# CORRELATIONS BETWEEN ZEROS AND SUPERSYMMETRY

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Abstract. In our previous work [BSZ2], we proved that the correlation functions for simultaneous zeros of random generalized polynomials have universal scaling limits and we gave explicit formulas for pair correlations in codimensions 1 and 2. The purpose of this paper is to compute these universal limits in all dimensions and codimensions. First, we use a supersymmetry method to express the $n$-point correlations as Berezin integrals. Then we use the Wick method to give a closed formula for the limit pair correlation function for the point case in all dimensions.

## 1. Introduction

This paper is a continuation of our articles [BSZ1, BSZ2, BSZ3] on the correlations between zeros of random holomorphic polynomials in $m$ complex variables and their generalization to holomorphic sections of positive line bundles $L \to M$ over general Kähler manifolds of dimension $m$ and their symplectic counterparts. These correlations are defined by the probability density $K^N_{nk}(z^1, \dots, z^n)$ of finding joint zeros of $k$ independent sections at the points $z^1, \dots, z^n \in M$ (see §2). To obtain universal quantities, we re-scale the correlation functions in normal coordinates by a factor of $\sqrt{N}$. Our main result from [BSZ2, BSZ3] is that the (normalized) correlation functions have a universal scaling limit

$$\widetilde{K}^\infty_{nkm}(z^1, \dots, z^n) = \lim_{N\to\infty} K^N_{1k}(z_0)^{-n} K^N_{nk}\Big(z_0 + \frac{z^1}{\sqrt{N}}, \dots, z_0 + \frac{z^n}{\sqrt{N}}\Big)\,, \tag{1}$$

which is independent of the manifold $M$, the line bundle $L$ and the point $z_0$; $\widetilde{K}^\infty_{nkm}$ depends only on the dimension $m$ of the manifold and the codimension $k$ of the zero set. The problem then arises of calculating these universal functions explicitly and analyzing their small distance and large distance behavior. In [BSZ1], [BSZ2], we gave explicit formulas for the pair correlation functions $\widetilde{K}^\infty_{2km}(z^1, z^2)$ in codimensions $k = 1, 2$, respectively. The purpose of this paper is to complete these results by giving explicit formulas for $\widetilde{K}^\infty_{nkm}$ in all dimensions and codimensions.

Our first formula expresses the correlation as a supersymmetric (Berezin) integral involving the matrices $\Lambda(z)$, $A^\infty(z)$ used in our prior formulas, as well as a matrix $\Omega$ of fermionic variables described below.

Theorem **1.1.** *The limit $n$-point correlation functions are given by*

$$\widetilde{K}^\infty_{nkm}(z^1, \dots, z^n) = \frac{[(m-k)!]^n}{(m!)^n[\det A^\infty(z)]^k} \int \frac{1}{\det[I + \Lambda(z)\Omega]} d\eta\,.$$

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Here, $\Omega$ is the $nkm \times nkm$ matrix

$$\Omega = \left(\Omega^{pjq}_{p'j'q'}\right) = \left(\delta^p_{p'}\delta^q_{q'}\eta^{p'}_{j'}\bar\eta^p_j\right) \qquad (1 \le p, p' \le n,\ 1 \le j, j' \le k,\ 1 \le q, q' \le m)\,, \tag{2}$$

where the $\eta_j, \bar\eta_j$ are anti-commuting (fermionic) variables, and $d\eta = \prod_{j,p} d\bar\eta^p_j d\eta^p_j$. The integral in Theorem 1.1 is a Berezin integral, which is evaluated by simply taking the coefficient of the top degree form of the integrand $\det[I + \Lambda(z)\Omega]^{-1}$ (see §3). Hence the formula in Theorem 1.1 is a purely algebraic expression in the coefficients of $\Lambda(z)$ and $A^\infty(z)$, which are given in terms of the Szegö kernel of the Heisenberg group and its derivatives (see §2). We remark that supersymmetric methods have also been applied to limit correlations in random matrix theory by Zirnbauer [Zi].

In the case $n = 2$, $\widetilde K^\infty_{2km}(z^1, z^2)$, depends only on the distance between the points $z^1, z^2$, since it is universal and hence invariant under rigid motions. Hence it may be written as:

$$\widetilde K^\infty_{2km}(z^1, z^2) = \kappa_{km}(|z^1 - z^2|)\,. \tag{3}$$

We refer to [BSZ2] for details. In [BSZ1] we gave an explicit formula for $\kappa_{1m}$ (using the "Poincaré-Lelong formula"), and in [BSZ2] we evaluated $\kappa_{2m}$. (The pair correlation function $\kappa_{11}(r)$ was first determined by Hannay [Ha] in the case of zeros of $SU(2)$ polynomials in one complex variable.) In §3.1, we use Theorem 1.1 to give the following new Berezin integral formula for $\kappa_{km}$:

Corollary **1.2.** *The pair correlation functions are given by*

$$\kappa_{km}(r) = \frac{(m-k)!^2}{m!^2(1-e^{-r^2})^k}\int \frac{1}{\Phi\Psi^{m-1}}d\eta\,,$$

*where*

$$\begin{aligned}
\Phi &= \det\left[I + P(\Omega_1 + \Omega_2) + T\Omega_1\Omega_2\right]\,,\\
&\quad P = 1 - \frac{r^2 e^{-r^2}}{1-e^{-r^2}}\,, \quad T = 1 - e^{-r^2} - \frac{r^4 e^{-r^2}}{1-e^{-r^2}}\,,\\
\Psi &= \det\left[I + \Omega_1 + \Omega_2 + (1-e^{-r^2})\Omega_1\Omega_2\right]\,.
\end{aligned}$$

Here, $\Omega_1,\ \Omega_2$ are the $k \times k$ matrices

$$\Omega_p = \left(\eta^p_{j'}\bar\eta^p_j\right)_{1\le j,j'\le k}\,, \qquad p = 1, 2\,.$$

We then expand the formula as a (finite) series (32), which we use to compute explicit formulas for $\kappa_{km}$.

The most vivid case is when $k = m$, where the simultaneous zeros of $k$-tuples of sections almost surely form a set of discrete points. Our second result is an explicit formula for the point pair correlation functions $\kappa_{mm}$ in all dimensions:

Theorem **1.3.** *The point pair correlation functions are given by*

$$\kappa_{mm}(r) = \frac{m(1-v^{m+1})(1-v)+r^2(2m+2)(v^{m+1}-v)+r^4\left[v^{m+1}+v^m+(\{m+1\}v+1)(v^m-v)/(v-1)\right]}{m(1-v)^{m+2}}\,, \qquad v = e^{-r^2}\,, \tag{4}$$

*for $m \geq 1$. For small values of $r$, we have*

$$\kappa_{mm}(r) = \frac{m+1}{4} r^{4-2m} + O(r^{8-2m}), \qquad \text{as } \ r \to 0\,. \tag{5}$$

We prove Theorem 1.3 in §4 without making use of supersymmetry. Our proof uses instead the Wick formula expansion of the Gaussian integral representation of the correlation.

It is interesting to observe the dimensional dependence of the short distance behavior of $\kappa_{mm}(r)$. When $m = 1, \kappa_{mm}(r) \to 0$ as $r \to 0$ and one has "zero repulsion." When $m = 2$, $\kappa_{mm}(r) \to 3/4$ as $r \to 0$ and one has a kind of neutrality. With $m \geq 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \to 0$ and there is some kind of attraction between zeros. More precisely, in dimensions greater than 2, one is more likely to find a zero at a small distance $r$ from another zero than at a small distance $r$ from a given point; i.e., zeros tend to clump together in high dimensions. Indeed, in all dimensions, the probability of finding another zero in a ball of small scaled radius $r$ about another zero is $\sim r^4$. We give below a graph of $\kappa_{33}$; graphs of $\kappa_{11}$ and $\kappa_{22}$ can be found in [BSZ2].

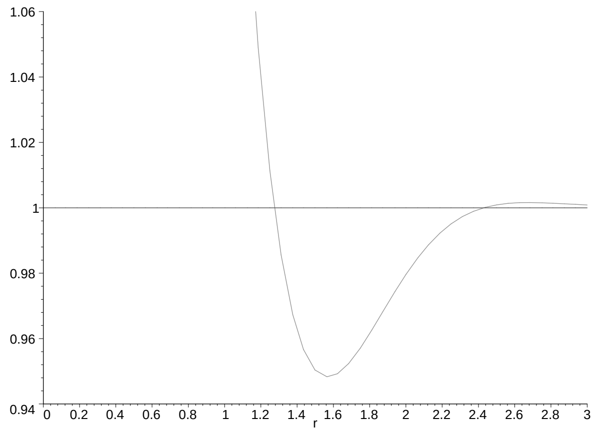


FIGURE 1. The limit pair correlation function $\kappa_{33}$

*Remark:* Theorem 1.3 says that the *expected number* of zeros in the punctured ball of scaled radius $r$ about a given zero is $\sim \int_0^r \kappa_{mm}(t) t^{2m-1} dt \sim r^4$. But, one can show that for balls of small scaled radii $r$, the expected number of zeros approximates the probability of finding a zero.

## 2. Background

We begin by recalling the scaling limit zero correlation formula of [BSZ2]. Consider a random polynomial $s$ of degree $N$ in $m$ variables. More generally, $s$ can be a random section of the $N^{\text{th}}$ power $L^N$ of a positive line bundle $L$ on an $m$-dimensional compact complex manifold $M$ (or a symplectic $2m$-manifold; see [SZ2, BSZ3]). We give $M$ the Kähler metric induced by the curvature form $\omega$ of the line bundle $L$. The probability measure on the space of sections is the complex Gaussian measure induced by the Hermitian inner product

$$\langle s_1, \bar{s}_2 \rangle = \int_M h^N(s_1, \bar{s}_2) dV_M \ ,$$

where $h^N$ is the metric on $L^N$ and $dV_M$ is the volume measure induced by $\omega$. (For further discussion of the topics of this section, see [BSZ2].) In particular, if $L$ is the hyperplane section bundle over $\mathbb{CP}^m$, then random sections of $L^N$ are polynomials of degree $N$ in $m$ variables of the form

$$P(z_1,\dots,z_m)=\sum_{|J|\le N}\frac{C_J}{\sqrt{(N-|J|)!j_1!\cdots j_m!}}z_1^{j_1}\cdots z_m^{j_m}\qquad (\,J=(j_1,\dots,j_m)\,)\,,$$

where the $C_J$ are i.i.d. Gaussian random variables with mean 0; they are called "SU$(m+1)$-polynomials."

We consider $k$-tuples $s=(s_1,\dots,s_k)$ of i.i.d. random polynomials (or sections) $s_j$ $(1\le k\le m)$. The zero correlation density $K^N_{nk}(z^1,\dots,z^n)$ is defined as the expected joint volume density of zeros of sections of $L^N$ at the points $z^1,\dots,z^n$. In the case $k=m$, where the zero sets are discrete points, $K^N_{nk}(z^1,\dots,z^n)$ can be interpreted as the probability density of finding simultaneous zeros at these points. For instance, the zero density function $K^N_{1k}(z)\approx c_kN^k$ as $N\to\infty$, where $c_k$ is independent of the point $z$ (see [SZ1]).

In [BSZ2, BSZ3], we gave generalized forms of the Kac-Rice formula [Kac, Ri], which we used to express $K^N_{nk}(z^1,\dots,z^n)$ in terms of the joint probability distribution (JPD) of the random variables $s(z^1),\dots,s(z^n),\nabla s(z^1),\dots,\nabla s(z^n)$. We then showed that the scaling limit correlation function $\widetilde K^\infty_{nkm}$ given by (1) can be expressed in terms of the scaling limit of the JPD.

The central result of [BSZ2] is that the limit JPD is universal and can be expressed in terms of the Szegö kernel $\Pi^{\mathbf H}_1$ for the Heisenberg group:

$$\Pi^{\mathbf H}_1(z,\theta;w,\varphi)=\frac{1}{\pi^m}e^{i(\theta-\varphi+\Im z\cdot\bar w)-\frac12|z-w|^2}=\frac{1}{\pi^m}e^{i(\theta-\varphi)+z\cdot\bar w-\frac12(|z|^2+|w|^2)}\,. \tag{6}$$

To be precise, the limit JPD is a complex Gaussian measure with covariance matrix $\Delta^\infty$ given by:

$$\Delta^\infty(z)=\frac{m!}{\pi^m}\begin{pmatrix}A^\infty(z) & B^\infty(z)\\ B^\infty(z)^* & C^\infty(z)\end{pmatrix}, \tag{7}$$

where

$$\begin{aligned}
\pi^{-m}A^\infty(z)^p_{p'} &= \Pi^{\mathbf H}_1(z^p,0;z^{p'},0)\,,\\
\pi^{-m}B^\infty(z)^p_{p'q'} &= \frac{\nabla}{\partial\bar z^{p'}_{q'}}\Pi^{\mathbf H}_1(z^p,0;z^{p'},0) \;=\; (z^p_{q'}-z^{p'}_{q'})\Pi^{\mathbf H}_1(z^p,0;z^{p'},0)\,,\\
\pi^{-m}C^\infty(z)^{pq}_{p'q'} &= \frac{\nabla^2}{\partial z^p_q\partial\bar z^{p'}_{q'}}\Pi^{\mathbf H}_1(z^p,0;z^{p'},0) \;=\; \Big(\delta_{qq'}+(\bar z^{p'}_q-\bar z^p_q)(z^p_{q'}-z^{p'}_{q'})\Big)\,\Pi^{\mathbf H}_1(z^p,0;z^{p'},0)\,.
\end{aligned} \tag{8}$$

(Here $A^\infty$, $B^\infty$, $C^\infty$ are $n\times n$, $n\times mn$, $mn\times mn$ matrices, respectively.) In the sequel, we shall use the matrix

$$\Lambda^\infty(z):=C^\infty(z)-B^\infty(z)^*A^\infty(z)^{-1}B^\infty(z)\,. \tag{9}$$

We note that $A^\infty(z)$ and $\Lambda^\infty(z)$ are positive definite whenever $z^1,\dots,z^n$ are distinct points.

In [BSZ2], we gave the following key formula for the limit correlation functions:

$$\widetilde{K}^{\infty}_{nkm}(z^1,\dots,z^n) = \frac{[(m-k)!]^n}{(m!)^n[\det A^{\infty}(z)]^k}\int_{\mathbb{C}^{kmn}}\prod_{p=1}^{n}\det_{1\le j,j'\le k}\left(\sum_{q=1}^{m}\xi^p_{jq}\bar\xi^p_{j'q}\right)d\gamma_{\Lambda(z)}(\xi)\,, \tag{10}$$

where $\gamma_{\Lambda(z)}$ is the Gaussian measure with $(nkm\times nkm)$ covariance matrix

$$\Lambda(z) := \left(\Lambda(z)^{pjq}_{p'j'q'}\right) = \left(\delta^j_{j'}\Lambda^{\infty}(z)^{pq}_{p'q'}\right)\,. \tag{11}$$

(I.e., $\langle\xi^p_{jq}\bar\xi^{p'}_{j'q'}\rangle_{\gamma_{\Lambda(z)}} = \Lambda(z)^{pjq}_{p'j'q'}$.) For the pair correlation case ($n=2$), equation (10) becomes:

$$\kappa_{km}(r) = \frac{1}{\left[\frac{m!}{(m-k)!}\right]^2\det A(r)^k}\int_{\mathbb{C}^{2km}}\det_{1\le j,j'\le k}\left(\sum_{q=1}^{m}\xi^1_{jq}\bar\xi^1_{j'q}\right)\det_{1\le j,j'\le k}\left(\sum_{q=1}^{m}\xi^2_{jq}\bar\xi^2_{j'q}\right)d\gamma_{\Lambda(r)}(\xi)\,, \tag{12}$$

where

$$A(r) = A^{\infty}(z^1,z^2)\,,\ \Lambda(r) = \Lambda(z^1,z^2)\,,\quad |z^1-z^2| = r\,.$$

The computations in this paper are all based on formula (10).

## 3. Supersymmetric approach to $n$-point correlations

We now prove Theorem 1.1 using our formula (10) for the limit $n$-point correlation function, which we restate as follows:

$$\widetilde{K}^{\infty}_{nkm}(z^1,\dots,z^n) = \frac{[(m-k)!]^n}{(m!)^n[\det A^{\infty}(z)]^k}G_{nkm}\,, \tag{13}$$

where

$$G_{nkm}(z) = \int_{\mathbb{C}^{kmn}}\prod_{p=1}^{n}\det_{1\le j,j'\le k}\left(\sum_{q=1}^{m}\xi^p_{jq}\bar\xi^p_{j'q}\right)d\gamma_{\Lambda(z)}(\xi)\,. \tag{14}$$

Our approach is to represent the determinant in (14) as a Berezin integral and then to exchange the order of integration.

We introduce anti-commuting (or "fermionic") variables $\eta^p_j,\bar\eta^p_j$ ($1\le j\le k,\ 1\le p\le n$), which can be regarded as generators of the Grassmann algebra $\bigwedge^\bullet\mathbb{C}^{2l} = \bigoplus_{t=0}^{2l}\bigwedge^t\mathbb{C}^{2l}$, $l=nk$. The *Berezin integral* on $\bigwedge^\bullet\mathbb{C}^{2l}$ is the linear functional $\mathcal{I}:\bigwedge^\bullet\mathbb{C}^{2l}\to\mathbb{C}$ given by

$$\mathcal{I}|_{\bigwedge^t\mathbb{C}^{2l}} = 0 \quad\text{for}\quad t<2l\,,\quad \mathcal{I}\left(\textstyle\prod_{j,p}\eta^p_j\bar\eta^p_j\right) = 1\,.$$

Elements $f\in\bigwedge^\bullet\mathbb{C}^{2l}$ are considered as functions of anti-commuting variables, and we write

$$\mathcal{I}(f) = \int f\,d\eta = \int f\ \textstyle\prod_{j,p}d\bar\eta^p_j d\eta^p_j\,.$$

(See for example [Ef, Chapter 2], [ID, §2.1].) If $H = \left(H^{pj}_{p'j'}\right)$ is an $l\times l$ complex matrix, we have the supersymmetric formula for the determinant:

$$\det H = \int e^{-\langle H\eta,\bar\eta\rangle}d\eta\,,\qquad \langle H\eta,\bar\eta\rangle = \sum_{j,p,j',p'}\eta^p_j H^{pj}_{p'j'}\bar\eta^{p'}_{j'}\,. \tag{15}$$

We now use (15) to compute $G_{nkm}$: let

$$\xi^p = \begin{pmatrix} \xi^p_{11} & \cdots & \xi^p_{1m} \\ \vdots & & \vdots \\ \xi^p_{k1} & \cdots & \xi^p_{km} \end{pmatrix}$$

(where $\{\xi^p_{jq}\}$ are ordinary "bosonic" variables). We also write $\xi = \xi^1 \oplus \cdots \oplus \xi^n : \mathbb{C}^{mn} \to \mathbb{C}^{kn}$. Then

$$\prod_{p=1}^{n} \det_{1\le j,j'\le k} \left( \sum_{q=1}^{m} \xi^p_{jq} \bar\xi^p_{j'q} \right) = \det(\xi\xi^*) = \det \begin{pmatrix} \xi^1\xi^{1*} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi^n\xi^{n*} \end{pmatrix} . \tag{16}$$

Applying (15) with $H = \xi\xi^*$, we have

$$\begin{aligned} G_{nkm} &= \frac{1}{\pi^{nkm}\det\Lambda} \int_{\mathbb{C}^{nkm}} \det(\xi\xi^*) e^{-\langle \Lambda^{-1}\xi, \bar\xi\rangle}\, d\xi \\ &= \frac{1}{\pi^{nkm}\det\Lambda} \int_{\mathbb{C}^{nkm}} \int e^{-\langle \Lambda^{-1}\xi,\bar\xi\rangle - \langle \xi\xi^*\eta,\bar\eta\rangle}\, d\eta d\xi\,, \qquad (17) \\ \langle \xi\xi^*\eta, \bar\eta\rangle &= \sum_{p,q,j,j'} \xi^p_{jq}\bar\xi^p_{j'q}\eta^p_{j'}\bar\eta^p_j = \langle \Omega\xi, \bar\xi\rangle\,, \qquad (18) \end{aligned}$$

where $\Omega$ is given by (2). Note that the entries of $\Omega$ commute, since they are of degree 2. Furthermore, adopting the supersymmetric definition of the conjugate [Ef],

$$(\eta^p_j)^- = \bar\eta^p_j, \quad (\bar\eta^p_j)^- = -\eta^p_j\,,$$

we see that the matrix $\Omega$ is superhermitian; i.e., $\Omega^* = \Omega$, where $\Omega^* = {}^t\Omega^-$.

Thus by (17)–(18), we have

$$G_{nkm} = \frac{1}{\pi^{nkm}\det\Lambda} \int_{\mathbb{C}^{nkm}} \int e^{-\left\langle (\Lambda^{-1}+\Omega)\xi, \bar\xi\right\rangle}\, d\eta d\xi\,. \tag{19}$$

We recall that

$$\frac{1}{\pi^{nkm}} \int_{\mathbb{C}^{nkm}} e^{-\langle P\xi,\bar\xi\rangle} d\xi = \det P^{-1}\,, \tag{20}$$

for a positive definite, Hermitian $(nkm \times nkm)$ matrix $P$. Furthermore, (20) holds when $P$ is the superhermitian matrix $\Lambda^{-1} + \Omega$; we give a short proof of this fact below. Reversing the order of integration in (19) and applying (20) with $P = \Lambda^{-1} + \Omega$, we have

$$\begin{aligned} G_{nkm} &= \frac{1}{\det\Lambda} \int \frac{1}{\det(\Lambda^{-1}+\Omega)} d\eta \\ &= \int \frac{1}{\det(I + \Lambda\Omega)} d\eta\,. \qquad (21) \end{aligned}$$

We now verify by formal substitution that (20) holds when $P = \Lambda^{-1} + \Omega$: Suppose that $\Theta_1, \cdots \Theta_t$ are even fermionic functions; i.e., $\Theta_j \in \bigwedge^{\text{even}} \mathbb{C}^{2l}$. Let $S_\Theta$ be the homomorphism from the algebra $\mathbb{C}\{z_1, \dots, z_t, \bar z_1, \dots, \bar z_t\}$ of convergent power series onto $\bigwedge^{\text{even}} \mathbb{C}^{2l}$ given by substitution:

$$S_\Theta[f(z_1, \dots, z_t, \bar z_1, \dots, \bar z_t)] = f(\Theta_1, \dots, \Theta_t, \bar\Theta_1, \dots, \bar\Theta_t)\,.$$

Let

$$f(Z,\bar Z) = \int e^{-\langle(\Lambda^{-1}+Z+Z^*)\xi,\bar\xi\rangle} d\xi = \int e^{-\langle(Z+Z^*)\xi,\bar\xi\rangle} e^{-\langle\Lambda^{-1}\xi,\bar\xi\rangle} d\xi = \frac{1}{\det(\Lambda^{-1}+Z+Z^*)}\,, \tag{22}$$

for $Z=(z_{\alpha\beta})\in \mathfrak{gl}(nkm,\mathbb{C})$, where the last equality is by (20). We easily see that $f$ is a convergent power series in $\{z_{\alpha\beta},\bar z_{\alpha\beta}\}$ and that the integrand $e^{-\langle(Z+Z^*)\xi,\bar\xi\rangle}$ can be written as an absolutely convergent power series in $\{z_{\alpha\beta},\bar z_{\alpha\beta}\}$ with values in $\mathcal{L}^1(e^{-\langle\Lambda^{-1}\xi,\bar\xi\rangle}d\xi)$. We now let $\Theta_{\alpha\beta}=\frac12\Omega_{\alpha\beta}=\frac12\bar\Omega_{\beta\alpha}$; the conclusion follows by applying $S_\Theta$ to (22). □

3.1. **Pair correlation.** In this section, we prove Corollary 1.2. To illustrate the computation, we consider first the case $k=m=1$ of zero correlations in dimension one: We have

$$\Lambda=\begin{pmatrix}\Lambda_1^1 & \Lambda_2^1\\ \Lambda_1^2 & \Lambda_2^2\end{pmatrix},\quad \Omega=\begin{pmatrix}\eta^1\bar\eta^1 & 0\\ 0 & \eta^2\bar\eta^2\end{pmatrix},\quad I+\Lambda\Omega=\begin{pmatrix}1+\Lambda_1^1\eta^1\bar\eta^1 & \Lambda_2^1\eta^2\bar\eta^2\\ \Lambda_1^2\eta^1\bar\eta^1 & 1+\Lambda_2^2\eta^2\bar\eta^2\end{pmatrix}.$$

We easily compute

$$\begin{aligned}
\det(I+\Lambda\Omega) &= 1+\Lambda_1^1\eta^1\bar\eta^1+\Lambda_2^2\eta^2\bar\eta^2+(\det\Lambda)\eta^1\bar\eta^1\eta^2\bar\eta^2\,,\\
\det(I+\Lambda\Omega)^{-1} &= 1-\Lambda_1^1\eta^1\bar\eta^1-\Lambda_2^2\eta^2\bar\eta^2+(2\Lambda_1^1\Lambda_2^2-\det\Lambda)\eta^1\bar\eta^1\eta^2\bar\eta^2\,,\\
\int \det(I+\Lambda\Omega)^{-1}d\bar\eta^1 d\eta^1 d\bar\eta^2 d\eta^2 &= 2\Lambda_1^1\Lambda_2^2-\det\Lambda \;=\; \Lambda_1^1\Lambda_2^2+\Lambda_2^1\Lambda_1^2\,.
\end{aligned}$$

Hence by Theorem 1.1, we have

$$\kappa_{11}(r)=\frac{\Lambda_1^1\Lambda_2^2+\Lambda_2^1\Lambda_1^2}{\det A}\,. \tag{23}$$

To obtain the explicit formula for $\kappa_{11}(r)$ [Ha, BSZ1], we set $z^1=(r,0,\dots,0)$, $z^2=0$ and substitute in (23) the resulting values of $\Lambda_{p'}^p$ and $\det A$ (see (24)–(26) below).

To obtain formulas for $\kappa_{km}(r)$ for higher $k,m$, we again set $z^1=(r,0,\dots,0)$, $z^2=0$. Using (6), (8) and (9), we see that $\Lambda^{pjq}_{p'j'q'}=0$ for $(j,q)\neq(j',q')$ and

$$\begin{pmatrix}\Lambda^{1j1}_{1j1} & \Lambda^{1j1}_{2j1}\\ \Lambda^{2j1}_{1j1} & \Lambda^{2j1}_{2j1}\end{pmatrix}=\begin{pmatrix}P & Q\\ Q & P\end{pmatrix},\quad \begin{pmatrix}\Lambda^{1jq}_{1jq} & \Lambda^{1jq}_{2jq}\\ \Lambda^{2jq}_{1jq} & \Lambda^{2jq}_{2jq}\end{pmatrix}=\begin{pmatrix}R & S\\ S & R\end{pmatrix}\ \text{for } q\ge 2\,, \tag{24}$$

where

$$\begin{aligned}
P &= \frac{1-e^{-r^2}-r^2e^{-r^2}}{1-e^{-r^2}}\,, & Q &= \frac{e^{-\frac12 r^2}\left(1-e^{-r^2}-r^2\right)}{1-e^{-r^2}}\,,\\
R &= 1\,, & S &= e^{-\frac12 r^2}\,.
\end{aligned} \tag{25}$$

We also have

$$A(r)=\begin{pmatrix}1 & e^{-\frac12 r^2}\\ e^{-\frac12 r^2} & 1\end{pmatrix}$$

and thus

$$\det A = 1-e^{-r^2}\,. \tag{26}$$

We can write the $2km \times 2km$ matrices $\Lambda, \Omega$ in block form:

$$\Lambda = \begin{pmatrix} \Lambda' & 0 & \cdots & 0 \\ 0 & \Lambda'' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda'' \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega' & 0 & \cdots & 0 \\ 0 & \Omega' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega' \end{pmatrix},$$

$$\Lambda' = \begin{pmatrix} PI_k & QI_k \\ QI_k & PI_k \end{pmatrix}, \quad \Lambda'' = \begin{pmatrix} RI_k & SI_k \\ SI_k & RI_k \end{pmatrix}, \quad \Omega' = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix},$$

where $I_k$ is the $k \times k$ identity matrix and

$$\Omega_p = \begin{pmatrix} \eta_1^p \bar\eta_1^p & \cdots & \eta_k^p \bar\eta_1^p \\ \vdots & & \vdots \\ \eta_1^p \bar\eta_k^p & \cdots & \eta_k^p \bar\eta_k^p \end{pmatrix}, \qquad p = 1, 2\,.$$

Hence

$$\det(I + \Lambda\Omega) = \Phi\Psi^{m-1}\,, \tag{27}$$

where

$$\Phi = \det(I + \Lambda'\,\Omega')\,, \quad \Psi = \det(I + \Lambda''\,\Omega')\,. \tag{28}$$

We note that

$$I + \Lambda'\Omega' = \begin{pmatrix} I + P\Omega_1 & Q\Omega_2 \\ Q\Omega_1 & I + P\Omega_2 \end{pmatrix},$$

and thus

$$\begin{aligned} \Phi &= \det(I + P\Omega_1) \det\left[I + P\Omega_2 - Q^2\Omega_1(I + P\Omega_1)^{-1}\Omega_2\right] \\ &= \det\left[I + P(\Omega_1 + \Omega_2) + (P^2 - Q^2)\Omega_1\Omega_2\right]\,. \end{aligned} \tag{29}$$

Similarly,

$$\Psi = \det\left[I + R(\Omega_1 + \Omega_2) + (R^2 - S^2)\Omega_1\Omega_2\right]\,. \tag{30}$$

We recall that by Theorem 1.1,

$$\kappa_{km}(r) = \frac{[(m-k)!]^2}{(m!)^2(1 - e^{-r^2})^k} \int \frac{1}{\det(I + \Lambda\Omega)} d\eta\,. \tag{31}$$

Combining (25)–(31), we obtain Corollary 1.2. $\square$

To obtain explicit formulas for the pair correlation in a fixed codimension $k$ (for all dimensions $m$), we write $\Psi = 1 - (1 - \Psi)$ and our formula becomes:

$$\kappa_{km}(r) = \frac{[(m-k)!]^2}{(m!)^2(1 - e^{-r^2})^k} \sum_{t=0}^{2k} \binom{m+t-2}{t} \int \Phi^{-1}(1 - \Psi)^t\, d\eta\,. \tag{32}$$

Using Maple™, we evaluate (32) to obtain the following pair correlation formulas:

$$\begin{aligned}
\kappa_{1m}(r) &= \frac{1}{m^2 \det A}\Big[P^2 + 2\,(m-1)\,PR + Q^2 + (m-1)^2\,R^2 + (m-1)\,S^2\Big]\,,\\
\kappa_{2m}(r) &= \frac{1}{m^2(m-1)\det A^2}\Big[4(m-1)P^2R^2 + 2P^2S^2 + 4\,(m-1)\,(m-2)\,PR^3\\
&\quad + 4(m-2)PRS^2 + 2\,(m-1)\,Q^2R^2 + 4Q^2S^2 + (m-1)\,(m-2)^2\,R^4\\
&\quad + 2\,(m-2)^2\,R^2S^2 + 2\,(m-2)\,S^4\Big]\,,\\
\kappa_{3m}(r) &= \frac{1}{m^2(m-1)(m-2)\det A^3}\Big[9\,(m-1)\,(m-2)\,P^2R^4 + 12\,(m-2)\,P^2R^2S^2\\
&\quad + 6P^2S^4 + 6\,(m-3)\,(m-1)\,(m-2)\,PR^5 + 12\,(m-3)\,(m-2)\,PR^3S^2\\
&\quad + 12\,(m-3)\,PRS^4 + 3\,(m-1)\,(m-2)\,Q^2R^4\\
&\quad + 12\,(m-2)\,Q^2R^2S^2 + 18Q^2S^4 + (m-1)\,(m-2)\,(m-3)^2\,R^6\\
&\quad + 3\,(m-2)\,(m-3)^2\,R^4S^2 + 6\,(m-3)^2\,R^2S^4 + 6\,(m-3)\,S^6\Big]\,.
\end{aligned}$$

Recalling (25)–(26), we then obtain power series expansions of the pair correlation function in codimensions $1, 2, 3$:

$$\begin{aligned}
\kappa_{1m}(r) &= \frac{m-1}{m}r^{-2} + \frac{m-1}{2\,m} + \frac{(m+2)\,(m+1)}{12\,m^2}r^2 - \frac{(m+4)\,(m+3)}{720\,m^2}r^6\\
&\quad + \frac{(m+6)\,(m+5)}{30240m^2}r^{10} - \frac{(m+8)\,(m+7)}{1209600\,m^2}r^{14}\cdots\,,\\
\kappa_{2m}(r) &= \frac{m-2}{m}r^{-4} + \frac{m-2}{m}r^{-2} + \frac{5m^2-7m+12}{12(m-1)m} + \frac{(m-2)(m+2)(m+1)}{12(m-1)m^2}r^2\\
&\quad + \frac{(m+3)(m+2)}{240(m-1)m}r^4 - \frac{(m-2)(m+4)(m+3)}{720(m-1)m^2}r^6 + \dots\,,\\
\kappa_{3m}(r) &= \frac{m-3}{m}r^{-6} + \frac{3(m-3)}{2\,m}r^{-4} + \frac{m^2-4\,m+6}{(m-2)\,m}r^{-2} + \frac{(m-3)\,(3\,m^2-m+8)}{8\,m\,(m-1)\,(m-2)}\\
&\quad + \frac{(m+2)\,(m+1)\,(19\,m^2-79\,m+120)}{240\,m^2\,(m-1)\,(m-2)}r^2\\
&\quad + \frac{(m-3)\,(m+3)\,(m+2)}{160\,m\,(m-1)\,(m-2)}r^4\cdots\,.
\end{aligned}$$

(The power series for $\kappa_{1m}$ and $\kappa_{2m}$ were given in [BSZ1] and [BSZ2] respectively.)

## 4. The point case

We now prove Theorem 1.3: For the case $k = m$, where the zero set is discrete, (12) becomes:

$$\kappa_{mm}(r) = \frac{G_m(r)}{(m!)^2 \det A(r)^m}\,, \tag{33}$$

where

$$G_m(r) = G_{2mm}(r) = \int_{\mathbb{C}^{2m^2}} \left| \det_{1\le j,q\le m} \left(\xi^1_{jq}\right) \det_{1\le j,q\le m} \left(\xi^2_{jq}\right) \right|^2 d\gamma_{\Lambda(r)}(\xi)\,. \tag{34}$$

We let $\left\langle \,\cdot\, \right\rangle_{\Lambda(r)} = \int \cdot\, d\gamma_{\Lambda(r)}$ denote the expected value with respect to the Gaussian probability measure $\gamma_{\Lambda(r)}$. Thus

$$\begin{aligned} G_m(r) &= \left\langle \det(\xi^1)\det(\xi^2)\det(\bar\xi^1)\det(\bar\xi^2)\right\rangle_{\Lambda(r)} \\ &= \sum_{\alpha,\beta,\mu,\nu} (-1)^{\alpha+\beta+\mu+\nu} \left\langle \left(\prod_q \xi^1_{\alpha_q q}\right)\left(\prod_q \xi^2_{\beta_q q}\right)\left(\prod_q \bar\xi^1_{\mu_q q}\right)\left(\prod_q \bar\xi^2_{\nu_q q}\right)\right\rangle_{\Lambda(r)}, \end{aligned} \tag{35}$$

where the sum is over all 4-tuples $\alpha, \beta, \mu, \nu \in \mathcal{S}_m$ (= permutations of $\{1, \dots, m\}$), and $(-1)^\sigma$ stands for the sign of the permutation $\sigma$. We shall compute the terms of (35) using the Wick formula rather than directly from the Berezin integral formula. The computations simplifies considerably since the matrix $\Lambda(r)$ is sparse. In fact, we shall see that the sign $(-1)^{\alpha+\beta+\mu+\nu}$ is positive whenever the corresponding moment is nonzero.

Let us now evaluate the moments of order $4m$ in (35):

$$\mathcal{M}_{\alpha\beta\mu\nu} := \left\langle \left(\prod_q \xi^1_{\alpha_q q}\right)\left(\prod_q \xi^2_{\beta_q q}\right)\left(\prod_q \bar\xi^1_{\mu_q q}\right)\left(\prod_q \bar\xi^2_{\nu_q q}\right)\right\rangle_{\Lambda(r)}. \tag{36}$$

Recall that the Wick formula expresses $\mathcal{M}_{\alpha\beta\mu\nu}$ as a sum of products of second moments with respect to the Gaussian measure $\gamma_{\Lambda(r)}$. Since this Gaussian is complex, these second moments come from pairings of $\xi$'s with $\bar\xi$'s. We write

$$\Lambda^{pjq}_{p'j'q'} = \delta^j_{j'}\Lambda^{pq}_{p'q'} = \left\langle \xi^p_{jq}\bar\xi^{p'}_{j'q'}\right\rangle_{\Lambda(r)}\,. \tag{37}$$

Hence the term $\mathcal{M}_{\alpha\beta\mu\nu}$ equals the permanent of the submatrix of $\left(\Lambda^{pjq}_{p'j'q'}\right)$ formed from the rows corresponding to the variables $\xi^1_{\alpha_1 1}, \dots, \xi^1_{\alpha_m m}, \xi^2_{\beta_1 1}, \dots, \xi^2_{\beta_m m}$ and columns corresponding to $\bar\xi^1_{\mu(1)1}, \dots, \bar\xi^1_{\mu(m)m}, \bar\xi^2_{\nu(1)1}, \dots, \bar\xi^2_{\nu(m)m}$:

$$\left(\begin{array}{ccc|ccc} \Lambda^{1\alpha_1 1}_{1\mu_1 1} & \cdots & \Lambda^{1\alpha_1 1}_{1\mu_m m} & \Lambda^{1\alpha_1 1}_{2\nu_1 1} & \cdots & \Lambda^{1\alpha_1 1}_{2\nu_m m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Lambda^{1\alpha_m m}_{1\mu_1 1} & \cdots & \Lambda^{1\alpha_m m}_{1\mu_m m} & \Lambda^{1\alpha_m m}_{2\nu_1 1} & \cdots & \Lambda^{1\alpha_m m}_{2\nu_m m} \\ \hline \Lambda^{2\beta_1 1}_{1\mu_1 1} & \cdots & \Lambda^{2\beta_1 1}_{1\mu_m m} & \Lambda^{2\beta_1 1}_{2\nu_1 1} & \cdots & \Lambda^{2\beta_1 1}_{2\nu_m m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Lambda^{2\beta_m m}_{1\mu_1 1} & \cdots & \Lambda^{2\beta_m m}_{1\mu_m m} & \Lambda^{2\beta_m m}_{2\nu_1 1} & \cdots & \Lambda^{2\beta_m m}_{2\nu_m m} \end{array}\right). \tag{38}$$

(Recall that permanent$(V_{ij}) = \sum_\sigma \prod_i V_{i\sigma_i}$, where the sum is over all permutations $\sigma$.)

To compute $\kappa_{mm}(r)$, we can set $z^1 = (r, 0, \dots, 0)$, $z^2 = 0$, as before. Recalling (24) and the fact that $\Lambda^{pjq}_{p'j'q'} = 0$ for $(j, q) \neq (j', q')$, we observe that (38) is made up of 4 diagonal matrices. For example, if $m = 3$, then (38) becomes

$$\left(\begin{array}{ccc|ccc} \delta^{\alpha_1}_{\mu_1}P & 0 & 0 & \delta^{\alpha_1}_{\nu_1}Q & 0 & 0 \\ 0 & \delta^{\alpha_2}_{\mu_2}R & 0 & 0 & \delta^{\alpha_2}_{\nu_2}S & 0 \\ 0 & 0 & \delta^{\alpha_3}_{\mu_3}R & 0 & 0 & \delta^{\alpha_3}_{\nu_3}S \\ \hline \delta^{\beta_1}_{\mu_1}Q & 0 & 0 & \delta^{\beta_1}_{\nu_1}P & 0 & 0 \\ 0 & \delta^{\beta_2}_{\mu_2}S & 0 & 0 & \delta^{\beta_2}_{\nu_2}R & 0 \\ 0 & 0 & \delta^{\beta_3}_{\mu_3}S & 0 & 0 & \delta^{\beta_3}_{\nu_3}R \end{array}\right).$$

We conclude that $\mathcal{M}_{\alpha\beta\mu\nu}$ is a product of $2\times 2$ permanents:

$$\mathcal{M}_{\alpha\beta\mu\nu} = \left(\delta^{\alpha_1}_{\mu_1}\delta^{\beta_1}_{\nu_1}P^2 + \delta^{\alpha_1}_{\nu_1}\delta^{\beta_1}_{\mu_1}Q^2\right)\prod_{q=2}^{m}\left(\delta^{\alpha_q}_{\mu_q}\delta^{\beta_q}_{\nu_q}R^2 + \delta^{\alpha_q}_{\nu_q}\delta^{\beta_q}_{\mu_q}S^2\right). \tag{39}$$

In particular, $\mathcal{M}_{\alpha\beta\mu\nu}$ vanishes unless

$$\{\mu_q,\nu_q\} = \{\alpha_q,\beta_q\} \qquad \text{for } 1\le\ q\le m\,. \tag{40}$$

We now claim that (40) implies that $(-1)^{\alpha+\beta+\mu+\nu}=1$: First of all, by multiplying the four permutations by $\alpha^{-1}$ on the left, we can assume without loss of generality that $\alpha_i = i$ for all $i$. Now write $\beta$ as a product of disjoint cycles. Then one sees that $\mu$ is a product of some of these cycles and $\nu$ is a product of the other cycles, and the positivity of the product of signs easily follows.

Hence equation (35) becomes

$$G_m(r) = \sum_{\alpha,\beta,\mu,\nu}\mathcal{M}_{\alpha\beta\mu\nu}\,. \tag{41}$$

We now use (35) to evaluate $G_m$ for arbitrary dimension $m$.

LEMMA **4.1.** $G_m = (m-1)!m!\big[P^2 f_m(R^2,S^2) + Q^2 f_m(S^2,R^2)\big]$,
*where*

$$\begin{aligned} f_m(x,y) &= y^{m-1} + 2xy^{m-2} + \cdots + (m-1)x^{m-2}y + mx^{m-1} \\ &= \frac{mx^{m+1} + y^{m+1} - (m+1)\,x^m y}{(x-y)^2}\,. \end{aligned}$$

*Proof.* We use induction on $m$. The identity holds trivially for $m=1$, since $f_1=1$ and $G_1 = P^2+Q^2$. Let $m\ge 2$ and suppose the identity has been verified for $1,\dots,m-1$. Since $\mathcal{M}_{\alpha\beta\mu\nu}$ is unchanged if we multiply all the permutations on the left by $\alpha^{-1}$, we have $G_m = m!G^0_m$, where $G^0_m = \sum_{\beta,\mu,\nu}\mathcal{M}_{e\beta\mu\nu}$ ($e$ = identity).

For $1\le i\le m$, let $\mathcal{C}_i\subset\mathcal{S}_m$ denote the collection of $i$-cycles of the form $(1a_2\dots a_i)$. For each $\sigma\in\mathcal{C}_i$, let $\sigma^\perp$ denote those permutations $\tau\in\mathcal{S}_m$ that fix the elements $1,a_1,\dots,a_i$. We claim that

$$\sum_{\beta\in\sigma^\perp}\sum_{\mu,\nu}\mathcal{M}_{e\beta\mu\nu} = (m-i)!(P^2R^{2i-2}+Q^2S^{2i-2})g_{m-i}(R^2,S^2)\,, \tag{42}$$

where $g_l(x,y) = x^l + x^{l-1}y + \cdots + xy^{l-1} + y^l$.

To verify (42), we can assume without loss of generality that $\sigma=(1\dots i)$. Recall that we need only consider permutations $\mu$ that are products of some of the cycles in $\beta$ (and $\nu$ is determined by the pair $(\beta,\mu)$, since $\nu$ is the product of the other cycles of $\beta$ when $\mathcal{M}_{e\beta\mu\nu}\neq 0$).

For the $P^2$-terms of the sum, $\nu$ contains the cycle $(1\dots i)$ so that $\mu_q = q,\ \nu_q = \sigma_q = \beta_q$ for $q = 1, \dots, i$. (For the $Q^2$-terms, $\mu$ contains $(1\dots i)$.) Hence by (39), we have

$$\sum_{\beta\in\sigma^\perp}\sum_{\mu,\nu}\mathcal{M}_{e\beta\mu\nu} = P^2R^{2i-2}\sum_{\beta\in\sigma^\perp}\sum_{\mu,\nu}{}'\prod_{q=i+1}^{m}\left(\delta^q_{\mu_q}\delta^{\beta_q}_{\nu_q}R^2 + \delta^q_{\nu_q}\delta^{\beta_q}_{\mu_q}S^2\right) + \text{terms with } Q^2\,, \tag{43}$$

where $\sum'$ is over those $\mu,\nu$ with $\mu_q = q,\ \nu_q = \beta_q$, for $1 \le q \le i$. To compute the double sum in the right side of (43), we notice by (39) and (41) that it equals $\frac{1}{(m-i)!}G_{m-i}$ with $P, Q$ replaced by $R, S$ respectively. Hence by our inductive assumption, we have

$$\begin{aligned}\sum_{\beta\in\sigma^\perp}\sum_{\mu,\nu}{}'\prod_{q=i+1}^{m}\left(\delta^q_{\mu_q}\delta^{\beta_q}_{\nu_q}R^2 + \delta^q_{\nu_q}\delta^{\beta_q}_{\mu_q}S^2\right) &= (m-i-1)![R^2 f_{m-i}(R^2,S^2) + S^2 f_{m-i}(S^2,R^2)]\\ &= (m-i)!\,g_{m-i}(R^2,S^2)\,.\end{aligned}$$

The computation of the $Q^2$ terms is similar, and hence (42) holds.

We now have by (42),

$$G_m = m!\sum_{i=1}^{m}\sum_{\sigma\in\mathcal{C}_i}\sum_{\beta\in\sigma^\perp}\sum_{\mu,\nu}\mathcal{M}_{e\beta\mu\nu} = m!\sum_{i=1}^{m}(\#\mathcal{C}_i)(m-i)!(P^2R^{2i-2} + Q^2S^{2i-2})g_{m-i}(R^2,S^2)\,.$$

Noting that $(\#\mathcal{C}_i)(m-i)! = (m-1)!$ and summing over $i$, we obtain the desired formula. ∎

We now complete the proof of Theorem 1.3. By (33) and Lemma 4.1, we have

$$\kappa_{mm}(r) = \frac{P^2 f_m(R^2,S^2) + Q^2 f_m(S^2,R^2)}{m(\det A(r))^m}\,. \tag{44}$$

Substituting (25)–(26) into (44), we obtain (4).

Finally, to verify (5), we note by (25) that $P = \frac{1}{2}r^2 + \cdots,\ Q = \frac{1}{2}r^2 + \cdots,\ R = 1,\ S = 1 + \cdots$, and hence

$$\kappa_{mm} = \frac{2f_m(1,1)(r^4/4) + \cdots}{mr^{2m} + \cdots} = \frac{f_m(1,1)}{2m}r^{4-2m} + \cdots = \frac{m+1}{4}r^{4-2m} + \cdots\,.$$

The following proposition yields the remainder estimate of (5).

**Proposition 4.2.** *If $m$ is odd, resp. even, then $\kappa_{mm}(r)$ is an odd, resp. even, function of $r^2$.*

*Proof.* Let $\widehat{P}, \widehat{Q}$ be the functions given by $P(r) = \widehat{P}(u),\ Q(r) = \widehat{Q}(u)$, where $u = r^2$. From (44), we have

$$\kappa_{mm} = \frac{\widehat{P}(u)^2 f_m(1, e^{-u}) + \widehat{Q}(u)^2 f_m(e^{-u}, 1)}{m(1-e^{-u})^m}\,.$$

We observe from (25) that $\widehat{P}(-u) = e^{u/2}\widehat{Q}(u)$ and thus

$$\kappa_{mm}(-u) = \frac{e^u\widehat{Q}(u)^2 f_m(1, e^u) + e^u\widehat{P}(u)^2 f_m(e^u, 1)}{m(1-e^u)^m} = (-1)^m\kappa_{mm}(u)\,,$$

since $f_m$ is homogeneous of order $m-1$. ∎

The expansions of (4) are easily obtained using Maple™:

$$\begin{aligned}
\kappa_{11}(r) &= \frac{1}{2}r^2 - \frac{1}{36}r^6 + \frac{1}{720}r^{10} - \frac{1}{16800}r^{14} + \frac{1}{435456}r^{18} - \frac{691}{8382528000}r^{22} \cdots \\
\kappa_{22}(r) &= \frac{3}{4} + \frac{1}{24}r^4 - \frac{1}{288}r^8 + \frac{1}{4800}r^{12} - \frac{1}{96768}r^{16} + \frac{691}{1524096000}r^{20} \cdots \\
\kappa_{33}(r) &= r^{-2} + \frac{1}{4}r^2 - \frac{11}{2160}r^6 - \frac{1}{50400}r^{10} + \frac{1}{80640}r^{14} - \frac{4871}{5029516800}r^{18} \cdots \\
\kappa_{44}(r) &= \frac{5}{4}r^{-4} + \frac{95}{144} + \frac{19}{576}r^4 - \frac{79}{40320}r^8 + \frac{7}{82944}r^{12} - \frac{6049}{2235340800}r^{16} \cdots \\
\kappa_{55}(r) &= \frac{3}{2}r^{-6} + \frac{4}{3}r^{-2} + \frac{55}{288}r^2 - \frac{19}{16800}r^6 - \frac{257}{1451520}r^{10} + \frac{21337}{1397088000}r^{14} \cdots \\
\kappa_{66}(r) &= \frac{7}{4}r^{-8} + \frac{7}{3}r^{-4} + \frac{5257}{8640} + \frac{407}{14400}r^4 - \frac{103}{82944}r^8 + \frac{38177}{1197504000}r^{12} \cdots
\end{aligned}$$

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