_\tau}{\sqrt{vn}} < y - \lambda \right\} \\ &\le P^{y\sqrt{vn}} \left\{ \frac{M_n}{\sqrt{vn}} > \lambda \right\} \\ &\le C_{3.36} \exp(-\lambda^2/2). \end{aligned} \tag{3.36}$$

To give an upper bound of $I_2$, we observe that there is $C_{3.37} = C_{3.37}(v) \in (1, \infty)$ such that

$$P^{x\sqrt{vn}} \left[ z_{n-k} : \frac{S_{n-k}}{\sqrt{vn}} \ge x \right] \ge 1/C_{3.37} \text{ if } x \ge 1 \text{ and } n-k \ge C_{3.37}. \tag{3.37}$$

In fact,

$$\begin{aligned} P^{x\sqrt{vn}}\left[z_{n-k}:\frac{S_{n-k}}{\sqrt{vn}}\geq x\right] &\geq P^0\left[\frac{S^{\min}_{n-k}}{\sqrt{nv}}>-x,\ \frac{S_{n-k}}{\sqrt{vn}}\geq 0\right]\\ &\geq P^0\left[\frac{S^{\min}_{n-k}}{\sqrt{v(n-k)}}>-1,\ \frac{S_{n-k}}{\sqrt{v(n-k)}}\geq 0\right]\end{aligned}$$

The last displayed probability converges, as $n-k\nearrow\infty$, to

$$P\left\{\min_{0\leq t\leq 1}B_t\geq -1,\ B_1\geq 0\right\}>0.$$

This proves (3.37). We now take $m=\lfloor C_{3.37}\rfloor+1$. We then have by (2.32) and (3.37) that

$$Z^{x\sqrt{vn}}_{n-k}\leq 1\leq C_{3.37}\left[z_{n-k}:\frac{S_{n-k}}{\sqrt{vn}}\geq x\right]\quad\text{for } x\geq 1,$$

and therefore that

$$\begin{aligned} I_2 &= \sum_{k\leq n-m}\int_{x\geq y+\lambda}P^{y\sqrt{vn}}\left[z_k:\tau=k,\frac{S_k}{\sqrt{vn}}\in dx\right]Z^{x\sqrt{vn}}_{n-k}\\ &\leq C_{3.37}\sum_{k\leq n-m}\int_{x\geq y+\lambda}P^{y\sqrt{vn}}\left[z_k:\tau=k,\frac{S_k}{\sqrt{vn}}\in dx\right]P^{x\sqrt{vn}}\left[z_{n-k}:\frac{S_{n-k}}{\sqrt{vn}}\geq x\right]\\ &= C_{3.37}P^{y\sqrt{vn}}\left[z_n:\frac{S_n}{\sqrt{vn}}\geq y+\lambda\right]. \end{aligned}\tag{3.38}$$

We divide the second term of (3.35) into

$$\begin{aligned} J_1 &= P^{y\sqrt{vn}}\left[z_n:A_n\cap\left\{\tau>n-m,\frac{|S_n-S_0|}{\sqrt{vn}}>\frac{\lambda}{2}\right\}\right],\\ J_2 &= P^{y\sqrt{vn}}\left[z_n:A_n\cap\left\{\tau>n-m,\frac{|S_n-S_0|}{\sqrt{vn}}\leq\frac{\lambda}{2}\right\}\right]. \end{aligned}$$

We have

$$\begin{aligned} J_2 &= \int_{x:|x-y|\leq\lambda}P^{y\sqrt{vn}}\left[z_{n-m}:\frac{S_{n-m}}{\sqrt{vn}}\in dx\right]P^{x\sqrt{vn}}\left[z_m:\max_{j\leq n}\left|\frac{S_j}{\sqrt{vn}}-y\right|\geq\lambda,\ \left|\frac{S_m}{\sqrt{vn}}-y\right|\leq\frac{\lambda}{2}\right]\\ &\leq e^{\beta m}\int_{x:|x-y|\leq\lambda}P^{y\sqrt{vn}}\left[z_{n-m}:\frac{S_{n-m}}{\sqrt{vn}}\in dx\right]P^0\left[\frac{M_m}{\sqrt{vn}}\geq\lambda\right]\\ &\leq C_{3.39}\exp\left(-\lambda^2 n/C_{3.39}\right), \end{aligned}\tag{3.39}$$

where $C_{3.39}=C_{3.39}(\beta)$. Combining (3.36), (3.38) and (3.39), we get

$$P^{y\sqrt{vn}}[z_n:A_n]\leq(1+C_{3.37})P^{y\sqrt{vn}}\left[z_n:\frac{|S_n-S_0|}{\sqrt{vn}}\geq\frac{\lambda}{2}\right]+C_{3.39}\exp\left(-\lambda^2 n/C_{3.39}\right).\tag{3.40}$$

This and (3.30) imply (3.33). On the other hand, we have by (1.11) that

$$\lim_{n\nearrow\infty}\mu_n^0\left\{\frac{S_n}{\sqrt{vn}}\geq\frac{\lambda}{2}\right\}=\exp(-\lambda^2/8).$$

This, together with (2.35) and (3.40), implies (3.34).

Step2: Proof of (3.28). It is easy to see that the probability on the left-hand-side of (3.28) is bounded from above by

$$\sum_{p=0}^{\lfloor n/l\rfloor}\mu_n^0(A_p), \tag{3.41}$$

where $A_p=\left\{M_{pl,(p+1)l}\geq\lambda\sqrt{vl}/3\right\}$, cf. [KaSh91, proof of (4.19)]. For $p\geq 1$, we have by (2.32), (3.33) and (2.35) that

$$\begin{aligned}
P^0\left[z_n:A_p\right] &\leq P^0\left[z_{(p+1)l}:A_p\right]\\
&= \int P^0\left[z_{pl}:\frac{S_{pl}}{\sqrt{vpl}}\in dx\right]P^{x\sqrt{vpl}}\left[z_l:M_l\geq\lambda\sqrt{vl}/3\right]\\
&\leq C_{3.42}Z^0_{(p+1)l}\exp\left(-\lambda/C_{3.42}\right) \qquad (3.42)\\
&\leq C_{3.43}l^{-1/2}\exp\left(-\lambda/C_{3.42}\right) \qquad (3.43)
\end{aligned}$$

For $p=0$, we use (3.34) to see that

$$P^0\left[z_n:A_0\right]\leq P^0\left[z_l:A_0\right]\leq C_{3.44}l^{-1/2}\exp\left(-\lambda/C_{3.44}\right). \tag{3.44}$$

Therefore,

$$\begin{aligned}
\sum_{p=0}^{\lfloor n/l\rfloor}\mu_n^0(A_p) &\leq C_{3.45}(n/l)n^{1/2}l^{-1/2}\exp\left(-\lambda/C_{3.43}\right)\\
&= C_{3.45}(n/l)^{-3/2}\exp\left(-\lambda/C_{3.43}\right). \qquad (3.45)
\end{aligned}$$

This proves (3.28). □

# 4 Supercritical phase

## 4.1 Proof of Theorem 1.1 (b)

The augument we persent here is similar to the one developed by van Leeuvan and Hilhorst [vLHi81, Section2]. Namely, we look at the matrix $Q$ and reduce the original problem to the spectral properties of the matrix.

**Lemma 4.1** *Let $s=1/\lambda\in(-1,+1)$. Then, for $\beta>\beta_c$, an equation*

$$Qu=\lambda u, \tag{4.1}$$

*given the value of $u(0)$, can uniquely be solved as*

$$u(x) = \frac{u(0)}{e^{\beta}\sqrt{\gamma(s)}}\left(\theta_{+}(s)\widehat{f^{x}}(s) + \theta_{-}(s)\widehat{f^{x}}(s)^{-1}\right), \quad x \geq 1. \tag{4.2}$$

*In particular, the equation (4.1) for $|\lambda| > 1$ has a bounded solution only if $\lambda = \lambda_{\pm}$.*

Proof: The equation (4.1) reads

$$e^{\beta} r u(0) + \tfrac{1}{2} e^{\beta} v u(1) = \lambda u(0), \tag{4.3}$$

$$r u(x) + \tfrac{1}{2} v\left(u(x-1) + u(x+1)\right) = \lambda u(x) \quad x \geq 1. \tag{4.4}$$

If $s = 1/\lambda \in (-1, +1)$, then the characteristic equation

$$rt + \tfrac{1}{2} v\left(1 + t^2\right) = \lambda t$$

for (4.4) has $\widehat{f^{\pm 1}}(s)$ as its two distinct real solutions ($|\widehat{f^{1}}(s)| < 1 < |\widehat{f^{-1}}(s)|$). It is therefore elementary to get (4.2). $\square$

For $\beta > \beta_c$, we set

$$\rho_2(x) = \rho(x)\langle\, \rho, \rho\, \rangle_{\beta}^{-1/2} \quad x \in \mathbf{N}, \tag{4.5}$$

so that $\langle\, \rho_2, \rho_2\, \rangle_{\beta} = 1$.

**Lemma 4.2** *For $\beta > \beta_c$, there is a constant $\varepsilon_{4.6} = \varepsilon_{4.6}(\beta, v) \in (0,1)$ such that*

$$|\lambda_{+}^{-n} P^{x}[f(S_n) z_n] - \langle\, f, \rho_2\, \rangle_{\beta} \rho_2(x)| \leq (1 + |x|)^{p} B_{4.6}(f) \varepsilon_{4.6}^{n} \tag{4.6}$$

*for all $x, n \in \mathbf{N}$ and a $p$-bounded function $f : \mathbf{N} \to \mathbf{R}$, where $B_{4.6}(f)$ is a constant which depends only on $\beta$, $v$ and $f$.*

Proof: Since $Q_{xy} = P^{x}[z_1^{y}]$, we have

$$\begin{aligned} P^{x}[f(S_n) z_n] &= \sum_{y \in \mathbf{N}} P^{x}[z_n^{y}] f(y) \\ &= \sum_{x_1, x_2, \ldots, x_{n-1}, y \in \mathbf{N}} Q_{x x_1} Q_{x_1 x_2} \cdots Q_{x_{n-1} y} f(y) \\ &= Q^{n} f(x). \end{aligned} \tag{4.7}$$

We will prove (4.6) by studying the matrix $Q$. To do so, let us introduce a Hilbert space

$$\ell_{\beta}^{2} = \{ f : \mathbf{N} \to \mathbf{R}\ ;\ \langle\, f, f\, \rangle_{\beta} < \infty \},$$

equipped with the inner product defined by (1.9). Let us first prove that

$$\|\lambda_{+}^{-n} Q^{n} f - \langle\, f, \rho_2\, \rangle_{\beta} \rho_2\|_{\beta} \leq \|f - \langle\, f, \rho_2\, \rangle_{\beta} \rho_2\|_{\beta} \varepsilon_{4.8}^{n} \quad \text{for } f \in \ell_{\beta}^{2}, \tag{4.8}$$

where $\| \ \cdot \ \|_\beta$ denots the $\ell^2_\beta$-norm. Consider $Q$ as a bounded operator on $\ell^2_\beta$. Then, $Q$ is symmetric and bounded. Therefore, $\sigma(Q)$: the set of $\ell^2_\beta$-spectra of $Q$, is a compact set in $\mathbf{R}$. We claim that

$$\begin{aligned} \sigma(Q) &\subset [-1,1] \cup \{\lambda_-, \lambda_+\}, && (4.9)\\ \lambda_+ &> \sup\{|\lambda| \in \sigma(Q)\backslash\{\lambda_+\}\}. && (4.10) \end{aligned}$$

We let $\sigma^{\text{disc.}}(Q)$ and $\sigma^{\text{ess.}}(Q)$ denote respectively, the discrete and the essential spectra of $Q$. Now, consider an operator

$$P = \begin{pmatrix} r & \frac{v}{2} & 0 & 0 & \cdots \\ \frac{v}{2} & r & \frac{v}{2} & 0 & \cdots \\ 0 & \frac{v}{2} & r & \frac{v}{2} & \ddots \\ 0 & 0 & \frac{v}{2} & r & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then, $P$ is $\ell^2_0$-symmetric and has the operator norm one on $\ell^2_0$. This implies $\sigma(P) \subset [-1,1]$, since the set of $\ell^2_\beta$-spectra of a bounded operator is independent of the choice of $\beta$. Note also that $Q-P$ is of finite rank and hence is a compact operator, which implies that $\sigma^{\text{ess.}}(Q) = \sigma^{\text{ess.}}(P) \subset [-1,1]$, cf. [ReSi78, page 111, Lemma 3]. On the other hand, we see from Lemma 4.1 that $\sigma^{\text{disc.}}(Q)\backslash[-1,+1] \subset \{\lambda_-, \lambda_+\}$. This, together with (1.6), implies (4.9) and (4.10). The claim (4.8) can now easily be seen by (4.9), (4.10) and and the spectral decomposition. The desired estimate (4.6) for polynomially bounded $f$ can be seen by applying (4.8) to a cut-off function $f_{x,n}(y) = f(y)1\{|x-y| \le n\}$ as follows. Since $Q^n f_{x,n}(x) = Q^n f(x)$, we have

$$\begin{aligned} &|\lambda_+^{-n} Q^n f(x) - \langle\, f, \rho_2 \,\rangle_\beta \rho_2(x)| \\ &\le \ |\lambda_+^{-n} Q^n f_{x,n}(x) - \langle\, f_{x,n}, \rho_2 \,\rangle_\beta \rho_2(x)| + |\langle\, f_{x,n} - f, \rho_2 \,\rangle_\beta |\rho_2(x) \qquad (4.11) \end{aligned}$$

Since $f_{x,n} \in \ell^2_\beta$, we see from (4.8) that the first term of (4.11) is not greater than $\varepsilon_{4.8}^n \|f_{x,n}\|_\beta$ which has a bound in the form of the right-hand-side of (4.6). On the other hand, it is easy to bound the second term of (4.11) by $B(f)\rho(n)$, where $B(f)$ is a constant which depends only on $\beta$, $v$ and $f$. $\Box$

We prove (1.13) in the following generalized form.

**Lemma 4.3** *For $\beta > \beta_c$, there are constants $C_{4.12} = C_{4.12}(\beta, v) > 0$ and $\varepsilon_{4.12} = \varepsilon_{4.12}(\beta, v) \in (0,1)$ as follows; if $l \ge k + (1+x)C_{4.12}$, $f : \mathbf{N}^k \to \mathbf{R}$ and $h : \mathbf{N} \to \mathbf{R}$ are polynomially bounded, $F = f(S_1, \ldots, S_k)$, then*

$$\left| \mu_l^x[F h(S_l)] - \frac{\langle\, h, \rho_2 \,\rangle_\beta}{\langle\, 1, \rho_2 \,\rangle_\beta} \mu_l^x[F] \right| \le B_{4.12}(f,h) \varepsilon_{4.12}^{l-k}. \qquad (4.12)$$

*where $B_{4.12}(f,h)$ is a constant which depends only on $\beta$, $v$, $f$ and $g$.*

Proof : Since $f$ is polynomially bounded, it is easy to see that there are constants $p > 0$ and $B_{4.13} = B_{4.13}(\beta, v, f) > 0$ such that

$$\sup_{k,x \in \mathbf{N}} (1+|x|)^{-p} P^x[|F| z_k] \lambda_+^{-k} \le B_{4.13}, \qquad (4.13)$$

It is also easy to see from (4.6) that there is a constant $C_{4.14} = C_{4.14}(\beta, v) > 0$ such that

$$1/2 \leq \frac{\lambda_+^{-n} Z_n^x}{\langle\, 1, \rho_2 \,\rangle_\beta \rho_2(x)} \leq 3/2 \quad \text{if } n \geq (1+x) C_{4.14}. \tag{4.14}$$

We take $C_{4.12} \geq C_{4.14}$. We see from (4.6) that

$$\begin{aligned}
&\left| P^y[h(S_l) z_l] - \frac{\langle\, h, \rho_2 \,\rangle_\beta}{\langle\, 1, \rho_2 \,\rangle_\beta} Z_l^y \right| \\
&\leq \lambda_+^l \left| \lambda_+^{-l} P^y[h(S_l) z_l] - \langle\, h, \rho_2 \,\rangle_\beta \rho_2(y) \right| + \lambda_+^l \left| \frac{\langle\, h, \rho_2 \,\rangle_\beta}{\langle\, 1, \rho_2 \,\rangle_\beta} \left( \lambda_+^{-l} Z_l^y - \langle\, 1, \rho_2 \,\rangle_\beta \rho_2(y) \right) \right| \\
&\leq (1+|y|)^p B_{4.15} \lambda_+^l \varepsilon_{4.6}^l,
\end{aligned} \tag{4.15}$$

for some $p = p(h) \geq 0$ and $B_{4.15} = B_{4.15}(\beta, v, g) > 0$. We have on the other hand that

$$\begin{aligned}
&\mu_l^x[F h(S_l)] \\
&= \frac{1}{Z_l^x} \sum_{y \in \mathbf{N}} P^x[F z_k^y] P^y[h(S_{l-k}) z_{l-k}] \\
&= \frac{1}{Z_l^x} \sum_{y \in \mathbf{N}} P^x[F z_k^y] \left( P^y[h(S_{l-k}) z_{l-k}] - \frac{\langle\, h, \rho_2 \,\rangle_\beta}{\langle\, 1, \rho_2 \,\rangle_\beta} Z_{l-k}^y \right) + \frac{\langle\, h, \rho_2 \,\rangle_\beta}{\langle\, 1, \rho_2 \,\rangle_\beta} \mu_l^x[F].
\end{aligned} \tag{4.16}$$

By (4.15) and (4.16),

$$\begin{aligned}
&\left| \mu_l^x[F h(S_l)] - \frac{\langle\, h, \rho_2 \,\rangle_\beta}{\langle\, 1, \rho_2 \,\rangle_\beta} \mu_l^x[F] \right| \\
&\leq B_{4.15} \frac{\lambda_+^{-k} P_k^x[|F|(1+|S_k|)^p z_k]}{\lambda_+^{-l} Z_l^x} \varepsilon_{4.6}^{l-k}
\end{aligned}$$

By (4.14) and (4.13), this is bounded from above by

$$B_{4.17} \varepsilon_{4.17}^{l-k} \tag{4.17}$$

for some $B_{4.17} = B_{4.17}(\beta, v, f, g) > 0$ and $\varepsilon_{4.17} = \varepsilon_{4.17}(\beta, v) \in (0,1)$ if $C_{4.12}$ is large enough. This completes the proof of the lemma. □

### 4.2 Proof of Theorem 1.3

Since $f$ is polynomially bounded, it is easy to see that there are constants $p > 0$ and $B_{4.18} = B_{4.18}(\beta, v, f) > 0$ such that

$$\sup_{k, x \in \mathbf{N}} (1+|x|)^{-p} \mu_k^x[|F|] \leq B_{4.18}. \tag{4.18}$$

We next observe that there are constants $C_{4.19} = C_{4.19}(\beta, v) > 0$ and $\varepsilon_{4.19} = \varepsilon_{4.19}(\beta, v) \in (0,1)$ such that if $n \geq k + (1+x) C_{4.19}$, then

$$\left| \frac{\lambda_+^{-(n-k)} Z_{n-k}^y}{\lambda_+^{-n} Z_n^x} - \frac{\rho(y)}{\rho(x)} \right| \leq C_{4.19} \varepsilon_{4.19}^{n-k} \quad \text{for all } y \in \mathbf{N}. \tag{4.19}$$

This can be seen by (4.6). We have

$$\begin{aligned}\mu_n^x[F] &= (Z_n^x)^{-1}\sum_{y\in\mathbf{N}} P^x[Fz_k^y]Z_{n-k}^y \\ &= \sum_{y\in\mathbf{N}} P^x[Fz_k^y]\lambda_+^{-k}\left(\frac{\lambda_+^{-(n-k)}Z_{n-k}^y}{\lambda_+^{-n}Z_n^x}-\frac{\rho(y)}{\rho(x)}\right)+\lambda_+^{-k}\sum_{y\in\mathbf{N}} P^x[Fz_k^y]\frac{\rho(y)}{\rho(x)}. \qquad (4.20)\end{aligned}$$

Since the second term of (4.20) equals $\nu^x[F]$, we see from (4.19), (4.20) and (4.13) that

$$\begin{aligned}|\mu_n^x[F]-\nu^x[F]| &\leq C_{4.19}\lambda_+^{-k}|P^x[Fz_k]|\varepsilon_{4.19}^{n-k} \\ &\leq C_{4.19}B_{4.13}(1+|x|)^p\varepsilon_{4.19}^{n-k}\end{aligned}$$

We therefore get (1.20) by taking $C_{1.20}$ large enough. □

### 4.3 Proof of Theorem 1.4

It is clear that (1.22) follows from (1.20) and (1.21). It is therefore, enough to prove (1.21).

We set $h(y)=h_{n,l}(y)=P^y[z_{n-l}g(S_1,\dots,S_m)]\lambda_+^{-(n-l)}$. Then, by (4.13), $h$ is polynomially bounded, uniformly in $n\geq 1$ and $1\leq l\leq n$. It is easy to see that

$$\mu_n^x[FG]=\frac{\lambda_+^{-l}Z_l^x}{\lambda_+^{-n}Z_n^x}\mu_l^x[Fh(S_l)] \text{ and } \mu_n^x[G]=\frac{\lambda_+^{-l}Z_l^x}{\lambda_+^{-n}Z_n^x}\mu_l^x[h(S_l)].$$

Therefore,

$$\begin{aligned}&\mu_n^x(F\,G)-\mu_n^x(F)\mu_n^x(G) \\ &= \frac{\lambda_+^{-l}Z_l^x}{\lambda_+^{-n}Z_n^x}\left(\mu_l^x[Fh(S_l)]-\mu_l^x[F]\mu_l^x[h(S_l)]\right) \qquad (4.21)\\ &+ \frac{\lambda_+^{-l}Z_l^x}{\lambda_+^{-n}Z_n^x}\mu_l^x[h(S_l)]\left(\mu_l^x[F]-\mu_n^x[F]\right). \qquad (4.22)\end{aligned}$$

We see from (1.20), (4.14) and (4.13) that if $C_{1.21}$ is large enough, then the absolute value of (4.22) is not greater than

$$B_{4.23}\varepsilon_{1.20}^{l-k}, \qquad (4.23)$$

for some $B_{4.23}=B_{4.23}(\beta,v,f,g)>0$.

We now turn to (4.21). We write

$$\begin{aligned}&\mu_l^x[Fh(S_l)]-\mu_l^x[F]\mu_l^x[h(S_l)] \\ &= \left(\mu_l^x[Fh(S_l)]-\frac{\langle\, h,\rho_2\,\rangle_\beta}{\langle\, 1,\rho_2\,\rangle_\beta}\mu_l^x[F]\right)+\mu_l^x[F]\left(\frac{\langle\, h,\rho_2\,\rangle_\beta}{\langle\, 1,\rho_2\,\rangle_\beta}-\mu_l^x[h(S_l)]\right). \qquad (4.24)\end{aligned}$$

We see from (4.12) and (4.18) that the both terms of (4.24) has the absolute value not greater than

$$B_{4.25}\varepsilon_{1.13}^{l} \qquad (4.25)$$

for some $B_{4.25}=B_{4.25}(\beta,v,f,g)>0$ if $C_{1.21}$ is large enough. □

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