Upper Bounds on Arnold Diffusion Time via Mather Theory

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Abstract

We consider several Hamiltonian systems for which the existence of Arnold's mechanism for diffusion (whiskered tori, transition ladder, etc.) has been proven. By means of Mather theory we show that the diffusion time may be bounded by a power of the homoclinic splitting.

Introduction

Let us consider the following Hamiltonian

$$H: \mathbf{T}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R} \qquad H(x, X) = h(X) + F(x, X) \tag{$*$}$$

where $\mathbf{T}^{n+1} = \frac{\mathbf{R}^{n+1}}{2\pi \mathbf{Z}^{n+1}}$ is the $(n+1)$- dimensional torus; h is an analytic function and F is analytic and of small norm; (x, X) are standard symplectic variables. Usually, H is referred to as a quasi-integrable Hamiltonian since it is a small perturbation of $h(X)$ whose motions are very simple to integrate. The aim of perturbation theory is to understand the orbit structure of H, particularly with regard to stability; for example one would like to provide bounds on $|X(t) - X(0)|$ for t as large as possible. In some problems of celestial mechanics, for instance, the variable X is related to the length of the semiaxes of the ellipses on which the planets run and strong oscillations of this variable could lead to collisions; in other models, X is related to the inclination of a planet's axis and determines which part of it receives the most light from the sun. Under certain conditions on h, there are two well-known theorems dealing with the stability of H, namely the KAM and Nekhorocheff theorems. Grossly, the KAM theorem asserts that, except for a set of small measure (of order $\|F\|^{\frac{1}{2}}$) of initial conditions, $X(t)$ remains for all times t in a $\|F\|^{\frac{1}{2}}$-neighbourhood of $X(0)$. The Nekhorocheff theorem asserts that, for all initial conditions, $X(t)$ remains close to $X(0)$ (to order $\|f\|^a$) for all times not exceeding $\exp \frac{D}{\|F\|^b}$, where a, b and D are positive constants depending only on h and on the dimension n.

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One would like to know whether these theorems are sharp; a simpler problem is to find perturbations F of arbitrarily small norm admitting orbits which satisfy, for some $T > 0$,

$$|X(T) - X(0)| \geq c \ , \qquad (**)$$

where c is a positive constant independent of F. There is a small class of analytic examples where $(**)$ has been proven. The first example was given in [A]; since [CG] (where a theory showing the existence of diffusion in general "a-priori unstable systems" is presented†) several generalizations have appeared; we quote, in particular, [G], [GGM1], [B]. The aim of this paper is to provide a variational method, based on Mather theory, apt to give bounds from above on the "diffusion time", i. e. the least time for which $(**)$ holds; in particular we consider the examples given in the quoted references and prove upper bound on the diffusion time for them.

We will consider the following five families of Hamiltonians:

$(\mathcal{CG})$
$$H(Q,q,I,p) = \frac{1}{2}|I|^2 + \frac{1}{2}p^2 + (\cos(q) - 1) + \epsilon f(Q,q)$$
$$(Q,I) \in \mathbf{T}^n \times \mathbf{R}^n \qquad (q,p) \in \mathbf{T}^1 \times \mathbf{R}^1$$

$(\mathcal{G})$
$$H(Q,q,I,p) = \langle \omega, I \rangle + \frac{1}{2}p^2 + (\cos(q) - 1) + \epsilon f(Q,q)$$
$$(Q,I) \in \mathbf{T}^n \times \mathbf{R}^n \qquad (q,p) \in \mathbf{T}^1 \times \mathbf{R}^1$$

$(\mathcal{GGM})$
$$H(Q,q,I,p) = \sqrt{\eta}\Omega_1 I_1 + \eta \frac{I_1^2}{2} + \eta^{-\frac{1}{2}}\Omega_2 I_2 + \frac{1}{2}p^2 + (\cos(q) - 1) + \epsilon f(Q_1, Q_2, q)$$
$$(Q_1, Q_2, I_1, I_2) \in \mathbf{T}^2 \times \mathbf{R}^2 \qquad (q,p) \in \mathbf{T}^1 \times \mathbf{R}^1 \qquad \eta > 0$$

$(\mathcal{B}1)$
$$H(Q,q,I,p) = \epsilon\langle \omega, I \rangle + \frac{1}{2}p^2 + \epsilon^d(\cos(q) - 1) + \epsilon^{d'} f(Q,q)$$
$$(Q,I) \in \mathbf{T}^n \times \mathbf{R}^n \qquad (q,p) \in \mathbf{T}^1 \times \mathbf{R}^1 \quad 1 \leq d \leq 2 \quad d' > 3 + d/2 \qquad \epsilon > 0$$

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† Roughly speaking, "a-priori unstable systems" are nearly–integrable hamiltonian systems, the integrable part of which carries separatrices. We remind that some flaws have been detected in [CG] (see the *Erratum* in [CG]). Obviously we are referring here to those parts of [CG] known to be correct: in particular the general analysis for a-priori unstable systems, i.e., §1 through §8 of [CG] (in §8 there is a minor mistake concerning the quantitative treatment of the construction of diffusing orbit: such mistake has been corrected, for example, in [CV]).

$$(\mathcal{B}2) \qquad H(Q,q,I,p) = \epsilon\frac{1}{2}|I|^2 + \frac{1}{2}p^2 + \epsilon^d(\cos(q)-1) + \epsilon^{d'} f(Q,q)$$

$$(Q,I) \in \mathbf{T}^n \times \mathbf{R}^n \qquad (q,p) \in \mathbf{T}^1 \times \mathbf{R}^1 \quad 1 \le d \le 2 \quad d' > 3 + d/2 \qquad \epsilon > 0$$

where $\langle \cdot, \cdot \rangle$ and by $|\cdot|$ denote respectively the standard inner product and norm in $\mathbf{R}^n$; f is a suitable trigonometric polynomial; the authors above usually choose

$$f(Q,q) = \sum_{i=0}^{n} a_i \cos(Q_i + q) \qquad a_i \neq 0 \ , \forall i, \qquad \sum_{i=0}^{n} |a_i| \le 1. \tag{1}$$

Since we want to use their perturbation results, f will satisfy (1) throughout the paper.

Some of the systems above represent simplified models of some hamiltonians of celestial mechanics; although none of them are in the form $(*)$ they are considered as a test ground for perturbation theory; for a full account of their origin and properties we refer the reader to the papers where these systems were introduced, [CG], [C], [G], [GGM1] and [B]. We call $(\mathcal{CG})$ the *a priori unstable* system, $(\mathcal{G})$ the *isochronous* system,$(\mathcal{GGM})$ the *three time scales* system, $(\mathcal{B}1)$ the *linear degenerate* system, and $(\mathcal{B}2)$ the *quadratic degenerate* system. We observe that for certain values of the parameters d and d' $(\mathcal{B}1)$ and $(\mathcal{B}2)$ coincide after rescaling with $(\mathcal{CG})$ and $(\mathcal{G})$. We note that $(\mathcal{CG})$ includes [A] as a particular case; see also [Be1] and [Be2] for related results.

We note that these systems consist in rotators coupled with a pendulum. The variables of the pendulum are the canonically conjugated coordinates p and q. The variables of the rotators are the canonically conjugated coordinates I and Q, and we call them "actions" and "angles", respectively.

If $\epsilon = 0$, the tori $\{I = const, q = p = 0\}$ are preserved by the Hamiltonian flow; it is easy to see that they have $n+1$ dimensional stable and unstable manifolds given by $\{I = const\}$ times the stable and unstable manifolds of the pendulum. When $\epsilon \neq 0$ these manifolds are perturbed and the stable manifold of one torus can intersect the unstable manifold of another torus. The proof of this can be very hard and much literature has been spawned by this problem; we will use the results of [CG], [C], [B], [G], and [GGM1] which show that, for any f and $\epsilon \neq 0$ small, each of the systems above has a family of invariant KAM tori of codimension 1, $\tau_1, \ldots, \tau_N$; on each τ_i the flow is conjugated to a rotation of frequency ω_i, with ω_i satisfying a diophantine condition of the type

$$|\langle \omega_i, k \rangle| \ge \frac{C}{|k|^{\Xi}} \quad \forall k \in \mathbf{Z}^n \setminus \{0\}.$$

The explicit value of the constants C and Ξ is stated in proposition 1. Each τ_i has an unstable manifold (christened "whisker" in [A]) which, if f satisfies (1), intersects transversally the stable manifold of τ_{i+1}; τ_1 and τ_N are at distance of order (at least) 1. Since the intersection is transversal, an angle between the two manifolds can be defined; this is commonly known as the "splitting" and its magnitude affects the time T in $(**)$. An easy and general proof of the existence of an orbit satisfying $(**)$ which covers the cases considered here is in [CV]: the aim of this paper is to obtain, using Mather theory, good bounds on the diffusion time

T. We remark that in all these examples the splitting between stable and unstable manifold is known and that our estimates on the "diffusion time" T are *polynomial* in the splitting. As a side remark this shows that the version of Nekhorocheff theorem given in [B] is optimal. Our result is the following theorem, the proof of which is presented at the end of section 1. Before stating it we note that the idea of using Mather theory in this context goes back at least to Bolotin, whose aim in [Bo] was to find homoclinics to a single invariant torus. Some of our hamiltonians have also been considered by Cresson ([CR]) who, by a different method, obtains a diffusion time polynomial in the splitting. We do not enter into further discussions on the literature: first, because it is enormous; second, because it is already available in the very good survey [LMS].

Theorem 1.

(i) Let H be as in $(\mathcal{CG})$ and let f be as in (1). Then for some $D>0$ and for all $\epsilon \neq 0$ small enough there are orbits of H whose energy is bounded independently on ϵ and such that

$$|I(T)-I(0)| \geq \frac{1}{D} \qquad 0<T<\frac{D}{\epsilon^{C_1+2\Xi+1}}.$$

Here, as in the following, Ξ, D and C_1 are positive constants, not depending on ϵ; Ξ and C_1 will be defined more precisely in proposition 1 below.

(ii) Let H be as in $(\mathcal{G})$ and let f be as in (1); let ω be such that

$$|\langle \omega, k\rangle| \geq \frac{C}{|k|^{\Xi}} \qquad \forall k \in \mathbf{Z}^n \setminus \{0\}.$$

Then for some $D>0$ and for all $\epsilon \neq$ small enough there are orbits of H whose energy is bounded independently on ϵ and such that

$$|I(T)-I(0)| \geq \frac{1}{D} \qquad 0<T<\frac{D}{\epsilon^{2\Xi+1}}.$$

(iii) Let H be as in $(\mathcal{GGM})$ and let f be as in (1); let $g, J, \Omega_1, \Omega_2 > 0$; then, for some $D>0$, for all $\eta \neq 0$ small enough and for $\epsilon \neq$ satisfying $|\epsilon| \leq \epsilon_0 = O(\eta^8)$ it is possible to find an orbit of H whose energy is bounded independently on ϵ and such that

$$|I(T)-I(0)| \geq \frac{1}{D} \qquad 0<T<\frac{D\epsilon^2}{\eta^{D\Xi} e^{-\frac{D}{\sqrt{\epsilon}}}}.$$

(iv) Let H be as in $(\mathcal{B}1)$ or $(\mathcal{B}2)$ and let f be as in (1). Then for some $D>0$ and for all $\epsilon \neq 0$ small enough there is an orbit of H whose energy is bounded independently on ϵ and satisfying

$$|I(T)-I(0)| \geq \frac{1}{D} \qquad 0<T\leq \frac{D}{\epsilon^{C_1+(2\Xi+1)(2d'-1-d/2)}},$$

where Ξ and C_1 are positive constants defined in proposition 1 below.

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We spend a few words on the proof, which is an almost immediate application of Mather theory. Our first step is to recall (proposition 1) all the results of the above-mentioned papers regarding the conservation of the KAM tori, their "whiskers" and the "splitting"; we translate these perturbative results in the language of the calculus of variations obtaining that some homoclinic orbits to an invariant torus are nondegenerate minima of the action functional. The diffusion orbit is built in proposition 2 as a local minimum of the action: it is close to a homoclinic to the first invariant torus on an interval $[0, T_1]$, to a homoclinic to the second torus on $[T_1, T_2]$, etc. This approach is similar to the one of Hadamard for the geodesic flow on manifolds of negative curvature; it depends strongly on the fact the the global minima are nondegenerate. The right notion of nondeneracy has been defined in [M2]; in our case it boils down to the fact that the Melnikoff function has a nondegenerate minimum. The main advantage of this approach is that, once the statements about stable and unstable manifolds are translated into variational language, the proof is a simple application of [M2].

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Section 1 — The variational setting

We will prove theorem 1 by a straightforward application of Mather theory ([M1], [M2]); no other work is needed than the translation of [CG], [C], [B], [G] and [GGM1] into variational terms.

Mather theory is formulated for the Euler-Lagrange flow (from now on the E-L flow) of a Lagrangian; of our systems, only $(\mathcal{CG})$ and $(\mathcal{B}2)$ are Lagrangian. To solve this problem, we will introduce in $(\mathcal{G})$, $(\mathcal{GGM})$ and $(\mathcal{B}1)$ a small kinetic energy, $\frac{1}{2}\kappa|I|^2$ and then we will let $\kappa \to 0$. All our estimates will be uniform in κ and we will recover theorem 1 by a limit argument.

We now introduce a family of Hamiltonians; its form is rather complicate because it is general enough to include, together with its limiting cases, $(\mathcal{CG})$, $(\mathcal{G})$, $(\mathcal{GGM})$, $(\mathcal{B}1)$ and $(\mathcal{B}2)$. We shall never need its precise expression; we will only need the facts about the invariant tori and their stable and unstable manifolds proven in the papers mentioned above. Let us consider

$$H(Q,q,I,p) =$$

$$\lambda_1\langle\omega_1, I_1\rangle + \kappa_1\frac{1}{2}|I_1|^2 + \lambda_2\langle\omega_2, I_2\rangle + \kappa_2\frac{1}{2}|I_2|^2 + \frac{1}{2}p^2 + g^2(\cos(q)-1) + \mu f(Q_1, Q_2, q) \qquad (Ham)$$

$$(Q_1, I_1) \in \mathbf{T}^{n_1} \times \mathbf{R}^{n_1}, \quad (Q_2, I_2) \in \mathbf{T}^{n_2} \times \mathbf{R}^{n_2}, \qquad (q,p) \in \mathbf{T}^1 \times \mathbf{R}^1$$

$$\lambda_1, \lambda_2 \in [0, +\infty), \qquad n = n_1 + n_2, \quad g, \kappa_1, \kappa_2, \mu \in (0, 1].$$

Since $\kappa_1, \kappa_2 > 0$ the Lagrangian corresponding to H is

$$\mathcal{L}(Q, q, \dot{Q}, \dot{q}) =$$

$$\frac{|\dot{Q}_1 - \lambda_1\omega_1|^2}{2\kappa_1} + \frac{|\dot{Q}_2 - \lambda_2\omega_2|^2}{2\kappa_2} + \frac{1}{2}\dot{q}^2 + g^2(1 - \cos(q)) - \mu f(Q_1, Q_2, q). \tag{Lag}$$

We recall that in the classical Legendre transform the correspondence between Hamiltonian and Lagrangian variables is given by

$$L: (Q_1, Q_2, I_1, I_2, q, p) \to (Q_1, Q_2, \lambda_1\omega_1 + \kappa_1 I_1, \lambda_2\omega_2 + \kappa_2 I_2, q, p). \tag{2}$$

In the following, $N_r(A)$ will denote a r-neighborhood of a set $A \subset \mathbf{T}^{n+1} \times \mathbf{R}^{n+1}$; D_i will always denote a constant greater than 1 and independent on the parameters appearing in (Ham). We will consider the cover of $\mathbf{T}^{n+1}$ given by $\mathbf{T}^n \times \mathbf{R}$, where we do not quotient in the q variable. The next proposition collects the perturbative KAM results and translates them into variational terms. Its essential point is that the local stable and unstable manifolds are graphs of exact 1-forms, $\mathrm{d}\Phi_{\mathrm{i,s}}$ and $\mathrm{d}\Phi_{\mathrm{i,u}}$; $\Phi_{i,s}$ and $\Phi_{i,u}$ represent the action functional of orbits lying on the stable and unstable manifold respectively. The statement is slightly involved because we consider two copies of each invariant torus, the one near $q = 0$ with superscript "$-$" and the one near $q = 2\pi$ with superscript "$+$". For the convenience of the reader, we make a comparison between our notations and those of [CG] in the appendix 1.

Proposition 1. *Let H be as in (Ham), let f be as in (1) and let one of the following hold:*
$(\mathcal{CG})$ $n = n_1$, $n_2 = 0$, λ_2 and κ_2 are absent, $\lambda_1 = 0$, $g = 1$, $\mu = \epsilon > 0$ is small and $\kappa_1 = 1$.

$(\mathcal{G})$ $n = n_1$, $n_2 = 0$, λ_2 and κ_2 are absent, $\lambda_1 = 1$, $g = 1$, $\mu = \epsilon > 0$ small and $\kappa_1 > 0$ sufficiently small.

$(\mathcal{GGM})$ $n_1 = n_2 = 1$, $\kappa_1 = \eta$, $\lambda_1 = \Omega_1\sqrt{\eta}$, $\lambda_2 = \Omega_2\eta^{-\frac{1}{2}}$, $g = 1$, $0 < \mu = \epsilon \le \epsilon_0 = O(\eta^8)$; we suppose that $\eta \neq 0$ is small and fixed and that $\kappa_2 > 0$ is sufficiently small.

$(\mathcal{B}1)$ $n = n_1$, $n_2 = 0$, λ_2 and κ_2 are absent, $\lambda_1 = \epsilon > 0$, $g^2 = \epsilon^d$, $\mu = \epsilon^{d'}$, with ϵ and $\kappa_1 > 0$ sufficiently small.

$(\mathcal{B}2)$ $n = n_1$, $n_2 = 0$, λ_2 and κ_2 are absent, $\lambda_1 = 0$, $g^2 = \epsilon^d$, $\mu = \epsilon^{d'}$, $\kappa_1 = \epsilon$, with $\epsilon > 0$ sufficiently small.

Then the following holds.

*(*1) There is a family of n-dimensional tori, $\tau_1, \dots, \tau_N \subset \mathbf{T}^{n+1} \times \mathbf{R}^{n+1}$, each of which is invariant for the Hamiltonian flow of H. Each of them has stable and unstable manifold, which we denote by W_i^s and W_i^u respectively. All the τ_i are contained in the same energy surface $\{H = \mathcal{E}\}$. Each τ_i projects diffeomorphically on $\tilde{\tau}_i \subset \mathbf{T}^{n+1}$. On the covering of $\mathbf{T}^{n+1}$ given by $\mathbf{T}^n \times \mathbf{R}$ each $\tilde{\tau}_i$ is the graph of a function $q_i: \mathbf{T}^n \to \mathbf{R}$; the C^1 norm of q_i tends to 0 as $\epsilon \to 0$. In particular, $\tilde{\tau}_i$ divides $\mathbf{T}^n \times \mathbf{R}$ in two connected components.*
*(*2) There are $r > 0$, $\omega_i \in \mathbf{R}^n$ and $\eta_i \ge \frac{g}{2}$ such that the flow of H on the local stable manifold in $N_r(\tau_i)$ is given by*

$$\zeta_i^s(\psi_0 + \omega_i t, y_0 e^{-\eta_i t})$$

where

$$\zeta_i^s \colon \mathbf{T}^n \times [-r,r] \to \mathbf{T}^n \times [-r,r] \times \mathbf{R}^{n+1}$$

is a Lipschitz function whose Lipschitz constant is bounded by D_1 for all i.
Analogously, the flow on the local unstable manifold of τ_i is given by

$$\zeta_i^u(\psi_0 + \omega_i t, y_0 e^{\eta_i t})$$

$$\zeta_i^u \colon \mathbf{T}^n \times [-r,r] \to \mathbf{T}^n \times [-r,r] \times \mathbf{R}^{n+1}$$

with ζ_i^u of Lipschitz constant at most D_1.
*(*3) $\exists C, \Xi > 0$ such that*

$$|\langle \omega_i, k\rangle| \geq \frac{C}{|k|^{\Xi}} \quad \forall k \in \mathbf{Z}^n \setminus \{0\}, \forall i \in (1, \ldots, N).$$

*(*4) Let us consider the covering of $\mathbf{T}^{n+1} \times \mathbf{R}^{n+1}$ given by $\mathbf{T}^n \times \mathbf{R} \times \mathbf{R}^{n+1}$ (i.e., we do not quotient in the q variable); let us denote by τ_i^- the pre-image of τ_i close to $\mathbf{T}^n \times \{0\} \times \mathbf{R}^{n+1}$ and by τ_i^+ the pre-image close to $\mathbf{T}^n \times \{2\pi\} \times \mathbf{R}^{n+1}$. We are going to state that the local stable and unstable manifolds of $\tau_i^{\pm}$ are graphs of functions from $\mathbf{T}^n \times \mathbf{R}$ to $\mathbf{R}^{n+1}$ (this is also, for instance, the situation of [A]). Since a Lagrangian submanifold which is a graph is the graph of a closed 1-form, we assert the following.*

*There are $a > 0$, $c_i \in \mathbf{R}^{n+1}$ and two smooth real-valued functions, $\Phi_{i,u}^-$ and $\Phi_{i,u}^+$, defined respectively on $\mathbf{T}^n \times [-\pi - a, \pi + a]$ and $\mathbf{T}^n \times [\pi - a, 3\pi + a]$ such that the graph of $c_i + \partial_x \Phi_{i,u}^{\pm}$ is contained in the unstable manifold of $\tau_i^{\pm}$ and contains $\tau_i^{\pm}$. The choice of c_i is not unique, since we can add to it $(0, \ldots, 0, \chi) \in (\mathbf{R}^n)^{\perp}$ and change $\Phi_{i,u}^{\pm}$ so that $c_i + \partial_x \Phi_{i,u}^{\pm}$ remains the same. The precise value of c_i will be chosen in (*6.)*
Analogously, there are $\Phi_{i,s}^{\pm}$ such that the graph of $c_i + \partial_x \Phi_{i,s}^{\pm}$ enjoys the same properties as before but with respect to the stable manifold.
*(*5) $\exists \bar{x}_i \in \mathbf{T}^n \times \{\pi\}$ such that*

$$c_{i+1} + \partial_x \Phi_{i+1,s}^+(\bar{x}_i) = c_i + \partial_x \Phi_{i,u}^-(\bar{x}_i)$$

and there are $\beta > 0$, $\delta \in (0, a)$ such that

$$[\Phi_{i,u}^-(x) - \Phi_{i+1,s}^+(x) + \langle c_i - c_{i+1}, x\rangle] - [\Phi_{i,u}^-(\bar{x}_i) - \Phi_{i+1,s}^+(\bar{x}_i) + \langle c_i - c_{i+1}, \bar{x}_i\rangle] \geq 0$$

$$\forall x \in \mathbf{T}^n \times \{\pi\}, \|x - \bar{x}_i\| \leq \delta$$

$$\inf \big\{ [\Phi_{i,u}^-(y) - \Phi_{i+1,s}^+(y) + \langle c_i - c_{i+1}, y\rangle] - [\Phi_{i,u}^-(x) - \Phi_{i+1,s}^+(x) + \langle c_i - c_{i+1}, x\rangle] :$$
$$\|y - \bar{x}_i\| = \delta, \|x - \bar{x}_i\| \leq \frac{\delta}{2}, \quad x, y \in \mathbf{T}^n \times \{\pi\} \big\} \geq \beta.$$

*(*6) Let us call P the projection of $\mathbf{T}^n \times \mathbf{R} \times \mathbf{R}^{n+1}$ onto $\mathbf{T}^{n+1} \times \mathbf{R}^{n+1}$; since $P(\tau_i^+) = P(\tau_i^-) = \tau_i$, also the projections of their stable and unstable manifolds coincide; thus we can choose the additive constants in $\Phi_{i,u}^{\pm}$ and in $\Phi_{i,s}^{\pm}$ so that*

$$\Phi_{i,u}^-(x - (0, \ldots, 0, 2\pi)) = \Phi_{i,u}^+(x), \qquad \Phi_{i,s}^-(x - (0, \ldots, 0, 2\pi)) = \Phi_{i,s}^+(x).$$

Clearly, $\Phi^{\pm}_{i,u}$ and $\Phi^{\pm}_{i,s}$ depend on the choice of c_i. For instance, if c_i is changed to $c_i+(0,\dots,0,\chi)$ then $\Phi^{\pm}_{i,u}$ is changed to $\Phi^{\pm}_{i,u}-\chi q\pm\pi\chi$.
We assert that it is possible to choose c_i in such a way that

$$\tilde{\Gamma}_{i,u}\equiv\{\Phi^{-}_{i,u}(x)=\Phi^{+}_{i,u}(x)\}$$

is a hypersurface contained in $\mathbf{T}^n\times(\pi,\pi+a]$ and

$$\tilde{\Gamma}_{i,s}\equiv\{\Phi^{-}_{i,s}(x)=\Phi^{+}_{i,s}(x)\}$$

is a hypersurface contained in $\mathbf{T}^n\times[\pi-a,\pi)$. Moreover, both $\tilde{\Gamma}_{i,s}$ and $\tilde{\Gamma}_{i,u}$ are graphs of functions from $\mathbf{T}^n$ to $\mathbf{R}$. We denote by $\Gamma^{-}_{i,s}$ the bounded component of $\mathbf{T}^n\times\mathbf{R}\setminus(\tilde{\Gamma}_{i,s}\cup\tilde{\tau}_i^{-})$, by $\Gamma^{+}_{i,s}$ the bounded component of $\mathbf{T}^n\times\mathbf{R}\setminus(\tilde{\Gamma}_{i,s}\cup\tilde{\tau}_i^{+})$. Analogously, we denote by $\Gamma^{-}_{i,u}$ the bounded component of $\mathbf{T}^n\times\mathbf{R}\setminus(\tilde{\Gamma}_{i,u}\cup\tilde{\tau}_i^{-})$, by $\Gamma^{+}_{i,s}$ the bounded component of $\mathbf{T}^n\times\mathbf{R}\setminus(\tilde{\Gamma}_{i,u}\cup\tilde{\tau}_i^{+})$.
From now on, c_i will be fixed in the above way; all the $\{c_i\}$ are bounded by a constant independent on the parameters.
*(*7) The functions $\Phi^{\pm}_{i,u}$ and $\Phi^{\pm}_{i,s}$ are Lipschitz with Lipschitz constants bounded by D_3. Moreover,*

$$\sup_{\mathbf{T}^n\times[-\pi-a,\pi+a]}|\partial_Q\Phi^{-}_{i,s}|+\sup_{\mathbf{T}^n\times[-\pi-a,\pi+a]}|\partial_Q\Phi^{-}_{i,u}|+\sup_{\mathbf{T}^n\times[\pi-a,3\pi+a]}|\partial_Q\Phi^{+}_{i,s}|+\sup_{\mathbf{T}^n\times[\pi-a,3\pi+a]}|\partial_Q\Phi^{+}_{i,u}|$$

tends to 0 as ϵ tends to 0.
*(*8) By points (*5) and (*4), there is an orbit $(Q_i(t),q_i(t))=x_i(t)$ such that $x_i(0)=\bar{x}_i$, $L^{-1}(x_i(t),\dot{x}_i(t))$ tends to τ_i for $t\to-\infty$ and to τ_{i+1} for $t\to+\infty$. We assert that this convergence is uniform in i:*

$$\exists b>0:\ L^{-1}(x_i(-b),\dot{x}_i(-b))\in N_r(\tau_i),\qquad L^{-1}(x_i(b),\dot{x}_i(b))\in N_r(\tau_{i+1}).$$

Moreover, in $(\mathcal{CG})$, $(\mathcal{G})$, $(\mathcal{GGM})$, $(\mathcal{B}1)$, $(\mathcal{B}2)$, we have that Ξ, r and δ are independent on the parameters and $|c_N-c_1|\geq\frac{1}{D}$ for some $D>0$ independent on the parameters. In particular, it is possible to fix any $\Xi>n-1$.

The constants g, β, b, C, N depend on the parameters in the following way.
$(\mathcal{CG})$ $g=1$, b is independent on ϵ, $C=C_0\epsilon^{C_1}$, with C_0 and C_1 positive and independent on ϵ, and $\beta\geq\beta_0\epsilon$ for some β_0 independent on ϵ; moreover, $N\leq\frac{D_0}{\beta}$.

$(\mathcal{G})$ $g=1$, b and C are independent on ϵ and $\beta\geq\beta_0\epsilon$ for some β_0 independent on ϵ; moreover, $N\leq\frac{D_0}{\beta}$.

$(\mathcal{GGM})$ $g=1$, b is independent on ϵ and η; for some $\Omega>0$, we have $C=\Omega e^{-s\eta^{-\frac{1}{2}}}$, $\beta\geq\epsilon^2\eta^{-D}e^{-\frac{D}{\sqrt{\eta}}}$ and $N\leq D_4\eta^{-\frac{1}{2}}\exp(D_4\eta^{-\frac{1}{2}})$.

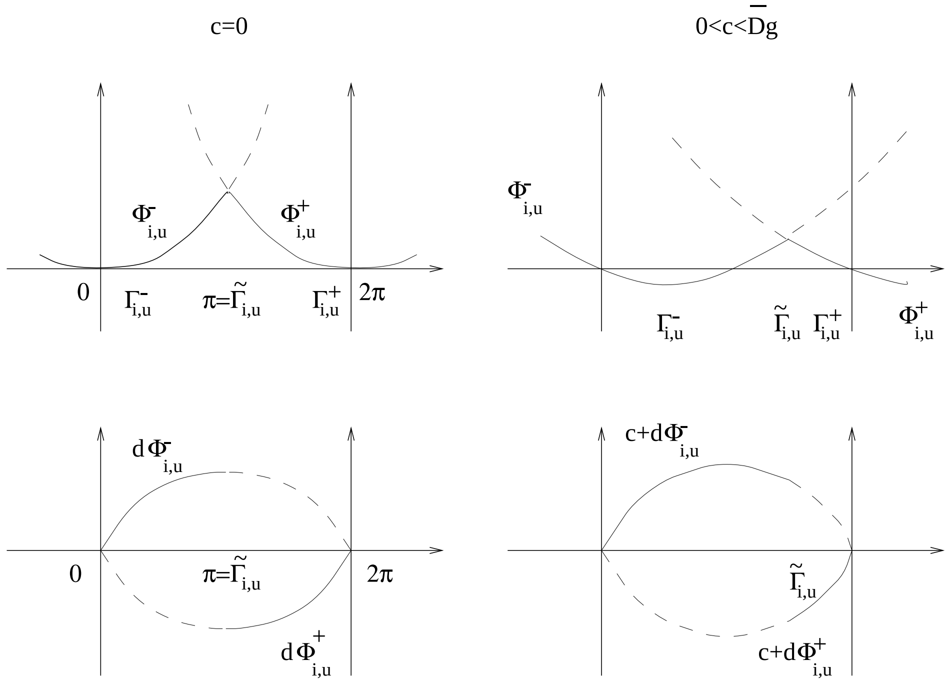
$(\mathcal{B}1)-(\mathcal{B}2)$ $g^2=\epsilon^d$, $C=C_0\epsilon^{C_1}$, with C_0 and C_1 positive and independent on ϵ, $b\leq\epsilon^{-d/2}$, $N\leq\frac{D_0}{\beta}$, $\beta\geq D_0\epsilon^{2d'-1-d/2}$.

Proof. Properties (*1), (*2) and (*3) are a consequence of KAM theorem for hyperbolic tori: see, for instance [CG], [Ge], [V]. In particular, in the case of $(\mathcal{G})$ they are stated in paragraphs 2 and 3 of [G]; in the case of $(\mathcal{GGM})$ they are part of formula 1.3 and theorem 1.4 of [GGM2]; in the case of $(\mathcal{B}1)$ and $(\mathcal{B}2)$ they follow from [C] and [B]. We remark that we are not exactly in the hypotheses of the above mentioned papers: for instance, [G] considers the Hamiltonian $(\mathcal{G})$ with $\kappa_1 = 0$. But in [Ge] it has been proven that, if $\kappa_1 > 0$ is sufficiently small, then the thesis of [G] continues to hold.

As we have already said, (*4) simply asserts that a certain portion of W_i^s and W_i^u projects diffeomorphically on $\mathbf{T}^n$; the theorems mentioned above imply that this is true in our cases.

In the light of (*4), the first formula of (*5) simply asserts that there is a heteroclinic intersection between the unstable manifold of τ_i and the stable manifold of τ_{i+1}. The second group of formulas of (*5) asserts that the intersection is transversal. The bulk of the papers quoted above consists in proving that that these formulas hold if the Melnikoff function has a nondegenerate minimum. We remark that in the above papers $\partial_{xx}(\Phi^-_{i,u} - \Phi^+_{i+1,s})(\bar{x}_i)$ is explicitly calculated; from the explicit expression it follows that in the points of minimum we have $\partial_{xx}(\Phi^-_{i,u} - \Phi^+_{i+1,s})(\bar{x}_i) \geq \beta Id$; point $(*5)$ follows from this and the Taylor formula. In the case of $(\mathcal{GGM})$ see also [GGM2], which gives the estimate on β and the number of tori N.

Before proving (*6), we recall what are $\Phi^\pm_u$ in the case of the separatrices of the simple pendulum, $\mathcal{L}(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + \epsilon[1 - \cos(q)]$. The reader should look at figure 1: for $c = 0$, Φ^-_u is a kind of parabola with vertex in $q = 0$, Φ^+_u a kind of parabola with vertex in $q = 2\pi$; the two curves intersect in $q = \pi$; these two functions depend on c as explained in (*6) and changing c moves the point of intersection left or right. The same c moves the point of intersection of Φ^-_s and Φ^+_s in the opposite direction; it is easy to see that, if we want the intersection of the graphs of Φ^-_s and Φ^+_s to lay on $[\pi - a, \pi)$, then we must choose c in $|c| \leq \bar{D}g$; indeed, if $|c| > \bar{D}g$ the intersection disappears. The reason for choosing c in this way is that the point of intersection will be a point of discontinuity for the functional we will minimize; thus we are interested in keeping it off $q = \pi$, the Poincaré section on which we will work. When we couple the pendulum to the rotators, these points of discontinuity become surfaces of discontinuity, $\tilde{\Gamma}_{i,s}$ and $\tilde{\Gamma}_{i,u}$, as shown in figure 2. We now prove (*6) in one case, $(\mathcal{GGM})$, since the others are similar. Let us consider $(\mathcal{GGM})$ with $\epsilon = 0$. In this case W_i^u is the product of $\mathbf{T}^2$ with the unstable manifold of the pendulum and (*6) follows by the considerations above. Indeed, we consider $\Phi^\pm_{i,u}$ when the third component of c_i is zero; if we choose χ suitably in $|\chi| < \bar{D}g$, then $\Phi^+_{i,u}(Q_1, Q_2, q) - \Phi^-_{i,u}(Q_1, Q_2, q) = 0$ is the two-dimensional torus $\{q = \pi + \frac{1}{2}a\}$; in other words, if we choose c_i with the third component equal to χ, $\tilde{\Gamma}_{i,u} = \{q = \pi + \frac{1}{2}a\}$. Since $\partial_q[\Phi^+_{i,u}(Q_1, Q_2, q) - \Phi^-_{i,u}(Q_1, Q_2, q)] \neq 0$, the implicit function theorem yields (*6) also when $|\epsilon| \leq \epsilon_0 = O(\eta^8)$. We also remark that it is easy to see that the first two components of c_i are bounded; from the argument above, it follows that the c_i are bounded.



Upstairs: graph of $\Phi^-_{i,u}$ and $\Phi^+_{i,u}$
for $c = 0$. The solid lines are the graph
of $h^\infty_0(0, q)$.
Downstairs: The solid lines represent
the initial conditions
of the orbits realizing $h^\infty_0(0, q)$.

Upstairs: graph of $\Phi^-_{i,u}$ and $\Phi^+_{i,u}$
for $0 < c < \bar{D}g$. The solid lines are the graph
of $h^\infty_c(0, q)$.
Downstairs: The solid lines representent
the initial conditions
of the orbits realizing $h^\infty_c(0, q)$.

Figure 1

As for (*7), we note that by (*4) $c_i + \partial_x \Phi^\pm_{i,u}$ and $c_i + \partial_x \Phi^\pm_{i,s}$ are bounded by the sup of $|(I, p)|$ on the local stable and unstable manifolds, which are uniformly bounded in the case of $(\mathcal{G})$, $(\mathcal{B}1)$ and $(\mathcal{B}2)$. Since by (*6) the c_i are bounded, we have that $\Phi^\pm_{i,u}$ and $\Phi^\pm_{i,s}$ are Lipschitz uniformly in i. The second formula of (*6) is a consequence of the explicit form of $\Phi^\pm_{i,u}$ and $\Phi^\pm_{i,s}$; for instance, in the case of $(\mathcal{GGM})$, these can be found at the beginning of section 3 of [GGM1]; there it is stated that they satisfy (*7).

We note that (*8) simply asserts that it takes a time b for the homoclinic to go from π to a neighborhood of the invariant torus; this follows considering the motion along the pendulum.

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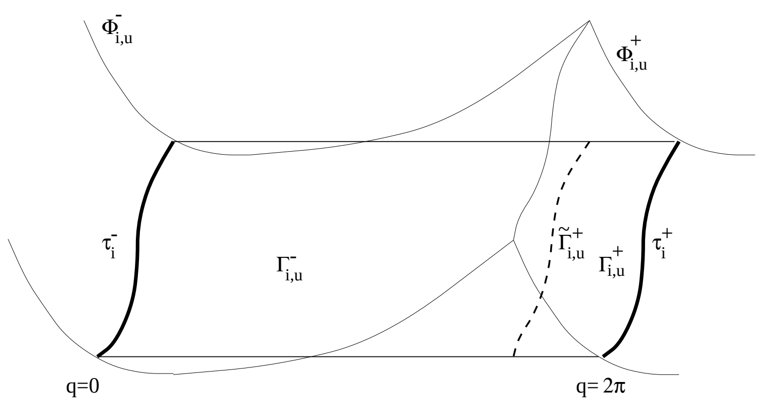


Figure 2

The proof of the following proposition is based on the variational argument of [M1] and [M2].

Proposition 2. *Let the system satisfy (*1)–(*8) of proposition 1 above, let $c_i \in \mathbf{R}^{n+1}$ be as in proposition 1 and let c'_i denote the first n components of c_i. Then there is an orbit satisfying*

$$|H(q(t), Q(t), p(t), I(t))| \leq M \quad \forall t \in (0, T), \qquad |I(T) - c'_N| + |I(0) - c'_0| \to 0 \quad \textit{as } \epsilon \to 0. \tag{3}$$

$$0 < T \leq 2N \left(b + \max \left(\frac{D_5 \cdot N^{\Xi}}{C\beta^{\Xi}}, \frac{D_5}{g} \log \frac{N}{\beta} \right) \right) \tag{4}$$

where D_5 and $M > 0$ are constants not depending on ϵ, η, κ_1 and κ_2.

This proposition will be proven in the next section.

Proof of Theorem 1: Essentially, it suffices to insert the constants of proposition 1 into the thesis of proposition 2. This does not yield immediately theorem 1: for instance, in proposition 1 we ask that the Hamiltonian $(\mathcal{G})$ has $\kappa_1 > 0$, while in case (i) of theorem 1 we consider the same Hamiltonian, but with $\kappa_1 = 0$. Since (3) and (4) are uniform in κ_1, we can pass to the limit in the following way. For $\kappa_1 > 0$ let us consider the orbit $(Q_{\kappa_1}, q_{\kappa_1}, I_{\kappa_1}, p_{\kappa_1})$ given by proposition 2; by formula (3) its initial conditions are bounded uniformly in κ_1; since T is bounded uniformly in κ_1 this implies that $(Q_{\kappa_1}, q_{\kappa_1}, I_{\kappa_1}, p_{\kappa_1})$ is equicontinuous on $[0, T]$. Thus we can pass to the limit for $\kappa_1 \to 0$ and get the thesis. The other cases are treated similarly.

\\\\\\

Section 2 — Proof of proposition 2

The proof consists in the variational argument of [M2]; as explained in the introduction, the diffusion orbits will be local minima of the action functional.

Let c_i and $\mathcal{E}$ be as in proposition 1; let S be a smooth function defined on $\mathbf{T}^{n+1}$, let A.C.$([0,T],\mathbf{T}^{n+1})$ denote the curves absolutely continuous on $[0,T]$ with image in $\mathbf{T}^{n+1}$ and let $\nabla S(x)=\partial_x S(x)$.

For $x,y\in\mathbf{T}^{n+1}$ we define

$$h^T_{c_i+\nabla S}(x,y)=$$
$$\min\{\int_0^T[\mathcal{L}(Q,q,\dot{Q},\dot{q})-\langle c_i+\nabla S,(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t:\ (Q,q)\in \text{A.C.}([0,T],\mathbf{T}^{n+1}),$$
$$(Q(0),q(0))=x,(Q(T),q(T))=y\}.$$

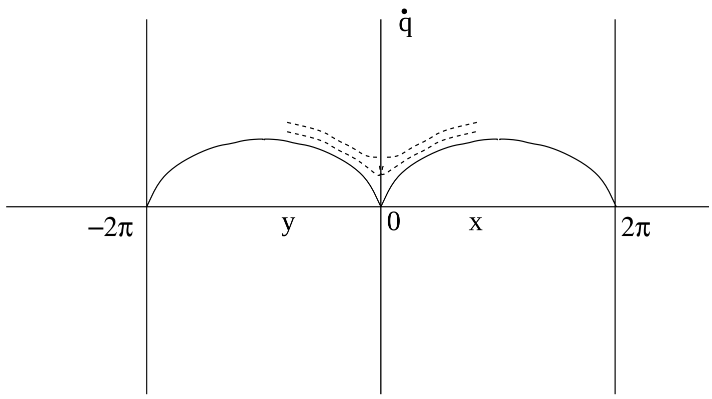
The minimum above exists by a theorem of Tonelli's (see for instance [M1]); it is a standard fact that the set of the orbits realizing $h^T_{c_i+\nabla S}(x,y)$ does not depend on the choice of S; moreover

$$h^T_{c_i+\nabla S}(x,y)=h^T_{c_i}(x,y)+S(x)-S(y). \tag{5}$$

We also define

$$h^\infty_{c_i+\nabla S}(x,y)=\liminf_{T\to\infty}h^T_{c_i+\nabla S}(x,y)$$

which can be considered as the least action of all orbits going from x to y in infinite time; in [M2] it has been proven that it is finite; in our particular case, this is part of the proof of lemma 1. In the above formula, let us consider the orbits $(Q_T,q_T)\in$ A.C.$([0,T],\mathbf{T}^{n+1})$ realizing the $\liminf$: it has been proven in [M1] that, up to a subsequence, they converge to an orbit (Q,q), defined on $[0,\infty)$ and with $(Q(0),q(0))=x$. Such an orbit need not necessarily have y or in its ω limit; we will say however that it realizes $h^\infty_{c_i+\nabla S}(x,y)$ (see figure 3 for the case of the pendulum). On the other side by a translation in time we can choose $(Q_T(-T),q_T(-T))=x$, $(Q_T(0),q_T(0))=y$; the orbits (Q_T,q_t) will converge, up to a subsequence, to (Q,q) with $(Q(0),q(0))=y$ and defined on $(-\infty,0]$.



The orbits realizing $h^T_0(x,y)$ for increasing T are shown as dotted lines.

The orbit realizing $h^\infty_0(x,y)$ has 0 in its ω-limit.

Figure 3

Heuristically, the term $-\langle c_i,(\dot{Q},\dot{q})\rangle$ keeps track of the path of the orbit (on its projection on c_i, actually); we will see in lemma 1 below that it forces the the orbits realizing $h^\infty_{c_i}$ to have asymptotic rotation number ω_i. The rôle of the energy $\mathcal{E}$ is to keep $h^\infty_{c_i}$ finite.

Let z_i belong to $\tilde\tau_i$, the projection of τ_i on $\mathbf{T}^{n+1}$. The next lemma shows that the orbits realizing $h^\infty_{c_i}(z_i,x)$ and $h^\infty_{c_i}(x,z_i)$ lie on the unstable and on the stable manifolds respectively. The reader can now see how the orbits realizing $h^\infty_{c_i}(z_i,x)$ depend on the choice of c_i: for instance, in the case of the pendulum, the initial conditions of the orbits realizing $h^\infty_c(0,x)$ are the solid lines in figure 1. In the following, L denotes the Legendre transform (2), $\tilde\tau_i$ is the torus defined in point (*1) of proposition 1 and $\Gamma^\pm_{i,s}$ is defined in (*6).

Lemma 1. *Let $z_i\in\tilde\tau_i$. Then: if $x\in\Gamma^\pm_{i,s}$, there is only one orbit realizing $h^\infty_{c_i}(x,z_i)$ and it is the one with initial condition $L(x,\partial_x\Phi^\pm_{i,s}(x)+c_i)$; if $x\in\Gamma^\pm_{i,u}$, there is only one orbit realizing $h^\infty_{c_i}(z_i,x)$ and it is the one with initial condition $L(x,\partial_x\Phi^\pm_{i,u}(x)+c_i)$.*

Also, the following holds.

i)
$$-\partial_x h^\infty_{c_i}(x,z_i)=\begin{cases}\partial_x\Phi^-_{i,s}(x) & \text{if}\quad x\in\Gamma^-_{i,s}\\ \partial_x\Phi^+_{i,s}(x) & \text{if}\quad x\in\Gamma^+_{i,s}\end{cases}$$

ii)
$$\partial_x h^\infty_{c_i}(z_i,x)=\begin{cases}\partial_x\Phi^-_{i,u}(x) & \text{if}\quad x\in\Gamma^-_{i,u}\\ \partial_x\Phi^+_{i,u}(x) & \text{if}\quad x\in\Gamma^+_{i,u}.\end{cases}$$

iii) $h^\infty_{c_i}(x_i,z_i)\le h^M_{c_i}(x_i,z_i)$ and $h^\infty_{c_i}(z_i,x_i)\le h^M_{c_i}(z_i,x_i)$ for all $M>0$.

iv)
$$h^\infty_{c_i}(x,y)+h^\infty_{c_i}(y,z)=h^\infty_{c_i}(x,z)\quad\forall x,y,z\in\tilde\tau_i.$$

Consequently, $h^\infty_{c_i}(z,z)=0$.

Proof. We begin to prove that, if $x\in\Gamma^\pm_{i,s}$, then there is a unique orbit (Q,q) which realizes $h^\infty_{c_i}(x,z_i)$; at time 0 this orbit has initial conditions $L(x,\partial_x\Phi^-_{i,s}(x)+c_i)$ if $x\in\Gamma^-_{i,s}$, and $L(x,\partial_x\Phi^+_{i,s}(x)+c_i)$ if $x\in\Gamma^+_{i,s}$. Since this orbit depends smoothly on x in $\Gamma^\pm_{i,s}$, it is then easy to differentiate $h^\infty_{c_i}(x,z_i)$ and get $i)$; $ii)$ is derived analogously.
Let us define
$$\phi(x)=\begin{cases}\Phi^-_{i,s}(x) & \text{if}\quad x\in\Gamma^-_{i,s}\\ \Phi^+_{i,s}(x) & \text{if}\quad x\in\Gamma^+_{i,s}.\end{cases}$$
We consider the following Lagrangian, discontinuous along $\tilde\Gamma_{i,s}$:

$$\tilde{\mathcal L}(Q,q,\dot Q,\dot q)=\mathcal L(Q,q,\dot Q,\dot q)-\langle\partial_x\phi(Q,q),(\dot Q,\dot q)\rangle.$$

We now sketch a standard computation (see for instance [Ca] or [M1]): if we fix (Q,q) and look for the minimum of $\tilde{\mathcal L}-\langle c_i,(\dot Q,\dot q)\rangle+\mathcal E$ in the variables $(\dot Q,\dot q)$ we obtain the following necessary condition, which is also sufficient since $\tilde{\mathcal L}$ is convex in $(\dot Q,\dot q)$:

$$\frac{\partial}{\partial(\dot Q,\dot q)}\mathcal L(Q,q,\dot Q,\dot q)=\partial_x\phi(Q,q)+c_i.$$

Thus the minimum of $\tilde{\mathcal L}-\langle c_i,(\dot Q,\dot q)\rangle+\mathcal E$ for (Q,q) fixed lies on the image of $L(Q,q,\partial_x\phi_{i,s}(Q,q)+c_i)$ where L is the Legendre transform defined in (2); we recall that $\langle c_i+\partial_x\phi,(\dot Q,\dot q)\rangle-\mathcal L$ restricted to this set is simply the Hamiltonian in different coordinates; thus if we want the minimum above to be constantly equal to 0, we need

$$H(Q,q,\partial_x\phi(Q,q)+c_i)=\mathcal{E}\,.$$

The last formula is true, since the energy is constant on W_i^s; vice versa, we see that $\tilde{\mathcal{L}}-\langle c_i,(\dot{Q},\dot{q})\rangle+\mathcal{E}$ is constantly equal to 0 on the graph of $L(Q,q,\partial_x\phi(Q,q)+c_i)$ and it is strictly larger than 0 elsewhere. Thus if $(\bar{Q}(t),\bar{q}(t))$ is a c_i-minimal orbit of $\tilde{\mathcal{L}}$ with $\bar{Q}(0)=x\in\Gamma^{\pm}_{i,s}$ which accumulates on τ_i, it must satisfy $(x,\dot{\bar{Q}}(0),\dot{\bar{q}}(0))=L(x,\partial_x\phi_{i,s}(x)+c_i)$: otherwise, the integral of $\tilde{\mathcal{L}}$ would be positive. If we prove that, for these boundary values, this orbit minimize also the integral of $\mathcal{L}$, we have done.
Let (Q,q) be any curve crossing $\tilde{\Gamma}_{i,s}$ at the times $t_1<t_2<\ldots<t_k$ and let $0<t_1$ and $t_k<T$. Let us suppose that, for γ small enough, $(Q(t_i-\gamma),q(t_i-\gamma))\in\Gamma^-_{i,s}$ and $(Q(t_i+\gamma),q(t_i+\gamma))\in\Gamma^+_{i,s}$; to fix ideas, let us also suppose that $(Q(0),q(0))\in\Gamma^-_{i,s}$ and $(Q(T),q(T))\in\Gamma^+_{i,s}$. Then we have that

$$\int_0^T[\tilde{\mathcal{L}}-\langle c_i,(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t=\int_0^T[\mathcal{L}-\langle c_i,(\dot{Q},\dot{q})\rangle-\langle\partial_x\phi(Q,q),(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t=$$

$$\int_0^T[\mathcal{L}-\langle c_i,(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t-\sum_{i=1}^k[-\Phi^+_{i,s}(Q(t_i^+),q(t_i^+))+\Phi^-_{i,s}(Q(t_i^-),q(t_i^-))]+\Phi^-_{i,s}(Q(0),q(0))-\Phi^+_{i,s}(Q(T),q(T))=$$

$$\int_0^T[\mathcal{L}-\langle c_i,(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t+\Phi^-_{i,s}(Q(0),q(0))-\Phi^+_{i,s}(Q(T),q(T))$$

where the last equality is a consequence of the fact that $\Phi^+_{i,s}|_{\tilde{\Gamma}_{i,s}}=\Phi^-_{i,s}|_{\tilde{\Gamma}_{i,s}}$ by the definition of $\tilde{\Gamma}_{i,s}$. By a standard approximation argument, the above formula holds also if (Q,q) crosses $\tilde{\Gamma}_{i,s}$ infinitely many times. The last formula implies that for all $T>0$ $h^T_{c_i}$ and $h^T_{c_i+\nabla\phi}$ are realized by the same orbits, since the corresponding action functionals only differ by a function of the boundary values; if (Q,q) accumulates on z_i, letting $T\to\infty$ we have the thesis.

We remark that from the same arguments it follows that, if $x\in\mathbf{T}^{n+1}$, then among all orbits connecting z_i to x in any time $T\in(0,+\infty]$, the minimal action one lays on the stable manifold; this proves *iii*).

We note that this also implies that $h^\infty_{c_i}(x,z_i)$ is finite: for instance, let $x\in\Gamma^-_{i,s}$ and let (Q,q) be the orbit which at time 0 has initial conditions $L(x,\partial_x\Phi^-_{i,s}+c_i)$. Then

$$\int_0^T[\mathcal{L}-\langle c_i,(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t=\int_0^T[\tilde{\mathcal{L}}-\langle c_i,(\dot{Q},\dot{q})\rangle+\mathcal{E}]\mathrm{d}t+\Phi^-_{i,s}(Q(0),q(0))-\Phi^-_{i,s}(Q(T),q(T))$$

$$=\Phi^-_{i,s}(Q(0),q(0))-\Phi^-_{i,s}(Q(T),q(T))$$

which is bounded; passing to the limit as $T\to+\infty$ we get that $h^\infty_{c_i}(x,z_i)$ is finite. Since $h^T_{c_i}(x,y)$ is Lipschitz in x and y uniformly for $T>1$, also $h^\infty_{c_i}$ is Lipschitz and being finite at one point by the previous formula, it is finite everywhere.

To prove *iv*), it suffices to note that, if $x,y,z\in\tau_i$, then

$$h^\infty_{c_i}(x,y)+h^\infty_{c_i}(y,z)=h^\infty_{c_i+\nabla\Phi^+_{i,s}}(x,y)+h^\infty_{c_i+\nabla\Phi^+_{i,s}}(y,z)+\Phi^+_{i,s}(x)-\Phi^+_{i,s}(y)+\Phi^+_{i,s}(y)-\Phi^+_{i,s}(z)=$$

$$\Phi^+_{i,s}(x)-\Phi^+_{i,s}(z)=h^\infty_{c_i}(x,z)$$

where the first equality is a consequence of (5), the second of the fact that $\tilde{\mathcal{L}}$ is constantly equal to zero on $L^{-1}(\tau_i)$ and the third of i).

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We consider the covering of $\mathbf{T}^{n+1}$ given by $\mathbf{T}^n\times\mathbf{R}$; for each $\bar{x}_i$ of (*5) we single out a point on its fiber, $\tilde{x}_i\in\mathbf{T}^n\times\{\pi+2i\pi\}$. For $i\in(1,\ldots,N-1)$ we consider a smooth function $S_i\colon\mathbf{T}^{n-1}\times\mathbf{R}\to\mathbf{R}$ which vanishes outside $\{x:\,|x-\tilde{x}_i|\le 2\delta\}$ and such that

$$\nabla S_i(x)=c_{i+1}-c_i\qquad\forall x:\,|x-\tilde{x}_i|\le\delta. \tag{6}$$

We set

$$\bar{c}_i(x)=c_i+\nabla S_i(x).$$

In $\mathbf{T}^n\times\mathbf{R}$ we choose the representative of $\tilde{\tau}_i$ close to $\mathbf{T}^n\times 2i\pi$; from this lift of $\tilde{\tau}_i$ we choose a point z_i. We fix $T>0$ and define

$$X=(\mathbf{T}^n\times\{3\pi\}\times\mathbf{R})\times(\mathbf{T}^n\times\{5\pi\}\times\mathbf{R})\times\ldots\times(\mathbf{T}^n\times\{\pi+2(N-1)\pi\}\times\mathbf{R})$$

$$Y=\{\quad\{(x_i,t_i)\}_{i=1}^{N-1}\in X:\,t_1=0,\quad t_{N-1}=T,\quad t_{i+1}>t_i\;\forall i\in(1,\ldots,N-2)\}$$

$$G((x_1,t_1),\ldots,(x_{N-1},t_{N-1}))=h^\infty_{\bar{c}_1}(z_1,x_1)+h^{t_2-t_1}_{\bar{c}_2}(x_1,x_2)+h^{t_3-t_2}_{\bar{c}_3}(x_2,x_3)+\ldots+$$
$$h^{t_{N-1}-t_{N-2}}_{\bar{c}_{N-1}}(x_{N-2},x_{N-1})+h^\infty_{\bar{c}_N}(x_{N-1},z_N).$$

We set

$$B=\big(B(\tilde{x}_1,\delta)\times\mathbf{R})\times(B(\tilde{x}_2,\delta)\times\mathbf{R})\times\ldots\times(B(\tilde{x}_{N-1},\delta)\times\mathbf{R}\big)\bigcap Y$$

$$B'=B(\tilde{x}_1,\delta)\times B(\tilde{x}_2,\delta)\times\ldots\times B(\tilde{x}_{N-1},\delta)$$

where $B(\tilde{x}_i,\delta)$ is the closed ball in $\mathbf{T}^n\times\{\pi+2i\pi\}$ centered in $\tilde{x}_i$ and of radius δ.

The next lemma explains the meaning of the functional G; its proof is relegated to appendix 2.

Lemma 2. *Let $((y_1,t_1),\ldots,(y_{N-1},t_{N-1}))$ be a local minimum of G in the interior of B and let (Q,q) be the function defined in the following way: on $[-\infty,t_1]$ (Q,q) is the orbit realizing $h^\infty_{\bar{c}_1}(z_1,y_1)$ with $(Q(t_1),q(t_1))=y_1$; on $[t_1,t_2]$ Q is the orbit realizing $h^{t_2-t_1}_{\bar{c}_2}(y_1,y_2)$ with $(Q(t_1),q(t_1))=y_1$, $(Q(t_2),q(t_2))=y_2$, etc...*
Then (Q,q) solves the E-L equation on $(t_1,t_{N-1})=(0,T)$ and satisfies the second formula of (3) in proposition 2.

From the above lemma we gather that to prove proposition 2 it suffices to prove that G has a minimum in the interior of B for some T satisfying (4). This is what we show in the next lemma.

Lemma 3. *There is*

$$T\le 2N\left(b+\max\left(\frac{D_5\cdot N^\Xi}{C\beta^\Xi},\frac{D_5}{g}\log\frac{N}{\beta}\right)\right) \tag{13}$$

such that G has a local minimum in the interior of B.

Proof. First of all we note that G has a minimum in B because its sublevels are compact: indeed, it is easy to see that $G(((x_1,t_1),\dots,(x_{N-1},t_{N-1}))) \to \infty$ if $t_{i+1}\to t_i$, $t_2\to 0$ or $t_{N-2}\to T$.

Thus it suffices to prove that the minimum is in the interior of B; to do this we will compare G with a functional F which has a strict minimum in the point $((\tilde{x}_1,t_1),\dots,(\tilde{x}_{N-1},t_{N-1}))$. We define

$$F\colon Y\to\mathbf{R}$$

$$F((x_1,t_1),\dots,(x_{N-1},t_{N-1})) = h^{\infty}_{\bar{c}_1}(z_1,x_1)+h^{\infty}_{\bar{c}_2}(x_1,z_2)+h^{\infty}_{\bar{c}_2}(z_2,x_2)+h^{\infty}_{\bar{c}_3}(x_2,z_3)+\dots+$$
$$h^{\infty}_{\bar{c}_{N-1}}(z_{N-1},x_{N-1})+h^{\infty}_{\bar{c}_N}(x_{N-1},z_N)$$

where the z_i are the same as in lemma 1. Clearly, F does not depend on the t_i and, roughly, it represents the action of a heteroclinic chain connecting τ_1 to τ_2 to τ_3, all the way to τ_N. We now note that, by lemma 1, (5) and the definition of $\bar{c}_i$,

$$h^{\infty}_{\bar{c}_i}(z_i,x_i)+h^{\infty}_{\bar{c}_{i+1}}(x_i,z_{i+1}) = const+\Phi^-_{i,u}(x_i)-\Phi^+_{i+1,s}(x_i)-S_i(x_i)+S_{i+1}(x_i).$$

Since S_{i+1} vanishes on $B(\tilde{x}_i,\delta)$, by (6) we get

$$h^{\infty}_{\bar{c}_i}(z_i,x_i)+h^{\infty}_{\bar{c}_{i+1}}(x_i,z_{i+1}) = const+\Phi^-_{i,u}(x_i)-\Phi^+_{i+1,s}(x_i)+\langle c_i-c_{i+1},x_i\rangle.$$

By the last formula and (*5) we have that the points $((\tilde{x}_1,t_1),\dots,(\tilde{x}_{N-1},t_{N-1}))$ are minima of F in B for all choice of $t_1<t_2<\dots<t_N$; moreover

$$\inf\{F((y_1,t_1),\dots,(y_{N-1},t_{N-1}))\,:\,((y_1,t_1),\dots,(y_{N-1},t_{N-1}))\in B,$$
$$(y_1,\dots,y_{N-1})\in\partial B',\tilde{t}_1<\tilde{t}_2<\dots<\tilde{t}_{N-1}\}\geq F((\tilde{x}_1,\tilde{t}_1),\dots,(\tilde{x}_{N-1},\tilde{t}_{N-1}))\beta. \tag{14}$$

We now show that, for some T satisfying (13), G is so close to F that it has a minimum inside B. Given $x_i\in B(\tilde{x}_i,\delta)$, $x_{i+1}\in B(\tilde{x}_{i+1},\delta)$ and $M^s_i,M^u_i>0$ we choose (Q^s,q^s) and (Q^u,q^u), orbits of $\mathcal{L}$ with initial conditions

$$(Q^s(-M^s_i),q^s(-M^s_i),\dot{Q}^s(-M^s_i),\dot{q}^s(-M^s_i)) = L(x_i,\partial_x\Phi^+_{i+1,s}(x_i)+c_{i+1})$$
$$(Q^u(M^u_i),q^u(M^u_i),\dot{Q}^u(M^u_i),\dot{q}^u(M^u_i)) = L(x_{i+1},\partial_x\Phi^+_{i+1,u}(x_{i+1})+c_{i+1})$$

where L denotes the Legendre transform, as in (2); in other words, these orbits lay one on the stable, one on the unstable manifold of τ^+_{i+1}. By lemma 1, (Q^s,q^s) and (Q^u,q^u) realize $h^{\infty}_{\bar{c}_{i+1}}(x_i,z_{i+1})$ and $h^{\infty}_{\bar{c}_{i+1}}(z_{i+1},x_{i+1})$ respectively. We note that, since $x_j\in\mathbf{T}^n\times(\pi+2j\pi)$ $\forall j$, both $\Phi^-_{i+1,s}$ and $\Phi^+_{i+1,u}$ are defined because of (*4). We choose M^s_i and M^u_i in the following way: they are the smallest times such that

$$|(Q^s(0),q^s(0))-z_{i+1}|\leq\frac{\beta}{D_3\cdot N\cdot 128}\qquad |(Q^u(0),q^u(0))-z_{i+1}|\leq\frac{\beta}{D_3\cdot N\cdot 128} \tag{15}$$

where D_3 was introduced in (*7) and the distances are those induced on $\mathbf{T}^n$ by its cover $\mathbf{R}^n$. We now recall the estimate on the time of ergodization of the torus (see theorem D of [BGW]): if ω satisfies (*3) the smallest T for which $\{\omega t\}_{t=0}^T$ is a ϵ-net can be estimated from above by $\frac{D_5}{C\epsilon^\Xi}$, where D_5 is a constant, depending only on the dimension n and on the Diophantine exponent Ξ. In symbols

$$\forall i, \quad \forall\epsilon>0, \quad \forall Q_0\in\mathbf{T}^n \quad \exists 0<M<\frac{D_5}{C\epsilon^\Xi}:\ |\omega_i M - Q_0|\le\epsilon.$$

Using this fact, (*2) and (*8) we see that there is $D_6>0$ such that

$$1<M_i^s, M_i^u \le b+\max\left(\frac{D_6\cdot N^\Xi}{C\beta^\Xi}, \frac{D_6}{g}\log\frac{N}{\beta}\right). \tag{16}$$

In the last formula, b accounts for the time it takes to reach the neighborhood of τ_{i+1} where the normal form (*2) holds; the second term in the max is due to the motion on the local stable or unstable manifold.

We now specialize the x_i in the following inductive way: we take $x_1=\tilde{x}_1$; if x_i is defined, we take $x_{i+1}\in B(\tilde{x}_{i+1},\frac{D_{11}\delta}{2N})$ such that $Q^u(0)=Q^s(0)$; this is possible since we have by (*1) and (*7) that the map which sends the first n coordinates of x_i into $Q^s(0)$ is bilipschitz with Lipschitz constant D_{11} independent on M_i^s. We give a short proof of this fact in the case ($\mathcal{B}2$). First of all, we note that it suffices to prove that the map $:Q^u(-M_i^u)\to Q^u(-M_i^u+b)$ is Lipschitz, since after that time the dynamics is given by (*1) and is surely Lipschitz. Let us consider the time-one map for the dynamics on the unstable manifold; from (*7) and the Hamilton equations, it follows that the map $:Q^u(0)\to Q^u(-1)$ is bilipschitz with Lipschitz constant $1+\epsilon o(2)$. By proposition 1 $b=0(\frac{1}{\epsilon})$ and thus the map $:Q^u(-M_i^u)\to Q^u(-M_i^u+b)$ is bilipschitz with Lipschitz constant D_{11}.

By (*1), $\tilde{\tau}_{i+1}^+$ is the graph of a Lipschitz function from $\mathbf{T}^n$ to $\mathbf{R}$; thus we can find q_0 near $2\pi(i+1)$ such that $(Q^s(0),q_0)=(Q^u(0),q_0)$ belongs to $\tilde{\tau}_{i+1}^+$ on $\mathbf{T}^{n+1}$ and, moreover, by (15),

$$|q^s(0)-q_0|\le\frac{\beta}{D_3\cdot N\cdot 64}, \qquad |(Q^s(0),q_0)-z_{i+1}|\le\frac{\beta}{D_3\cdot N\cdot 64}. \tag{17}$$

Let us now define

$$\bar{q}(t)=\begin{cases} q_0+(q^u(\frac{\beta}{D_3\cdot N\cdot 64})-q_0)\cdot t\cdot\frac{D_3\cdot N\cdot 64}{\beta} & 0\le t\le\frac{\beta}{D_3\cdot N\cdot 64}\\ q^u(t) & t\ge\frac{\beta}{D_3\cdot N\cdot 64}.\end{cases}$$

We get that

$$h_{\bar{c}_{i+1}}^{M_i^u}((Q^u(0),q_0),x_{i+1})\le\int_0^{M_i^u}\mathcal{L}(Q^u,\bar{q},\dot{Q}^u,\dot{\bar{q}})\mathrm{d}t\le$$

$$\int_0^{M_i^u}\mathcal{L}(Q^u,q^u,\dot{Q}^u,\dot{q}^u)\mathrm{d}t+\frac{\beta}{D_3\cdot N\cdot 32}=h_{\bar{c}_{i+1}}^{M_i^u}((Q^u(0),q^u(0)),x_{i+1})+\frac{\beta}{D_3\cdot N\cdot 32} \tag{18}$$

where the first inequality is a consequence of the definition of $h_{\bar{c}_{i+1}}^{M_i^u}$ and the second follows from a standard calculation; the equality is a consequence of lemma 1. Since in lemma 1 it is proven that $h_{\bar{c}_{i+1}}^\infty(z_{i+1},z_{i+1})=0$, we have that

$$|h_{\bar{c}_{i+1}}^\infty(z_{i+1},Q^u(0),q^u(0)))|=|h_{\bar{c}_{i+1}}^\infty(z_{i+1},Q^u(0),q^u(0)))-h_{\bar{c}_{i+1}}^\infty(z_{i+1},z_{i+1})|=$$

$$|\Phi^+_{i+1,u}(Q^u(0),q^u(0)) - \Phi^+_{i+1,u}(z_{i+1})| \leq \frac{\beta}{N\cdot 64} \tag{19}$$

where the second equality is a consequence of lemma 1, the inequality of (*7) and of (15). We now recall that, if (Q,q) is c-minimal, then for all $t_1,t_2>0$,

$$h_c^{t_1+t_2}((Q,q)(0),(Q,q)(t_1+t_2)) = h_c^{t_1}((Q,q)(0),(Q,q)(t_1)) + h_c^{t_2}((Q,q)(t_1),(Q,q)(t_2)).$$

Therefore, since (Q^u,q^u) is $\bar c_i$-minimal on $(-\infty,0]$ we have that

$$h^\infty_{\bar c_{i+1}}(z_{i+1},x_{i+1}) = h^\infty_{\bar c_{i+1}}(z_{i+1},(Q^u(0),q^u(0))) + h^{M_i^u}_{\bar c_{i+1}}((Q^u(0),q^u(0)),x_{i+1}) \geq$$

$$h^{M_i^u}_{\bar c_{i+1}}((Q^u(0),q^u(0)),x_{i+1}) - \frac{\beta}{N\cdot 64} \geq h^{M_i^u}_{\bar c_{i+1}}((Q^u(0),q_0),x_{i+1}) - \frac{\beta}{N\cdot 16}$$

where the first inequality is a consequence of (19), the second of (18). Analogously, we get

$$h^\infty_{\bar c_{i+1}}(x_i,z_{i+1}) \geq h^{M_i^s}_{\bar c_{i+1}}(x_i,(Q^s(0),q_0)) - \frac{\beta}{N\cdot 16}.$$

Since $(Q^s(0),q_0)=(Q^u(0),q_0)$we have that

$$h^{M_i^s+M_i^u}_{\bar c_{i+1}}(x_i,x_{i+1}) \leq h^{M_i^s}_{\bar c_{i+1}}(x_i,(Q^s(0),q_0)) + h^{M_i^u}_{\bar c_{i+1}}((Q^u(0),q_0),x_{i+1})$$

which from the last formula implies

$$h^{M_i^s+M_i^u}_{\bar c_{i+1}}(x_i,x_{i+1}) \leq h^\infty_{\bar c_{i+1}}(x_i,z_{i+1}) + h^\infty_{\bar c_{i+1}}(z_{i+1},x_{i+1}) + \frac{\beta}{N\cdot 8}.$$

The last formula implies that, setting

$$\tilde t_1 = 0, \qquad \tilde t_{i+1} = \tilde t_i + (M_i^s + M_i^u), \qquad 1\leq i\leq N-2$$

then

$$G((x_1,\tilde t_1),\ldots,(x_{N-1},\tilde t_{N-1})) \leq F((x_1,\tilde t_1),\ldots,(x_{N-1},\tilde t_{N-1})) + \frac{\beta}{8}. \tag{20}$$

Let us now consider

$$(y_i,y_{i+1}) \in (\mathbf{T}^n\times\pi + 2i\pi)\times(\mathbf{T}^n\times\pi + 2(i+1)\pi)$$

and let us suppose that the orbit (Q,q) realizing $h^{t_{i+1}-t_i}_{\bar c_{i+1}}(y_i,y_{i+1})$ crosses $\tilde\tau^+_{i+1}$ at time $t\in(t_i,t_{i+1})$. Then

$$h^{t_{i+1}-t_i}_{\bar c_{i+1}}(y_i,y_{i+1}) = h^{t-t_i}_{\bar c_{i+1}}(y_i,(Q,q)(t)) + h^{t_{i+1}-t}_{\bar c_{i+1}}((Q,q)(t),y_{i+1}) \geq$$

$$h^\infty_{\bar c_{i+1}}(y_i,(Q,q)(t)) + h^\infty_{\bar c_{i+1}}((Q,q)(t),y_{i+1}) =$$

$$h^\infty_{\bar c_{i+1}}(y_i,(Q,q)(t)) + h^\infty_{\bar c_{i+1}}((Q,q)(t),z_{i+1}) + h^\infty_{\bar c_{i+1}}(z_{i+1},(Q,q)(t)) + h^\infty_{\bar c_{i+1}}((Q,q)(t),y_{i+1}) =$$

$$h^\infty_{\bar c_{i+1}}(y_i,z_{i+1}) + h^\infty_{\bar c_{i+1}}(z_{i+1},y_{i+1}) \tag{21}$$

where the inequality follows from $iii)$ of lemma 1 and the second equality from the fact, shown in lemma 1, that

$$0 = h^\infty_{\bar c_{i+1}}((Q,q)(t),(Q,q)(t)) = h^\infty_{\bar c_{i+1}}((Q,q)(t),z_{i+1}) + h^\infty_{\bar c_{i+1}}(z_{i+1},(Q,q)(t)).$$

We have by (20) that

$$\inf\{G((y_1,t_1),\dots,(y_{N-1},t_{N-1})) : ((y_1,t_1),\dots,(y_{N-1},t_{N-1})) \in B, (y_1,\dots,y_{N-1}) \in \partial B'\} \geq$$

$$\inf\{F((y_1,t_1),\dots,(y_{N-1},t_{N-1})) : ((y_1,t_1),\dots,(y_{N-1},t_{N-1})) \in B, (y_1,\dots,y_{N-1}) \in \partial B'\}$$

and by (13) that

$$\inf\{F((y_1,t_1),\dots,(y_{N-1},t_{N-1})) : ((y_1,t_1),\dots,(y_{N-1},t_{N-1})) \in B, (y_1,\dots,y_{N-1}) \in \partial B'\} \geq$$

$$F((\tilde{x}_1,\tilde{t}_1),\dots,(\tilde{x}_{N-1},\tilde{t}_{N-1})) + \beta.$$

Since $x_1 \in B(\tilde{x}_i, \frac{D_{11}\delta}{N})$ and the Melnikoff function has a quadratic minimum in $(\tilde{x}_i, t_i)$, we have that

$$F((\tilde{x}_1,\tilde{t}_1),\dots,(\tilde{x}_{N-1},\tilde{t}_{N-1})) + \beta \geq F((\tilde{x}_1,t_1),\dots,(\tilde{x}_{N-1},t_{N-1})) + \beta - N\left(\frac{D_{11}\delta}{N}\right)^2$$

and by (19)

$$F((\tilde{x}_1,t_1),\dots,(\tilde{x}_{N-1},t_{N-1})) + \beta - N\left(\frac{D_{11}\delta}{N}\right)^2 \geq G((\tilde{x}_1,t_1),\dots,(\tilde{x}_{N-1},t_{N-1})) + \frac{7}{8}\beta - N\left(\frac{D_{11}\delta}{N}\right)^2.$$

If we put together all these inequalities and recall that $N \simeq \frac{1}{\beta}$ we get that

$$\inf\{G((y_1,t_1),\dots,(y_{N-1},t_{N-1})) : ((y_1,t_1),\dots,(y_{N-1},t_{N-1})) \in B, (y_1,\dots,y_{N-1}) \in \partial B'\} >$$

$$F((x_1,\tilde{t}_1),\dots,(x_{N-1},\tilde{t}_{N-1}))$$

which implies that G has a local minimum in the interior of B.
Since $T = \tilde{t}_{N-1}$, formula (4) now derives from the definition of $\tilde{t}_{N-1}$ and (16).

The estimate on T follows from its definition and formula (15); we now prove the estimate on the energy. We note that $t_{N-1} - t_0 > N - 1$ by (16) and the definition of $t_{N-1} - t_0 = T$. By the mean value theorem there is $0 < t_{op} < T$ such that $\dot{q}(t_{op}) = p(t_{op}) < 2\pi$; lemma 2 of [BK] yields that $|I(t_{op})| \leq \tilde{M}$ and thus $|H(q(t), Q(t), p(t), I(i))| \leq M$ with M independent on the parameters; this yields the first formula of point (3) of proposition 2.

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Appendix 1 — Comparison with the notations of [CG]

In Lemma 1 and 1' of §5 of [CG] the variables in the phase space, called here (Q, I, q, p), are named $(\vec{\alpha}, \vec{A}, \varphi, I)$. The perturbative parameter, that we denoted ϵ here in $(\mathcal{CG})$, was called μ in [CG].

The tori τ_i in (*1) of our proposition 1 correspond to $\mathcal{T}_\mu(s)$ of [CG], where s varies in the KAM Cantor set, called Σ_μ in [CG].

The quantities ω_i and η_i of (*2) correspond to $(1+\gamma)\vec{\omega}_s$ and $g_s(1+\gamma')$, respectively, where $s \in \Sigma_\mu$ as above.

In the notations of formula (5.5) of [CG], our $\zeta_i^s(\psi, y)$ corresponds to

$$\vec{A} = \vec{A}' + \vec{\Xi}(\psi, y, 0, s, \mu) \qquad \vec{\alpha} = \psi + \vec{\Delta}(\psi, y, 0, s, \mu) + \vec{\delta}(\vec{A}', y, 0, \mu)$$

$$I = R(\vec{A}', y, 0, \mu) + \Lambda(\psi, y, 0, s, \mu) \qquad \phi = S(\vec{A}', y, 0, \mu) + \Theta(\psi, y, 0, s, \mu)\,,$$

with $\vec{A}' = \vec{A}'_s(0, \mu)$ corresponding to the first n components of c_i (the last component is determined in (*6) of our proposition 1).

Beware of the very different use of the symbol s: as mentioned above, in [CG] s belongs to the KAM Cantor set Σ_μ, while here it just means "stable".

Also, the KAM canonical transformation sending the variables $(\vec{\alpha}, \vec{A}, \varphi, I)$ into the "normal" coordinates $(\vec{\alpha}', \vec{A}', q', p')$ has naturally associated a generating function

$$\langle \vec{A}', \vec{\alpha} \rangle + p'\varphi + \Phi^\star_\infty(\vec{\alpha}, \vec{A}', \varphi, p')\,.$$

Such a $\Phi^\star_\infty$ is not explicitly introduced in [CG] and it is not the same as the $\tilde{\Phi}_\infty$ introduced in §5 of [CG] after formula (5.66), since the last does not take into account the transformation of Lemma 0 of [CG]. However, such a $\Phi^\star_\infty$ essentially agree with the $\Phi^\pm_{i,u}$ introduced here in proposition one, in the sense that

$$\Phi^-_{i,u}(Q, q) = \Phi^\star_\infty(Q, \vec{A}'_i, q, 0) + \chi_i q\,,$$

where $\vec{A}'_i$ is the action corresponding to τ_i, i.e. τ_i corresponds to $\mathcal{T}_\mu(s_i)$ in the notations of [CG], and $\vec{A}'_i = \vec{A}'_{s_i}(0, \mu)$, and χ_i above is chosen in order to fulfill (*6).

Appendix 2 — Proof of lemma 2

Clearly, (Q, q) satisfies the E-L equation on each (t_i, t_{i+1}) since on these intervals it minimizes the action functional. It is somewhat more delicate to show that (Q, q) solves the E-L equation also in t_i, $2 \le i \le N-2$: the problem is that a small variation at t_i could bring (Q, q) into an orbit on which we have no information (see figure 4). We begin to note that it suffices to prove that $\dot{q}(t_i-) > 0$ and $\dot{q}(t_i+) > 0$ for $2 \le i \le N-2$. Indeed, let $m > 0$ be such that

$$|(Q(t), q(t)) - \tilde{x}_i| < \delta \quad \forall t \in [t_i - m, t_i + m]$$

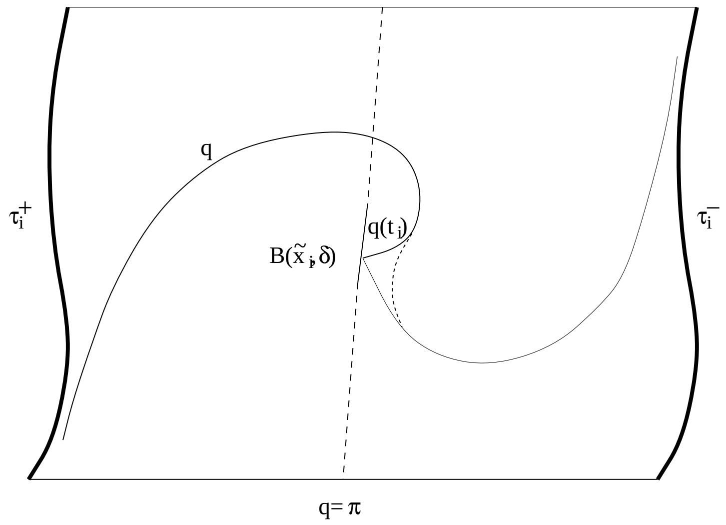
and let $(\bar{Q}, \bar{q})$ be a test function supported in $[t_i - m, t_i + m]$. Let us consider $(Q, q) + \gamma(\bar{Q}, \bar{q})$; by the implicit function theorem, if γ is small enough, we can find a continuous $t(\gamma)$ such that $q(t(\gamma)) + \gamma\bar{q}(t(\gamma)) = q(t_i)$ and thus

$$(Q(t(\gamma)) + \gamma\bar{Q}(t(\gamma)), q(t(\gamma)) + \gamma\bar{q}(t(\gamma))) \in B(\tilde{x}_i, \delta).$$

Since $((y_1, t_1), \ldots, (y_{N-1}, t_{N-1}))$ is a local minimum we have that for γ small enough

$$G((y_1, t_1), \ldots, (y_i, t_i), \ldots, (y_{N-1}, t_{N-1})) \le G((y_1, t_1), \ldots, ((Q + \gamma\bar{Q}, q + \gamma\bar{q})(t(\gamma)), t(\gamma)), \ldots, (y_{N-1}, t_{N-1}))$$

which implies that the action functional of $(Q, q) + \gamma(\bar{Q}, \bar{q})$ is greater or equal than the action functional of (Q, q); the usual argument now tells us that the E-L equation holds also in t_i.



A small variation of q is represented as a dotted line.

We do not have any information on its action, since it does not pass through $B(\tilde{x}_i, \delta)$.

Figure 4

This argument does not apply at $t_1 = 0$ and at $t_{N-1} = T$ since at these two points we are fixing *both* the times and the Poincaré sections, $q = 3\pi$ and $q = \pi + 2(N-1)\pi$ respectively. In other words, there are not enough variations for q and thus $\dot{q}$ can be discontinuous at these two times. But we can still vary Q by an arbitrary test function and thus Q will satisfy $\frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{Q}}\mathcal{L} = \partial_Q\mathcal{L}$; in particular, $\dot{Q}$ will be continuous at $t = 0$ and at $t = T$. Since on $(-\infty, 0]$ (Q, q) realizes $h^{\infty}_{\bar{c}_1}(z_1, x_1)$, by lemma 1 $(Q(0), q(0), I(0), p(0-)) =$ $L^{-1}(Q(0), q(0), \dot{Q}(0), \dot{q}(0-))$ stays on the unstable manifold of τ_1 and $(Q(T), q(T), I(T), p(T+)) =$ $L^{-1}(Q(T), q(T), \dot{Q}(T), \dot{q}(T+))$ stays on the stable manifold of τ_N. Thus, by (*7), $|I(0) - c'_1|$ and $|I(T) - c'_N|$ are small for ϵ small, where c'_i denotes the first n coordinates of c_i; this yields the second formula of point (3) of proposition 2.

Let us now prove that $\dot{q}(t_i-) > 0$ for $i \in (2, \ldots, N-1)$; the proof for $\dot{q}(t_i+)$ is analogous. The only information we can use is the fact that (Q, q) minimizes the action; we will show by contradiction that if $\dot{q}(t_i-) \le 0$ then (Q, q) cannot minimize. We begin to prove the cases $(\mathcal{CG})$, $(\mathcal{G})$, $(\mathcal{GGM})$; to prove $(\mathcal{B}1)$ and $(\mathcal{B}2)$ it suffices to change the constants.

Let us suppose by contradiction that $\dot{q}(t_i-) \le 0$. We fix a constant $\sigma > 0$; in the following we will require that it is sufficiently small; its choice does not depend on ϵ and $t_i - t_{i-1}$.

From the E-L equation we see that for ϵ small enough this implies that

$$\exists \hat{t} \in [t_i - 1, t_i - \frac{1}{2}] \quad \text{such that} \quad q(\hat{t}) \ge q(t_i) - \sigma$$

which implies by direct computation that, for σ small enough,

$$\int_{\hat{t}}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q))]\mathrm{d}t \ge \theta \tag{7}$$

$$\int_{t_{i-1}}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1-\cos(q))]\mathrm{d}t -$$

$$\min\{\int_{t_{i-1}}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1-\cos(\bar{q}))]\mathrm{d}t : \bar{q}(t_{i-1}) = q(t_{i-1}), \bar{q}(t_i) = q(t_i)\} \geq \theta \tag{8}$$

for some $\theta > 0$ independent on $t_i - t_{i-1}$. Let α be half the action of the homoclinic of the pendulum

$$\alpha = \min\{\int_{-\infty}^{0} \frac{1}{2}|\dot{q}|^2 + (1-\cos(q))\mathrm{d}t : q(-\infty) = 0, q(0) = \pi\}.$$

A simple computation with the pendulum functional shows that we can choose $\sigma \in (0, \min(\frac{1}{64}\alpha, \frac{\theta}{64}))$ and $M \geq 4$ such that

$$\forall T \geq M, \qquad \forall q_0 = \pm\sigma$$

$$\min\{\int_0^T \frac{1}{2}|\dot{q}|^2 + (1-\cos(q))\mathrm{d}t : q(0) = q_0, q(T) = \pi\} \leq \alpha + 2\sigma \tag{9}$$

$$\forall T > 0 \qquad \min\{\int_0^T \frac{1}{2}|\dot{q}|^2 + (1-\cos(q))\mathrm{d}t : q(0) = q_0, q(T) = \pi\} \geq \alpha - 2\sigma \tag{10}$$

$$\forall T \geq 0 \qquad \min\{\int_0^T \frac{1}{2}|\dot{q}|^2 + (1-\cos(q))\mathrm{d}t :$$

$$q(0) = \sigma, q(T) \geq \pi - \sigma, \sigma \leq q(t) \leq 2\pi - \sigma \quad \forall t \in [0,T]\} \geq \alpha - 6\sigma + D_9 T + \frac{D_{10}}{T}. \tag{11}$$

In the sequel, σ is fixed in the above way; we will feel free to increase M. We note that D_{10} is independent on the choice of σ, so we can assume $0 < \sigma < \frac{D_{10}}{64}$.

We begin to consider the case $t_i - t_{i-1} \leq 2M$. We denote by $\bar{q}(t)$ the orbit of the pendulum satisfying $\bar{q}(t_{i-1}) = q(t_{i-1})$, $\bar{q}(t_i) = q(t_i)$. Then we have that

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q})\mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}})\mathrm{d}t =$$

$$\int_{t_{i-1}}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1-\cos(q)) + \epsilon f(Q,q)]\mathrm{d}t - \int_{t_{i-1}}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1-\cos(\bar{q})) + \epsilon f(Q,\bar{q})]\mathrm{d}t \geq$$

$$\int_{t_{i-1}}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1-\cos(q))]\mathrm{d}t - \int_{t_{i-1}}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1-\cos(\bar{q}))]\mathrm{d}t - 4\epsilon M$$

where the inequality is a consequence of condition (1) in the introduction. By (8) we get that, for ϵ small enough,

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q})\mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}})\mathrm{d}t \geq \theta - 4\epsilon M > 0$$

contradicting the minimality of (Q, q) on (t_{i-1}, t_i).

Let us now suppose that $t_i - t_{i-1} \geq 2M$; let t' be the maximum time in (t_{i-1}, t_i) such that $q(t') = 2\pi i + \sigma$. We divide again into two cases: if

$$\forall t \in [t', t_i] : 2\pi i + \sigma \leq q(t) < 2\pi(i+1) - \sigma$$

(the first inequality is automatic from the definition of t') then we define

$$\bar{q}(t) = \begin{cases} q(t) & t \in [t_{i-1}, t'] \\ \tilde{q}(t) & t \in [t', t_i] \end{cases}$$

where $\tilde{q}$ is the orbit of the pendulum with boundary conditions $\tilde{q}(t') = 2\pi i + \sigma$, $\tilde{q}(t_i) = q(t_i)$. A little computation yields

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q})\mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}})\mathrm{d}t \geq$$

$$\int_{t'}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q))]\mathrm{d}t - \int_{t'}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1 - \cos(\bar{q}))]\mathrm{d}t - 4\epsilon(t_i - t'). \tag{12}$$

We consider two subcases: if $t_i - t' \geq M$ we evaluate both functionals and we get

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q})\mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}})\mathrm{d}t \geq$$

$$\theta + \alpha - 2\sigma + D_9(\hat{t} - t') + \frac{D_{10}}{\hat{t} - t'} - (\alpha + 2\sigma) - 4\epsilon(t_i - t')$$

where θ is the contribution of q on $[\hat{t}, t_i]$ due to formula (7), the next four terms come from (11), the sixth term from (9) and the last is the contribution of the perturbation. For ϵ small enough, the last formula contradicts the minimality of (Q, q).
In the other subcase, $t_i - t' \leq M$, we get from (8)

$$\int_{t'}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q))]\mathrm{d}t - \int_{t'}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1 - \cos(\bar{q}))]\mathrm{d}t \geq \theta$$

which for ϵ small by (12) implies

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q})\mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}})\mathrm{d}t > 0$$

a contradiction.
The other case is when

$$\exists \bar{t} \in [t', t_i] : \; q(\bar{t}) = 2\pi(i+1) - \sigma.$$

We denote by $\bar{t}$ the maximum t with the above property and by $\tilde{t}$ the minimum one. We divide again into two subcases: if $\bar{t} - \tilde{t} \geq 4$, we define

$$\psi = \min\left(1, \frac{\tilde{t} - t'}{4}\right)$$

and

$$\bar{q}(t) = \begin{cases} q(t) & t \in [t_{i-1}, t'] \\ 2\pi i - \sigma(t - t' - \psi)\frac{1}{\psi} & t \in [t', t' + \psi] \\ 2\pi i & t \in [t' + \psi, \tilde{t} - \psi] \\ 2\pi i - \sigma(t - \tilde{t} + \psi)\frac{1}{\psi} & t \in [\tilde{t} - \psi, \tilde{t}] \\ q(t) - 2\pi & t \in [\tilde{t}, \bar{t} - \psi] \\ (q(\bar{t} - \psi) - 2\psi)(1 - \frac{t - \bar{t} + \psi}{\psi}) + (2\pi i + \sigma)\frac{t - \bar{t} + \psi}{\psi} & t \in [\bar{t} - \psi, \bar{t}] \\ \tilde{q}(t) & t \in [\bar{t}, t_i] \end{cases}$$

where $\tilde{q}$ is the orbit of the pendulum with boundary conditions $\tilde{q}(\bar{t}) = 2\pi i - \sigma$, $\tilde{q}(t_i) = q(t_i)$. It follows easily with an argument like the one at the end of proposition 1 that $\dot{q}$ is bounded by 1 on $[\tilde{t}, \bar{t}]$ and consequently that $\dot{\bar{q}}$ is bounded by 1 on $[\bar{t} - \psi, \bar{t}]$. With a small computation we see that

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q}) \mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}}) \mathrm{d}t =$$

$$\int_{[t', \tilde{t}] \cup [\bar{t}, t_i]} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q)) + \epsilon f(Q, q)] \mathrm{d}t - \int_{[t', \tilde{t} - \psi] \cup [\bar{t} - \psi, t_i]} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1 - \cos(\bar{q})) + \epsilon f(Q, \bar{q})] \mathrm{d}t.$$

If $t_i - \bar{t} \geq M$, we get that

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q}) \mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}}) \mathrm{d}t \geq$$

$$3(\alpha - 6\sigma) + D_9(\tilde{t} - t' + t_i - \bar{t}) + \frac{D_{10}}{\tilde{t} - t'} + \frac{D_{10}}{t_i - \bar{t}} - (\alpha + 2\sigma) - 4\sigma \frac{1}{\psi} - 2\epsilon(\tilde{t} - t' + t_i - \bar{t})$$

where the first four terms, due to (11), are the contribution of q, which goes from $2\pi i$ to $2\pi(i+1)$ and back to $2\pi i + \pi$; the next term is the pendulum action of $\bar{q}$ on $[\bar{t}, t_i]$ estimated by (9) since $t_i - \bar{t} \geq M$ and the last term accounts for the perturbation. For ϵ small enough the last formula contradicts the minimality of (Q, q).

If $t_i - \bar{t} < M$ we see that, for the same $\bar{q}$ defined above,

$$\int_{\bar{t}}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q))] \mathrm{d}t - \int_{\bar{t}}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1 - \cos(\bar{q}))] \mathrm{d}t \geq 0$$

since the boundary conditions are the same up to a reflection around π.If in the above formula we change correspondingly the estimate on the pendulum action of q and $\bar{q}$ on $[\bar{t}, t_i]$, we get

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q}) \mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}}) \mathrm{d}t \geq$$

$$2(\alpha - 2\sigma) + D_9(\tilde{t} - t') + \frac{D_{10}}{\tilde{t} - t'} - 4\sigma \frac{1}{\psi} - 2\epsilon(\tilde{t} - t') - 2\epsilon M$$

which for ϵ small enough contradicts the minimality of (Q, q).

If $\bar{t} - \tilde{t} < 4$, we define

$$\bar{q}(t) = \begin{cases} q(t) & t \in [t_{i-1}, t'] \\ \tilde{q}(t) & t \in [t', t_i] \end{cases}$$

where $\tilde{q}$ is the orbit of the pendulum with boundary conditions $\tilde{q}(t') = q(t') = 2\pi i + \sigma$, $\tilde{q}(t_i) = q(t_i) = 2\pi i + \pi$. We divide again in two subcases. If $t_i - \bar{t} \geq M$ a small computation shows that

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q}) \mathrm{d}t - \int_{t_{i-1}}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}}) \mathrm{d}t \geq$$

$$\int_{t'}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q))] \mathrm{d}t - \int_{t'}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1 - \cos(\bar{q}))] \mathrm{d}t - 2\epsilon(t_i - t') \geq$$
$$3(\alpha - 2\sigma) + D_9(t_i - t' - 4) - (\alpha + 2\sigma) - 2\epsilon(t_i - t')$$

where the first two terms are the contributions of the pendulum action of q, due to (11), the third is the pendulum action of $\bar{q}$ and the last accounts for the perturbation. Their sum is positive if ϵ is small enough, contradicting minimality. Let now $t_i - \bar{t} < M$; since on $[\bar{t}, t_i]$ q covers the same distance as $\bar{q}$ on $[t', t_i]$, and since $t_i - \bar{t} \le t_i - t'$, it is easy to see that

$$\int_{\bar{t}}^{t_i} [\frac{1}{2}|\dot{q}|^2 + (1 - \cos(q))]\mathrm{d}t - \int_{t'}^{t_i} [\frac{1}{2}|\dot{\bar{q}}|^2 + (1 - \cos(\bar{q}))]\mathrm{d}t \ge 0$$

which implies by (11)

$$\int_{t'}^{t_i} \mathcal{L}(Q, q, \dot{Q}, \dot{q})\mathrm{d}t - \int_{t'}^{t_i} \mathcal{L}(Q, \bar{q}, \dot{Q}, \dot{\bar{q}})\mathrm{d}t \ge 2(\alpha - 2\sigma) + D_9(\tilde{t} - t') - 2\epsilon(\tilde{t} - t' + M) > 0$$

contradicting minimality.

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