Ruelle operators: Functions which are harmonic with respect to a transfer operator

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Contents

Abstract. Let $N \in \mathbb{N}$, $N \geq 2$, be given. Motivated by wavelet analysis, we consider a class of normal representations of the C^*-algebra $\mathfrak{A}_N$ on two unitary generators U, V subject to the relation

$$UVU^{-1} = V^N.$$

The representations are in one-to-one correspondence with solutions $h \in L^1(\mathbb{T})$, $h \geq 0$, to $R(h) = h$ where R is a certain transfer operator (positivity-preserving) which was studied previously by D. Ruelle. The representations of $\mathfrak{A}_N$ may also be viewed as representations of a certain (discrete) N-adic $ax+b$ group which was considered recently by J.-B. Bost and A. Connes.

CHAPTER 1

Introduction

In multiresolution wavelet theory, there is a fundamental interplay and interconnection between the following two operators: M and R, where M is the *cascade refinement operator* and R is the corresponding *transfer operator.* We also denote the second of these, R, the *Ruelle operator,* because of its close connection to an operator that David Ruelle used first in his study of phase transitions in quantum statistical mechanics lattice models; see [**Rue68**], [**May80**], and [**Rue78a**]. We first recall the two operators here in a simple wavelet context, but the scope will be widened later (Chapter 6): Let M be the operator in $L^2(\mathbb{R})$ given by

$$(M\psi)(x) = \sqrt{N}\sum_{k\in\mathbb{Z}} a_k\psi(Nx-k), \tag{1.1}$$

where $N \geq 2$ is integral, and $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$, are given subject to $\sum_k |a_k|^2 = 1$; $\psi \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. If

$$m_0(z) := \sum_{k\in\mathbb{Z}} a_k z^k, \qquad z \in \mathbb{T}, \tag{1.2}$$

then the condition $m_0 \in L^\infty(\mathbb{T})$ implies that M is a bounded operator in $L^2(\mathbb{R})$. With the usual identification

$$\mathbb{R}/2\pi\mathbb{Z} \ni \omega \longmapsto e^{-i\omega} = z \in \mathbb{T} \tag{1.3}$$

we have the corresponding identification $m_0(\omega) = m_0(z)$ of 2π-periodic functions on $\mathbb{R}$ with functions on $\mathbb{T}$, and we shall use the same letter denoting the function either way. Introducing the Fourier transform $\psi \mapsto \hat{\psi}$ in $L^2(\mathbb{R})$, we get (1.1) in the equivalent form:

$$(M\psi)\hat{\ }(\omega) = \frac{m_0\left(\frac{\omega}{N}\right)}{\sqrt{N}}\hat{\psi}\left(\frac{\omega}{N}\right), \qquad \omega \in \mathbb{R}. \tag{1.4}$$

The Ruelle transfer operator is defined on $L^1(\mathbb{T})$ by

$$(Rf)(z) = \frac{1}{N}\sum_{w^N=z} |m_0(w)|^2 f(w), \qquad f \in L^1(\mathbb{T}),\ z \in \mathbb{T}, \tag{1.5}$$

where the summation is over the N roots w, i.e., the N solutions to $w^N = z$.

For the *quadrature wavelet filters,* there is the further restriction

$$\sum_{w^N=z} |m_0(w)|^2 = N, \tag{1.6}$$

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or equivalently

$$R(\mathbb{1}) = \mathbb{1} \tag{1.7}$$

where $\mathbb{1}$ denotes the constant function in $L^2(\mathbb{T})$. But our analysis will *not* be restricted to this special case.

When $m_0 \in L^\infty(\mathbb{T})$ is given, and R is the corresponding Ruelle operator, we study the eigenvalue problem

$$h \in L^1(\mathbb{T}), \qquad h \geq 0, \qquad R(h) = h. \tag{1.8}$$

But without (1.6), a nonzero solution h to (1.8) is not then guaranteed. The problem (1.8) is closely connected to the problem

$$\varphi \in L^2(\mathbb{R}), \qquad M\varphi = \varphi, \tag{1.9}$$

whose nonzero solutions (if any) are the *scaling functions* (or *father functions*) in wavelet theory.

Suppose (1.6) is given: then a famous argument of Mallat [**Mal89**] states that the $L^2(\mathbb{R})$-norm of the functions F_n,

$$F_n(\omega) = \prod_{k=1}^{n} \frac{m_0\left(\frac{\omega}{N^k}\right)}{\sqrt{N}} \chi_{[-\pi,\pi\rangle}\left(\frac{\omega}{N^n}\right) \tag{1.10}$$

is constant. In fact $\|F_n\|_{L^2(\mathbb{R})} = 1$, $n = 1, 2, \dots$. If moreover $\omega \mapsto m_0(\omega)$ is Lipschitz near $\omega = 0$, and $m_0(0) = \sqrt{N}$ (the low-pass condition), then

$$F(\omega) = \prod_{k=1}^{\infty} \frac{m_0\left(\frac{\omega}{N^k}\right)}{\sqrt{N}} \tag{1.11}$$

is pointwise convergent. If the eigenspace $\{h \mid Rh = h\}$ is further given to be one-dimensional, then $F \in L^2(\mathbb{R})$ and $\lim_{n\to\infty} \|F - F_n\|_{L^2(\mathbb{R})} = 0$. In view of (1.4), the inverse Fourier transform $\varphi = \check{F}$ will then solve the eigenvalue problem (1.9), and $\hat{\varphi}(0) = 1$.

To motivate the more general problem (1.8), we note that, if $h \in L^1(\mathbb{T})$ solves (1.8), then

$$\int_{-\pi N^n}^{\pi N^n} h\left(\frac{\omega}{N^n}\right) \prod_{k=1}^{n} \frac{\left|m_0\left(\frac{\omega}{N^k}\right)\right|^2}{N}\, d\omega = \int_0^{2\pi} R^n h(\omega)\, d\omega = \int_0^{2\pi} h(\omega)\, d\omega \geq 0.$$

So again, if $h \neq 0$, then the sequence F_n, now defined *via* h by

$$F_n(\omega) := \chi_{[-\pi,\pi\rangle}\left(\frac{\omega}{N^n}\right) \left(h\left(\frac{\omega}{N^n}\right)\right)^{\frac{1}{2}} \prod_{k=1}^{n} \frac{m_0\left(\frac{\omega}{N^k}\right)}{\sqrt{N}} \tag{1.12}$$

has constant norm in $L^2(\mathbb{R})$. If, for example, $h(0) = 1$, then it can be checked that $F_n \to F$ in $L^2(\mathbb{R})$ for $n \to \infty$, where F has a (1.11)-representation.

For more background references on wavelets, filters, and scaling functions, from the operator-theoretic viewpoint, we give [**CoDa96**], [**CoRy95**], [**DaLa**], [**Dau92**], [**Hör95**], [**MePa93**], [**Mey98**], and [**Vil94**]. However, a summary of some main ideas is included for the convenience of the reader, and to make the paper more self-contained. Our terminology is close to that of the listed references.

One of the aims of our paper is to widen the scope of the quadrature analysis and to study the more general eigenvalue problem (1.8). For that, it is helpful to adopt a representation-theoretic viewpoint, and not to insist on $L^2(\mathbb{R})$ as the Hilbert space for the eigenvalue problem (1.9). We will look for abstract Hilbert spaces $\mathcal{H}$ which admit solutions $\varphi \in \mathcal{H}$, $\varphi \neq 0$, to $M\varphi = \varphi$ in a way that naturally generalizes (1.9). This will also lead to results on multiresolutions which give solutions *up to unitary equivalence,* as well as conditions for equivalence. This approach dictates another slight modification: if $\mathcal{H}$ is given only abstractly, we must specify a unitary operator $U\colon \mathcal{H} \to \mathcal{H}$ which corresponds to the scaling operator

$$U\colon \psi \longmapsto \frac{1}{\sqrt{N}}\psi\left(\frac{x}{N}\right) \tag{1.13}$$

for the special case when $\mathcal{H} = L^2(\mathbb{R})$. Similarly, we must specify a representation π of $L^\infty(\mathbb{T})$ on $\mathcal{H}$ such that

$$U\pi(f) = \pi\left(f\left(z^N\right)\right)U, \qquad f \in L^\infty(\mathbb{T}) \tag{1.14}$$

as a commutation relation for operators on $\mathcal{H}$. In this wider setting, the problem (1.9) then takes the form:

$$\varphi \in \mathcal{H}\colon \qquad U\varphi = \pi(m_0)\varphi. \tag{1.15}$$

This means that $(U, \pi, \mathcal{H}, \varphi)$ are specified, and satisfy (1.14)–(1.15). The symbol U denotes both an element in $\mathfrak{A}_N$ (i.e., the C^*-algebra on two unitary generators U and V subject to $UVU^{-1} = V^N$), and a unitary operator in $\mathcal{H}$. If $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ is the corresponding representation, then $\tilde{\pi}(U) = U$ (with the double meaning for U).

The restricting condition we place on $\left(U, \pi, L^2(\mathbb{R})\right)$ is that there is some $\varphi \in L^2(\mathbb{R})$, which satisfies $U\varphi = \pi(m_0)\varphi$, and $\hat{\varphi}(0) = 1$, where $\hat{\varphi}$ is the Fourier transform, i.e.,

$$\hat{\varphi}(\omega) := \int_{\mathbb{R}} e^{-i\omega x}\varphi(x)\,dx. \tag{1.16}$$

A system $\left(U, \pi, L^2(\mathbb{R}), \varphi, m_0\right)$ with these properties will be called a *wavelet representation,* and φ scaling function (or *father function*). They are used in the construction of wavelets via *multiresolutions;* see [**BrJo97**] and [**Dau92**].

If a wavelet representation is given with scaling function, we form the $L^1(\mathbb{T})$-function h_φ as follows: $\mathbb{T} = \mathbb{R}\diagup 2\pi\mathbb{Z}$, $z = e^{-i\omega} \in \mathbb{T}$ a parametrization, and

$$h_\varphi(z) = \sum_{n\in\mathbb{Z}} z^n \left\langle \pi(e_n)\varphi \,\middle|\, \varphi\right\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\omega + 2\pi k)\right|^2.$$

If the *Ruelle transfer operator* is defined as

$$(Rf)(z) := \frac{1}{N}\sum_{\substack{w\in\mathbb{T}\\ w^N = z}} \left|m_0(w)\right|^2 f(w), \tag{1.17}$$

then

$$R(h_\varphi) = h_\varphi. \tag{1.18}$$

(See Lemma 3.3.)

In this paper, we prove a converse to this result: Every solution h to $Rh = h$ arises this way as $h = h_{\varphi_\pi}$ for some representation π. (See Theorems 2.4 and 6.7.)

We further show in Chapter 7 that the solutions to (1.15) may be represented by a family of finite (positive) Borel measures $\nu = \nu(m_0, h)$ on a certain Bohr compactification K_N of $\mathbb{R}$. It is simply the Bohr–Besicovitch compactification (see [**Bes55**]) corresponding to frequencies of the form

$$\sum_i \frac{n_i}{N^i}, \qquad n_i \in \mathbb{Z}, \text{finite sums.} \tag{1.19}$$

Hence, in this case, $\mathcal{H} = L^2(K_N, \nu)$. But, more surprisingly, ν may be chosen (depending on m_0, h as in (1.8)) such that the solution φ to (1.15), in the $L^2(K_N, \nu)$ representation, is simply $\varphi = \mathbb{1}$, i.e., the constant unit function in $L^2(K_N, \nu)$.

A second advantage of the present wider scope is that it includes applications outside wavelet theory. We note in Chapter 6 that Ruelle operators of the form (1.5) arise naturally (in fact first!) in statistical mechanics [**Rue68**] and ergodic theory [**Kea72**], and our representation-theoretic results are applied there.

Our motivation for the study of the interplay between the two operators M and R in (1.1) and (1.5) came from the now familiar connection between the spectral analysis of R and the convergence properties of the iterations

$$\psi^{(0)},\ M\psi^{(0)},\ M^2\psi^{(0)},\ \dots \tag{1.20}$$

when $\psi^{(0)}$ is given. Similarly the spectral theory of R dictates directly the regularity properties of the solution φ to (1.9). (See [**Dau92**], [**CoDa96**], [**BEJ97**], [**Vil94**], and Chapter 4 below for more details on that point.)

Until recently, there was only very little in the literature on notions of *equivalence* for wavelets, or for the filter functions which are used to generate them, or even *isomorphism,* or *invariants,* for these objects; see, however, [**BrJo97**].

A second motivation centers around the cocycle equivalence problems for wavelet filters. Let m_0 and m_0' be in $L^\infty(\mathbb{T})$. We say that they are *cocycle equivalent* if there is an $f \in L^2(\mathbb{T})$ such that

(i) $f(z^N)\, m_0'(z) = m_0(z) f(z)$ and
(ii) $\sum_{w^N=z} |m_0'(w)|^2 = N$ for a.a. $z \in \mathbb{T}$.

We stress that condition (ii) is only assumed to hold for m_0' and not for m_0. Let m_0 be fixed, and let R be the corresponding Ruelle operator of order N, i.e.,

$$(Rf)(z) := \frac{1}{N} \sum_{w^N=z} |m_0(w)|^2 f(w). \tag{1.21}$$

In Chapter 5, we show the following result: If $f \in L^2(\mathbb{T})$ defines a cocycle equivalence, then $h(z) := |f(z)|^2$ solves $R(h) = h$. If conversely $h \in L^1(\mathbb{T})$, $h \geq 0$, is given to satisfy $R(h) = h$, then $f(z) := h(z)^{\frac{1}{2}}$ defines a cocycle equivalence for some m_0', i.e., (i)–(ii) hold.

The relation $UfU^{-1} = f(z^N)$ for the abelian algebra $L^\infty(\mathbb{T})$ and a single unitary U may be rewritten as $Uf = f(z^N)\,U$, and then U may possibly not be unitary. A representation of the more general relation is then a pair (π, U) where π is a representation of $L^\infty(\mathbb{T})$ in some Hilbert space $\mathcal{H}$, and U is a bounded operator in $\mathcal{H}$ such that

$$U\pi(f) = \pi(f(z^N))\,U \tag{1.22}$$

holds as an identity on $\mathcal{H}$ for all $f \in L^\infty(\mathbb{T})$. Let $m_0 \in L^\infty(\mathbb{T})$, and let $h \in L^1(\mathbb{T})$ be given such that $h \geq 0$ and $R_{m_0}(h) = h$. Then a representation (π, U) results as follows: take $\mathcal{H} = L^2(\mathbb{T}, h\,d\mu)$ and $U = S_0$ given by

$$(S_0\xi)(z) = m_0(z)\,\xi(z^N), \qquad \xi \in L^2(h) := L^2(\mathbb{T}, h\,d\mu), \tag{1.23}$$

i.e., $\int_{\mathbb{T}} |\xi(z)|^2 h(z)\, d\mu(z) < \infty$ where μ is the normalized Haar measure on $\mathbb{T}$, and

$$\pi_0(f)\xi(z) = f(z)\xi(z), \qquad f \in L^\infty(\mathbb{T}),\ \xi \in L^2(h). \tag{1.24}$$

We show in Theorem 5.6 that, if m_0 is a *non-singular* (defined below) wavelet filter, then S_0 (of (1.23)) is a *pure shift,* i.e., it is isometric in $L^2(h)$, and

$$\bigcap_{n=1}^{\infty} S_0^n\left(L^2(h)\right) = \{0\}. \tag{1.25}$$

In fact, for the wavelet filters, we show in Chapter 3 that $L^2(h)$ embeds isometrically into $L^2(\mathbb{R})$; that is, there is an *intertwining isometry* $L^2(h) \overset{W}{\hookrightarrow} L^2(\mathbb{R})$ such that

$$WS_0 = UW \tag{1.26}$$

where

$$U\psi(x) := N^{-\frac{1}{2}}\psi\left(\frac{x}{N}\right), \qquad \psi \in L^2(\mathbb{R}), \tag{1.27}$$

and

$$W(\mathbb{1}) = \varphi. \tag{1.28}$$

In that case, φ (in $L^2(\mathbb{R})$) is the scaling function, and

$$\varphi = M\varphi \tag{1.29}$$

with $M = U^{-1}\pi(m_0)$. Moreover, M satisfies

$$M^*\pi(f)M = \pi\left(|m_0|^2 f\left(z^N\right)\right), \qquad f \in L^\infty(\mathbb{T}). \tag{1.30}$$

More generally, if (M, π) is given in a Hilbert space $\mathcal{H}$, subject to (1.30), we say that M is a π-*isometry,* and the previous paper [**Jor98**] gives a complete structure result for π-isometries, starting with the following Wold decomposition:

$$\mathcal{H} = \sum_{n=0}^{\infty}{}^{\oplus}\left[M^k\mathcal{L}\right] \oplus \bigcap_{1}^{\infty}\left[M^n\mathcal{H}\right], \tag{1.31}$$

where $\mathcal{L} := \ker(M^*)$ and $\left[M^k\mathcal{L}\right]$ denotes the closure in $\mathcal{H}$ of $\left\{M^k l \mid l \in \mathcal{L}\right\}$. Moreover, the components in the decomposition are mutually orthogonal, and each one reduces the representation π.

For the case of non-singular wavelet filters, it follows from (1.25) that U is then unitarily equivalent to the *bilateral shift* which extends S_0. (Recall S_0 is then a *unilateral* shift.) If the wavelet filter is singular, i.e., if m_0 vanishes on a subset of $\mathbb{T}$ of positive measure, then $\mathcal{L} = \ker M^* \neq \{0\}$, and the decomposition (1.31) then has the shift part

$$\left[M^k\mathcal{L}\right] \xrightarrow{M} \left[M^{k+1}\mathcal{L}\right].$$

But the scaling function φ must be in $\bigcap_1^\infty\left[M^n\mathcal{H}\right]$. For representations more general than wavelets, we show in Chapter 7 that there is an *intertwining isometry* W_B (analogous to W above):

$$L^2(h) \overset{W_B}{\hookrightarrow} L^2(K_N, \nu) \tag{1.32}$$

where K_N is the compact Bohr group, and ν is a Borel probablilty measure on K_N which depends on m_0 and h. The relations which correspond to (1.26)–(1.29) are:

$$W_B S_0 = U W_B, \tag{1.33}$$

$$(U\psi)(\chi) = m_0(\chi)\,\psi\left(\chi^N\right), \qquad \psi \in L^2(K_N,\nu),\ \chi \in K_N, \tag{1.34}$$

where $K_N = \left(\mathbb{Z}\left[\frac{1}{N}\right]\right)^{\widehat{}}$, $\chi^n(\lambda) = \chi(n\lambda)$, $\lambda \in \mathbb{Z}\left[\frac{1}{N}\right]$, $n \in \mathbb{Z}$, and

$$m_0(\chi) := \sum_{n\in\mathbb{Z}} a_n \chi^n \tag{1.35}$$

is the natural extension of m_0 from $\mathbb{T}$ to K_N.

But, in this case, W_B sends the constant function in $\mathbb{T}$ to that of K_N, i.e.,

$$W_B(\mathbb{1}) = \mathbb{1}, \tag{1.36}$$

so that φ is now represented by the constant function $\mathbb{1}$ in $L^2(K_N,\nu)$, i.e.,

$$M(\mathbb{1}) = \mathbb{1} \tag{1.37}$$

with $M = U^{-1}\pi_0(m_0)$ now taking the form

$$M\psi(\chi) = \psi\left(\chi^{-N}\right), \qquad \psi \in L^2(K_N,\nu),\ \chi \in K_N. \tag{1.38}$$

In the next two chapters, we introduce a certain discrete solvable group G_N and its C^*-algebra $\mathfrak{A}_N$; and we prove the representation theorem alluded to above. The scaling function is formulated in an abstract setting, and we identify a corresponding class of representations which are defined from solutions h to $Rh = h$, $h \in L^1$, $h \geq 0$. Conversely, we show that every such eigenfunction h defines a representation with an (abstract) scaling vector.

One of our motivations for the analysis of (1.1) or (1.9) was recent work on structural properties of the solutions φ, and aimed at giving new invariants for them. Our own papers [**BEJ97**] and [**Jor98**] establish such representation-theoretic invariants. Analysis of (1.1) in a variety of guises can also be found in [**CDM91**], [**ChDe60**], [**CoDa96**], [**CoRa90**], [**Ho96**], [**Her95**], [**JLS98**], [**LMW96**], and [**LWC95**].

CHAPTER 2

A discrete $ax+b$ group

Let $N \in \{2,3,\dots\}$ be given, and let $\Lambda_N := \mathbb{Z}\left[\frac{1}{N}\right]$ be the ring obtained from $\mathbb{Z}$ by extending with the fraction $\frac{1}{N}$, i.e., Λ_N contains $\mathbb{Z}$ and all powers $\left\{\frac{1}{N^k} \mid k=1,2,\dots\right\}$. We will then consider the group $G = G_N$ of all 2×2 matrices

$$\left\{ \left(\begin{smallmatrix} N^j & \lambda \\ 0 & 1 \end{smallmatrix}\right) \mid j \in \mathbb{Z},\ \lambda \in \Lambda_N \right\}. \tag{2.1}$$

We showed in [**BreJo91**] that there is a one-to-one correspondence between the unitary representations of G_N and the $*$-representations of the C^*-algebra $\mathfrak{A}$ on two unitary generators U, V, subject to the relations

$$UVU^{-1} = V^N. \tag{2.2}$$

By a representation of (2.2), we mean a realization of U and V as unitary operators on some Hilbert space $\mathcal{H}$, say, such that (2.2) also holds for those operators. Let $f \in L^\infty(\mathbb{T})$; then $f(V)$ is defined by the spectral theorem, applied to V, and $\pi_V(f) := f(V)$ is a representation of $L^\infty(\mathbb{T})$ in the sense that $\pi_V(f_1 f_2) = \pi_V(f_1)\,\pi_V(f_2)$, and $\pi_V(f)^* = \pi_V\left(\bar f\right)$ where $\bar f(z) := \overline{f(z)}$, $z \in \mathbb{T}$, $f_1, f_2, f \in L^\infty(\mathbb{T})$. Moreover, (2.2) then takes the form

$$U\pi_V(f)\,U^{-1} = \pi_V\left(f\left(z^N\right)\right), \tag{2.3}$$

and conversely, every pair (π, U) where π is a representation of $L^\infty(\mathbb{T})$ on $\mathcal{H}$, and U is a unitary operator on $\mathcal{H}$ such that

$$U\pi(f)\,U^{-1} = \pi\left(f\left(z^N\right)\right), \tag{2.4}$$

is of this form for some V. In fact, let $e_n(z) = z^n$, $n \in \mathbb{Z}$, and set $V := \pi(e_1)$.

Since V is unitary, it has a spectral resolution $V = \int_{\mathbb{T}} \lambda\, E(d\lambda)$ with a projection-valued spectral measure $E(\,\cdot\,)$ on $\mathbb{T}$. We will study cyclic vectors $\varphi \in \mathcal{H}$. If the measure $\left\|E(\,\cdot\,)\varphi\right\|^2$ on $\mathbb{T}$ is absolutely continuous with respect to Haar measure on $\mathbb{T}$, we say that the corresponding representation is *normal.* The normal representations will be denoted $\operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$.

The C^*-algebra on the relations (2.2), introduced in [**BreJo91**], will be denoted $\mathfrak{A}_N$, and we shall always use the correspondence between the representation $\operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ on some Hilbert space $\mathcal{H}$, and the corresponding unitary representations of G_N.

We also showed in [**BreJo91**] that the discrete (solvable) group G_N has representations which are not predicted from Mackey's theory of semidirect products of *continuous* groups. In fact, the simplest discrete group constructions lead to type III representations, even in cases where the analogous continuous groups have type I representations; see also [**Bla77**] and [**BoCo95**].

A state σ on $\mathfrak{A}_N$ is said to be *normal* if there is an $L^1(\mathbb{T})$-function h, $h \geq 0$, such that

$$\sigma(f) = \int_{\mathbb{T}} fh\,d\mu, \qquad f \in L^\infty(\mathbb{T}),$$

where μ denotes the normalized Haar measure on $\mathbb{T}$, and we view $\mathfrak{A}_N$ as the C^*-algebra on generators $f \in L^\infty(\mathbb{T})$ and a single unitary element U, such that

$$UfU^{-1} = f\left(z^N\right).$$

We say that the state σ is *U-invariant* if

$$\sigma\left(UAU^{-1}\right) = \sigma(A), \qquad A \in \mathfrak{A}_N.$$

If σ is normal, then U-invariance is equivalent to the condition

$$\int_{\mathbb{T}} fh\,d\mu = \int_{\mathbb{T}} f\left(z^N\right)h(z)\,d\mu(z), \qquad f \in L^\infty(\mathbb{T}). \tag{2.5}$$

LEMMA 2.1. *A normal state σ on $\mathfrak{A}_N$ with density h $(\in L^1(\mathbb{T}))$ is U-invariant if and only if h is the constant function.*

PROOF. One direction is immediate, so let $h \in L^1(\mathbb{T})$, $h \geq 0$, be a density for a fixed state σ on $\mathfrak{A}_N$, and assume U-invariance. Then apply (2.5) to $f(z) = e_n(z) = z^n$, $z \in \mathbb{T}$, $n \in \mathbb{Z}$. Let $\tilde{h}(n) = \int_{\mathbb{T}} \bar{e}_n h\,d\mu$ be the Fourier coefficients of h. We get

$$\tilde{h}(n) = \tilde{h}(Nn), \qquad n \in \mathbb{Z}.$$

The operator

$$(R_N f)(z) = \frac{1}{N}\sum_{w^N = z} f(w)$$

satisfies

$$(R_N f)^\sim(n) = \tilde{f}(Nn), \qquad n \in \mathbb{Z},$$

and a standard argument yields

$$R_N^n(f) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{T}} f\,d\mu; \tag{2.6}$$

see [**BrJo97**] for details. If the state σ is U-invariant, h must therefore satisfy $R_N(h) = h$, and by (2.6), h must be a constant function, which completes the proof. □

Our interest in $\mathrm{Rep}(\mathfrak{A}_N, \mathcal{H})$ started with the following example:

$$\mathcal{H} = L^2(\mathbb{R}), \qquad (U\psi)(x) = \frac{1}{\sqrt{N}}\psi\left(\frac{x}{N}\right), \tag{2.7}$$

and

$$\pi(e_n)\psi(x) := \psi(x-n), \qquad \psi \in L^2(\mathbb{R}),\ x \in \mathbb{R},\ n \in \mathbb{Z}. \tag{2.8}$$

DEFINITION 2.2. Let $m_0 \in L^\infty(\mathbb{T})$ be given, and assume the following three properties:

(i)

$$\sum_{k=0}^{N-1}\left|m_0\left(e^{i\frac{2\pi k}{N}}z\right)\right|^2 = N, \tag{2.9}$$

(ii) m_0 is continuous on $\mathbb{T}$ near $z = 1$, and
(iii) $m_0(1) = \sqrt{N}$.

These functions are called *low-pass filters* and are central to the theory of orthogonal wavelets.

The transfer operator R of Ruelle (1.21) plays a crucial rôle in the study of the regularity properties of the wavelets which are defined from some given filter m_0. (See, e.g., [**CoDa96**].) When m_0 is given, subject to (i)–(iii) above, there is an algorithm due to S.G. Mallat [**Mal89, Dau92**] which exhibits φ as the inverse Fourier transform of the infinite product

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{m_0\left(\frac{\omega}{N^k}\right)}{\sqrt{N}} \chi_{[-\pi N^n, \pi N^n\rangle}(\omega). \tag{2.10}$$

The limit is known to be well defined, and specifying an $L^2(\mathbb{R})$-function $F(\omega)$. Moreover $\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega x} F(\omega)\, d\omega$ will then be in $L^2(\mathbb{R})$ and satisfy

$$U\varphi = \pi(m_0)\varphi, \tag{2.11}$$

φ depending on the representation. Let $m_0(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Then recall the right-hand side is

$$\sum_{n\in\mathbb{Z}} a_n \varphi(x - n).$$

Equivalently, the scaling identity (2.11) may be rephrased as

$$\varphi(x) = \sqrt{N} \sum_{n\in\mathbb{Z}} a_n \varphi(Nx - n), \qquad x \in \mathbb{R}. \tag{2.12}$$

LEMMA 2.3. *The wavelet representation $\left(L^2(\mathbb{R}), \pi, U\right)$ which is given by a wavelet filter m_0 and a scaling function φ is irreducible.*

PROOF. Recall m_0 satisfies conditions (i)–(iii) of Definition 2.2, and the corresponding scaling function $\varphi \in L^2(\mathbb{R})$ is then determined by the Mallat algorithm; see (2.10). But (2.10) also shows that any other function φ_1, say, which satisfies $U\varphi_1 = \pi(m_0)\varphi_1$, or equivalently

$$\varphi_1(x) = \sqrt{N} \sum_{n} a_n \varphi_1(Nx - n),$$

where $m_0(z) = \sum_n a_n z^n$, must be a constant times φ, i.e., $\varphi_1 = c\varphi$. If P is an operator on $L^2(\mathbb{R})$ which commutes with both U and $\pi(L^\infty(\mathbb{T}))$, then $MP\varphi = P\varphi$ where $M = U^{-1}\pi(m_0)$. It follows that $P\varphi = c\varphi$ for some constant c. But φ is a cyclic vector for the von Neumann algebra $\mathfrak{A}$ generated by U and $\pi(L^\infty(\mathbb{T}))$, so

$$PA\varphi = AP\varphi = Ac\varphi = cA\varphi \qquad \text{for all } A \in \mathfrak{A},$$

and we conclude that P is a scalar times the identity operator in $L^2(\mathbb{R})$, concluding the proof of irreducibility. ☐

THEOREM 2.4.

(i) *Let $m_0 \in L^\infty(\mathbb{T})$, and suppose m_0 does not vanish on a subset of $\mathbb{T}$ of positive measure. Let*

$$(Rf)(z) = \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 f(w), \qquad f \in L^1(\mathbb{T}). \tag{2.13}$$

Then there is a one-to-one correspondence between the data (a) *and* (b) *below, where* (b) *is understood as equivalence classes under unitary equivalence:*

(a) $h \in L^1(\mathbb{T})$, $h \geq 0$, *and*

$$R(h) = h. \tag{2.14}$$

(b) $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$, $\varphi \in \mathcal{H}$, *and the unitary* U *from* $\tilde{\pi}$ *satisfying*

$$U\varphi = \pi(m_0)\varphi. \tag{2.15}$$

(ii) *From* (a)$\to$(b), *the correspondence is given by*

$$\langle \varphi \mid \pi(f)\varphi \rangle_{\mathcal{H}} = \int_{\mathbb{T}} f h \, d\mu, \tag{2.16}$$

where μ *denotes the normalized Haar measure on* $\mathbb{T}$.

From (b)$\to$(a), *the correspondence is given by*

$$h(z) = h_\varphi(z) = \sum_{n \in \mathbb{Z}} z^n \langle \pi(e_n)\varphi \mid \varphi \rangle_{\mathcal{H}}. \tag{2.17}$$

(iii) *When* (a) *is given to hold for some* h, *and* $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ *is the corresponding cyclic representation with* $U\varphi = \pi(m_0)\varphi$, *then the representation is unique from* h *and* (2.16) *up to unitary equivalence: that is, if* $\pi' \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H}')$, $\varphi' \in \mathcal{H}'$ *also cyclic and satisfying*

$$\langle \varphi' \mid \pi'(f)\varphi' \rangle = \int_{\mathbb{T}} f h \, d\mu$$

and

$$U'\varphi' = \pi'(m_0)\varphi',$$

then there is a unitary isomorphism W *of* $\mathcal{H}$ *onto* $\mathcal{H}'$ *such that* $W\pi(A) = \pi'(A)W$, $A \in \mathfrak{A}_N$, *and* $W\varphi = \varphi'$.

In the setup for the theorem, we are *not* assuming that $\frac{1}{N}\sum_{w^N=z} |m_0(w)|^2 = 1$, although this will be the case for the applications to wavelets. Hence the existence of solutions to the eigenvalue problem

$$Rf = f, \qquad f \in L^1(\mathbb{T}), \ f \geq 0, \ f \neq 0, \tag{2.18}$$

is not guaranteed.

We note further that the mapping $z \mapsto z^N$ of $\mathbb{T}$ into $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ is a special case of the following more general setup. We will show in Chapter 6 that Theorem 2.4 carries over to the more general setting.

Let $I = [0,1]$, and let $T \colon I \to I$ be a *piecewise expanding* C^2 *surjective Markov map,* i.e., there is

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$$

such that the restriction of T to each of the subintervals is monotone. Further there exists $\beta > 1$ such that $\inf_{x \in I} |T'(x)| \geq \beta$; and the implication

$$\text{if } T((x_{i-1}, x_i)) \cap (x_{j-1}, x_j) \neq \varnothing, \text{ then } (x_{j-1}, x_j) \subset T((x_{i-1}, x_i)) \tag{2.19}$$

holds. Set

$$Rf(x) = \sum_{Ty=x} \frac{f(y)}{|T'(y)|}.$$

It is easy to see that each solution to the eigenvalue problem

$$Rf = f, \qquad f \in L^1(I), \ f \geq 0,$$

defines a measure $f\,dx$ on I which is T-invariant.

PROPOSITION 2.5. (Pollicott–Yuri) *Let T and R be as described. Then there exists $f \in L^1(I)$, $f \geq 0$, $f \neq 0$ with $Rf = f$.*

PROOF. [**PoYu98**, p. 127]. ☐

REMARK 2.6. (*Moments of Representations.*) An element $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ is generated by the operators $\{\pi(f) \mid f \in L^\infty(\mathbb{T})\}$, and the unitary operator $U\colon \mathcal{H} \to \mathcal{H}$, and the commutation relation is

$$U\pi(f)U^{-1} = \pi\left(f\left(z^N\right)\right).$$

The theorem is concerned with solutions $\varphi \in \mathcal{H}$ to $U\varphi = \pi(m_0)\varphi$ when m_0 is given. For a given representation, we have a *spectral measure* ν_φ on $\mathbb{T}$ such that

$$\langle \varphi \mid \pi(f)\varphi \rangle = \int_{\mathbb{T}} f(z)\, d\nu_\varphi(z),$$

and we noted that $R(\nu_\varphi) = \nu_\varphi$. So, if ν_φ is absolutely continuous, with Radon–Nikodym derivative $\frac{d\nu_\varphi}{d\mu} = h$, then $R(h) = h$, and

$$\langle \varphi \mid \pi(f)\varphi \rangle = \int_{\mathbb{T}} fh\, d\mu,$$

and we say that the right-hand side represents *the moments* of π in the state φ; specifically,

$$\langle \varphi \mid \pi(e_n)\varphi \rangle = \int_{\mathbb{T}} z^n h(z)\, d\mu(z), \qquad n \in \mathbb{Z}, \tag{2.20}$$

where μ as usual denotes the normalized Haar measure on $\mathbb{T}$. *The other moments* are

$$\langle \varphi \mid U^n\varphi \rangle = \int_{\mathbb{T}} m_0^{(n)}(z)\, h(z)\, d\mu(z), \qquad n = 0, 1, 2, \dots, \tag{2.21}$$

where

$$m_0^{(n)}(z) := m_0(z)\, m_0\left(z^N\right) \cdots m_0\left(z^{N^{n-1}}\right).$$

The mixed moments

$$\omega_\varphi\left(U^{-k} f U^n\right) = \left\langle \varphi \,\middle|\, U^{*\,k}\pi(f)\,U^n\varphi \right\rangle = \left\langle U^k\varphi \,\middle|\, \pi(f)\,U^n\varphi \right\rangle \qquad \text{for } 0 \leq k \leq n$$

involve the Ruelle operator R through the formula

$$\omega_\varphi\left(U^{-k} f U^n\right) = \int_{\mathbb{T}} m_0^{(n-k)} R^k(fh)\, d\mu. \tag{2.22}$$

We show that a cyclic representation, in this case $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$, is uniquely determined by its moments (2.20) and (2.22); *cf.*, e.g., (2.7) and (2.8).

While, in the statement of Theorem 2.4, we are not assuming that m_0 satisfy

$$\sum_{w^N = z} |m_0(w)|^2 = N,$$

we then cannot be guaranteed solutions $h \in L^1(\mathbb{T})$, $h \geq 0$, $h \neq 0$, to $R_{m_0}(h) = h$, where

$$R_{m_0}(f)(z) := \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 f(w).$$

However, there is a recent such existence theorem due to L. Hervé [**Her95**] with a direct wavelet application. It is assumed in [**Her95**] that $m_0 \in C^\infty(\mathbb{T})$, $m_0(1) = \sqrt{N}$. This means that the infinite product

$$\prod_{k=1}^{\infty} \frac{m_0\left(e^{-i\frac{\omega}{N^k}}\right)}{\sqrt{N}}$$

is well defined pointwise as a function f of $\omega \in \mathbb{R}$, and, of course,

$$\sqrt{N} f(\omega) = m_0\left(e^{-i\frac{\omega}{N}}\right) f\left(\frac{\omega}{N}\right), \qquad \omega \in \mathbb{R}.$$

For the wavelet problem, we need $f \in L^2(\mathbb{R})$ such that the inverse Fourier transform $\varphi = \check{f}$ may serve as an $L^2(\mathbb{R})$-scaling function. The theorem of Hervé states that, under the given conditions, $f \in L^2(\mathbb{R})$ if and only if there is a solution $h \in C^\infty(\mathbb{T})$, $h \geq 0$, $h \neq 0$, to $R_{m_0}(h) = h$, in fact $h(1) > 0$.

The Ruelle operator, also called the Perron–Frobenius–Ruelle operator, or the transfer operator, is based on a simple but powerful idea. In addition to the diverse applications given in [**Rue76**], [**Rue78b**], [**Rue79**], [**Rue88**], and [**Rue90**], it has also found applications in ergodic theory [**Sin72**], [**Wal75**], and harmonic analysis [**JoPe98**], [**Jor98**], [**Sch74**], and in statistical mechanics [**Rue68**], [**Mey98**].

CHAPTER 3

Proof of Theorem 2.4

The present chapter is entirely devoted to the proof of Theorem 2.4, and we begin with five lemmas.

An alternative proof of Theorem 2.4 would be to get the cyclic representation (which is asserted in the theorem) from the Gelfand–Naimark–Segal (GNS) construction. But then we would have to show first that the data which are given in the theorem either define a positive definite function on the N-adic $ax+b$ group, or alternatively a positive linear functional (state) on $\mathfrak{A}_N$, and there is *not* a direct approach to doing that. It turns out to be shorter to first directly construct the representation, and then, *a posteriori,* to conclude the positive definite properties of an associated function on the group, or a functional on $\mathfrak{A}_N$. This is also discussed in detail in Chapter 6, which in fact provides representations in a context which is more general than that of Theorem 2.4. Our general reference for oparator algebras and the GNS construction is [**BrRoI**], but our particular application in Chapter 7 is closer to the viewpoint taken in [**GlJa87**].

LEMMA 3.1. *If* $m_0 \in L^\infty(\mathbb{T})$ *then the Ruelle operator in* (1.17) *maps* $L^1(\mathbb{T})$ *into itself, and has* $L^1 \to L^1$ *operator norm equal to* $\|m_0\|_\infty^2$.

PROOF. Let $f \in L^1(\mathbb{T})$. Then

$$\begin{aligned}\int_{\mathbb{T}} |(Rf)(z)| \, d\mu(z) &\leq \frac{1}{N} \int_{\mathbb{T}} \sum_{w^N = z} |m_0(w)|^2 |f(w)| \, d\mu(z) \\ &= \int_{\mathbb{T}} |m_0(z)|^2 |f(z)| \, d\mu(z) \\ &\leq \|m_0\|_\infty^2 \cdot \int_{\mathbb{T}} |f(z)| \, d\mu(z).\end{aligned}$$

The $L^1 \to L^1$ norm is in fact *equal to* $\|m_0\|_\infty^2$, and this is based on an argument in [**BrJo98**] to which we refer. □

LEMMA 3.2. *Let* $h \in L^1(\mathbb{T})$, $h \geq 0$, *be given, and let* $L^2(h)$ *denote the* L^2*-space of functions on* $\mathbb{T}$ *defined relative to the absolutely continuous measure* $h\,d\mu$ (*where* μ *is Haar measure on* $\mathbb{T}$), *i.e.,*

$$\|f\|_h^2 := \int_{\mathbb{T}} |f|^2 \, h \, d\mu. \tag{3.1}$$

On $L^2(h)$, *we have the representation* π_0, *and the operator* S_0, *defined as follows:*

$$(\pi_0(f)\xi)(z) := f(z)\xi(z), \qquad f \in L^\infty(\mathbb{T}),\ \xi \in L^2(h), \tag{3.2}$$

and

$$(S_0\xi)(z) := m_0(z)\xi(z^N). \tag{3.3}$$

Then

$$S_0\pi_0(f) = \pi_0(f(z^N))S_0; \tag{3.4}$$

and S_0 is isometric in $L^2(h)$ if and only if $R(h) = h$.

Proof. The properties in the lemma are clear except for the criteria for S_0 to be isometric: Let $h \in L^1(\mathbb{T})$, $h \geq 0$ be given, and let $f_1, f_2 \in L^\infty(\mathbb{T})$. Then

$$\begin{aligned}\langle S_0 f_1 \,|\, S_0 f_2\rangle_{L^2(h)} &= \int_{\mathbb{T}} |m_0(z)|^2\, \overline{f_1(z^N)}\, f_2(z^N)\, h(z)\, d\mu(z) \\ &= \frac{1}{N}\int_{\mathbb{T}} \overline{f_1(z)}\, f_2(z) \sum_{w^N=z} |m_0(w)|^2\, h(w)\, d\mu(z) \\ &= \int_{\mathbb{T}} \overline{f_1(z)}\, f_2(z)\,(Rh)(z)\, d\mu(z),\end{aligned}$$

and it follows that S_0 is $L^2(h)$-isometric if $Rh = h$. But taking $f_1 = e_{n_1} = z^{n_1}$, $f_2 = e_{n_2} = z^{n_2}$, $n_1, n_2 \in \mathbb{Z}$, we can see that the identity

$$\langle S_0 e_{n_1} \,|\, S_0 e_{n_2}\rangle_{L^2(h)} = \langle e_{n_1} \,|\, e_{n_2}\rangle_{L^2(h)}$$

implies that the two functions $R(h)$ and h must have identical Fourier coefficients. Since we have Fourier uniqueness for $L^1(\mathbb{T})$, the result follows, i.e., $Rh = h$ must hold when it is given that S_0 is isometric on $L^2(h)$. $\square$

Lemma 3.3. *Let $h \in L^1(\mathbb{T})$, $h \geq 0$, be given, and let $\pi \in \mathrm{Rep}(\mathfrak{A}_N, \mathcal{H})$, $\varphi \in \mathcal{H}$, satisfy*

$$\langle \varphi \,|\, \pi(f)\varphi\rangle_{\mathcal{H}} = \int_{\mathbb{T}} f h\, d\mu, \qquad f \in L^\infty(\mathbb{T}). \tag{3.5}$$

Then

$$h(z) := \sum_{n\in\mathbb{Z}} z^n \langle \pi(e_n)\varphi \,|\, \varphi\rangle \tag{3.6}$$

is in $L^1(\mathbb{T})$; and if further $U\varphi = \pi(m_0)\varphi$, then $Rh = h$.

Proof. Clearly, the expression (3.6) makes sense as a distribution on $\mathbb{T}$, and its Fourier coefficients are $n \mapsto \langle \pi(e_n)\varphi \,|\, \varphi\rangle_{\mathcal{H}}$. But substituting $f = e_{-n}$, $n \in \mathbb{Z}$, into (3.5) shows that $\langle \pi(e_n)\varphi \,|\, \varphi\rangle = \int_{\mathbb{T}} \overline{e_n}\, h\, d\mu$, which are the Fourier coefficients for the given $L^1(\mathbb{T})$-function h. Hence the right-hand side of (3.6) must be h, again by Fourier uniqueness. If $U\varphi = \pi(m_0)\varphi$, then we calculate the h-Fourier coefficients as follows:

$$\begin{aligned}
\tilde{h}(n) &= \int_{\mathbb{T}} \overline{e_n}\, h\, d\mu \\
&= \langle \pi(e_n)\varphi \mid \varphi\rangle \\
&= \langle U\pi(e_n)\varphi \mid U\varphi\rangle \\
&= \langle \pi(e_{Nn}) U\varphi \mid \pi(m_0)\varphi\rangle \\
&= \langle \pi(e_{Nn} m_0)\varphi \mid \pi(m_0)\varphi\rangle \\
&= \left\langle \varphi \middle| \pi\left(e_{-Nn}\left|m_0\right|^2\right)\varphi \right\rangle \\
&= \int_{\mathbb{T}} e_{-Nn}\left|m_0\right|^2 h\, d\mu \\
&= \frac{1}{N}\int_{\mathbb{T}} e_{-n}(z) \sum_{w^N=z} \left|m_0(w)\right|^2 h(w)\, d\mu(z) \\
&= \int_{\mathbb{T}} z^{-n}(Rh)(z)\, d\mu(z) \\
&= (Rh)^{\sim}(n), \qquad n \in \mathbb{Z}.
\end{aligned}$$

Hence the two $L^1(\mathbb{T})$-functions h and Rh have the same Fourier coefficients, and therefore $h = R(h)$ holds as claimed ☐

The correspondence (b)→(a) in Theorem 2.4 follows now directly from the lemmas, and we turn to (a)→(b).

Let $h \in L^1(\mathbb{T})$ be given satisfying $h \geq 0$ and $Rh = h$. The conditions on m_0 are just $m_0 \in L^\infty(\mathbb{T})$ and that m_0 does not vanish on a subset of $\mathbb{T}$ of positive measure. In Lemma 3.2, we already did the first step in a recursive algorithm for constructing the desired representation $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ and cyclic vector $\varphi \in \mathcal{H}$ such that $U\varphi = \pi(m_0)\varphi$. This construction will be a *unitary* dilation (or lifting) of the *isometric* properties (3.1)–(3.3) in Lemma 3.2. The construction is also a generalized multiresolution. In the case $\mathcal{H} = L^2(\mathbb{R})$, it is directly related to the traditional multiresolution from wavelet theory, and we shall follow up on this in Chapters 4–5 below.

Since h is given, we have the Hilbert space $L^2(h) = L^2(\mathbb{T}, h\,d\mu)$ from (3.1) in Lemma 3.2, and we set $\mathcal{H}_0 = L^2(h)$. It is of course the completion of $L^\infty(\mathbb{T})$ in the norm $\|\cdot\|_h$ from (3.1), and we write $\mathcal{H}_0 = \widetilde{\mathcal{V}_0}$ with

$$\mathcal{V}_0 := \{(\xi, 0) \mid \xi \in L^\infty(\mathbb{T})\}.$$

Then on

$$\mathcal{V}_n := \{(\xi, n) \mid \xi \in L^\infty(\mathbb{T})\}$$

we set

$$\|(\xi, n)\|_{\mathcal{H}}^2 := \int_{\mathbb{T}} R^n\left(\left|\xi\right|^2 h\right) d\mu \tag{3.7}$$

and

$$\langle (\xi, n) \mid (\eta, n)\rangle_{\mathcal{H}} = \int_{\mathbb{T}} R^n\left(\bar{\xi}\eta h\right) d\mu \qquad \text{for } n = 1, 2, \dots,$$

where μ is the usual normalized Haar measure on $\mathbb{T}$, i.e., $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cdots\, d\omega$ relative to $z = e^{-i\omega}$, $\omega \in \mathbb{R}$, when functions on $\mathbb{T}$ are identified with 2π-periodic functions on $\mathbb{R}$. We now let $\mathcal{H}_n$ be the completion of $\mathcal{V}_n$ in this norm, and we construct $\mathcal{H}$ itself as an inductive limit of the Hilbert spaces $\mathcal{H}_n$, $n = 0, 1, 2, \dots$. To do this, we construct a system of isometries

$$\begin{tikzcd} \mathcal{H}_n \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{H}_{n+k+l} \\ & \mathcal{H}_{n+k} \arrow[ur, hook] & \end{tikzcd} \tag{3.8}$$

and we get it from the completion of a corresponding isometry diagram

$$\begin{tikzcd} \mathcal{V}_n \arrow[rr, hook] \arrow[dr, hook] & & \mathcal{V}_{n+k+l}. \\ & \mathcal{V}_{n+k} \arrow[ur, hook] & \end{tikzcd}$$

When n, k are given, $n \geq 0$, $k \geq 1$, we construct the isometry $\mathcal{V}_n \hookrightarrow \mathcal{V}_{n+k}$ by iteration of the one from $\mathcal{V}_n$ to $\mathcal{V}_{n+1}$, i.e.,

$$\mathcal{V}_n \hookrightarrow \mathcal{V}_{n+1} \hookrightarrow \mathcal{V}_{n+2} \hookrightarrow \cdots \hookrightarrow \mathcal{V}_{n+k},$$

where $J\colon \mathcal{V}_n \to \mathcal{V}_{n+1}$ is defined by

$$J\left(\left(\xi, n\right)\right) := \left(\xi\left(z^N\right), n+1\right). \tag{3.9}$$

The isometric property of this operator is proved in the following lemma:

Lemma 3.4. *The mapping J defined in* (3.9) *is for each n isometric from $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$.*

Proof. Let $\xi \in L^\infty(\mathbb{T})$. Then

$$\begin{aligned} \left\|\left(\xi\left(z^N\right), n+1\right)\right\|_{\mathcal{H}}^2 &= \int_{\mathbb{T}} R^{n+1}\left(\left|\xi\left(z^N\right)\right|^2 h(z)\right)(z)\, d\mu(z) \\ &= \int_{\mathbb{T}} R^n\left(R\left(\left|\xi\left(z^N\right)\right|^2 h(z)\right)\right) d\mu(z) \\ &= \int_{\mathbb{T}} R^n\left(\left|\xi\right|^2 Rh\right) d\mu \\ &= \int_{\mathbb{T}} R^n\left(\left|\xi\right|^2 h\right) d\mu \\ &= \left\|(\xi, n)\right\|_{\mathcal{H}}^2, \end{aligned}$$

where we used the properties

$$R\left(f\left(z^{N}\right)g\left(z\right)\right)=fR\left(g\right)$$

and

$$Rh=h$$

for the Ruelle operator R in (1.17). □

We now define

$$U\left(\xi,0\right):=\left(S_{0}\xi,0\right)=\left(m_{0}\left(z\right)\xi\left(z^{N}\right),0\right), \tag{3.10}$$

$$U\left(\xi,n+1\right):=\left(m_{0}\left(z^{N^{n}}\right)\xi\left(z\right),n\right), \tag{3.11}$$

and

$$\pi\left(f\right)\left(\xi,n\right):=\left(f\left(z^{N^{n}}\right)\xi\left(z\right),n\right), \tag{3.12}$$

for $f,\xi\in L^{\infty}\left(\mathbb{T}\right)$, and $n=0,1,\dots$. A direct substitution of the definitions then leads to commutativity of the following three commutative diagrams (Figures 1a–1c):

$$\begin{tikzcd}
\mathcal{V}_0 \arrow[rr, "U"'] \arrow[d, hook, "J"] & & \mathcal{V}_0 \arrow[d, hook, "J"] \\
\mathcal{V}_1 \arrow[r, "U"'] & \mathcal{V}_0 \arrow[r, "J"'] & \mathcal{V}_1,
\end{tikzcd}$$

FIGURE 1a

$$\begin{tikzcd}
\mathcal{V}_n \arrow[r, "U"'] \arrow[d, hook, "J"] & \mathcal{V}_{n-1} \arrow[d, hook, "J"] \\
\mathcal{V}_{n+1} \arrow[r, "U"'] & \mathcal{V}_n,
\end{tikzcd}$$

FIGURE 1b

for $n=1,2,\dots$, and

$$\begin{tikzcd}
\mathcal{V}_n \arrow[r, "\pi(f)"'] \arrow[d, hook, "J"] & \mathcal{V}_n \arrow[d, hook, "J"] \\
\mathcal{V}_{n+1} \arrow[r, "\pi(f)"'] & \mathcal{V}_{n+1}.
\end{tikzcd}$$

FIGURE 1c

In fact, substitution of (3.10)–(3.12) into the diagrams leads to the following, easily verified, identities (Figures 2a–2c):

$$\begin{tikzcd}
 & \mathcal{V}_0 \ni (\xi, 0) \arrow[rr, "U"] \arrow[dl, hook, "J"'] & & (S_0\xi, 0) \in \mathcal{V}_0 \arrow[dr, hook', "J"] & \\
\mathcal{V}_1 \ni \left(\xi\left(z^N\right), 1\right) \arrow[drr, "U"'] & & & & \left(\left(S_0\xi\right)\left(z^N\right), 1\right) \in \mathcal{V}_1 \\
 & & \begin{matrix} \left(m_0\left(z\right)\xi\left(z^N\right), 0\right) \\ \rotatebox{-90}{$\in$} \\ \mathcal{V}_0, \end{matrix} \arrow[urr, hook, "J"'] & &
\end{tikzcd}$$

FIGURE 2a

$$\begin{tikzcd}
\mathcal{V}_n \ni (\xi, n) \arrow[r, "U"'] \arrow[d, hook, "J"] & \left(m_0\left(z^{N^{n-1}}\right)\xi\left(z\right), n-1\right) \in \mathcal{V}_{n-1} \arrow[d, hook, "J"] \\
\mathcal{V}_{n+1} \ni \left(\xi\left(z^N\right), n+1\right) \arrow[r, "U"'] & \left(m_0\left(z^{N^n}\right)\xi\left(z^N\right), n\right) \in \mathcal{V}_n,
\end{tikzcd}$$

FIGURE 2b

and the last simpler diagram, which does not involve a horizontal shift in the n-index:

$$\begin{tikzcd}
\mathcal{V}_n \ni (\xi, n) \arrow[r, "\pi(f)"'] \arrow[d, hook, "J"] & \left(f\left(z^{N^n}\right)\xi\left(z\right), n\right) \in \mathcal{V}_n \arrow[d, hook, "J"] \\
\mathcal{V}_{n+1} \ni \left(\xi\left(z^N\right), n+1\right) \arrow[r, "\pi(f)"'] & \left(f\left(z^{N^{n+1}}\right)\xi\left(z^N\right), n+1\right) \in \mathcal{V}_{n+1}.
\end{tikzcd}$$

FIGURE 2c

The purpose of the diagrams is to verify that the operator U and the representation π (of $L^\infty(\mathbb{T})$), as defined from (3.10)–(3.12), pass to the inductive limit construction which is obtained by the identification of $\mathcal{H}_n$ with a closed subspace in $\mathcal{H}_{n+1}$ for each n, and therefore, by iteration, in $\mathcal{H}_{n+k}$ for all $k = 1, 2, \dots$. When the inductive limit

$$\mathcal{H} = \varinjlim_n \mathcal{H}_n \tag{3.13}$$

is then formed, we get a well defined operator U on $\mathcal{H}$, and a representation π of $L^\infty(\mathbb{T})$ on $\mathcal{H}$. (Vectors ξ in $\mathcal{H}$ may be characterized by the following orthogonal expansion: $\xi = \sum_{n=0}^\infty \xi_n$, where $\xi_0 \in \mathcal{H}_0$, and $\xi_n \in \mathcal{H}_n \ominus \mathcal{H}_{n-1}$, $n = 1, 2, \dots$; and $\sum_0^\infty \|\xi_n\|^2 < \infty$. For an alternative and purely function-theoretic characterization of the Hilbert space $\mathcal{H}$, see also Corollary 3.9 below.) From Lemma 3.2, and a direct verification, we also get the identity

$$U\pi(f) = \pi\left(f\left(z^N\right)\right)U$$

for the corresponding operators on $\mathcal{H}$.

A final lemma now completes the proof of Theorem 2.4, (a)→(b).

LEMMA 3.5. *If $m_0 \in L^\infty(\mathbb{T})$ and $Rh = h$, then U is isometric, and if also m_0 vanishes on at most a subset of $\mathbb{T}$ of measure zero, then U is a unitary operator in $\mathcal{H}$.*

PROOF. From the inductive construction (3.10)–(3.11), we have $U\colon \mathcal{V}_{n+1} \to \mathcal{V}_n$, and we wish to pass U to the completion $U\colon \mathcal{H}_{n+1} \to \mathcal{H}_n$. That can be done if we check first that U is isometric from $\mathcal{V}_{n+1}$ to $\mathcal{V}_n$ for all $n = 0, 1, 2, \dots$. We already checked, in fact, that U is isometric on $\mathcal{V}_0$, and therefore on the completion $L^2(h)$; that was Lemma 3.2.

Let $\xi \in L^\infty(\mathbb{T})$. Then

$$\begin{aligned}
\|U(\xi, n+1)\|_{\mathcal{H}}^2 &= \left\|\left(m_0\left(z^{N^n}\right)\xi(z), n\right)\right\|_{\mathcal{H}}^2 \\
&= \int_{\mathbb{T}} R^n\left(\left|m_0\left(z^{N^n}\right)\xi(z)\right|^2 h(z)\right) d\mu(z) \\
&= \int_{\mathbb{T}} |m_0|^2 R^n\left(|\xi|^2 h\right) d\mu \\
&= \frac{1}{N}\int_{\mathbb{T}} \sum_{w^N = z} |m_0(w)|^2 R^n\left(|\xi|^2 h\right)(w)\, d\mu(z) \\
&= \int_{\mathbb{T}} R^{n+1}\left(|\xi|^2 h\right)(z)\, d\mu(z) \\
&= \|(\xi, n+1)\|_{\mathcal{H}}^2,
\end{aligned}$$

which is the desired isometric property.

Using again the inductive limit construction of $\mathcal{H}$, we note that U will be unitary on $\mathcal{H}$, i.e., $U(\mathcal{H}) = \mathcal{H}$ if and only if

$$U(\mathcal{H}_{n+1}) = \mathcal{H}_n \tag{3.14}$$

for $n = 0, 1, 2, \dots$. Equivalently, we must show that the spaces $\mathcal{H}_n \ominus U(\mathcal{H}_{n+1})$ vanish for $n = 0, 1, \dots$. To do this we need the following

CLAIM 3.6. The completion $\mathcal{H}_n = \widetilde{\mathcal{V}}_n$ in the norm on $\mathcal{V}_n$ consists of measurable functions ξ on $\mathbb{T}$ satisfying

$$\int_{\mathbb{T}} R^n\left(|\xi|^2 h\right) d\mu < \infty. \tag{3.15}$$

PROOF. Let $\xi_i \in L^\infty(\mathbb{T})$, $i = 1, 2, \dots$, and suppose

$$\lim_{i,j\to\infty} \int_{\mathbb{T}} R^n\left(|\xi_i - \xi_j|^2 h\right) d\mu = 0.$$

Let

$$m_0^{(n)}(z) = m_0(z)\, m_0\left(z^N\right) \cdots m_0\left(z^{N^{n-1}}\right).$$

Then

$$\int_{\mathbb{T}} R^n \left(|\xi_i - \xi_j|^2 h \right) d\mu = \int_{\mathbb{T}} \left| m_0^{(n)} \right|^2 |\xi_i - \xi_j|^2 h \, d\mu.$$

We conclude that there is a pointwise a.e. convergent subsequence $\xi_{i_1}, \xi_{i_2}, \dots$ with limit ξ, say. We have

$$\int_{\mathbb{T}} \left| m_0^{(n)} \right|^2 |\xi_{i_k} - \xi|^2 h \, d\mu \underset{k \to \infty}{\longrightarrow} 0,$$

and

$$\int_{\mathbb{T}} \left| m_0^{(n)} \right|^2 |\xi|^2 h \, d\mu < \infty.$$

Since

$$\int_{\mathbb{T}} \left| m_0^{(n)} \right|^2 |\xi|^2 h \, d\mu = \int_{\mathbb{T}} R^n \left(|\xi|^2 h \right) d\mu,$$

the claim follows. □

To prove the unitarity assertion of the lemma, we must show that if ξ satisfies (3.15) of Claim 3.6, and if

$$\int_{\mathbb{T}} R^n \left(\overline{\xi (z)} m_0 \left(z^{N^n} \right) \eta (z) h (z) \right) d\mu (z) = 0 \tag{3.16}$$

for all $\eta \in L^\infty (\mathbb{T})$, then ξ must vanish a.e. on $\mathbb{T}$, and therefore

$$\int_{\mathbb{T}} \left| m_0^{(n)} \right|^2 |\xi|^2 h \, d\mu = \int_{\mathbb{T}} R^n \left(|\xi|^2 h \right) d\mu = 0.$$

Since $m_0 \in L^\infty (\mathbb{T})$, the function $z \mapsto \overline{\xi (z)} m_0 \left(z^{N^n} \right)$ also satisfies condition (3.15) in the Claim. Since $\mathcal{H}_n = \widetilde{\mathcal{V}}_n$, we conclude that

$$\int_{\mathbb{T}} R^n \left(\left| \xi (z) m_0 \left(z^{N^n} \right) \right|^2 h (z) \right) d\mu = 0.$$

But the integral is also

$$\int_{\mathbb{T}} |m_0|^2 R^n \left(|\xi|^2 h \right) d\mu,$$

and, if we now (finally!) use that m_0 does not vanish on a subset of $\mathbb{T}$ of positive measure, we see that $R^n \left(|\xi|^2 h \right)$ must vanish pointwise a.e. on $\mathbb{T}$, and therefore $\int_{\mathbb{T}} R^n \left(|\xi|^2 h \right) d\mu = 0$, concluding the proof that

$$\mathcal{H}_n \ominus U (\mathcal{H}_{n+1}) = 0. \tag{3.17}$$

□

To conclude the proof of Theorem 2.4(ii), we only need to identify $\varphi \in \mathcal{H}$ such that $U\varphi = \pi (m_0) \varphi$. Take $\varphi = (1\!\!1, 0) \sim (1\!\!1, 1) \sim (1\!\!1, 2) \sim \cdots$, identification *via* the isometry J of (3.9). Then

$$U\varphi = (S_0 1\!\!1, 0) = (m_0, 0) = \pi (m_0) (1\!\!1, 0) = \pi (m_0) \varphi.$$

It is clear from the construction that φ is *cyclic* for the particular representation, i.e., $\tilde{\pi} \in \operatorname{Rep} (\mathfrak{A}_N, \mathcal{H})$, which corresponds to the pair (U, π) where π is the $L^\infty (\mathbb{T})$-representation which satisfies (2.4).

The final assertion in Theorem 2.4(iii) is that $\tilde{\pi}$ is *unique up to unitary equivalence.* The proof of this is somewhat similar to the standard uniqueness part in the GNS construction: see, e.g., [**BrRoI**]. Since there are some differences as well, we sketch the details.

REMARK 3.7. Suppose m_0 does not vanish on a set of positive measure. The function $\xi(z) := \frac{1}{m_0(z)}$ represents an element in $\mathcal{H}_1$ *via* $(\xi, 1)$, even though generally $\frac{1}{m_0(z)}$ is not in $L^\infty(\mathbb{T})$; see, e.g., Examples 4.3–4.5. We have

$$\|(\xi,1)\|_{\mathcal{H}_1}^2 = \int_{\mathbb{T}} R\left(\frac{1}{|m_0(z)|^2} h(z)\right) d\mu(z) = \int_{\mathbb{T}} h(z)\, d\mu(z) = \|\varphi\|_{\mathcal{H}}^2,$$

where φ is the cyclic vector which corresponds to a given $h \in L^1(\mathbb{T})$, $h \geq 0$, $Rh = h$. If $(\mathcal{H}_h, \pi, U)$ denotes the representation of $\mathfrak{A}_N$ which is induced from h *via* Theorem 2.4, then a simple calculation shows that

$$U^*(\varphi) = U^{-1}(\varphi) = \left(\frac{1}{m_0}, 1\right) \in \mathcal{H}_1. \tag{3.18}$$

PROOF OF UNIQUENESS IN THEOREM 2.4. The uniqueness up to unitary equivalence is only asserted when U is unitary, i.e., when the representation $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ is constructed from a given $m_0 \in L^\infty(\mathbb{T})$, and an $h \in L^1(\mathbb{T})$, $h \geq 0$, and $R_{m_0}(h) = h$. The determining conditions for $\tilde{\pi}$ are:

(i) *cyclicity,*
(ii) $\langle \varphi \,|\, \pi(f)\varphi \rangle = \int_{\mathbb{T}} f h \, d\mu$, and
(iii) $U\varphi = \pi(m_0)\varphi$.

But we just established that

$$U^*(\mathcal{H}_n) = \mathcal{H}_{n+1},$$

$n = 0, 1, 2, \dots$. Hence

$$U^{*\,n}(\mathcal{H}_0) = \mathcal{H}_n, \tag{3.19}$$

where $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots$ is the resolution which defines $\mathcal{H}$ as an inductive limit. In Lemma 3.2, we saw that $\mathcal{H}_0 \simeq L^2(h)$ with the isomorphism defined by $W\colon \mathcal{H}_0 \underset{\simeq}{\longrightarrow} L^2(h)$, $W\pi(f)\varphi = f$, $f \in L^\infty(\mathbb{T})$. This was based on the computation

$$\|\pi(f)\varphi\|_{\mathcal{H}}^2 = \int |f|^2 R_{m_0}(h)\, d\mu. \tag{3.20}$$

If $R_{m_0} h = h$, then we get

$$\|\pi(f)\varphi\|_{\mathcal{H}} = \|f\|_{L^2(h)}.$$

But we also saw that W intertwines U and S_0, i.e., that

$$WU = S_0 W. \tag{3.21}$$

Using the identity (3.19), we conclude that the intertwining property on the $\mathcal{H}_0$ level extends to the $\mathcal{H}_n$-spaces for all $n \geq 0$, and therefore that any two $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ and $\tilde{\pi}' \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H}')$ which both satisfy (i)–(iii) must be unitarily equivalent. This completes the last part of the proof of Theorem 2.4. □

We conclude with a lemma which illustrates the above construction, and which will be needed in the sequel.

LEMMA 3.8. *The orthogonal projection of $\mathcal{H}_1$ onto $\mathcal{H}_0$ is given by*

$$J^*(\xi,1)=\left(\frac{R(\xi h)}{h},0\right). \tag{3.22}$$

The condition on ξ characterizing membership in $\mathcal{H}_1$ is

$$\int_{\mathbb{T}} R\left(|\xi|^2 h\right)\,d\mu=\int_{\mathbb{T}}|m_0|^2|\xi|^2 h\,d\mu<\infty, \tag{3.23}$$

and implied in (3.22) *is the assertion that the expression $\frac{R(\xi h)(z)}{h(z)}$ is then well defined, and that it is in $L^2(h)$, i.e., that*

$$\int_{\mathbb{T}}\frac{|R(\xi h)(z)|^2}{h(z)}\,d\mu(z)<\infty. \tag{3.24}$$

PROOF. To see that the expressions under the integral make sense pointwise, the following estimate is needed (obtained by an iteration of Schwarz's estimate!):

$$|R(\xi h)(z)|\le R\left(|\xi|^2 h\right)(z)^{\frac12}h(z)^{\frac12}\le\cdots\le\left(R\left(|\xi|^{2^n}h\right)(z)\right)^{\frac{1}{2^n}}h(z)^{\frac12+\frac14+\cdots+\frac{1}{2^n}}.$$

To prove (3.22), we use the definition $J\colon\ (\eta,0)\mapsto\left(\eta\left(z^N\right),1\right)$, and further compute that:

$$\begin{aligned}\langle J^*(\xi,1)\,|\,(\eta,0)\rangle&=\int_{\mathbb{T}}R\left(\bar\xi(z)\,\eta\left(z^N\right)h(z)\right)\,d\mu\\&=\int_{\mathbb{T}}\eta(z)\,R\left(\bar\xi h\right)(z)\,d\mu(z)\\&=\left\langle\frac{R(\xi h)}{h}\,\middle|\,\eta\right\rangle_{L^2(h)},\end{aligned}$$

and (3.22) follows from this. Since we saw that J is isometric when $R(h)=h$ holds (*cf.* Lemma 3.4), it follows that JJ^* is the desired projection of $\mathcal{H}_1$ onto $\mathcal{H}_0$; but we may use J in making an isometric identification. $\square$

As a corollary, we get

COROLLARY 3.9. *The elements in the Hilbert space $\mathcal{H}$, which is constructed from the resolution $\mathcal{H}_0\subset\mathcal{H}_1\subset\mathcal{H}_2\subset\cdots$, may be given by precisely the sequences of measurable functions (ξ_n), $n=0,1,\dots$, such that*

$$\sup_n\int_{\mathbb{T}}R^n\left(|\xi_n|^2 h\right)\,d\mu<\infty,$$

and

$$R(\xi_{n+1}h)=\xi_n h,\qquad n=0,1,\dots.$$

A system like this is also called a martingale.

CHAPTER 4

Wavelet filters

In Chapter 2 we showed that if m_0 is a wavelet filter (see Definition 2.2) then *one of* the representations from Theorem 2.4 may be realized in $\mathcal{H} = L^2(\mathbb{R})$. Specifically, there is a cyclic vector $\varphi \in L^2(\mathbb{R})$ and a corresponding $h_\varphi \in L^1(\mathbb{T})$ such that $h_\varphi \geq 0$ and $R(h_\varphi) = h_\varphi$. The assertion here is that, in this case, the solution φ to $U\varphi = \pi(m_0)\varphi$ may be found in $L^2(\mathbb{R})$. For this, we need the normalization (2.9), i.e.,

$$\sum_{k=0}^{N-1} \left| m_0 \left(e^{i\frac{2\pi k}{N}} z \right) \right|^2 \equiv N \qquad \text{(a.e. on } \mathbb{T}), \tag{4.1}$$

which is one of the defining conditions on a wavelet filter. For $N = 2$, it reads:

$$|m_0(z)|^2 + |m_0(-z)|^2 = 2, \qquad z \in \mathbb{T}. \tag{4.2}$$

In that case, set

$$m_1(z) := z\,\overline{m_0(-z)}, \qquad z \in \mathbb{T}, \tag{4.3}$$

and

$$(S_j f)(z) = m_j(z) f\left(z^2\right), \qquad f \in L^2(\mathbb{T}). \tag{4.4}$$

LEMMA 4.1. *The operators S_0 and S_1 are isometries in $L^2(\mathbb{T})$ and satisfy*

$$S_i^* S_j = \delta_{ij} I \qquad \text{and} \qquad \sum_{i=0}^{1} S_i S_i^* = I, \tag{4.5}$$

where I denotes the identity operator.

PROOF. See [**BrJo97**]. The conclusion of the lemma may be rephrased as the assertion that the operators in (4.4) define a representation of the Cuntz C^*-algebra $\mathcal{O}_2$. There is a similar conclusion for $\mathcal{O}_N$. We use standard notation from [**Cun77**] and [**BrRoI**] for the Cuntz algebras and their representations. $\square$

LEMMA 4.2. *Let $h \in L^1(\mathbb{T})$, $h \geq 0$, satisfy $R(h) = h$. Then S_0 is isometric in $L^2(h) := L^2(\mathbb{T}, h\,d\mu)$, but S_1 is generally not isometric in $L^2(h)$.*

PROOF. We already saw in Chapter 2 that, if $h \in L^1(\mathbb{T})$, $h \geq 0$, is given, then S_0 is isometric in $L^2(h)$ if and only if $R(h) = h$. The same argument shows that the condition for S_1 to be isometric in $L^2(h)$ is $R\left(\check{h}\right) = h$, where $\check{h}(z) := h(-z)$, $z \in \mathbb{T}$, and there are easy examples where this is not satisfied.

EXAMPLE 4.3. Let

$$m_0(z) := \frac{1}{\sqrt{2}}\left(1 + z^3\right), \tag{4.6}$$

and let R be the corresponding Ruelle operator. It is easy to see that the scaling function φ for the wavelet representation is

$$\varphi(x) = \frac{1}{3}\chi_{[0,3\rangle}(x), \qquad x \in \mathbb{R}. \tag{4.7}$$

The corresponding harmonic function h_φ (i.e., $R(h_\varphi) = h_\varphi$) is computed as follows:

$$\begin{aligned} h_\varphi(z) &= \sum_n z^n \langle \pi(e_n)\varphi \,|\, \varphi\rangle \\ &= \frac{1}{9}\left(z^{-2} + 2z^{-1} + 3 + 2z + z^2\right) \\ &= \frac{1}{9}(1 + 2\cos\omega)^2, \end{aligned} \tag{4.8}$$

where $z := e^{-i\omega}$, $\omega \in \mathbb{R}$. Then $\check{h}_\varphi(z) = h_\varphi(-z) \approx h_\varphi(\omega + \pi) = \frac{1}{9}(1 - 2\cos\omega)^2$, and a calculation shows that

$$R\left(\check{h}_\varphi\right)\left(e^{-i\omega}\right) = h_\varphi\left(e^{-i\omega}\right) - \frac{4}{9}\cos\left(\frac{\omega}{2}\right)\left(1 + \cos\frac{3\omega}{2}\right).$$

In this case, therefore, $h = h_\varphi$ does not satisfy $R\left(\check{h}\right) = h$, and so S_1 is not isometric in $L^2\left(\frac{1}{9}(1 + 2\cos\omega)^2\right)$. ☐

REMARK 4.4. The structure of the states on $\mathfrak{A}_2$ which are induced from solutions

$$R(h) = h, \qquad h \in L^1(\mathbb{T}),\ h \geq 0,$$

is not yet well understood. But these states are clearly *not tracial,* not even for the simplest wavelet representations such as the one described above in Example 4.3. Recall, if φ is the scaling function (4.7) in $L^2(\mathbb{R})$, then the corresponding state σ_φ is

$$\sigma_\varphi(A) := \langle \varphi \,|\, \tilde{\pi}_\varphi(A)\varphi\rangle_{L^2(\mathbb{R})}, \qquad A \in \mathfrak{A}_2,$$

and

$$h_\varphi\left(e^{-i\omega}\right) = \frac{1}{9}(1 + 2\cos\omega)^2.$$

The state σ_φ is *not tracial* because

$$\sigma_\varphi\left(UVU^{-1}\right) \neq \sigma_\varphi(V)$$

where

$$(V\psi)(x) = \psi(x - 1),$$

and

$$(U\psi)(x) = \frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right), \qquad \psi \in L^2(\mathbb{R}).$$

Since $UVU^{-1} = V^2$, we need only check that $\sigma_\varphi\left(V^2\right) \neq \sigma_\varphi(V)$; and a direct calculation yields

$$\sigma_\varphi\left(V^2\right) = \int_{\mathbb{T}} e_2 h_\varphi \, d\mu = \frac{1}{9},$$

while

$$\sigma_{\varphi}(V)=\int_{\mathbb{T}} e_1 h_{\varphi}\,d\mu=\frac{2}{9}.$$

EXAMPLE 4.5. We now turn to the representation associated with the solution $R\mathbb{1}=\mathbb{1}$. (Recall for the wavelet filters, property (1.6) or (2.9) ensures that the constant function $\mathbb{1}$ is also an eigenfunction.) For the representation $(\mathcal{H},\pi,U)$ induced from $h=\mathbb{1}$, we may take

$$\varphi=\frac{1}{\sqrt{3}}\left(\chi_{[0,1\rangle}\oplus\chi_{[1,2\rangle}\oplus\chi_{[2,3\rangle}\right) \tag{4.9}$$

in $\mathcal{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})\oplus L^2(\mathbb{R})$. On the direct sum $\mathcal{H}:=\sum^{\oplus}L^2(\mathbb{R})$ we introduce the usual representation $\tilde{\pi}=(\pi,U)$:

$$\begin{cases}\pi(f)(\psi_i):=(\pi(f)\psi_i), & f\in L^\infty(\mathbb{T}),\\ U(\psi_i):=(U\psi_i),\end{cases}$$

where π and U on $L^2(\mathbb{R})$ are given by the usual formulas (2.7) and (2.8). It is clear that there is an isometry

$$\sum\nolimits^{\oplus}L^2(\mathbb{R})\xrightarrow{W}L^2(\mathbb{R})$$

which intertwines the respective representations of $L^\infty(\mathbb{T})$. The vector φ in $\mathcal{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ then satisfies

$$UW\varphi=W\pi(m_0)\varphi \tag{4.10}$$

or equivalently

$$W^*UW\varphi=\pi(m_0)\varphi. \tag{4.11}$$

It is immediate from (4.9) that $h_\varphi=\mathbb{1}$, and we have identified the representation from Theorem 2.4 induced by $h_\varphi=\mathbb{1}$. (We show later that $\tilde{\pi}$ is the sum of *two* mutually inequivalent irreducible representations.)

Before getting to the structure theorem for the wavelet filters, we need a lemma:

LEMMA 4.6. *Consider the wavelet representation $\left(L^2(\mathbb{R}),\pi,U\right)$ defined from a given wavelet filter m_0. Then for every $\varphi,\psi\in L^2(\mathbb{R})$, there is an $L^1(\mathbb{T})$-function $h(z)=H(\varphi,\psi)(z)$ such that*

$$\langle\varphi\,|\,\pi(f)\psi\rangle=\int_{\mathbb{T}} f(z)\,H(\varphi,\psi)(z)\,d\mu(z), \tag{4.12}$$

and it is represented by the Fourier expansion

$$H(\varphi,\psi)(z)=\sum_{n\in\mathbb{Z}} z^n\langle\pi(e_n)\varphi\,|\,\psi\rangle. \tag{4.13}$$

PROOF. Let

$$Z\colon L^2(\mathbb{R})\longrightarrow L^2(\mathbb{T}\times[0,1\rangle)$$

denote the Zak transform; see [**Dau92**, p. 109] for details. We have

$$(Z\psi)(z,x)=\sum_{n\in\mathbb{Z}} z^n\psi(x+n),\qquad \psi\in L^2(\mathbb{R}),\ x\in\mathbb{R},$$

and

$$\int_{\mathbb{T}} \int_0^1 |Z\psi(z,x)|^2 \, dx \, d\mu(z) = \int_{\mathbb{R}} |\psi(x)|^2 \, dx, \tag{4.14}$$

and Z maps $L^2(\mathbb{R})$ *onto* $L^2(\mathbb{T} \times [0,1\rangle)$. It follows, from (4.14), polarization, and Fubini's theorem, that the function

$$H(\varphi,\psi)(z) = \int_0^1 \overline{Z\varphi(z,x)}\, Z\psi(z,x)\, dx \tag{4.15}$$

is in $L^1(\mathbb{T})$ for all $\varphi, \psi \in L^2(\mathbb{R})$, and that this function satisfies the two desired properties (4.12) and (4.13) stated in the lemma. ☐

REMARK 4.7. The connection between the two operators M and R of the Introduction may be expressed with (4.15) as follows:

$$R(H(\varphi,\psi)) = H(M\varphi, M\psi).$$

This identity, which is equivalent to (1.30) above, also provides the direct link between spectral theory of R and approximation properties of $\{M^n\psi \mid n = 1, 2, \dots\}$ as $n \to \infty$. For details, see [**BrJo98**], [**Jor98**], and [**BJR97**].

Let m_0 be a wavelet filter with corresponding scaling function $\varphi \in L^2(\mathbb{R})$ and Ruelle operator R in $L^1(\mathbb{T})$. Then

$$h_\varphi(z) = \sum_{n\in\mathbb{Z}} z^n \langle \pi(e_n)\varphi \mid \varphi \rangle_{L^2(\mathbb{R})}$$

is the solution (in $L^1(\mathbb{T})$) to $R(h_\varphi) = h_\varphi$ which generates the wavelet representation $\tilde{\pi}_\varphi$ of $\mathfrak{A}_N$ on $L^2(\mathbb{R})$.

Similarly, if $h \in L^1(\mathbb{T})$, $h \geq 0$, is given such that $R(h) = h$, then, by Theorem 2.4, there is a (unique up to unitary equivalence) cyclic representation $\tilde{\pi}_h \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H}_h)$ such that

$$\langle \Phi \mid \pi_h(f)\Phi \rangle = \int_{\mathbb{T}} f h \, d\mu \tag{4.16}$$

and $U\Phi = \pi(m_0)\Phi$ where $\Phi \in \mathcal{H}_h$ is the cyclic vector.

THEOREM 4.8. *If $h \in L^1(\mathbb{T})$ is given such that $R(h) = h$, and if, for some $c \in \mathbb{R}_+$, we have $h_\varphi \leq ch$, then the wavelet representation $\tilde{\pi}_\varphi$ is contained in $\tilde{\pi}_h$, i.e., we have $\mathcal{H}_h = L^2(\mathbb{R}) \oplus \mathcal{K}$ and $\tilde{\pi}_h = \tilde{\pi}_\varphi \oplus \tilde{\pi}_{\mathcal{K}}$ for some $\tilde{\pi}_{\mathcal{K}} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{K})$.*

REMARK 4.9. When the theorem is applied to Example 4.5, i.e., the representation $\tilde{\pi}_1$ of $\mathfrak{A}_2$ which is induced by the pair $m_0 = \frac{1}{\sqrt{2}}(1 + z^3)$, $h_1 = \mathbb{1}$, we see that $\tilde{\pi}_1$ is the sum of two mutually inequivalent irreducible representations, $\tilde{\pi}_\varphi$ (the wavelet representation) being one of them. The other one in $\mathcal{K} \subset \mathcal{H}_1$ is in fact irreducible.

Let $\varphi \in L^2(\mathbb{R})$ be given by (4.7), i.e., $\varphi = \frac{1}{3}\chi_{[0,3\rangle}$, and let

$$h(z) = \frac{1}{9}\left(3 + 2\left(z + z^{-1}\right) + z^2 + z^{-2}\right)$$

be the function (4.8) which defines the wavelet representation $\tilde{\pi}_\varphi$ in $L^2(\mathbb{R})$ for $m_0 = \frac{1}{\sqrt{2}}(1 + z^3)$. Recall $\tilde{\pi}_\varphi = (U, \pi_0)$ is given by $U\psi(x) = \frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right)$, and $\pi_0(e_n)\psi = \psi(x - n)$, $\psi \in L^2(\mathbb{R})$. Let $\mathcal{K} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and $\rho = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Set $U_{\mathcal{K}} = U \oplus U$ on $\mathcal{K} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, $\alpha_\rho(f)(z) = f(\rho z)$, $f \in L^\infty(\mathbb{T})$, and $\pi_{\mathcal{K}}(f) = \pi_0(\alpha_\rho(f)) \oplus \pi_0\left(\alpha_{\rho^2}(f)\right)$. Let $\tilde{\pi}_{\mathcal{K}} \in \operatorname{Rep}(\mathfrak{A}_2, \mathcal{K})$ be the corresponding representation. We make the following claims:

CLAIMS. If $\tilde{\pi}_1$ is the representation induced from $h_1 \equiv \mathbb{1}$, i.e., the representation of Example 4.5, then

(i) $\tilde{\pi}_1 = \tilde{\pi}_\varphi \oplus \tilde{\pi}_{\mathcal{K}}$, and
(ii) $\tilde{\pi}_{\mathcal{K}}$ is irreducible on $\mathcal{K} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

PROOF. The formulas for m_0 and h show that $m_0(\rho z) = m_0(z)$, $z \in \mathbb{T}$, and

$$h(z) + h(\rho z) + h\left(\rho^2 z\right) \equiv \mathbb{1}. \tag{4.17}$$

Moreover, we have

(a) $R(h) = h$,
(b) $R(h(\rho\,\cdot))(z) = h\left(\rho^2 z\right)$, and
(c) $R\left(h\left(\rho^2\,\cdot\right)\right)(z) = h(\rho z)$.

The first assertions, (4.17), and (a) are immediate from substitution, so we check only the last:

$$(Ad\ (\mathrm{b}))\qquad \begin{aligned} R(h(\rho\,\cdot))(z) &= \frac{1}{2}\sum_{w^2=z} |m_0(w)|^2 h(\rho w) \\ &= \frac{1}{2}\sum_{w^2=\rho^2 z} |m_0(w)|^2 h(w) \\ &= R(h)\left(\rho^2 z\right) \\ &= h\left(\rho^2 z\right), \end{aligned}$$

where we used $R(h) = h$ in the computation. The proof of (c) is the same.

It follows that

$$h_{\mathcal{K}}(z) := 1 - h(z) = h(\rho z) + h\left(\rho^2 z\right)$$

or

$$h_{\mathcal{K}} = \alpha_\rho(h) + \alpha_{\rho^2}(h)$$

satisfies $R(h_{\mathcal{K}}) = h_{\mathcal{K}}$ and $h_{\mathcal{K}} \geq 0$. Let $\tilde{\pi}_{\mathcal{K}}$ be the corresponding representation which is induced from $h_{\mathcal{K}}$ *via* Theorem 2.4. It then follows from (4.17) that Claim (i) holds. In fact,

$$\begin{aligned} \langle \varphi \oplus \varphi \,|\, \pi_{\mathcal{K}}(f)(\varphi \oplus \varphi) \rangle &= \int_{\mathbb{T}} f h_{\mathcal{K}}\, d\mu \\ &= \int_{\mathbb{T}} f\left(\alpha_\rho(h) + \alpha_{\rho^2}(h)\right) d\mu \\ &= \int_{\mathbb{T}} \left(\alpha_{\rho^{-1}}(f) + \alpha_{\rho^{-2}}(f)\right) h\, d\mu \\ &= \int_{\mathbb{T}} \left(\alpha_{\rho^2}(f) + \alpha_\rho(f)\right) h\, d\mu \\ &= \left\langle \varphi \,|\, \pi_0\left(\alpha_{\rho^2}(f)\right)\varphi \right\rangle + \langle \varphi \,|\, \pi_0(\alpha_\rho(f))\varphi \rangle, \end{aligned}$$

which is the assertion we made about the $\pi_{\mathcal{K}}$-representation of $L^\infty(\mathbb{T})$ on $\mathcal{K} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

It remains to prove the irreducibility assertion in (ii). Each component in the sums

$$U_{\mathcal{K}} = U \oplus U$$

and

$$\pi_{\mathcal{K}}(f) = \pi_0(\alpha_\rho(f)) \oplus \pi_0\left(\alpha_{\rho^2}(f)\right)$$

may be viewed separately, but they are not representations of $\mathfrak{A}_2$ since $\alpha_\rho(h)$ is not an eigenfunction, i.e., $R(\alpha_\rho(h)) \neq \alpha_\rho(h)$; in fact, $R(\alpha_\rho(h)) = \alpha_{\rho^2}(h)$. But an operator in the commutant of $\tilde{\pi}_{\mathcal{K}} = (U_{\mathcal{K}}, \pi_{\mathcal{K}})$, i.e., $Q \in \tilde{\pi}_{\mathcal{K}}(\mathfrak{A}_2)'$, $Q = (Q_{ij})$, $1 \leq i, j \leq 2$, relative to the decomposition $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ must have Q_{ij} determined by the obvious matrix identities, i.e., each Q_{ij} commuting with U, $Q_{11}\pi_0(\alpha_\rho(f)) = \pi_0(\alpha_\rho(f))\,Q_{11}$, $Q_{21}\pi_0(\alpha_\rho(f)) = \pi_0\left(\alpha_{\rho^2}(f)\right)Q_{21}$, etc. A direct check shows that $Q_{ij} = \delta_{ij}\lambda$ for some $\lambda \in \mathbb{C}$. To see that $Q_{21} = 0$, for example, use that the stated representations of $L^\infty(\mathbb{T})$ are inequivalent. To see this, note that the functions $\alpha_\rho(h)$ and $\alpha_{\rho^2}(h)$ have different zero sets: in fact,

$$\{z \in \mathbb{T} \mid h(\rho z) = 0\} = \{1, \rho\},$$

and

$$\left\{z \in \mathbb{T} \mid h\left(\rho^2 z\right) = 0\right\} = \left\{1, \rho^2\right\}. \qquad \square$$

PROOF OF THEOREM 4.8. We recall that $\mathfrak{A}_N$ is the C^*-algebra generated by $L^\infty(\mathbb{T})$ and a unitary symbol U subject to the relation

$$UfU^{-1} = f\left(z^N\right), \qquad f \in L^\infty(\mathbb{T}), \tag{4.18}$$

and a representation $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H})$ amounts to a representation π of $L^\infty(\mathbb{T})$ on $\mathcal{H}$, and a unitary operator, also denoted U, on $\mathcal{H}$ such that

$$U\pi(f)\,U^{-1} = \pi\left(f\left(z^N\right)\right), \qquad f \in L^\infty(\mathbb{T}).$$

In the computations for $\tilde{\pi}_\varphi$ and $\tilde{\pi}_h$ we use the fact that $\mathfrak{A}_N$ contains an increasing family of abelian subalgebras, *viz.*, $U^{-n}fU^n$, $n = 0, 1, \dots$. To see that the algebra for n is contained in the next one, use

$$U^{-n}fU^n = U^{-(n+1)}UfU^{-1}U^{n+1} = U^{-(n+1)}f\left(z^N\right)U^{n+1}.$$

This structure also allows the interpretation of $\mathfrak{A}_N$ as an inductive limit C^*-algebra; see [**BreJo91**] for more details.

LEMMA 4.10. *We have the estimate*

$$\|\tilde{\pi}_\varphi(A)\,\varphi\|_{L^2(\mathbb{R})} \leq \sqrt{c}\,\|\tilde{\pi}_h(A)\,\Phi\|_{\mathcal{H}_h} \tag{4.19}$$

for all $A \in \mathfrak{A}_N$*, where* $\tilde{\pi}_\varphi \in \operatorname{Rep}\left(\mathfrak{A}_N, L^2(\mathbb{R})\right)$ *and* $\tilde{\pi}_h \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H}_h)$ *are the two cyclic representations described above, and where* $R(h) = h$*, and* $h_\varphi \leq ch$*.*

PROOF. There is a dense $*$-subalgebra in $\mathfrak{A}_N$ consisting of finite sums

$$\sum_{n \geq 0} U^{-n}\alpha_n + \beta_n U^n, \qquad \alpha_0, \alpha_1, \dots, \beta_0, \beta_1, \dots \in L^\infty(\mathbb{T}), \tag{4.20}$$

so we will estimate

$$\|\tilde{\pi}_h(A)\,\Phi\|^2_{\mathcal{H}_h} = \langle \Phi \mid \tilde{\pi}_h(A^*A)\,\Phi \rangle$$

for general elements $A \in \mathfrak{A}_N$ of this form. In fact, we show that there is a positive function F_A in $L^\infty(\mathbb{T})$ which only depends on A such that

$$\left\|\tilde{\pi}_h(A)\Phi\right\|^2_{\mathcal{H}_h} = \int_{\mathbb{T}} F_A h\,d\mu. \tag{4.21}$$

So this also applies to the wavelet representation $\tilde{\pi}_\varphi$, and we get

$$\left\|\tilde{\pi}_\varphi(A)\varphi\right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{T}} F_A h_\varphi\,d\mu.$$

Since $h_\varphi \leq ch$, the estimate in the lemma will follow.

If $A \in \mathfrak{A}_N$ is given as above, we may compute A^*A and find the following expression (finite sum!):

$$\begin{aligned}\left\langle \Phi \mid \tilde{\pi}_h(A^*A)\Phi\right\rangle &= \left\|\tilde{\pi}_h(A)\Phi\right\|^2_{\mathcal{H}_h}\\ &= \left\|\pi(\xi_0)\Phi + U^{-1}\pi(\xi_1)\Phi + \cdots + U^{-n}\pi(\xi_n)\Phi\right\|^2_{\mathcal{H}_h},\end{aligned}$$

where $\xi_0, \xi_1, \dots, \xi_n \in L^\infty(\mathbb{T})$. Introducing

$$S_0\xi(z) = m_0(z)\,\xi\left(z^N\right), \qquad \xi \in L^\infty(\mathbb{T}),$$

the argument from the proof of Lemma 3.8 yields

$$\begin{aligned}\left\|\sum_{k=0}^{n} U^{-k}\pi(\xi_k)\Phi\right\|^2_{\mathcal{H}_h} &= \left\|\sum_{k=0}^{n}\pi\left(S_0^k\xi_{n-k}\right)\Phi\right\|^2_{\mathcal{H}_h}\\ &= \left\|\sum_{k=0}^{n} S_0^k\xi_{n-k}\right\|^2_{L^2(h)}\\ &= \int_{\mathbb{T}}\left|\sum_{k=0}^{n} S_0^k\xi_{n-k}\right|^2 h\,d\mu.\end{aligned}$$

The formula for F_A may be read off from this, and the lemma follows. □

The lemma states that there is a well defined bounded operator $W\colon \mathcal{H}_h \to L^2(\mathbb{R})$ which is given by

$$W\tilde{\pi}_h(A)\Phi := \tilde{\pi}_\varphi(A)\varphi, \qquad A \in \mathfrak{A}_N;$$

and it is clear that W will intertwine the two representations, i.e., that

$$W\tilde{\pi}_h(A) = \tilde{\pi}_\varphi(A)W, \qquad A \in \mathfrak{A}_N, \tag{4.22}$$

or $W\tilde{\pi}_h = \tilde{\pi}_\varphi W$ for short. But then WW^* commutes with $\tilde{\pi}_\varphi$. Specifically,

$$WW^*\tilde{\pi}_\varphi = W\tilde{\pi}_h W^* = \tilde{\pi}_\varphi WW^*.$$

Since $\tilde{\pi}_\varphi$ is irreducible by Lemma 2.3, we conclude that $WW^* = \text{const.}\, I_{L^2(\mathbb{R})}$. It follows from the assumptions that the constant, c, say, is nonzero. Hence $\left(\sqrt{c}\right)^{-1}W^*$ is an isometry from $L^2(\mathbb{R})$ into $\mathcal{H}_h$ which intertwines, i.e.,

$$\tilde{\pi}_h\left(\sqrt{c}\right)^{-1}W^* = \left(\sqrt{c}\right)^{-1}W^*\tilde{\pi}_\varphi.$$

The assertion of the theorem follows from this. □

The argument from the theorem also gives the following corollary which provides us with a numerical index for the convex cone of solutions $h \in L^1(\mathbb{T})$, $h \geq 0$, $Rh = h$.

COROLLARY 4.11. *Let h and h_φ be as described in Theorem 4.8; then the intertwining operators*

$$W\colon \mathcal{H}_h \longrightarrow L^2(\mathbb{R})$$

form a Hilbert space, and the dimension of this Hilbert space is the multiplicity of $\tilde{\pi}_\varphi$ in $\tilde{\pi}_h$.

PROOF. The only argument in the proof of the corollary which is not already in the theorem is the assertion of a Hilbert space structure on the intertwiners $W\colon \mathcal{H}_h \to L^2(\mathbb{R})$. If we have two such, W_1, W_2, say, then $W_1 W_2^*$ commutes with the wavelet representation $\tilde{\pi}_\varphi \in \operatorname{Rep}\left(\mathfrak{A}_N, L^2(\mathbb{R})\right)$. Since the latter is irreducible, $W_1 W_2^*$ must be a scalar times $I_{L^2(\mathbb{R})}$. Denoting this scalar $\langle W_1, W_2\rangle$, we have the desired inner product. We leave the rest of the details to the reader. ☐

COROLLARY 4.12. *Let $m_0 = \frac{1}{\sqrt{2}}\left(1+z^3\right)$, and $h_e\left(e^{-i\omega}\right) = \frac{1}{9}\left(1+2\cos\omega\right)^2$. Since the wavelet representation in $L^2(\mathbb{R})$ is induced by h_e and irreducible, we conclude that the only solutions h to*

$$0 \leq h \leq \text{const.}\, h_e, \qquad R_{m_0}(h) = h,$$

are of the form

$$h = \lambda h_e, \qquad \lambda \geq 0.$$

PROOF. We saw in Example 4.3 that $h_e = \frac{1}{9}\left(1+2\cos\omega\right)^2$ is the solution to $R(h_e) = h_e$ which corresponds to the wavelet representation for $m_0 = \frac{1}{\sqrt{2}}\left(1+z^3\right)$, and so it is irreducible by Lemma 2.3. But we saw that irreducibility of the representation $\tilde{\pi}_h \in \operatorname{Rep}(\mathfrak{A}_2, \mathcal{H}_h)$ corresponds to extremality of the state ω_h on $\mathfrak{A}_2$ which is given by

$$\omega_h\left(fU^n\right) = \int_{\mathbb{T}} f m_0^{(n)} h \, d\mu,$$

where $m_0^{(n)}(z) = m_0(z)\, m_0\left(z^2\right)\cdots m_0\left(z^{2^{n-1}}\right)$, $n = 0, 1, \dots$, $f \in L^\infty(\mathbb{T})$. We also saw that the estimate $\omega_h \leq \text{const.}\, \omega_{h'}$ on the positive elements in $\mathfrak{A}_2$ is equivalent to

$$\int_{\mathbb{T}} |f|^2 h \, d\mu \leq \text{const.} \int_{\mathbb{T}} |f|^2 h' \, d\mu, \qquad f \in L^\infty(\mathbb{T}).$$

The conclusion of the corollary is immediate from this since we noted that the wavelet representation is induced by h_e and irreducible. ☐

The argument which we used in the proofs of Lemma 4.10 and Corollary 4.11 yields a more general result about the commutant of our representations of $\mathfrak{A}_N$. Let $m_0 \in L^\infty(\mathbb{T})$ and suppose m_0 satisfies identity (2.9), and moreover that it is non-singular. Our result will apply to wavelet filters, but the other properties of wavelet filters will not be needed. Let $R = R_{m_0}$ be the corresponding Ruelle operator, and consider a solution h to $R(h) = h$, $h \in L^1(\mathbb{T})$, $h \geq 0$. Let $\tilde{\pi} \in \operatorname{Rep}(\mathfrak{A}_N, \mathcal{H}_h)$ be the cyclic representation which is induced *via* Theorem 2.4, and let $\varphi \in \mathcal{H}_h$ denote the cyclic vector. Recall that $\tilde{\pi}$ is determined by a representation π of $L^\infty(\mathbb{T})$, and a unitary operator U, on $\mathcal{H}_h$ such that

$$U\pi(f)U^{-1} = \pi\left(f\left(z^N\right)\right), \tag{4.23}$$

$$\langle\varphi \mid \pi(f)\varphi\rangle = \int_{\mathbb{T}} fh\,d\mu, \tag{4.24}$$

and

$$U\varphi = \pi(m_0)\varphi. \tag{4.25}$$

THEOREM 4.13. *Let (m_0, h) be as described above. Then there is a one-to-one correspondence between positive elements in the commutant of $\tilde{\pi}(\mathfrak{A}_N)$ and solutions*

$$h_Q \in L^1(\mathbb{T}), \qquad R(h_Q) = h_Q, \tag{4.26}$$

satisfying the pointwise estimate

$$0 \leq h_Q \leq ch \tag{4.27}$$

for some constant c.

PROOF. We introduce the notation $\mathcal{M}' = \tilde{\pi}(\mathfrak{A}_N)'$ for the commutant, i.e., the bounded operators Q in $\mathcal{H}_h$ such that

$$Q\tilde{\pi}(A) = \tilde{\pi}(A)Q, \qquad A \in \mathfrak{A}_N. \tag{4.28}$$

Positivity of Q means: $\langle\psi \mid Q\psi\rangle \geq 0$, $\psi \in \mathcal{H}_h$. If Q is positive, the square root $Q^{\frac{1}{2}}$ is well defined by the spectral theorem, and $Q^{\frac{1}{2}}$ is in $\mathcal{M}'$ if Q is.

Let $Q \in \mathcal{M}'$ be given and positive. We then have

$$\langle\varphi \mid Q\tilde{\pi}(A^*A)\varphi\rangle = \left\|\tilde{\pi}(A)Q^{\frac{1}{2}}\varphi\right\|^2, \qquad A \in \mathfrak{A}_N,$$

so $A \mapsto \langle\varphi \mid Q\tilde{\pi}(A)\varphi\rangle$ is a positive linear functional on $\mathfrak{A}_N$, and it will be denoted ω_Q, i.e.,

$$\omega_Q(A) := \langle\varphi \mid Q\tilde{\pi}(A)\varphi\rangle, \qquad A \in \mathfrak{A}_N. \tag{4.29}$$

A standard estimate from operator theory (see, e.g., [**BrRoI**]) yields the estimate

$$\omega_Q(A^*A) \leq \|Q\|\,\|\tilde{\pi}(A)\varphi\|^2 \tag{4.30}$$

Applying this to $A = \pi(f)$, $f \in L^\infty(\mathbb{T})$, we see that the measure determined by $f \mapsto \omega_Q(\pi(f))$ is absolutely continuous. Let $h_Q \in L^1(\mathbb{T})$ be the Radon–Nikodym derivative, i.e.,

$$\omega_Q(\pi(f)) = \int_{\mathbb{T}} fh_Q\,d\mu,$$

where μ is Haar measure on $\mathbb{T}$. Setting $\varphi_Q = Q^{\frac{1}{2}}\varphi$, we see that

$$\langle\varphi_Q \mid \pi(f)\varphi_Q\rangle = \int_{\mathbb{T}} fh_Q\,d\mu, \qquad f \in L^\infty(\mathbb{T}). \tag{4.31}$$

Since $Q^{\frac{1}{2}} \in \mathcal{M}'$, we also have

$$U\varphi_Q = \pi(m_0)\varphi_Q. \tag{4.32}$$

Combining (4.31) and (4.32), we conclude that $R(h_Q) = h_Q$. Lemma 3.3, or Theorem 2.4, yields that conclusion. In view of (4.31), a second application of (4.30) yields the pointwise estimate

$$0 \leq h_Q \leq \|Q\|\,h, \tag{4.33}$$

concluding the proof in one direction.

Conversely, suppose h_Q is given and satisfies (4.26)–(4.27) of the theorem. Since the representation $\tilde{\pi} = (\pi, U)$ is cyclic, $\mathcal{H}_h$ is spanned (after taking closure) by vectors of the form

$$U^{-n}\pi(f)\varphi, \qquad n = 0, 1, \dots, \qquad f \in L^{\infty}(\mathbb{T}),$$

and we may define an operator Q on $\mathcal{H}_h$ by the matrix entries

$$\begin{aligned}(4.34)\quad & \left\langle U^{-n_1}\pi(f_1)\varphi \,\middle|\, QU^{-n_2}\pi(f_2)\varphi\right\rangle \\ &= \begin{cases} \int_{\mathbb{T}} \overline{f_1\left(z^{N^{n_2-n_1}}\right)} f_2(z)\,\overline{m_0^{(n_2-n_1)}(z)} h_Q(z)\,d\mu(z) & \text{if } n_2 \geq n_1, \\ \int_{\mathbb{T}} \overline{f_1(z)} f_2\left(z^{N^{n_1-n_2}}\right) m_0^{(n_1-n_2)}(z)\,h_Q(z)\,d\mu(z) & \text{if } n_2 \leq n_1, \end{cases}\end{aligned}$$

where $m_0^{(k)}(z) = m_0(z)\,m_0\left(z^N\right)\cdots m_0\left(z^{N^{k-1}}\right)$. The argument from Lemma 4.10 shows that Q is well defined, and bounded. In fact, using (4.34) for $n_1 = n_2 = n$ and $f_1 = f_2 = f$, we get

$$\left\langle U^{-n}\pi(f)\varphi \,\middle|\, QU^{-n}\pi(f)\varphi\right\rangle = \int_{\mathbb{T}} |f|^2\,h_Q\,d\mu \leq c\int_{\mathbb{T}} |f|^2\,h\,d\mu = c\left\|U^{-n}\pi(f)\varphi\right\|^2_{\mathcal{H}_h},$$

which is the desired boundedness. But the definition of the operator Q also entails that it is in the commutant, i.e., that it satisfies (4.28). The second implication is proved. □

COROLLARY 4.14. *Let m_0 and h be as described in Theorem 4.13, and let $\tilde{\pi}_h$ be the corresponding representation of $\mathfrak{A}_N$ on $\mathcal{H}_h$. Then the commutant $\tilde{\pi}_h(\mathfrak{A}_N)'$ is abelian, and so $\tilde{\pi}_h$ is the direct integral of a family of mutually inequivalent irreducible representations of $\mathfrak{A}_N$.*

PROOF. The commutativity is a direct consequence of the formula $Q \mapsto h_Q$ in Theorem 4.13, and, more specifically, of equation (4.34), which expresses a fixed $Q \in \tilde{\pi}_h(\mathfrak{A}_N)'$ in terms of h_Q, where $R(h_Q) = h_Q$ and $h_Q \leq ch$. The assertion about $\tilde{\pi}_h$ being a direct integral of a family of mutually inequivalent irreducible representations follows from a standard fact in representation theory: If $\tilde{\pi}_h$ contains two equivalent irreducibles, then the commutant $\tilde{\pi}_h(\mathfrak{A}_N)'$ would contain a copy of the 2-by-2 complex matrices, or have such a copy of $M_2(\mathbb{C})$ in a direct integral.

If m_0 is continuous, then the eigenspace $\left\{h \in L^1(\mathbb{T}) \mid R_{m_0}(h) = h\right\}$ is finite-dimensional by a theorem in [**CoRa90**], so in that case, $\tilde{\pi}_h$ is a finite direct sum of a finite number of mutually inequivalent irreducible representations. By a result in the next chapter, these irreducible representations must be wavelet representations in the case when m_0 is a wavelet filter which is non-singular. □

CHAPTER 5

Cocycle equivalence of filter functions

Let $h \in L^1(\mathbb{T})$, $h \geq 0$, be given, and form the Hilbert space $L^2(h) := L^2(\mathbb{T}, h\,d\mu)$ as usual. We saw in Lemma 4.2 that the operator

$$(S_0\xi)(z) := m_0(z)\,\xi\left(z^N\right) \tag{5.1}$$

is isometric in $L^2(h)$ if and only if $R(h) = h$ where $R = R_{m_0}$ is the Ruelle operator formed from m_0. We will make the standing assumption that the function m_0 is a wavelet filter. It follows that $R(\mathbb{1}) = \mathbb{1}$, so S_0 is an isometry in $L^2(\mathbb{T})$, which is the special case $h = \mathbb{1}$.

Definition 5.1. We say that two wavelet filters m_0 and m_0' are *cocycle equivalent* if there is a nonzero measurable function f on $\mathbb{T}$ such that

$$f\left(z^N\right) m_0'(z) = f(z)\, m_0(z), \qquad z \in \mathbb{T}. \tag{5.2}$$

It is immediate that m_0 and m_0' satisfy (5.2) for some f if and only if the multiplication operator $(M_f\xi)(z) = f(z)\,\xi(z)$ intertwines the two isometries

$$S_0\xi(z) = m_0(z)\,\xi\left(z^N\right) \tag{5.3}$$

and

$$S_0'\xi(z) = m_0'(z)\,\xi\left(z^N\right) \tag{5.4}$$

on $L^2(\mathbb{T})$, i.e., if and only if

$$M_f S_0 = S_0' M_f. \tag{5.5}$$

Lemma 5.2.

(i) *Let f be a function on $\mathbb{T}$ which defines a cocycle equivalence for wavelet filters m_0 and m_0', and set $h = |f|^2$. Then*

$$R_{m_0}(h) = h. \tag{5.6}$$

(ii) *Conversely, if m_0 is given and $R_{m_0}(h) = h$, $h \geq 0$, nonzero, then $f = \sqrt{h}$ defines a cocycle equivalence.*

Proof. *Ad* (i): Let $h = |f|^2$ be as described in (i). Then

$$\begin{aligned}
(Rh)(z) &= \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2\, h(w) \\
&= \frac{1}{N} \sum_{w^N = z} |m_0'(w)|^2\, h\left(w^N\right) \\
&= h(z) \underbrace{\frac{1}{N} \sum_{w^N = z} |m_0'(w)|^2}_{\equiv 1} = h(z).
\end{aligned}$$

Note that the normalization condition $\frac{1}{N}\sum_{w^N=z}\left|m_0'(w)\right|^2=1$ was needed only for the function m_0', not for the second filter function m_0 of Definition 5.1.

Ad (ii): Let h be as in (ii), and set $f=\sqrt{h}$, and

$$v(z):=(h(z))^{\frac{1}{2}}\,m_0(z). \tag{5.7}$$

Then

$$\frac{1}{N}\sum_{w^N=z}|v(w)|^2=\frac{1}{N}\sum_{w^N=z}h(w)\,|m_0(w)|^2=(Rh)(z)=h(z).$$

In particular,

$$h\left(z^N\right)=\frac{1}{N}\sum_{k=0}^{N-1}\left|v\left(e^{i\frac{2\pi k}{N}}z\right)\right|^2;$$

so if $h\left(z^N\right)=0$, then all N terms on the right-hand side must vanish. In particular, $v(z)=0$. It follows that the function

$$m_0'(z):=\frac{v(z)}{\left(h\left(z^N\right)\right)^{\frac{1}{2}}} \tag{5.8}$$

is well defined, and satisfies

$$\frac{1}{N}\sum_{w^N=z}\left|m_0'(w)\right|^2=1,$$

which is to say that m_0' is a wavelet filter. It is clear from (5.7) and (5.8) that $f=\sqrt{h}$ does define a cocycle equivalence between the two wavelet filters m_0 and m_0' as claimed. $\square$

DEFINITION 5.3. We say that a measurable function on $\mathbb{T}$ is *non-singular* if it does not vanish on a subset of $\mathbb{T}$ of positive Lebesgue measure.

The next result (Lemma 5.4) establishes a general property of positive harmonic functions defined from a Ruelle operator, in the non-singular case: and it is an analogue of a classical result for positive harmonic functions defined in the usual way from a Laplace operator; see, e.g., [**Rud90**] or [**SzFo70**].

LEMMA 5.4. *If m_0 is a wavelet filter, and if $R_{m_0}h=h$, $h\in L^1(\mathbb{T})$, $h\geq 0$, and $h\neq 0$. Then h is non-singular whenever m_0 is.*

PROOF. Let m_0 and h be as stated. Let $Z(h)$ be the complement in $\mathbb{T}$ of the support of h. We show that $\mu(Z(h))>0$ leads to a contradiction. Since m_0 is non-singular, and $h(z)=\frac{1}{N}\sum_{w^N=z}|m_0(w)|^2\,h(w)$, it follows that the set $Z^{(1)}(h)=\left\{z\in\mathbb{T}\mid z^N\in Z(h)\right\}$ is contained in $Z(h)$ except possibly for a zero-measure subset in $\mathbb{T}$. Moreover, $\mu\left(Z^{(1)}(h)\right)=\mu(Z(h))$. Suppose now that $\mu(Z(h))>0$, i.e., that h is not non-singular. Then continue recursively, defining

$$Z^{(n)}(h)=\left\{z\in\mathbb{T}\mid z^{N^n}\in Z(h)\right\}. \tag{5.9}$$

We get the inclusions

$$Z^{(n+1)}(h)\subset Z^{(n)}(h)\subset\cdots\subset Z(h)\subset\mathbb{T},$$

except possibly for points of a subset of zero measure, and $\mu\left(Z^{(n)}(h)\right)=\mu(Z(h))>0,\qquad n=1,2,\dots$. Defining $Z^{(\infty)}(h):=\bigcap_n Z^{(n)}(h)$, we note that $\mu\left(Z^{(\infty)}(h)\right)=$

$\mu(Z(h))$. Since $Z^{(\infty)}(h)$ is invariant under $z \mapsto z^N$, we conclude from ergodicity that $\mu\left(Z^{(\infty)}(h)\right) = \mu(Z(h)) = 1$ relative to normalized Haar measure μ on $\mathbb{T}$. But this contradicts the assumption $h \neq 0$ in $L^1(\mathbb{T})$, and the proof is concluded. $\square$

Assume that m_0 is non-singular. If $h \in L^1(\mathbb{T})$, $h \geq 0$, and $h \neq 0$ solves $R_{m_0}h = h$, then h will also be non-singular by the lemma. It follows that the multiplication operator $\Delta := M_{h^{\frac{1}{2}}}$ is then an isometric isomorphism between the Hilbert spaces $L^2(h) = L^2(\mathbb{T}, h\,d\mu)$ and $L^2(\mathbb{T})$, i.e., we have, for $\xi \in L^2(h)$,

$$\|\Delta\xi\|_{L^2(\mathbb{T})} = \|\xi\|_{L^2(h)}, \quad L^2(h) \xrightarrow{\Delta} L^2(\mathbb{T}), \quad \text{and} \quad \Delta\left(L^2(h)\right) = L^2(\mathbb{T}). \tag{5.10}$$

LEMMA 5.5. *Let $(S_0\xi)(z) = m_0(z)\,\xi\left(z^N\right)$ and suppose S_0 is isometric in $L^2(h)$, and further that m_0 is non-singular. Then*

$$\left(\Delta S_0 \Delta^{-1}\xi\right)(z) = m_0(z)\left(\frac{h(z)}{h\left(z^N\right)}\right)^{\frac{1}{2}} \xi\left(z^N\right) \tag{5.11}$$

is an isometry in $L^2(\mathbb{T})$. Moreover, the two conditions

$$\sum_{w^N = z} |m_0(w)|^2 = N \tag{5.12}$$

and

$$m_0(1) = \sqrt{N}$$

are preserved under the cocycle operation

$$m_0 \longmapsto m_0(z)\left(\frac{h(z)}{h\left(z^N\right)}\right)^{\frac{1}{2}}.$$

PROOF. The proof follows essentially from the previous lemmas, but we recall that m_0 is assumed to be continuous near $z = 1$, and if $R_{m_0}(h) = h$, $h \in L^1(\mathbb{T})$, $h \geq 0$, it can be checked that h will then also be continuous near $z = 1$ so that the evaluation of

$$m_0^{(h)}(z) = m_0(z)\left(\frac{h(z)}{h\left(z^N\right)}\right)^{\frac{1}{2}} \tag{5.13}$$

at $z = 1$ is well defined.

It remains to check that $m_0^{(h)}$ satisfies (5.12). But

$$\begin{aligned}
\frac{1}{N}\sum_{w^N = z}\left|m_0^{(h)}(w)\right|^2 &= \frac{1}{N}\sum_{w^N = z}|m_0(w)|^2 \frac{h(w)}{h\left(w^N\right)} \\
&= \frac{1}{h(z)}\frac{1}{N}\sum_{w^N = z}|m_0(w)|^2 h(w) \\
&= \frac{1}{h(z)}(Rh)(z) = \frac{h(z)}{h(z)} = 1,
\end{aligned}$$

which completes the proof. Recall $R(h) = h$ holds since S_0 is isometric in $L^2(h)$. $\square$

If $h \in L^1(\mathbb{T})$, $h \geq 0$, $h \neq 0$, is given and if m_0 is a non-singular wavelet filter, we showed that S_0 is an isometry in $L^2(h)$ if and only if $R_{m_0}(h) = h$, and we will now show that this isometry is pure, i.e., that $\bigcap_{n=1}^{\infty} S_0^n\left(L^2(h)\right) = \{0\}$. The proof is based on the previous lemmas and one of the main results in [**BrJo97**].

THEOREM 5.6. *Let* m_0 *and* h *be as described above, i.e.,* m_0 *a non-singular wavelet filter and* $R_{m_0}(h) = h$, $h \in L^1(\mathbb{T})$, $h \geq 0$, $h \neq 0$. *Then* $S_0\xi(z) = m_0(z)\,\xi(z^N)$, $\xi \in L^2(h)$, *satisfies*

$$\bigcap_{n=1}^{\infty} S_0^n\left(L^2(h)\right) = \{0\}. \tag{5.14}$$

If U *is the unitary operator in* $\mathcal{H}_h$ *with*

$$U\pi(f)\,U^{-1} = \pi\left(f\left(z^N\right)\right)$$

which defines the h*-induced cyclic representation, i.e.,* $U\varphi = \pi(m_0)\,\varphi$, *and* φ *denoting the cyclic vector, then*

$$\bigcap_{n=1}^{\infty} U^n\left(V_0(\varphi)\right) = \{0\}, \tag{5.15}$$

where

$$V_0(\varphi) := \overline{\operatorname{span}}\left\{\pi(f)\,\varphi \mid f \in L^\infty(\mathbb{T})\right\}. \tag{5.16}$$

PROOF. We already showed that $\|U^n\pi(f)\,\varphi\|_{\mathcal{H}_h} = \|S_0^n f\|_{L^2(h)} = \|f\|_{L^2(h)}$, $n = 0, 1, \dots$, so it follows that the two intersection properties (5.14) and (5.15) are equivalent. But, in view of Lemma 5.5, we may check equivalently that

$$\left(S_0^{(h)}\xi\right)(z) = m_0(z)\left(\frac{h(z)}{h(z^N)}\right)^{\frac{1}{2}}\xi\left(z^N\right),$$

as an isometry in $L^2(\mathbb{T})$, satisfies

$$\bigcap_{n=1}^{\infty}\left(S_0^{(h)}\right)^n L^2(\mathbb{T}) = \{0\}. \tag{5.17}$$

Because of Lemma 5.5, this in turn is immediate from [**BrJo97**, Theorem 3.1], and the proof is concluded. ☐

The significance of the result is that $\left(S_0, L^2(h)\right)$ will then be unitarily equivalent to $\left(S, H^2(\mathbb{T}, \mathcal{K})\right)$ with

$$(SF)(z) = zF(z), \tag{5.18}$$

$F\colon \mathbb{T} \to \mathcal{K}$ satisfying $F(z) = \sum_{n=0}^{\infty} z^n k_n$, and

$$\int_{\mathbb{T}} \|F(z)\|_{\mathcal{K}}^2\, d\mu(z) = \sum_{n=0}^{\infty} \|k_n\|_{\mathcal{K}}^2,$$

where

$$k_n \in \mathcal{K} = \ker\left(S_0^*\right) = \left\{\xi \in L^2(h) \mid S_0^*\xi = 0\right\}.$$

This is a simple application of the standard Wold decomposition (see [**SzFo70**]) for the isometry $\left(S_0, L^2(h)\right)$.

If $N = 2$, set

$$(S_1\xi)(z) := z\,\overline{m_0(-z)}\left(\frac{h(-z)}{h(z)}\right)^{\frac{1}{2}}\xi\left(z^2\right), \qquad \text{for } \xi \in L^2(h). \tag{5.19}$$

Then we have

$$S_i^* S_j = \delta_{ij} I_{L^2(h)}, \qquad \text{and} \qquad \sum_{i=0}^{1} S_i S_i^* = I_{L^2(h)} \tag{5.20}$$

in view of Lemma 5.2. We therefore get $\mathcal{K} = \ker(S_0^*) = S_1\left(L^2(h)\right)$.

The relations (5.20) are called the Cuntz relations and correspond to representations of the corresponding C^*-algebra $\mathcal{O}_2$ (see [**Cun77**]) which is known to be simple. They are the representations which act on $L^2(h)$, and *via* $Wf = \pi(f)\varphi$, they intertwine with operators on $\mathcal{H}_h$, in particular $WS_0 f = UWf$, $f \in L^2(h)$. *Via* Lemma 5.5, they correspond to representations of $\mathcal{O}_2$ on $L^2(\mathbb{T})$, but these representations are different from those of Lemma 4.1 and Lemma 5.2.

In fact, the representation on $L^2(\mathbb{T})$ which intertwines with (5.20) is given by $\left(S_i^{(h)}\xi\right)(z) = m_i^{(h)}(z)\,\xi\left(z^2\right)$, $\xi \in L^2(\mathbb{T})$, $i = 0, 1$, where

$$m_0^{(h)}(z) = m_0(z)\left(\frac{h(z)}{h(z^2)}\right)^{\frac{1}{2}}$$

and

$$m_1^{(h)}(z) = z\,\overline{m_0(-z)}\left(\frac{h(-z)}{h(z^2)}\right)^{\frac{1}{2}}, \qquad z \in \mathbb{T}.$$

COROLLARY 5.7. *Let m_0 and h be as above, and let $(\pi, \mathcal{H}_h, \varphi)$ be the corresponding cyclic representation. Then the operator given by $f \mapsto WS_1 f$, from $L^2(h)$ into $\mathcal{H}_h$ (see (5.19)), maps $L^2(h)$ onto the space $V_0(\varphi) \ominus U(V_0(\varphi))$ where $V_0(\varphi)$ is the closed cyclic subspace in $\mathcal{H}_h$ which is generated by φ under the representation $\pi(L^\infty(\mathbb{T}))$.*

PROOF. The details are contained in the discussion above. $\square$

CHAPTER 6

The transfer operator of Keane

Let m_0 be a wavelet filter, and let R be the corresponding Ruelle operator. Then we showed in [**BEJ97**] and [**Jor98**] that, for $\lambda \in \mathbb{T} \setminus \{1\}$, i.e., $\lambda \in \mathbb{C}$, $|\lambda| = 1$, the eigenvalue problem $R(h) = \lambda h$ does not have nonzero solutions h in $L^1(\mathbb{T})$ if the scaling function has orthogonal $\mathbb{Z}$-translates. In this chapter, we consider a more general framework which admits such solutions, and which also includes problems in the theory of iteration, other than the wavelet problems, e.g., iteration of conformal transformations. The non-trivial solutions to $R(h) = \lambda h$ will be interpreted as functionals on a C^*-algebra analogous to $\mathfrak{A}_N$, but they will not be positive if $\lambda \neq 1$.

Let $(X, \mathcal{B}, \mu)$ be a finite measure space, i.e., μ will be assumed to be *finite* positive measure which is defined on a σ-algebra of subsets of X. If X is a topological space, we assume that $\mathcal{B}$ includes the Borel subsets of X. Let $T\colon X \to X$ be a measurable mapping of X *onto* X which is N-to-one, i.e., for μ-a.a. x in X, $T^{-1}(x) = \{y \in X \mid Ty = x\}$ is of cardinality N, and assume further that μ is T-invariant, i.e., that

$$\mu\left(T^{-1}(E)\right) = \mu(E), \qquad E \in \mathcal{B}. \tag{6.1}$$

We shall further need a selection of measurable inverses $\sigma_i\colon X \to X$, $i = 1, \dots, N$, such that $T(\sigma_i(x)) = x$, μ-a.a. x in X, $i = 1, \dots, N$, and such that

$$\mu\left(\sigma_i(X) \cap \sigma_j(X)\right) = 0 \tag{6.2}$$

for all $i \neq j$. The following invariance condition (which is slightly stronger than (6.1)) will be needed throughout the chapter. It may be stated in the following three equivalent forms:

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \mu \circ \sigma_i^{-1}, \tag{6.3}$$

$$\mu(E) = \frac{1}{N} \sum_{i=1}^{N} \mu\left(\sigma_i^{-1}(E)\right), \qquad E \in \mathcal{B}, \tag{6.4}$$

$$\int_X f \, d\mu = \frac{1}{N} \sum_{i=1}^{N} \int_X f \circ \sigma_i \, d\mu \qquad \text{for all } \mathcal{B}\text{-measurable } f \text{ on } X. \tag{6.5}$$

Example 6.1. Let $X = \mathbb{T}$, $T(z) = z^N$, and let $\{\sigma_i\}$ be the choice of N-th roots specified by, e.g.,

$$\sigma_k\left(e^{-i\omega}\right) = e^{-\frac{i\omega k}{N}}, \qquad k = 0, 1, \dots, N-1; \tag{6.6}$$

and let μ be Haar measure on $\mathbb{T}$, i.e., $\frac{1}{2\pi}\int_{-\pi}^{\pi} \cdots \, d\omega$. Then it is easy to check that conditions (6.1)–(6.5) hold.

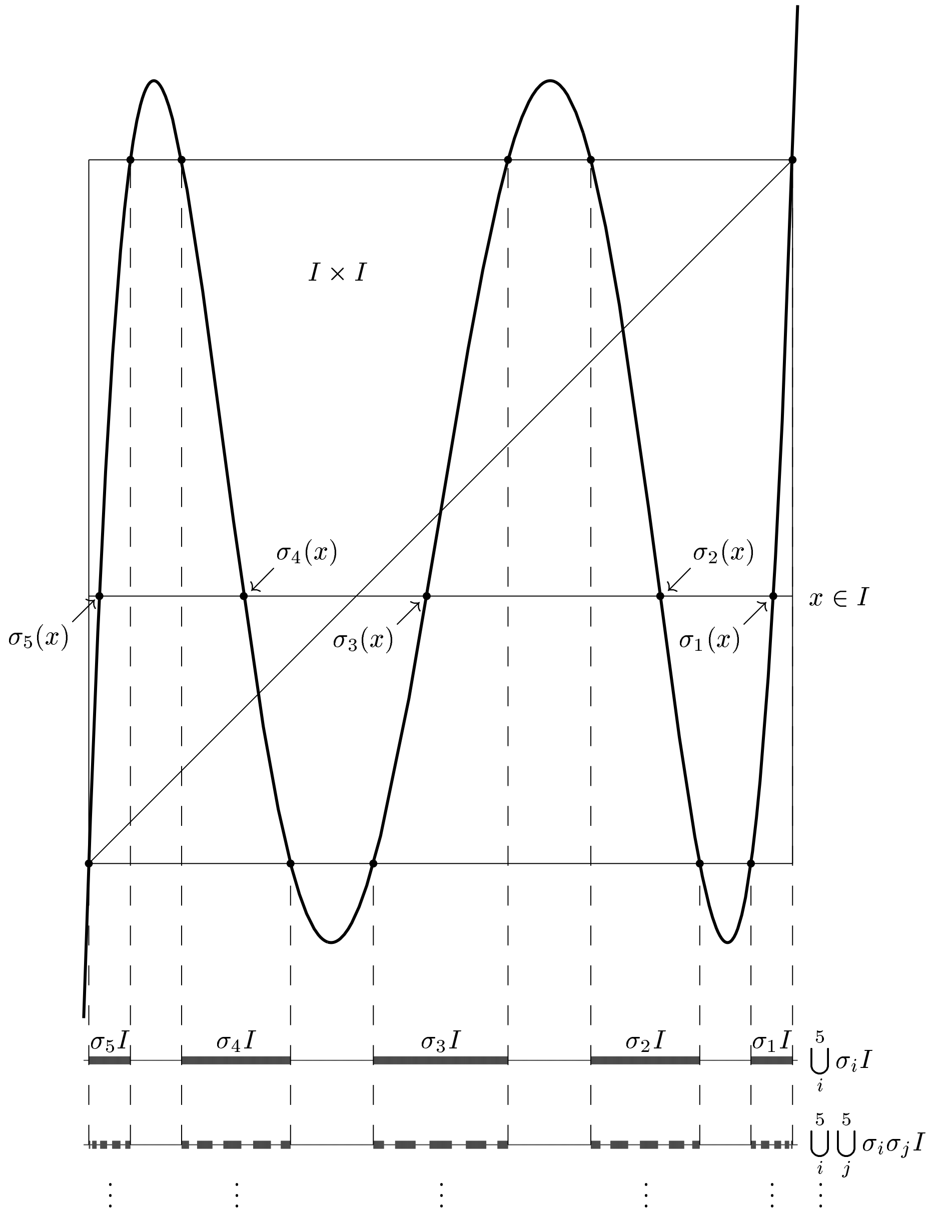

FIGURE 3. Example 6.2, case 1, $p(a) = a$: $p(t) = t^5 - 5\alpha^2 t^3 + 5\alpha^4 t$, $\alpha = 1.05$, $I := [a, b]$, $a = -b$, $b \approx 2.08411$. (Figure designed by Brian Treadway.)

EXAMPLE 6.2. (The *Julia set* of a polynomial) Let $p(z) = z^N + p_1 z^{N-1} + \cdots + p_N$ be a polynomial with real coefficients p_i, and leading coefficient 1. When $z \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is given, define $z_0 = z$, and $z_{n+1} = p(z_n)$. The fixed points are the solutions ζ to $p(\zeta) = \zeta$, and of course $\zeta = \infty$ is a fixed point. If ζ is a fixed point, the *domain of attraction* $\Omega(\zeta)$ is $\Omega(\zeta) = \left\{z_0 \in \overline{\mathbb{C}} \mid z_n \to \zeta\right\}$, and we say that ζ is *attracting* if $\Omega(\zeta)$ contains a neighborhood of ζ. Clearly $\zeta = \infty$

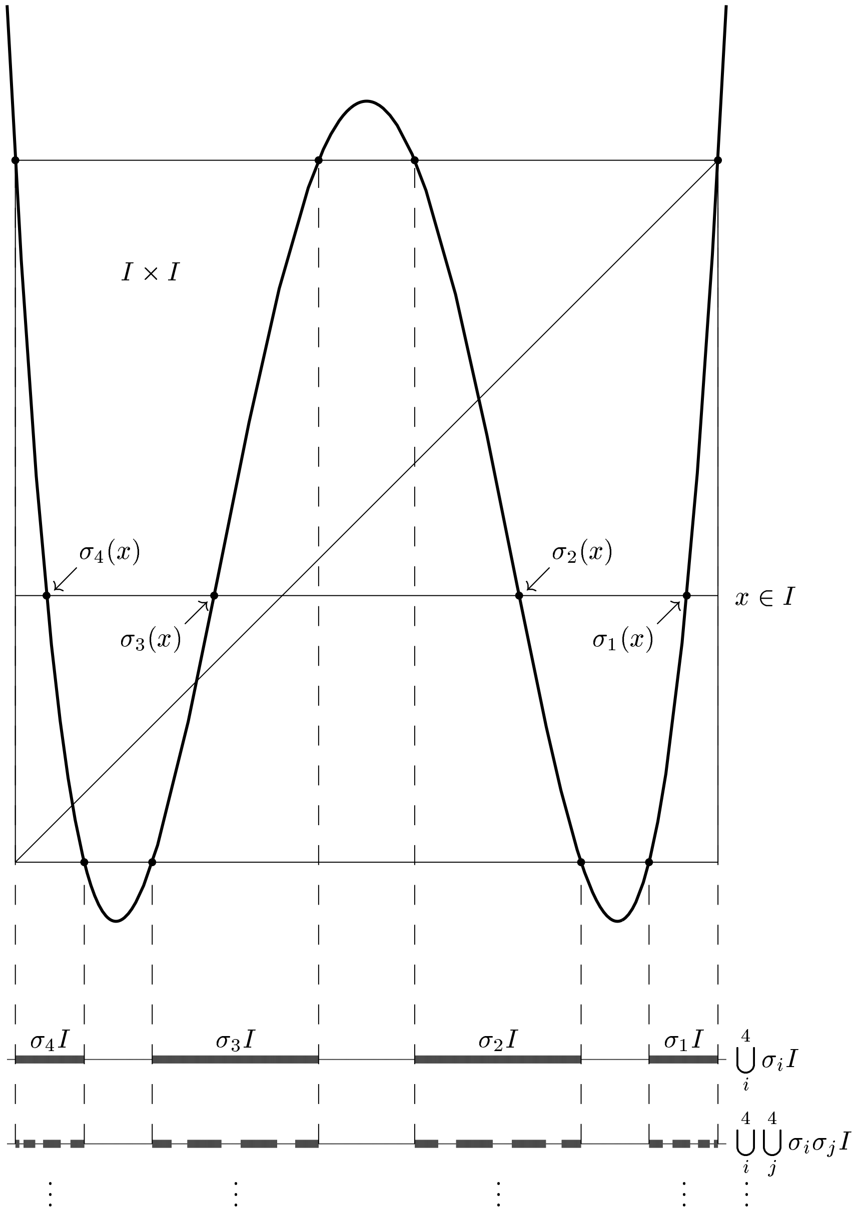

FIGURE 4. Example 6.2, case 2, $p(a) = b$: $p(t) = t^4 - 4\alpha^2 t^2 + 2\alpha^4$, $\alpha = 1.05$, $I := [a, b]$, $a = -b$, $b \approx 2.08064$. (Figure designed by Brian Treadway.)

is attracting. The Julia set $X = X(p)$ is the complement of the union of the $\Omega(\zeta)$'s over all the fixed points; and it follows that $X(p)$ is compact. For spectral theory, it is enough to restrict to the case when $X(p)$ is contained in $\mathbb{R}$, and it is known that the convex hull of $X(p)$ is of the form $[a, b]$ where $a \in X(p)$, and b is an unstable fixed point. The equations $p(y) = a$, or $p(y) = b$, have exactly N solutions in $[a, b]$ (see Figures 3 and 4), and so p must have $N - 1$ critical points in

$[a,b]$. So if $x \in [a,b]$, the equation $p(y) = x$ has N solutions $y_i \in [a,b]$, $y_i = \sigma_i(x)$, $i = 1, \dots, N$, $a \le \sigma_N(x) < \sigma_{N-1}(x) < \cdots < \sigma_1(x) \le b$, and

$$X(p) = \bigcap_{k=1}^{\infty} \bigcup \sigma_{i_1}(\sigma_{i_2}(\cdots(\sigma_{i_k}([a,b]))\cdots)).$$

Reasoning from the definitions, and insisting on $X(p) \subset \mathbb{R}$, we see that necessarily $p(b) = b$ with b an unstable fixed point, while for a there are only two possible cases, $p(a) = a$, or $p(a) = b$. In the first case, a is then also an unstable fixed point. In the real case, only ∞ is attracting. (Of course, the possibilities are more varied in the case of complex Julia sets.)

Defining μ by

$$\mu(\sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_k}([a,b])) = N^{-k} \tag{6.7}$$

and extending to the Borel σ-algebra, we get a measure μ on $X(p)$ which satisfies conditions (6.1)–(6.5); see, e.g., [**Bel92**] and [**Brö65**] for details.

Note that condition (2.19) is not satisfied in this example and that μ is not absolutely continuous.

Let $(X, \mathcal{B}, \mu)$ be as described above, and let $g \colon X \to [0,1]$ be a fixed measurable function satisfying

$$\sum_{i=1}^{N} g(\sigma_i(x)) = 1 \qquad \text{a.a. } x \in X, \tag{6.8}$$

or, equivalently,

$$\sum_{Ty=x} g(y) = 1 \qquad \text{a.a. } x \in X. \tag{6.9}$$

Following Keane [**Kea72**], we define a more general transfer operator as follows:

$$(Rf)(x) = \sum_{Ty=x} g(y) f(y), \tag{6.10}$$

and we note that $R(\mathbb{1}) = \mathbb{1}$ when $\mathbb{1}$ is the constant function 1 on X. It will be assumed that a measure μ is chosen (and that it exists, see [**Kea72**]) satisfying condition (6.3), and we will normalize μ such that $\mu(X) = 1$.

The paper [**JoPe98**] considers the orthonormal basis problem in $L^2(\mu)$ for the measure μ described above in (6.7). The question of when a particular orthogonal family of functions $\mathcal{F}$ in $L^2(\mu)$ is total is shown to reduce to when the eigenspace $\{h \mid R_g(h) = h\}$ is one-dimensional. In that application, $g = g_{\mathcal{F}}$ will depend on the particular function family $\mathcal{F}$ to be tested.

Let $(X, \mathcal{B}, T, \mu)$ be as described above. Following [**Kea72**] we place some additional restrictions on the system:

- (X, d) is a compact metric space;
- T is a local homeomorphism;
- there are $\rho, \delta \in \mathbb{R}_+$ such that $d(Tx, Ty) \ge \rho d(x,y)$ if $d(x,y) \le \delta$;
- for each $\varepsilon \in \mathbb{R}_+$ there is n_ε such that $\bigcup_{n \ge n_\varepsilon} T^{-n}(x_0)$ is ε-dense in X for all $x_0 \in X$.

A measure $\nu = \nu_g$ on X is said to be a g-measure if

$$\int_X (R_g f)(x)\ d\nu_g(x) = \int_X f\,d\nu_g$$

for all $f \in C(X)$. Keane showed that, under the above restrictions, g-measures exist, but they are generally not unique. The examples of g-measures that we shall need here are constructed as follows: A finite set of points $x_1, x_2, \dots, x_k$ in X is called a cycle if $x_{i+1} = Tx_i$, $x_1 = Tx_k$, and if $g(x_i) = 1$, $i = 1, \dots, k$.

LEMMA 6.3. *Let the g system be given as above, and let $x_1, \dots, x_k$ be a cycle. Then $\nu := \frac{1}{k}\sum_{i=1}^{k} \delta_{x_i}$ is a g-measure, where δ_{x_i} denotes the point mass at x_i.*

PROOF. Let $f \in C(X)$, and let ν be as stated. Then

$$\int_X R(f)\ d\nu = \frac{1}{k}\sum_i R(f)(x_i) = \frac{1}{k}\sum_i \sum_{Ty=x_i} g(y)\,f(y)\,.$$

But $T^{-1}(x_i)$ includes x_{i-1}, and g vanishes on all other points in $T^{-1}(x_i)$. Hence

$$\int_X R(f)\ d\nu = \frac{1}{k}\sum_i g(x_{i-1})\,f(x_{i-1}) = \frac{1}{k}\sum_i f(x_{i-1}) = \int_X f\,d\nu. \qquad \square$$

LEMMA 6.4. *Returning to the general setup, let the operator R be as in* (6.10). *Then R leaves invariant all the spaces $L^p(X,\mu)$, $1 \le p \le \infty$, and we have*

$$N\int_X g(x)\,\xi(Tx)\,f(x)\ d\mu(x) = \int_X \xi(x)\,(Rf)(x)\ d\mu(x) \tag{6.11}$$

for all $\xi \in L^\infty(X)$, and $f \in L^1(X,\mu)$.

PROOF. The argument from Chapter 3 shows that $L^p(X,\mu)$ is invariant under R for $p = 1, 2$ and $p = \infty$. The other cases follow from a standard interpolation argument for L^p-norms. The proof of (6.11) is a simple application of (6.10) and (6.5). Specifically, we have:

$$\begin{aligned} N\int_X g(x)\,\xi(Tx)\,f(x)\ d\mu(x) &\underset{(\text{by }(6.5))}{=\!=} \int_X \sum_{Ty=x} g(y)\,\xi(x)\,f(y)\ d\mu(x) \\ &\underset{(\text{by }(6.10))}{=\!=} \int_X \xi(x)\,(Rf)(x)\ d\mu(x)\,, \end{aligned}$$

which is the desired identity (6.11) of the lemma. $\square$

It turns out that almost all the results of Chapters 2–5 carry over, *mutatis mutandis,* to the more general transfer operators of Keane. But now, instead of the C^*-algebra $\mathfrak{A}_N$ from Chapter 2, we need the following one, $\mathfrak{A}(X,T)$: It is generated by $L^\infty(X)$ and a single (abstract) unitary element U subject to the relation

$$UfU^{-1} = f \circ T. \tag{6.12}$$

Note that a main difference between $L^\infty(\mathbb{T})$, which was used in $\mathfrak{A}_N$ from Chapter 2, and the general case, is that $L^\infty(X)$ is generally not singly generated, so $\mathfrak{A}(X,T)$ does not have an equivalent formulation as $UVU^{-1} = V^N$, and similarly, there is not a discrete group formulation which is parallel to the one we used for the $ax+b$ group in Chapter 2.

DEFINITION 6.5. A representation $\tilde{\pi}$ of $\mathfrak{A}(X,T)$ in a Hilbert space $\mathcal{H}$ is a pair (π, U) where π is a representation of $L^{\infty}(X)$ on $\mathcal{H}$, and U a unitary operator $U\colon \mathcal{H} \to \mathcal{H}$, i.e., $U^* = U^{-1}$, such that

$$U\pi(f)U^{-1} = \pi(f \circ T), \qquad f \in L^{\infty}(X). \tag{6.13}$$

Note that $\mathfrak{A}(X,T)$ is the norm-closure of the linear span of elements of the form $U^{-n}fU^k$ where $n, k \in \{0, 1, \dots\}$, and $f \in L^{\infty}(X)$, and we set

$$\tilde{\pi}\left(U^{-n}fU^k\right) = U^{*\,n}\pi(f)U^k$$

with the slight abuse of notation, denoting by U both an abstract element in $\mathfrak{A}(X,T)$ and an operator in $\mathcal{H}$. The representation is *cyclic* if there is a vector $\varphi \in \mathcal{H}$ such that $\{\tilde{\pi}(A)\varphi \mid A \in \mathfrak{A}(X,T)\}$ is norm-dense in $\mathcal{H}$.

LEMMA 6.6. *Let $(X, \mathcal{B}, \mu, g)$ be as described above, and set $m_0 = \sqrt{Ng}$, i.e.,*

$$m_0(x) = \sqrt{N}\,(g(x))^{\frac{1}{2}}, \qquad x \in X. \tag{6.14}$$

Let $\tilde{\pi} = (\pi, U)$ be a normal *representation of $\mathfrak{A}(X,T)$ in $\mathcal{H}$, and let $\varphi \in \mathcal{H}$ satisfy*

$$U\varphi = \pi(m_0)\varphi, \tag{6.15}$$

i.e., $U\varphi = \sqrt{N}\pi\left(g^{\frac{1}{2}}\right)\varphi$; then the spectral density $h \in L^1(X,\mu)$ satisfies

$$R(h) = h. \tag{6.16}$$

(*Recall the spectral density h is the Radon–Nikodym derivative of the measure ν_φ defined by $f \mapsto \langle \varphi \,|\, \pi(f)\varphi \rangle$ with respect to μ, i.e., $h = \frac{d\nu_\varphi}{d\mu}$. The absolute continuity of ν_φ with respect to μ is part of the definition of a normal representation.*)

PROOF. For arbitrary $f \in L^{\infty}(X)$, we have

$$\begin{aligned}
\int_X fh\,d\mu &= \langle \varphi \,|\, \pi(f)\varphi \rangle_{\mathcal{H}} \\
&= \langle U\varphi \,|\, U\pi(f)\varphi \rangle_{\mathcal{H}} \\
&= \langle \pi(m_0)\varphi \,|\, \pi(f \circ T)U\varphi \rangle_{\mathcal{H}} \\
&= \langle \pi(m_0)\varphi \,|\, \pi(f \circ T)\pi(m_0)\varphi \rangle_{\mathcal{H}} \\
&= N\langle \varphi \,|\, \pi(g(f \circ T))\varphi \rangle_{\mathcal{H}} \\
&= N\int_X g(f \circ T)h\,d\mu \\
&= \int_X fR(h)\,d\mu,
\end{aligned}$$

where Lemma 6.4 was used in the last step. Since $L^{\infty}(X)$ separates points in $L^1(X,\mu)$ the desired identity $R(h) = h$ follows from this, and the proof is completed. ☐

Our main theorem below is a converse of this lemma.

THEOREM 6.7. *Let $(X, \mathcal{B}, \mu, g)$ be as described above, i.e., we have a measure space $(X, \mathcal{B}, \mu)$ where μ satisfies* (6.5) *and a function $g\colon X \to [0,1]$ which satisfies the normalization* (6.8) *of Keane for some $N \geq 2$, and let R be the corresponding transfer operator, see* (6.10).

Then there is a one-to-one correspondence between (a) *and* (b) *below:*

(a) *$h \in L^1(X,\mu)$, $h \geq 0$, $R(h) = h$, and*
(b) *positive linear functionals ω_h on $\mathfrak{A}(X,T)$ such that*

$$\omega_h\left(fU^n\right) = N^{\frac{n}{2}} \int_X f\sqrt{g^{(n)}}\, h\, d\mu \tag{6.17i}$$

and

$$\omega_h\left(U^{-1}fU^n\right) = N^{\frac{n-1}{2}} \int_X \sqrt{g^{(n-1)}}\, R(fh)\, d\mu, \qquad n \geq 1. \tag{6.17ii}$$

If g is non-singular and if μ is T-ergodic, then the isometry U induced from the Gelfand–Naimark–Segal (GNS) construction applied to (6.17) is unitary as an operator in the Hilbert space $\mathcal{H}_h$ of the GNS representation. The representation is unique up to unitary equivalence.

PROOF. The structure of the present proof is very close to that of Theorem 2.4 and we refer to Chapter 3 for details. Here we will only sketch some points which are specific to the present more general situation. By the terminology in (6.17), $g^{(n)}$ is

$$g^{(n)}(x) = g(x)\, g(Tx) \cdots g\left(T^{n-1}x\right), \tag{6.18}$$

and we have

$$N^n \int_X g^{(n)}\left(\xi \circ T^n\right) f\, d\mu = \int_X \xi R^n(f)\, d\mu$$

as a simple generalization of Lemma 6.4 ($\xi \in L^\infty(X)$, $f \in L^1(X,\mu)$).

In passing from (a)$\rightarrow$(b),we adopt the inductive limit construction for the Hilbert space $\mathcal{H}_h$ of the representation. Once the representation of $\mathfrak{A}(X,T)$ is identified, it is clear that the linear functional ω_h in (6.17) will be positive; indeed

$$\omega_h(A^*A) = \langle \varphi \mid \tilde{\pi}(A^*A)\varphi \rangle = \|\tilde{\pi}(A)\varphi\|^2 \geq 0, \qquad A \in \mathfrak{A}(X,T).$$

The inductive limit construction from Figures 1–2 in Chapter 3 will be briefly reviewed as it applies to the present case: Let

$$(L^\infty(X), n) = \{(\xi, n) \mid \xi \in L^\infty(X)\} \tag{6.19}$$

for $n = 0, 1, \ldots$, and let

$$\|(\xi,n)\|^2_{\mathcal{H}_h} := \int_X R^n\left(|\xi|^2 h\right) d\mu. \tag{6.20}$$

Then as in (3.9)–(3.12), we have that

$$J\colon\ (\xi, n) \longmapsto (\xi \circ T, n+1) \tag{6.21}$$

is isometric from $(L^\infty(X), n)$ into $(L^\infty(X), n+1)$, and if we define U by

$$U(\xi, 0) = \sqrt{N}\left(g^{\frac{1}{2}}(\xi \circ T), 0\right)$$

and

$$U(\xi, n+1) = \sqrt{N}\left((g \circ T^n)^{\frac{1}{2}}\xi, n\right),$$

then U is isometric in the Hilbert space which results from the inductive limit construction applied to (6.21). The argument from Chapter 3 and Figures 1–2 also shows that U is unitary, if g is non-singular (i.e., does not vanish on a subset $E \subset X$, $\mu(E) > 0$), and if T is ergodic relative to μ. The ergodicity is needed to guarantee that a nonzero solution to (a) must automatically be non-singular if g is; see Chapter 3 and Lemma 5.4 for details.

In any case, even if U is only isometric we have the formula

$$U\pi(f) = \pi(f \circ T)U$$

when the representation π of $L^{\infty}(X)$ is defined by

$$\pi(f)(\xi, n) = ((f \circ T^{n})\xi, n)$$

for $n = 0, 1, \dots$ and $f, \xi \in L^{\infty}(X)$.

Finally, the cyclic vector $\varphi \in \mathcal{H}_h$ (the inductive limit Hilbert space) will be given by $\varphi = (\mathbb{1}, 0) \in (L^{\infty}(X), 0)$, and by (6.21),

$$\varphi = (\mathbb{1}, 0) \sim (\mathbb{1}, 1) \sim (\mathbb{1}, 2) \sim \cdots,$$

so

$$U\varphi = \sqrt{N}\pi\left(g^{\frac{1}{2}}\right)\varphi$$

and

$$\langle \varphi \mid \pi(f)\varphi \rangle = \int_X fh\,d\mu,$$

which concludes the construction of the representation from h as in (a), i.e., $R(h) = h$.

It remains to prove that if a representation $\tilde{\pi} = (\pi, U)$ of $\mathfrak{A}(X, T)$ results from a positive functional ω_h in (6.17), then

$$U\varphi = \pi(m_0)\varphi, \tag{6.22}$$

where $m_0 := \sqrt{N}g^{\frac{1}{2}}$ and φ is the cyclic vector of this GNS construction. Introducing

$$\begin{aligned} m_0^{(n)}(x) &= m_0(x)\,m_0(Tx)\cdots m_0\left(T^{n-1}x\right) \\ &= N^{\frac{n}{2}}\left(g(x)\,g(Tx)\cdots g\left(T^{n-1}x\right)\right)^{\frac{1}{2}} \\ &= N^{\frac{n}{2}}\left(g^{(n)}(x)\right)^{\frac{1}{2}}, \end{aligned}$$

we claim that

$$\langle \tilde{\pi}(A)\varphi \mid U\varphi \rangle_{\mathcal{H}_h} = \langle \tilde{\pi}(A)\varphi \mid \pi(m_0)\varphi \rangle_{\mathcal{H}_h} \tag{6.23}$$

for all $A \in \mathfrak{A}(X, T)$ if (6.17) is given. The result follows from this and cyclicity, i.e., (6.22) must hold. In checking (6.23), it is enough to consider $A = fU^{n}$, $n = 0, 1, \dots$, $f \in L^{\infty}(X)$, and then

$$\langle \tilde{\pi}(fU^{n})\varphi \mid U\varphi \rangle = \int_X \overline{m_0^{(n-1)}}\, R\left(\bar{f}h\right)\,d\mu, \qquad \text{by (6.17ii)},$$

while

$$\begin{aligned}
\left\langle \tilde{\pi}\left(fU^{n}\right)\varphi \,\middle|\, \pi\left(m_{0}\right)\varphi\right\rangle &= \int_{X} \bar{f}\, \overline{m_{0}^{(n)}}\, m_{0} h \, d\mu \\
&= \int_{X} \left|m_{0}\right|^{2} \overline{m_{0}^{(n-1)} \circ T}\, \bar{f} h \, d\mu \\
&= \int_{X} \left|m_{0}\left(x\right)\right|^{2} \bar{m}_{0}\left(Tx\right) \cdots \bar{m}_{0}\left(T^{n-1}x\right) \bar{f}\left(x\right) h\left(x\right)\, d\mu\left(x\right) \\
&= \int_{X} R^{*}\left(\overline{m_{0}^{(n-1)}}\right) \bar{f} h \, d\mu \\
&= \int_{X} \overline{m_{0}^{(n-1)}}\, R\left(\bar{f} h\right)\, d\mu, \qquad \text{by (6.17i)},
\end{aligned}$$

which proves that the two sides in (6.23) are identical, and so the desired (6.22) must hold. The proof is completed. □

CHAPTER 7

A representation theorem for R-harmonic functions

While the result in this chapter may be formulated for the representations which correspond to the general N-to-1 transformations $T\colon X \to X$ of the previous chapter, we shall restrict attention here (for simplicity) to the case from Chapters 2–3 above, i.e., $X = \mathbb{T}$, and $T\colon z \mapsto z^N$ when $N \geq 2$ is fixed. As in the previous chapters we consider a fixed wavelet filter m_0 of order N and the corresponding Ruelle operator $R = R_{m_0}$. From Theorem 2.4, we know that each solution, $h \in L^1(\mathbb{T})$, $h \geq 0$, $Rh = h$, defines a representation (π, U) on a Hilbert space $\mathcal{H}$ with cyclic vector φ such that $U\varphi = \pi(m_0)\varphi$. We show in this chapter that this $\mathcal{H}$ may be taken to be the L^2-space $L^2(K_N, \nu)$, where $K_N = (\Lambda_N)\widehat{\ }$ (the Pontryagin compact dual of $\Lambda_N = \mathbb{Z}\left[\frac{1}{N}\right]$), and where $\nu = \nu(m_0, h)$ is a measure on K_N, depending on (m_0, h), i.e., $\mathcal{H} \simeq L^2(K_N, \nu)$, in such a way that φ is the constant function in $L^2(K_N, \nu)$. The construction of the measure $\nu(m_0, h)$ uses an inductive limit procedure for subalgebras of $\mathfrak{A}_N$ which is somewhat analogous to (but different from) one used recently in [**Bre96**], [**Lac98**], [**Mur95**], and [**Sta93**].

In the proof of Theorem 4.8, we saw that $\mathfrak{A}_N$ contains an abelian subalgebra which is generated by elements of the form

$$U^{-i} f U^{i}, \qquad f \in C(\mathbb{T}), \tag{7.1}$$

$i \in \{0, 1, 2, \dots\}$. Recall that $\mathfrak{A}_N$ is defined from the relation

$$U f U^{-1} = f\left(z^N\right) \tag{7.2}$$

on the generators $C(\mathbb{T})$ and U. Hence

$$U^{-i} f U^{i} = U^{-(i+1)} f\left(z^N\right) U^{i+1}. \tag{7.3}$$

Let $\mathcal{A}_N \subset \mathfrak{A}_N$ be the (abelian) subalgebra which is generated by the elements in (7.1), and let K_N be the compact *Gelfand space* of $\mathcal{A}_N$, i.e., $\mathcal{A}_N \simeq C(K_N)$. In Chapter 2, we introduced $\Lambda_N = \mathbb{Z}\left[\frac{1}{N}\right]$, and we consider Λ_N as a discrete abelian group. The corresponding dual compact group $\widehat{\Lambda_N}$ (Pontryagin dual) consists of all characters on Λ_N, i.e., all one-dimensional representations

$$\chi\colon \Lambda_N \longrightarrow \mathbb{T} \qquad \text{such that } \begin{cases} \chi(\lambda + \lambda') = \chi(\lambda)\chi(\lambda'), \\ \chi(-\lambda) = \overline{\chi(\lambda)}, \end{cases} \qquad \lambda, \lambda' \in \Lambda_N, \tag{7.4}$$

with group operation

$$(\chi\chi')(\lambda) = \chi(\lambda)\chi'(\lambda).$$

THEOREM 7.1.

(i) *The Gelfand space K_N of $\mathcal{A}_N$ is the compact group $\widehat{\Lambda_N}$.*

(ii) *If m_0 is a non-singular wavelet filter, and $h \in L^1(\mathbb{T})$, $h \geq 0$, solves $R_{m_0}(h) = h$, then there is a unique measure $\nu = \nu(m_0, h)$ on $K_N \simeq \widehat{\Lambda_N}$ such that*

$$\int_{K_N} U^{-i} f U^i \, d\nu = \int_{\mathbb{T}} R^i(fh) \, d\mu, \tag{7.5}$$

with μ denoting the Haar measure on $\mathbb{T}$.

(iii) *Let $e\left(\frac{n}{N^k}\right)$ be identified with the $(2\pi) N^k$-periodic function $x \mapsto \exp\left(i \frac{nx}{N^k}\right)$ on $\mathbb{R}$; then the vectors*

$$\left\{ e\left(\frac{n}{N^k}\right) \;\middle|\; n \in \mathbb{Z},\, k \in \{0, 1, 2, \dots\} \right\} \tag{7.6}$$

span $L^2(K_N, \nu)$, and the representation (π, U) of $\mathfrak{A}_N$ which corresponds to (m_0, h) is given by:

$$\pi(e_n) e(\lambda) = e(n + \lambda), \qquad \lambda \in \Lambda_N, \tag{7.7}$$

$$U e\left(\frac{n}{N^{k+1}}\right) = \sum_{j \in \mathbb{Z}} a_j e\left(j + \frac{n}{N^k}\right), \tag{7.8}$$

where

$$m_0(z) = \sum_{j \in \mathbb{Z}} a_j e_j(z) = \sum_{j \in \mathbb{Z}} a_j z^j$$

and

$$U e_0 = m_0 = \sum_{n \in \mathbb{Z}} a_n e_n, \tag{7.9}$$

where $e_0 = \mathbb{1}$.

Proof. *Ad* (i): We introduced the function $e\left(\frac{n}{N^k}\right)$ in (7.6) as a function on $\mathbb{R}$ of period $(2\pi) N^k$, and by virtue of (7.2) it is identified with $U^{-k} e_n U^k$. This corresponds to the case $f = e_n$ in (7.1). From the theory of almost periodic functions (see, e.g., [**Rud90**] or [**Bes55**]), functions in $C(K_N)$ or $L^2(K_N)$ may be identified with the corresponding functions on $\mathbb{R}$ spanned by the frequencies from $\mathbb{Z}\left[\frac{1}{N}\right]$. For $C(K_N)$, the completion is in the sup-norm, and for $L^2(K_N)$ the completion is in the norm which is defined as the limit, $T \to \infty$,

$$\frac{1}{2T} \int_{-T}^{T} |f(x)|^2 \, dx. \tag{7.10}$$

The dual of K_N will be given by $\left\{ \frac{n}{N^k} \mid n \in \mathbb{Z},\, k \in \{0, 1, 2, \dots\} \right\}$ subject to the following equivalence relation:

$$\frac{n}{N^k} \sim \frac{l}{N^i} \qquad \text{if and only if } N^i n = N^k l. \tag{7.11}$$

In defining operators on $e\left(\frac{n}{N^k}\right)$ we must then check consistency with respect to the equivalence relation (7.11). The conclusion in (i) follows if we check that $U^{-k} e_n U^k = U^{-i} e_l U^i$ when (7.11) holds. The proof of this identity is based on (7.1), (7.3), and an induction. The first step is consideration of n, k, l such that $n = Nl$. But then $U^{-1} e_n U = U^{-1} e_{Nl} U = U^{-1} e_l\left(z^N\right) U = U^{-1} U e_l U^{-1} U = e_l$, where we used (7.2) in the second-to-last step. The induction is left to the reader.

Ad (ii): We first check that the right-hand side in formula (7.5) defines a linear functional L on $\mathcal{A}_N$, i.e.,

$$L\left(U^{-i}fU^{i}\right) := \int_{\mathbb{T}} R^{i}\left(fh\right)\,d\mu, \tag{7.12}$$

$i \in \{0,1,2,\dots\}$, $f \in C\left(\mathbb{T}\right)$. Hence if $U^{-i}fU^{i}$ is represented in two ways, as for example in (7.3), we check that L takes the same value either way: that amounts to checking that

$$\int_{\mathbb{T}} R^{i+1}\left(f\left(z^{N}\right)h\right)\,d\mu = \int_{\mathbb{T}} R^{i}\left(fh\right)\,d\mu. \tag{7.13}$$

The other cases will then follow from this and induction. The proof of (7.13) is based on the eigenvalue property, $Rh = h$. In fact,

$$\begin{aligned}\int_{\mathbb{T}} R^{i+1}\left(f\left(z^{N}\right)h\right)\,d\mu &= \int_{\mathbb{T}} R^{i}\left(R\left(f\left(z^{N}\right)h\right)\right)\,d\mu \\ &= \int_{\mathbb{T}} R^{i}\left(fRh\right)\,d\mu \\ &= \int_{\mathbb{T}} R^{i}\left(fh\right)\,d\mu,\end{aligned}$$

which is (7.13). Note that $Rh = h$ was used in the last step.

We now specialize to $f = e_n$, and define

$$L\left(\frac{n}{N^{k}}\right) := \int_{\mathbb{T}} R^{k}\left(e_{n}h\right)\,d\mu. \tag{7.14}$$

It follows that L may be viewed as a function on $\Lambda_N = \mathbb{Z}\left[\frac{1}{N}\right]$, and we must check that it is positive definite. We will then get the desired measure $\nu = \nu\left(m_0, h\right)$ as a solution to the corresponding *moment problem.* We claim that, for all finite sequences $\lambda \to A\left(\lambda\right)$ (i.e., at most a finite number of nonzero scalar terms), we have

$$\sum_{\lambda\in\Lambda_N}\sum_{\lambda'\in\Lambda_N} \overline{A\left(\lambda\right)}\,L\left(\lambda'-\lambda\right)A\left(\lambda'\right) \geq 0. \tag{7.15}$$

Let $\lambda = l + \frac{n}{N^k}$, $\lambda' = l' + \frac{n'}{N^k}$, $l, l', n, n' \in \mathbb{Z}$, $k \in \{0,1,2,\dots\}$. (The general case may be reduced to this by (7.11).) Then

$$L\left(\lambda'-\lambda\right) = \int_{\mathbb{T}} R^{k}\left(e\left(N^{k}\left(l'-l\right)+n'-n\right)h\right)\,d\mu,$$

so

$$\begin{aligned}&\sum_{l,n}\sum_{l',n'} \overline{A\left(l,n\right)}\,A\left(l',n'\right)L\left(\frac{N^{k}\left(l'-l\right)+n'-n}{N^{k}}\right) \\ &\quad= \sum_{l,n}\sum_{l',n'} \overline{A\left(l,n\right)}\,A\left(l',n'\right)\int_{\mathbb{T}} R^{k}\left(e\left(N^{k}\left(l'-l\right)+n'-n\right)h\right)\,d\mu \\ &\quad= \int_{\mathbb{T}} R^{k}\left(\left|\sum_{l,n} A\left(l,n\right)e\left(N^{k}l+n\right)\right|^{2} h\right)\,d\mu,\end{aligned}$$

and this last term is ≥ 0 since R^k takes nonnegative functions to nonnegative functions, i.e., R^k is positivity-preserving.

It then follows from a theorem of Akhiezer, [**Akh65**], [**Nel59**], and Kolmogorov, [**Kol77**] or [**CFS82**], that there is a unique measure $\nu = \nu(m_0, h)$ on K_N such that

$$\int_{K_N} e\left(\frac{n}{N^k}\right)\, d\nu = L\left(\frac{n}{N^k}\right),$$

and therefore

$$\int_{K_N} e_\lambda(x)\, d\nu(x) = L(\lambda) \tag{7.16}$$

for all $\lambda \in \Lambda_N$ $\left(= \mathbb{Z}\left[\frac{1}{N}\right]\right)$. Details of the construction will be given below.

Ad (iii): Let $\nu = \nu(m_0, h)$ be the measure from (ii). The proof that the operator U on $L^2(K_N, \nu)$, and the representation π of $L^\infty(\mathbb{T})$ on $L^2(K_N, \nu)$, are given by formulas (7.7)–(7.9) follows from the corresponding assertion in the proof of Theorem 2.4; see Chapter 3 above. The correspondence which makes the connection to Chapter 3 is the identification

$$U^{-k} e_n U^k \sim e\left(\frac{n}{N^k}\right) \tag{7.17}$$

from (i), and the basis property of $\{e_\lambda \mid \lambda \in \Lambda_N\}$ in $L^2(K_N, \nu)$.

To show that the Kolmogorov construction applies, and to prove the uniqueness part of the theorem, we must identify a projective system of measures, as described, for example, in [**Par77**, Proposition 27.8, page 124]. Since, for each k, we have the exponentials $\left\{e\left(\frac{n}{N^k}\right) \mid n \in \mathbb{Z}\right\}$ span the functions on $\mathbb{R}$ with period $(2\pi)N^k$, we will work with this scale of periodic functions, $k = 0, 1, 2, \dots$. Functions with period $(2\pi)N^k$ will be identified with functions on $\mathbf{X}_k = \mathbb{R}\diagup(2\pi)N^k\mathbb{Z} \simeq \left[-\pi N^k, \pi N^k\right\rangle$ with the case $k = 0$, $\mathbf{X}_0 \simeq \mathbb{T}$. Restricting to the continuous case, we note that every $f \in C(\mathbf{X}_k)$ has the representation $f\left(z^{N^k}\right) = F(z)$ for $F \in C(\mathbb{T})$. The natural maps $\varphi_{k,k+l}$ defined by

$$\varphi_{k,k+l} : \mathbf{X}_{k+l} \ni z \longmapsto z^{N^l} \in \mathbf{X}_k$$

then yield a commutative diagram of maps

$$\begin{tikzcd}
\mathbf{X}_{k+l+m} \arrow[rr, "\varphi_{k,k+l+m}"] \arrow[dr, "\varphi_{k+m,k+l+m}"'] & & \mathbf{X}_k . \\
& \mathbf{X}_{k+m} \arrow[ur, "\varphi_{k,k+m}"'] &
\end{tikzcd}$$

Recalling the moment sequence in (7.14), i.e., $L\left(\frac{n}{N^k}\right) = \int_{\mathbb{T}} R^k(e_n h)\, d\mu$, we see that, for each k, a measure ν_k on $\mathbf{X}_k$ is determined uniquely from the trigonometric moment problem (see [**Akh65**]). The measure ν_k is *unique,* as it is known that the trigonometric moment problem is *determined,* i.e., the measure is determined uniquely from its moments. The consistency condition which is required in the Kolmogorov construction may be stated in the form, as an identity for $f \in C(\mathbf{X}_k)$:

$$\int_{\mathbf{X}_{k+l}} f\left(z^{N^l}\right)\, d\nu_{k+l} = \int_{\mathbf{X}_k} f\, d\nu_k.$$

Recalling that each of the measures ν_k derives from a moment problem, this identity is equivalent to an identity on $F \in C(\mathbb{T})$, *viz.:*

$$\int_{\mathbb{T}} R^{k+l}\left(F\left(z^{N^l}\right)h(\,\cdot\,)\right)(z)\,d\mu(z) = \int_{\mathbb{T}} R^k(Fh)(z)\,d\mu(z),$$

where μ is the Haar measure on $\mathbb{T}$. But this is the identity which we derived above (by induction, starting with $l = 1$ in (7.13).) Hence the Kolmogorov construction determines a measure ν uniquely. It is a measure on the projective limit of the systems $(\mathbf{X}_k, \nu_k)$. But the compact group K_N was identified earlier with this *projective* limit, $\varprojlim \{\mathbf{X}_k\}$. Of course, $C(K_N)$ will then be the *injective* limit of the system $\{C(\mathbf{X}_k)\}$. To see that the measure ν on K_N is unique, it only remains to invoke the uniqueness part of the Kolmogorov construction for systems of probability measures; see [**Par77**, Prop. 27.8] for the details on that point.

Let $\pi = \pi_\nu$ be the representation of $\mathfrak{A}_N$ on $L^2(K_N, \nu)$ which is induced by the measure $\nu = \nu(m_0, h)$ which we just constructed. Recall m_0 and h are given. If $R = R_{m_0}$ is the Ruelle operator, we have $R_{m_0}(h) = h$ at the outset. From the results in Chapter 5, this means that

$$(S_0 f)(z) := m_0(z)\, f\left(z^N\right)$$

defines an *isometry* in $L^2(h)$ $(= L^2(\mathbb{T}, h\,d\mu)$ where μ is the Haar measure on $\mathbb{T})$. We will show that the unitary operator $\pi(U)$ on $L^2(K_N, \nu)$ arises as an extension of S_0 when $L^2(h)$ is identified (isometrically) with an invariant subspace in $L^2(K_N, \nu)$. Since clearly $S_0 \mathbb{1} = m_0$, it will follow immediately that $\pi(U)\mathbb{1} = m_0$ $(\in L^2(h) \subset L^2(K_N, \nu))$. $\square$

LEMMA 7.2. *Let m_0 and h be as described in the statement of Theorem* 7.1*, and assume that m_0 is non-singular. Let $\nu = \nu(m_0, h)$ be the corresponding measure on $K_N = \left(\mathbb{Z}\left[\frac{1}{N}\right]\right)^{\widehat{}}$. Then $L^2(h) = L^2(\mathbb{T}, h\,d\mu)$ embeds isometrically in $L^2(K_N, \nu)$ and the unitary operator $\pi_\nu(U)$ is an extension* (*or power dilation*) *to $L^2(K_N, \nu)$ of the isometry S_0 in $L^2(h)$.*

PROOF. Since $\mathbb{Z} \hookrightarrow \mathbb{Z}\left[\frac{1}{N}\right]$ by the natural inclusion, we have $K_N \hookrightarrow \mathbb{T} = (\mathbb{Z})^{\widehat{}}$ with the embedding from Pontryagin duality; see [**Rud90**]. Restricting functions on $\mathbb{T}$ to K_N, we then get the identification of $L^\infty(\mathbb{T})$ with a subspace of $L^\infty(K_N)$. Since, for $f \in L^\infty(\mathbb{T})$, we have

$$\|f\|^2_{L^2(h)} = \int_{\mathbb{T}} |f|^2\, h\,d\mu = \|f\|^2_{L^2(K_N,\nu)},$$

it follows that $L^2(h)$ embeds isometrically into $L^2(K_N, \nu)$ as claimed. Since $R_{m_0}(h) = h$, S_0 will be isometric on $L^2(h)$, and therefore identify with a partial isometry in $L^2(K_N, \nu)$.

Recall that the elements $\left\{U^{-i} f U^i \mid f \in L^\infty(\mathbb{T}),\ i = 0, 1, 2, \dots\right\}$ in $\mathfrak{A}_N$ generate the abelian algebra $\mathcal{A}_N$, and we showed that K_N is the Gelfand space of $\mathcal{A}_N$. Hence we may identify $\mathcal{A}_N$ also with a linear subspace in $L^2(K_N, \nu)$, and on this subspace we set

$$\pi_\nu(U)\left(U^{-(i+1)} f U^{i+1}\right) := \left(U^{-i} f U^i\right) m_0 = U^{-i}\left(f(z)\, m_0\left(z^{N^i}\right)\right) U^i.$$

To see that $\pi_\nu(U)$ is well defined and isometric (in $L^2(K_N, \nu)$) we must check that

$$\int_{K_N} \left|U^{-(i+1)} f U^{i+1}\right|^2 d\nu = \int_{K_N} \left|U^{-i} f U^i m_0\right|^2 d\nu,$$

which is equivalent to

$$\int_{\mathbb{T}} R^{i+1}\left(|f|^2 h\right) d\mu = \int_{\mathbb{T}} |m_0|^2 R^i \left(|f|^2 h\right) d\mu.$$

Since we already checked this identity, the result follows. It follows from the formula for $\pi_\nu(U)$ that it maps *onto* $L^2(K_N, \nu)$ if m_0 is given to be non-singular, i.e., does not vanish on a subset of positive measure in $\mathbb{T}$. Hence $\pi_\nu(U)$ is a unitary extension (or *power dilation*) as claimed in the lemma. □

REMARK 7.3. Alternatively, $L^2(K_N)$ may be defined relative to the Haar measure μ_N on K_N. This Haar measure in turn is determined uniquely by the *ansatz* $(\lambda \in \Lambda_N)$:

$$\int_{K_N} e_\lambda \, d\mu_N = \delta_\lambda = \begin{cases} 0 & \text{if } \lambda \neq 0, \\ 1 & \text{if } \lambda = 0. \end{cases} \tag{7.18}$$

The fact that (7.18) determines a unique measure on K_N follows from the same argument which we used in (ii) above. This measure μ_N will be translation-invariant on K_N by the following calculation. (Hence it must be the Haar measure by the uniqueness theorem!) We have for $\chi_0 \in K_N = (\Lambda_N)\widehat{\ }$:

$$\int_{K_N} (\chi_0 \chi)(\lambda) \, d\mu_N(\chi) = \chi_0(\lambda) \int_{K_N} e_\lambda \, d\mu_N = \chi_0(\lambda)\, \delta_\lambda = \int_{K_N} e_\lambda \, d\mu_N.$$

CHAPTER 8

Signed solutions to $R(f) = f$

In Example 4.5 we considered a wavelet filter m_0 of order 2 and a positive solution g to $R_{m_0}(g) = g$ such that g had the following order-3 symmetry:

$$\sum_{k=0}^{2} g\left(e^{i\frac{2\pi k}{3}} z\right) = 1. \tag{8.1}$$

This means that g satisfies a special case of the condition from Chapter 6 above, corresponding to $N = 3$. Specifically, let

$$\mathbb{T} \ni z \underset{T_3}{\longmapsto} z^3 \in \mathbb{T};$$

then

$$\sum_{T_3 w = z} g(w) = 1, \qquad z \in \mathbb{T}.$$

In this chapter, we will study a more general scaling duality which relates scaling of order N to that of order p, where N and p are positive integers which are given and mutually prime, $(N, p) = 1$, i.e., no common divisors other than 1. We will then have a pair of Ruelle operators and a specific duality between the eigenvalue problems for the respective operators. It is also a concrete instance of a case when the dimension

$$\dim \left\{ f \in L^1(\mathbb{T}) \mid R(f) = f \right\} \tag{8.2}$$

can be calculated; in this case, it is shown to be equal to the number of orbits for a certain finite dihedral group action.

The setting of this duality will be two given wavelet filters m_0 and m_p related as follows: It will be assumed that $(N, p) = 1$,

$$\sum_{w^N = z} |m_0(w)|^2 = N, \tag{8.3}$$

and

$$m_p(z) = m_0(z^p), \qquad z \in \mathbb{T}. \tag{8.4}$$

Since $(N, p) = 1$, it follows that m_p will also satisfy the scale-N condition (8.3). (The simplest case of this is the one in Example 4.3, when $N = 2$, $p = 3$, $m_0(z) = \frac{1}{\sqrt{2}}(1 + z)$, $m_3(z) = \frac{1}{\sqrt{2}}\left(1 + z^3\right)$.) We shall need both Ruelle operators R_0 and R_p constructed from m_0 and m_p when the two filters are related through (8.4), i.e.,

$$R_0 f(z) = \frac{1}{N} \sum_{w^N = z} |m_0(w)|^2 f(w)$$

and

$$R_p f(z) = \frac{1}{N} \sum_{w^N = z} |m_p(w)|^2 f(w).$$

LEMMA 8.1. *Let N, p be positive integers ≥ 2 such that $(N, p) = 1$, and let m_0, m_p be given wavelet filters of order N and related* via (8.4). *Let R_0 and R_p be the respective Ruelle operators. Let $\alpha_N \in \operatorname{Aut}(\mathbb{Z}_p)$ be the automorphism $i \mapsto Ni$, passed to $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, and let $\rho_p := e^{i\frac{2\pi}{p}}$.*

(i) *Let $f \in L^1(\mathbb{T})$ satisfy $R_p f = f$, and set*

$$F_0(z) = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\rho_p^j z\right).$$

Then F_0 is of the form $F_0(z) = H_0(z^p)$, $H_0 \in L^1(\mathbb{T})$, and $R_0(H_0) = H_0$.

(ii) *Let $\langle j, k \rangle := e^{i\frac{2\pi jk}{p}}$, and*

$$F_k(z) = \frac{1}{p} \sum_{j=0}^{p-1} \overline{\langle j, k \rangle} f\left(\rho_p^j z\right). \tag{8.5}$$

Then F_k is of the form

$$F_k(z) = z^k H_k(z^p), \qquad z \in \mathbb{T},\ H_k \in L^1(\mathbb{T}), \tag{8.6}$$

with

$$R_0(H_k) = H_{\alpha_N^{-1}(k)}, \qquad k \in \mathbb{Z}_p. \tag{8.7}$$

PROOF. Let $f_j(z) := f\left(\rho_p^j z\right)$. Then

$$R_p(f_j) = f_{\alpha_N(j)}. \tag{8.8}$$

Indeed

$$\begin{aligned} R_p(f_j)(z) &= \frac{1}{N} \sum_{w^N = z} |m_0(w^p)|^2 f\left(\rho_p^j w\right) \\ &= \frac{1}{N} \sum_{w^N = \rho_p^{Nj} z} |m_0(w^p)|^2 f(w) \\ &= R_p(f)\left(\rho_p^{Nj} z\right) \\ &= f\left(\rho_p^{Nj} z\right) = f_{\alpha_N(j)}(z), \end{aligned}$$

which is the assertion (8.8). Since

$$\sum_{j=0}^{p-1} \left(z \rho_p^j\right)^n = \begin{cases} p z^n & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n, \end{cases}$$

there is an $H_0 \in L^1(\mathbb{T})$ such that $F_0(z) = H_0(z^p)$, and

$$\begin{aligned}
H_0(z^p) = F_0(z) &= \sum_{j=0}^{p-1} f_j(z) \\
&= \sum_{j=0}^{p-1} f_{\alpha_N(j)}(z) = \sum_{j=0}^{p-1} R_p(f_j)(z) = (R_p F_0)(z) \\
&= \frac{1}{N} \sum_{w^N = z} |m_p(w)|^2 F_0(w) \\
&= \frac{1}{N} \sum_{w^N = z} |m_0(w^p)|^2 H_0(w^p) \\
&= \frac{1}{N} \sum_{w^N = z^p} |m_0(w)|^2 H_0(w) \\
&= R_0(H_0)(z^p),
\end{aligned}$$

which yields the desired identity $R_0(H_0) = H_0$ in (i).

The proof of (ii) is quite similar and will only be sketched. We have

$$\frac{1}{p} \sum_{j=0}^{p-1} \rho_p^{-jk} \left(\rho_p^j z\right)^n = \begin{cases} z^n & \text{if } n \equiv k \mod p, \\ 0 & \text{if } n \not\equiv k \mod p. \end{cases}$$

Hence

$$F_k(z) = \sum_l c_l z^{k+lp} = z^k H_k(z^p)$$

for some $H_k \in L^1(\mathbb{T})$. The argument from above yields $F_k = R_p\left(F_{\alpha_N(k)}\right)$, and therefore

$$\begin{aligned}
H_k(z^p) = z^{-k} F_k(z) &= z^{-k} R_p\left(F_{\alpha_N(k)}\right)(z) \\
&= z^{-k} \frac{1}{N} \sum_{w^N = z} |m_0(w^p)|^2 F_{\alpha_N(k)}(w) \\
&= \frac{1}{N} \sum_{w^N = z^p} |m_0(w)|^2 H_{\alpha_N(k)}(w) \\
&= R_0\left(H_{\alpha_N(k)}\right)(z^p),
\end{aligned}$$

which is the desired identity $R_0\left(H_{\alpha_N(k)}\right) = H_k$, or equivalently $R_0(H_k) = H_{\alpha_N^{-1}(k)}$, $k \in \mathbb{Z}_p$. □

Consider the action τ_p on $L^1(\mathbb{T})$ given by

$$(\tau_p f)(z) = f(\rho_p z), \qquad z \in \mathbb{T},$$

and

$$\tau_p^j f(z) = f\left(\rho_p^j z\right), \qquad j \in \mathbb{Z}_p.$$

If $V \subset L^1(\mathbb{T})$ is a given subspace, we set

$$V^{\tau_p} = \{f \in V \mid \tau_p f = f\}, \tag{8.9}$$

and

$$V(z^p) = \left\{ f \in L^1(\mathbb{T}) \mid \exists h \in V \text{ s.t. } f(z) = h(z^p) \right\}. \tag{8.10}$$

Clearly then

$$V^{\tau_p} = \left\{ f \in V \;\middle|\; \exists F \in L^1(\mathbb{T}) \text{ s.t. } f = \frac{1}{p} \sum_{j=0}^{p-1} \tau_p^j(F) \right\}.$$

Return to the setting of the lemma, i.e., two given wavelet filters m_0, m_p related *via* (8.4) and corresponding Ruelle operators R_0 and R_p.

We have the following reciprocity for the eigenspaces.

COROLLARY 8.2. *Let* $m_p(z) = m_0(z^p)$, *and let*

$$V_0 = \left\{ f \in L^1(\mathbb{T}) \mid R_0 f = f \right\}$$

and

$$V_p = \left\{ f \in L^1(\mathbb{T}) \mid R_p f = f \right\}.$$

Then

$$V_p^{\tau_p} = V_0(z^p). \tag{8.11}$$

PROOF. The inclusion $\subset$ already follows from Lemma 8.1(i). To prove $\supset$, let $f(z) = h(z^p)$ where $h \in L^1(\mathbb{T})$ and $R_0 h = h$. Then

$$\begin{aligned}
(R_p f)(z) &= \frac{1}{N} \sum_{w^N = z} |m_p(w)|^2 f(w) \\
&= \frac{1}{N} \sum_{w^N = z} |m_0(w^p)|^2 h(w^p) \\
&= \frac{1}{N} \sum_{w^N = z^p} |m_0(w)|^2 h(w) \\
&= R_0(h)(z^p) = h(z^p) \\
&= f(z),
\end{aligned}$$

proving $f \in V_p$. Since $\tau f = f$ is clear, the result follows. □

In many examples, scaling functions constructed from given wavelet filters may not have orthogonal translates $\{\varphi(\cdot - k) \mid k \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$. It is known **[Hör95]** that if φ is constructed from m_0, then the orthogonality holds if and only if the eigenspace $\{f \mid R_{m_0} f = f\}$ is one-dimensional. In this case, we say that m_0 is *pure.* Let N and p be given, $(N, p) = 1$, and let $m_p(z) := m_0(z^p)$ with m_0 pure. Then it follows from Corollary 8.2 that $V_p^{\tau_p}$ $(= \{f \in V_p \mid \tau_p f = f\})$ must be one-dimensional. We saw in Lemma 8.1 that every $f \in V_p$ decomposes uniquely as

$$f = \sum_{j \in \mathbb{Z}_p} f_j, \qquad \text{where } \tau_p f_j = \rho_p^j f_j,\ j \in \mathbb{Z}_p.$$

The explicit form of this decomposition is spelled out in Lemma 8.1(ii). Specifically, $f_j(z) = z^j h_j(z^p)$ with $R_0(h_j) = h_{\alpha_N^{-1}(j)}$, $j \in \mathbb{Z}_p$. When m_0 is pure, it follows that a basis for V_p may be labelled by the finite orbits $j \mapsto Nj \mapsto N^2 j \mapsto \cdots \to N^k j \to j$ in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. Specifically, let $\mathcal{O}_k$ be such an orbit and let $f \in V_p$; then $f_{\mathcal{O}_k}(z) =$

$\sum_{s\in\mathcal{O}_k} f\left(\rho_p^s z\right)$ is in V_p, and the finite Fourier analysis on $\mathbb{Z}_p$ of the lemma shows that the orbits parametrize a basis for V_p when m_0 is pure and $(N, p) = 1$.

These same orbits were found in [**BrJo96**] to parametrize a class of permutation representations of the Cuntz algebras. Specifically, [**BrJo96**, Proposition 8.2 and Remark 8.3] lists these orbits for $N = 2$ and $p \in \mathbb{N}$ odd, for selected values of p. While there is a pattern to this orbit counting, we do not have a general formula valid for all odd values of p. This is just the case $N = 2$, and more general pairs N, p such that $(N, p) = 1$ may be more difficult. For $N = 2$, it can be seen that the period of an orbit starting at $j \in \mathbb{Z}_p$ is the order of 2 modulo $p \diagup \gcd(j, p)$.

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