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Note on the Paper "The Norm Convergence of the Trotter–Kato Product Formula with Error Bound" by Ichinose and Tamura

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Abstract

The norm convergence of the Trotter–Kato product formula is established with ultimate *optimal* error bound for the self-adjoint semigroup generated by the operator sum of two self-adjoint operators. A generalization is also given to the operator sum of several self-adjoint operators.

Keywords : Trotter–Kato product formula, self-adjoint semigroups, error bound.

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1 Introduction

The present note is an addendum to the recent paper [1] by the first two authors. The aim is to prove the norm convergence of the Trotter–Kato product formula for the self-adjoint semigroup with *ultimate* error bound.
To refer to some items in that paper we shall write, for instance, Lemma 2.1 in [1] as Lemma I.2.1, Eq. (3.2) in [1] as (I.3.2), Ref. [4] in [1] as [I 4], and so on.

To formulate our new theorems, again consider real-valued, Borel measurable functions f on $[0, \infty)$ satisfying

$$0 \leq f(s) \leq 1, \quad f(0) = 1, \quad f'(0) = -1. \tag{1.1}$$

Some examples of functions satisfying (1.1) are

$$f(s) = e^{-s}, \quad f(s) = (1 + k^{-1}s)^{-k}, \quad k > 0. \tag{1.2}$$

We are interested in those functions f which satisfy not only (1.1) but also that for every small $\varepsilon > 0$ there exists a positive constant $\delta = \delta(\varepsilon) < 1$ such that

$$f(s) \leq 1 - \delta(\varepsilon), \quad s \geq \varepsilon, \tag{1.3}$$

and that for some fixed constant κ with $1 < \kappa \leq 2$,

$$[f]_\kappa := \sup_{s>0} \frac{|f(s) - 1 + s|}{s^\kappa} < \infty. \tag{1.4}$$

A function $f(s)$ satisfying (1.1) has property (1.3), if it is non-increasing. Of course, the functions in (1.2) have properties (1.3) and (1.4). Condition (1.3) is necessary. For this account and some further remarks on conditions (1.3) and (1.4) we refer to [1].
Then we can show the following theorem.

Theorem 1 Let f and g be functions having properties (1.3) and (1.4) with $\kappa = 2$ as well as (1.1). If A and B are nonnegative self-adjoint operators in a Hilbert space $\mathcal{H}$ with domains $D[A]$ and $D[B]$ such that the operator sum $C := A + B$ is self-adjoint on $D[C] = D[A] \cap D[B]$, then it holds in operator norm that

$$\begin{aligned} &\|[g(tB/2n)f(tA/n)g(tB/2n)]^n - e^{-tC}\| = O(n^{-1}), \\ &\|[f(tA/n)g(tB/n)]^n - e^{-tC}\| = O(n^{-1}), \quad n \to \infty. \end{aligned} \tag{1.5}$$

The convergence is uniform on each compact t-interval in the closed half line $[0, \infty)$, and further, if C is strictly positive, i.e. $C \geq \eta$ for some constant $\eta > 0$, uniform on the whole closed half line $[0, \infty)$.

Taking, for instance, $f(s) = g(s) = e^{-s}$ in (1.2), we have the following

Corollary For nonnegative self-adjoint operators A and B whose operator sum $C := A + B$ is self-adjoint on $D[C] = D[A] \cap D[B]$ it holds in operator norm that

$$\begin{aligned} &\|(e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n - e^{-tC}\| = O(n^{-1}), \\ &\|(e^{-tA/n}e^{-tB/n})^n - e^{-tC}\| = O(n^{-1}), \quad n \to \infty, \end{aligned} \tag{1.6}$$

uniformly on each compact t-interval in $[0,\infty)$, and further, if C is strictly positive, uniformly on $[0,\infty)$.

It is with error bound $O(n^{-1/2})$ that this theorem has first been proved in [1] when f and g satisfy (1.4) with $3/2 \le \kappa \le 2$ as well as (1.1) and (1.3). The error bound $O(n^{-1})$ obtained in Theorem 1 turns out to be optimal and ultimate, though with $\kappa = 2$.
Therefore Theorem 1 *does* properly extend and contain, now with optimal error bound, all the known related results, not only in the abstract case such as in Rogava [I 21], Ichinose–Tamura [I 11], [I 13] and Neidhardt–Zagrebnov [I 16], [I 17], [I 18], but also for the Schrödinger operators such as in Helffer [I 6], Dia–Schatzman [I 3], Ichinose–Takanobu [I 7], [I 8], Doumeki–Ichinose–Tamura [I 4], Ichinose–Tamura [I 12] and Ichinose–Takanobu [I 9], [I 10]. Indeed, in all these cases, the operator sum of two self-adjoint operators concerned there is self-adjoint. A little more detailed account of these facts is referred to Introduction in [1].

Next, we want to give a generalization to the case of the sum of m self-adjoint operators A_1, A_2, ..., A_m in $\mathcal{H}$. Then the product formula

$$\lim_{n\to\infty} (e^{-tA_1/n} e^{-tA_2/n} \cdots e^{-tA_m/n})^n = e^{-tC}, \quad n \to \infty,$$

in strong operator topology was already shown by Kato–Masuda [2], when C is even the form sum of A_1, A_2, $\cdots$, A_m which is self-adjoint.
In this note we content ourselves to show the following theorem, though it only deals with the symmetric product case.

Theorem 2 Let f_1, $\cdots$, f_m be functions having properties (1.3) and (1.4) with $\kappa = 2$ as well as (1.1). If $A_1, \cdots, A_m$ are m nonnegative self-adjoint operators in a Hilbert space $\mathcal{H}$ with domains $D[A_1], \cdots, D[A_m]$ such that the operator sum $C := A_1 + \cdots + A_m$ is self-adjoint on $D[C] = D[A_1] \cap \cdots \cap D[A_m]$, then it holds in operator norm that

$$\begin{aligned}\|[f_m(tA_m/2n) \cdots f_2(tA_2/2n) f_1(tA_1/n) f_2(tA_2/2n) \quad \cdots \quad & f_m(tA_m/2n)]^n - e^{-tC}\| \\ &= O(n^{-1}), \ n \to \infty. \qquad (1.7)\end{aligned}$$

The convergence is uniform on each compact t-interval in the closed half line $[0,\infty)$, and further, if C is strictly positive, i.e. $C \ge \eta$ for some constant $\eta > 0$, uniform on the whole closed half line $[0,\infty)$.

To prove our theorems we make essential use of Lemma I.2.1, the operator-norm version of Chernoff's theorem with error bound, proved in [1].
Theorems 1 and 2 are shown in Sections 2 and 3. Section 4 remarks optimality of the new error bound $O(n^{-1})$.

2 Proof of Theorem 1

(a) *The symmetric product case.*

We are quickly jumping to the circumstances around (I.3.6) in the proof of this case of the theorem in [1]. So recall the notation $S_t = t^{-1}(1 - F(t))$ with $F(t) = g(tB/2)f(tA)g(tB/2)$ as well as (I.3.2).
By Lemma I.2.1, it suffices to show that

$$\|(1 + S_t)^{-1} - (1 + C)^{-1}\| = O(t), \quad t \downarrow 0. \tag{2.1}$$

We have

$$(1 + S_t)^{-1} - (1 + C)^{-1} = (1 + S_t)^{-1}(C - S_t)(1 + C)^{-1}. \tag{2.2}$$

The new idea is to iterate this formula (2.2) with help of its adjoint form. Then we get

$$\begin{aligned} &(1+S_t)^{-1} - (1+C)^{-1} = ((1+C)^{-1} + [(1+S_t)^{-1} - (1+C)^{-1}])(C - S_t)(1+C)^{-1} = \\ &(1 + C)^{-1}(C - S_t)(1 + C)^{-1} + [(C - S_t)(1 + C)^{-1}]^*(1 + S_t)^{-1}(C - S_t)(1 + C)^{-1} \\ &\equiv R_1'(t) + R_2'(t). \end{aligned} \tag{2.3}$$

First, for the second term in the last member of (2.3), we have, using (I.3.8),

$$R_2'(t) = [K_t^{-1/2}(C - S_t)(1 + C)^{-1}]^*(1 + Q_t)^{-1}[K_t^{-1/2}(C - S_t)(1 + C)^{-1}].$$

What is actually proved in Lemma I.3.2 is

$$\|K_t^{-1/2}(C - S_t)(1 + C)^{-1}\| = O(t^{1/2}).$$

By this bound and Lemma I.3.1, we have the bound

$$\|R_2'(t)\| = O(t). \tag{2.4}$$

Next, the first term can be represented (cf. (I.3.12)) as

$$\begin{aligned} R_1'(t) &= (1 + C)^{-1}(A - A_t)(1 + C)^{-1} + (1 + C)^{-1}(B - B_{t/2})(1 + C)^{-1} \\ &+ (1 + C)^{-1}(\tfrac{t}{4}B_{t/2}(1 - tA_t)B_{t/2} + \tfrac{t}{2}(A_tB_{t/2} + B_{t/2}A_t))(1 + C)^{-1} \\ &\equiv R_{11}'(t) + R_{12}'(t) + R_{13}'(t)\,. \end{aligned} \tag{2.5}$$

In the next lemma we prove all the three $R_{1j}'(t)$ in the last line of (2.5) have norm of order $O(t)$.

Lemma 2.1

$$\|R_{11}'(t)\| \le a^2[f]_2 t, \quad \|R_{12}'(t)\| \le 2^{-1}a^2[g]_2 t, \quad \|R_{13}'(t)\| \le da^2 t, \tag{2.6}$$

with a constant d independent of $t > 0$.

Proof.

I. Just in the same way as in (I.3.1), since C is a self-adjoint and so closed operator, by the closed graph theorem there is a positive constant a such that

$$\|(1+A)(1+C)^{-1}\| = \|(1+C)^{-1}(1+A)\| \le a, \|(1+B)(1+C)^{-1}\| = \|(1+C)^{-1}(1+B)\| \le a. \tag{2.7}$$

Therefore $R'_{11}(t)$ is rewritten as

$$R'_{11}(t) = [(1+C)^{-1}(1+A)][(1+A)^{-1} - (1+A)^{-2}(1+A_t)][(1+A)(1+C)^{-1}],$$

so that by (2.7)

$$\|R'_{11}(t)\| \le a^2\|(1+A)^{-1} - (1+A)^{-2}(1+A_t)\|.$$

On the other hand, we have by our assumption on f

$$\begin{aligned} \|(1+A)^{-1} \ &- \ (1+A)^{-2}(1+A_t)\| = \sup_{\lambda\ge 0}\Big|\frac{1}{1+\lambda} - \Big(1+\frac{1-f(t\lambda)}{t}\Big)/(1+\lambda)^2\Big| \\ &= \ t\sup_{\lambda\ge 0}\Big(\frac{\lambda}{1+\lambda}\Big)^2\frac{|f(t\lambda)-1+t\lambda|}{t^2\lambda^2} \le [f]_2 t. \end{aligned}$$

Thus we get the bound $\|R'_{11}(t)\| \le a^2[f]_2 t$.

II. For $R'_{12}(t)$, the proof is the same as for $R'_{11}(t)$. We have only to note that

$$R'_{12}(t) = [(1+C)^{-1}(1+B)][(1+B)^{-1} - (1+B)^{-2}(1+B_{t/2})][(1+B)(1+C)^{-1}].$$

III. For $R'_{13}(t)$ we have

$$\begin{aligned} R'_{13}(t) \ &= \ \frac{t}{4}[(1+C)^{-1}(1+B)][(1+B)^{-1}B_{t/2}]f(tA)[B_{t/2}(1+B)^{-1}][(1+B)(1+C)^{-1}] \\ &+ \ \frac{t}{2}[(1+C)^{-1}(1+A)][(1+A)^{-1}A_t][B_{t/2}(1+B)^{-1}][(1+B)(1+C)^{-1}] \\ &+ \ \frac{t}{2} \end{aligned}$$

$[(1+\mathrm{C})^{-1}(1+B)][(1+B)^{-1}B_{t/2}][A_t(1+A)^{-1}][(1+A)(1+C)^{-1}]$.Then by (2.7) with the constants a_0, b_0 introduced in (I.3.14), we get the bound

$$\|R'_{13}(t)\| \le (a_0 b_0 + b_0^2/4)a^2 t.$$

This completes the proof of Lemma 2.1, ending the proof of the symmetric product case.

(b) *The non-symmetric product case.*
The proof in this case in [1] also is valid, as mentioned at the beginning of the proof of this case there, because Lemma I.2.1 holds for every α with $0 < \alpha \le 1$.
This ends the proof of Theorem 1.

3 Proof of Theorem 2

Let

$$C_j = A_1 + A_2 + \cdots + A_j, \qquad j = 1, 2, \cdots, m. \tag{3.1}$$

Here C_j may be understood simply as the operator sum of j self-adjoint operators $A_1,\ A_2,\ \cdots,\ A_j$ with domain $D[C_j] := D[A_1] \cap D[A_2] \cap \cdots D[A_j]$, which may not be self-adjoint if $1 < j < m$, or as the form sum of these j operators which is self-adjoint. Note that $C_1 = A_1$ and $C_m = C$. Put, with the notations (I.3.2),

$$
\begin{aligned}
A_{j,t} &= t^{-1}[1 - f_j(tA_j)], \\
C_{j,t} &= t^{-1}[1 - f_j(tA_j/2) \cdots f_2(tA_2/2) f_1(tA_1) f_2(tA_2/2) \cdots f_j(tA_j/2)], \\
K_{j,t} &= 1 + C_{j-1,t} + A_{j,t/2} - \frac{t}{4} A_{j,t/2}^2.
\end{aligned}
\tag{3.2}
$$

for $j = 2, 3, \cdots, m$. There will be below no C_t, which $C_{m,t}$ differs from. Moreover, we put

$$
Q_{j,t} = \frac{t^2}{4} K_{j,t}^{-1/2} A_{j,t/2} C_{j-1,t} A_{j,t/2} K_{j,t}^{-1/2} - \frac{t}{2} K_{j,t}^{-1/2} (C_{j-1,t} A_{j,t/2} + A_{j,t/2} C_{j-1,t}) K_{j,t}^{-1/2}.
$$

Then we have the identity

$$
1 + C_{j,t} = K_{j,t}^{1/2} (1 + Q_{j,t}) K_{j,t}^{1/2}, \tag{3.3}
$$

and the following estimate may be proved by the same reasoning as in the proof of Lemma I.3.1:

$$
\|(1 + Q_{j,t})^{-1}\| \leq 2/(3 - \sqrt{5}). \tag{3.4}
$$

Similar to the proof of our Theorem 1, all we have to do now is to show the following two estimates:

$$
\|(1 + C)^{-1}(C - C_{m,t})(1 + C)^{-1}\| = O(t),
$$

and

$$
\|K_{m,t}^{-1/2}(C - C_{m,t})(1 + C)^{-1}\| = O(t^{1/2}).
$$

To this end, we prove for each $j = 1, 2, \cdots, m$ the following estimates:

$$
\begin{aligned}
&(\mathrm{a})_j \qquad \|(1 + C)^{-1}(C_j - C_{j,t})(1 + C)^{-1}\| = O(t), \\
&(\mathrm{b})_j \qquad \|K_{j,t}^{-1/2}(C_j - C_{j,t})(1 + C)^{-1}\| = O(t^{1/2}), \\
&(\mathrm{c})_j \qquad \|C_{j,t}(1 + C)^{-1}\| = O(1).
\end{aligned}
$$

The proof is done by induction on j. Notice that for each inductive step j, we have, similarly to (I.3.8) and (I.3.12), the following identity:

$$
\begin{aligned}
C_{j+1} - C_{j+1,t} &= (C_j - C_{j,t}) + (A_{j+1} - A_{j+1,t/2}) + \frac{t}{4} A_{j+1,t/2}(1 - tC_{j,t}) A_{j+1,t/2} \\
&+ \frac{t}{2}(C_{j,t} A_{j+1,t/2} + A_{j+1,t/2} C_{j,t})
\end{aligned}
\tag{3.5}
$$

as operators on $D[C]$ and the estimate

$$
\|K_{j,t}^{-1/2} K_{j-1,t}^{1/2}\| \leq \sqrt{2/(3 - \sqrt{5})}. \tag{3.6}
$$

Here we can see (3.6) from (3.3), (3.4) with definitions (3.1) and (3.2) for $j = 2, \cdots m$, noting $K_{j,t} \geq 1 + C_{j-1,t}$ and setting $K_{1,t} = 1 + C_{1,t} = 1 + A_{1,t}$. As in the proof of Theorem 1, since C is a self-adjoint and so closed operator, we have again by the closed graph theorem

$$
\|(1 + A_j)(1 + C)^{-1}\| \leq a, \qquad j = 1, 2, \cdots, m, \tag{3.7}
$$

with some positive constant a.
For $j = 1$, the estimates $(a)_1$ and $(b)_1$ are trivial. The estimate $(c)_1$ is also obvious. Assume that the estimates $(a)_j$, $(b)_j$ and $(c)_j$ are valid. Then we use

$$C_{j+1,t}(1+C)^{-1} = C_{j+1}(1+C)^{-1} + (C_{j+1,t} - C_{j+1})(1+C)^{-1}. \tag{3.8}$$

to show the estimate $(c)_{j+1}$. The first term on the right-hand side of (3.8) is bounded in view of (3.7) and by induction hypothesis $(c)_j$. We use (3.5) and $(c)_j$ to estimate the second term. By analogous arguments used to prove our Theorem 1, the identity (3.5) gives us the estimate $(a)_{j+1}$ and, together with (3.6), the estimate $(b)_{j+1}$. Details of these estimates are now some routine calculations. Thus we have proved the estimates $(a)_j$, $(b)_j$ and $(c)_j$ for all $j = 1, 2, \cdots, m$, ending the proof of Theorem 2.

4 Optimality of the Error Bound

In this section, we want to note that the new error bound $O(n^{-1})$ in Theorem 1 is optimal. We consider with $f(s) = g(s) = e^{-s}$ first the non-symmetric and next symmetric product case.

(a) *The non-symmetric product case.*

For the time being, let A and B be simply bounded operators. Then by the Baker–Campbell–Hausdorff formula (e.g. [5], [3]) we have for small $|t|$

$$e^{-tA}e^{-tB} = \exp[-t(A+B) + \tfrac{t^2}{2}[A,B] - \tfrac{t^3}{6}[A-B, \tfrac{1}{2}[A,B]] + O_p(|t|^4)], \tag{4.1}$$

with $[A,B] = AB - BA$, where and below $O_p(|t|^k)$, for $k > 0$, means some bounded operator with norm of order $O(|t|^k)$. Then

$$N(t) := (e^{-tA}e^{-tB})^{1/t} = \exp[-(A+B) + \tfrac{t}{2}[A,B] + O_p(|t|^2)].$$

We understand $N(0) = e^{-(A+B)}$ and have

$$E_1(A;B) := \frac{d}{dt}N(t)|_{t=0} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\prod_{j=1}^{k}(A+B)^{j-1}[A,B](A+B)^{k-j}, \tag{4.2}$$

of which the right-hand side is norm-convergent and can be a non-zero operator with bound $2^{-1}\|[A,B]\|e^{\|A+B\|}$, if A and B do not commute with each other. It follows that

$$N(t) = e^{-(A+B)} + tE_1(A;B) + O_p(t^2),$$

so that with $t = 1/n$

$$(e^{-A/n}e^{-B/n})^n = N(1/n)^n = e^{-(A+B)} + n^{-1}E_1(A;B) + O_p(n^{-2}). \tag{4.3}$$

Thus we have seen that, in the non-symmetric case, the error bound $O(n^{-1})$ is optimal.

(b) *The symmetric product case.*

We have from (4.1)

$$e^{-tB/2}e^{-tA}e^{-tB/2} = \exp[-t(A+B) - \tfrac{t^3}{24}[2A+B,[A,B]] + O_p(|t|^4)]. \tag{4.4}$$

Similarly it follows that

$$S(t) := (e^{-tB/2}e^{-tA}e^{-tB/2})^{1/t} = \exp[-(A+B) - \tfrac{t^2}{24}[2A+B,[A,B]] + O_p(|t|^3)].$$

Here we understand $S(0) = e^{-(A+B)}$ and have $dS(t)/dt|_{t=0} = 0$, so that with $t = 1/n$

$$(e^{-B/2n}e^{-A/n}e^{-B/2n})^n = S(1/n)^n = e^{-(A+B)} + O_p(n^{-2}). \tag{4.5}$$

Hence, in the symmetric case, the optimal error bound would appear to be of order $O(n^{-2})$. But it is *not*.
In fact, in this case also the optimal error bound *is* just of order $O(n^{-1})$. In the following example we shall see that there exist unbounded nonnegative self-adjoint operators A and B in a Hilbert space $\mathcal{H}$ such that the operator sum $A+B$ is self-adjoint on $D[A]\cap D[B]$ and the following lower estimate holds for n large:

$$\|e^{-t(A+B)} - (e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n\| \geq L(t)n^{-1},$$

where $L(t)$ is a positive continuous function of $t>0$, independent of n.
We are using the same idea as in [4].

Example.

Let $\mathcal{H} = \oplus_{k=1}^{\infty}\mathcal{H}_k$ be the direct sum of a countable family of Hilbert spaces $\mathcal{H}_k := \mathbf{R}^2$ with inner product $(\cdot\,,\cdot)_k$. It has the inner product $(z,w) = \sum_{k=1}^{\infty}(z_k,w_k)_k$, for $z = (z_k)_{k\in\mathbf{N}}$ and $w = (w_k)_{k\in\mathbf{N}}$ in $\mathcal{H}$.
Let S, T and E be the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.6}$$

Note that

$$ST + TS = 0, \qquad S^2 = T^2 = E. \tag{4.7}$$

For each k, define two bounded non-negative self-adjoint operators

$$A_k = k(S+E), \qquad B_k = k(S\cos\theta_k + T\sin\theta_k + E) \tag{4.8}$$

on $\mathcal{H}_k$, where the parameters $\theta_k \in (0,\pi/2]$ are so chosen that

$$\cos\theta_k = 1 - \varepsilon_k, \quad \varepsilon_k = 1/2k^2. \tag{4.9}$$

Then consider two unbounded non-negative self-adjoint operators

$$A = (A_k)_{k\in\mathbf{N}}, \qquad B = (B_k)_{k\in\mathbf{N}} \tag{4.10}$$

in $\mathcal{H}$ with domains

$$D[A] = \{z = (z_k)_{k\in\mathbf{N}}; \sum_k \|A_k z_k\|^2 < \infty\}, \;\; D[B] = \{z = (z_k)_{k\in\mathbf{N}}; \sum_k \|B_k z_k\|^2 < \infty\},$$

and their operator sum $A+B = (A_k + B_k)_{k\in\mathbf{N}}$ with domain $D[A]\cap D[B]$, which is symmetric and nonnegative.
In the following two propositions we shall see that these operators constitute an example where the lower error bound in the symmetric product case is also just $L(t)n^{-1}$ with a positive continuous function $L(t)$ of $t>0$.

Proposition 1 A and B have the same domain, and the operator sum $A+B$ is self-adjoint on $D[A]\cap D[B] = D[A] = D[B]$.

Proof. For each k we have

$$A_k = 2k\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_k = 2k\begin{pmatrix} \cos^2(\theta_k/2) & \cos(\theta_k/2)\sin(\theta_k/2) \\ \cos(\theta_k/2)\sin(\theta_k/2 & \sin^2(\theta_k/2) \end{pmatrix}$$

so that

$$A_k + B_k = 2k[E + \cos(\theta_k/2)(S\cos(\theta_k/2) + T\sin(\theta_k/2))]. \tag{4.11}$$

$A_k + B_k$ has eigenvalues $2k(1\pm\cos(\theta_k/2))$, and can be diagonalized with the orthogonal matrix

$$P_k = \begin{pmatrix} \cos(\theta_k/4) & \sin(\theta_k/4) \\ -\sin(\theta_k/4) & \cos(\theta_k/4) \end{pmatrix}$$

as

$$P_k(A_k+B_k)P_k^{-1} = 2k\begin{pmatrix} 1+\cos(\theta_k/2) & 0 \\ 0 & 1-\cos(\theta_k/2) \end{pmatrix},$$

so that

$$P_k(A_k+B_k)^2P_k^{-1} = (4k)^2\begin{pmatrix} \cos^4(\theta_k/4) & 0 \\ 0 & \sin^4(\theta_k/4) \end{pmatrix}.$$

To show Proposition 1, we have only to show that

$$\begin{aligned} &\|Bz\|^2 \le 2\|Az\|^2 + 2\|z\|^2, \quad z\in D[A], \\ &\|Az\|^2 \le \tfrac{1}{2}\|(A+B)z\|^2 + \tfrac{1}{2}\|z\|^2, \quad z\in D[A]\cap D[B], \end{aligned} \tag{4.12}$$

for it also follows from (4.12) that

$$\|Az\| \le (\sqrt{2}+1)(\|Bz\| + \|z\|), \quad z\in D[B].$$

To do so it suffices to show that for $z_k = {}^t(x_k, y_k) \in \mathbf{R}^2$

$$\begin{aligned} &\|B_k z_k\|_k^2 \le 2\|A_k z_k\|_k^2 + 2\|z_k\|_k^2, \\ &\|A_k z_k\|_k^2 \le \tfrac{1}{2}\|(A_k+B_k)z_k\|_k^2 + \tfrac{1}{2}\|z_k\|_k^2. \end{aligned} \tag{4.13}$$

We get the first estimate of (4.13) for $z_k = {}^t(x_k, y_k) \in \mathbf{R}^2$ as

$$\begin{aligned} \|B_k z_k\|_k^2 &= (z_k, B_k^2 z_k)_k = (2k)^2(x_k\cos(\theta_k/2) + y_k\sin(\theta_k/2))^2 \\ &\le 2(2k)^2(x_k)^2 + \varepsilon_k(2k)^2(y_k)^2 \le 2\,\|A_k z_k\|_k^2 + 2\,\|z_k\|_k^2. \end{aligned}$$

We get the second estimate of(4.13) with $w_k = {}^t(u_k, v_k) = P_k z_k$ as

$$\begin{aligned}
\|A_k z_k\|_k^2 &= (w_k, P_k A_k^2 P_k^{-1} w_k)_k \leq 2(2k)^2((u_k)^2\cos^2(\theta_k/4) + (v_k)^2\sin^2(\theta_k/4)) \\
&\leq \tfrac{1}{2}((4k)^2\cos^4(\theta_k/4)+1)(u_k)^2 + \tfrac{1}{2}((4k)^2\sin^4(\theta_k/4)+1)(v_k)^2 \\
&= \tfrac{1}{2}(w_k, P_k(A_k+B_k)^2 P_k^{-1} w_k)_k + \tfrac{1}{2}(w_k, w_k)_k \\
&= \tfrac{1}{2}\|(A_k+B_k)z_k\|_k^2 + \tfrac{1}{2}\|z_k\|_k^2.
\end{aligned}$$

Proposition 2 There is a positive bounded continuous function $L(t)$ of $t > 0$ independent of n such that the lower estimate

$$\|e^{-t(A+B)} - (e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n\| \geq L(t)n^{-1}. \tag{4.14}$$

holds for every $t > 0$ and $n \geq 1$.
Proof.
Note that the inequalities

$$\begin{aligned}
&\|e^{-t(A+B)} - (e^{-tB/2n}e^{-tA/n}e^{-tB/2n})^n\| \geq \\
&\|e^{-t(A_n+B_n)} - (e^{-tB_n/2n}e^{-tA_n/n}e^{-tB_n/2n})^n\|_n \geq \\
&\tfrac{1}{2}|\mathrm{Tr}[e^{-t(A_n+B_n)} - (e^{-tB_n/2n}e^{-tA_n/n}e^{-tB_n/2n})^n]|
\end{aligned} \tag{4.15}$$

hold, where the norm in the first member means the operator norm of bounded operators on $\mathcal{H}$, that in the second member the operator norm on $\mathcal{H}_n = \mathbf{R}^2$ and Tr in the third member the trace of 2×2 matrices.
For later use, let us note for $\Lambda = S\cos\theta + T\sin\theta$ the following formulas:

$$e^{-s\Lambda} = E\cosh s - \Lambda\sinh s, \qquad \mathrm{Tr}e^{-s\Lambda} = 2\cosh s. \tag{4.16}$$

To get the first formula in (4.16), expand the exponential and use $\Lambda^2 = E$, a consequence of (4.7). The second formula follows from the first one and $\mathrm{Tr}\Lambda = 0$.

Thanks to (4.11) and the above formulas, we get

$$\mathrm{Tr}e^{-t(A_n+B_n)} = 2e^{-2nt}\cosh(2nt\cos(\theta_n/2)) \tag{4.17}$$

for n large. On the other hand, the second trace in the last member of (4.15) is, by (4.16), equal to

$$\begin{aligned}
\mathrm{Tr}(e^{-tA_n/2n}e^{-tB_n/n}e^{-tA_n/2n})^n &= e^{-2nt}\mathrm{Tr}(e^{-tS/2}e^{-t(S\cos\theta_n+T\sin\theta_n)}e^{-tS/2})^n \\
&= e^{-2nt}\mathrm{Tr}(a_nE - b_nS - c_nT)^n,
\end{aligned}$$

where a_n, b_n and c_n are positive numbers defined by

$$\begin{aligned}
&a_n = \cosh^2 t + \sinh^2 t\cos\theta_n = \cosh 2t - \varepsilon_n \sinh^2 t, \\
&b_n = \sinh t\cosh t(1+\cos\theta_n), \\
&c_n = \sinh t\sin\theta_n.
\end{aligned} \tag{4.18}$$

Since they satisfy the identity

$$a_n^2 - b_n^2 - c_n^2 = 1,$$

there exist positive numbers K_n and Θ_n such that

$$a_n = \cosh K_n, \quad b_n = \sinh K_n \cos\Theta_n, \quad c_n = \sinh K_n \sin\Theta_n. \tag{4.19}$$

Setting $s = K_n$ and $\theta = \Theta_n$ in (4.16), we have

$$\mathrm{Tr}(e^{-tB_n/2n}e^{-tA_n/n}e^{-tB_n/2n})^n = e^{-2nt}\mathrm{Tr}(e^{-K_n(S\cos\Theta_n + T\sin\Theta_n)})^n = 2e^{-2nt}\cosh nK_n. \tag{4.20}$$

Now, with θ_n or ε_n in (4.9) let us introduce positive numbers δ_n such that

$$\delta_n = 2 - 2\cos(\theta_n/2), \quad \text{or} \quad \varepsilon_n = 2\delta_n - \tfrac{1}{2}\delta_n^2. \tag{4.21}$$

Note that $\varepsilon_n/2 \le \delta_n \le \varepsilon_n$. By (4.18) and (4.19) we have

$$\begin{aligned}
&\cosh K_n - \cosh(2-\delta_n)t = \\
&(1 - \tfrac{1}{2}\varepsilon_n)\cosh 2t + \tfrac{1}{2}\varepsilon_n - \cosh(2-\delta_n)t = \\
&\int_0^t (t-s)\tfrac{d^2}{ds^2}[(1-\tfrac{1}{2}\varepsilon_n)\cosh 2s + \tfrac{1}{2}\varepsilon_n - \cosh(2-\delta_n)s]ds = \\
&\int_0^t (t-s)[4(1-\tfrac{1}{2}\varepsilon_n)\cosh 2s - (2-\delta_n)^2\cosh(2-\delta_n)s]ds \\
\ge \quad & (2\delta_n - 2\delta_n^2 + \tfrac{1}{2}\delta_n^3)\int_0^t (t-s)(\cosh 2s - 1)ds.
\end{aligned} \tag{4.22}$$

Here the last step is due to the convexity of the function $\cosh s$:

$$\cosh((1-\tfrac{1}{2}\delta_n)2s + \tfrac{1}{2}\delta_n 0s) \le (1-\tfrac{1}{2}\delta_n)\cosh 2s + \tfrac{1}{2}\delta_n\cosh 0s.$$

We see from (4.9) and (4.21)

$$2\delta_n - 2\delta_n^2 + \tfrac{1}{2}\delta_n^3 = \delta_n(2-\varepsilon_n) \ge \varepsilon_n(1-\tfrac{1}{2}\varepsilon_n) \ge 3/8n^2,$$

and

$$\int_0^t (t-s)(\cosh 2s - 1)ds = \tfrac{1}{4}(\cosh 2t - 1 - 2t^2).$$

We are about to use the mean value theorem: Let $b > a$. Then for real-valued smooth functions $\varphi(s)$ and $\psi(s)$ there exists ξ with $a < \xi < b$ such that

$$\frac{\varphi(b)-\varphi(a)}{\psi(b)-\psi(a)} = \frac{\varphi'(\xi)}{\psi'(\xi)}.$$

Note (4.22) implies that $K_n > (2-\delta_n)t$ for $t > 0$. Then we get with (4.17) and (4.20) that, for some M_n with $(2-\delta_n)t < M_n < K_n$,

$$\begin{aligned}
&\tfrac{1}{2}\mathrm{Tr}[(e^{-tB_n/2n}e^{-tA_n/n}e^{-tB_n/2n})^n - e^{-t(A_n+B_n)}] = \\
&e^{-2nt}(\cosh nK_n - \cosh n(2-\delta_n)t) = \\
&e^{-2nt}n\frac{\sinh nM_n}{\sinh M_n}(\cosh K_n - \cosh(2-\delta_n)t) \ge \\
&e^{-2nt}e^{(n-1)M_n}\tfrac{3}{32n}(\cosh 2t - 1 - 2t^2),
\end{aligned}$$

where we have used (4.22) and the inequality $\sinh ns/\sinh s \geq e^{(n-1)s}$. Since

$$(n-1)M_n \geq (n-1)(2-\delta_n)t \geq (n-1)(2-1/2n^2)t \geq 2(n-1)t - t/8,$$

we have proved (4.14) with $L(t) = \frac{3}{32}e^{-17t/8}(\cosh 2t - 1 - 2t^2)$. This ends the proof of Proposition 2.

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