Representations of Cuntz algebras, loop groups and wavelets

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Abstract. A theorem of Glimm states that representation theory of an NGCR C^*-algebra is always intractable, and the Cuntz algebra $\mathcal{O}_N$ is a case in point. The equivalence classes of irreducible representations under unitary equivalence cannot be captured with a Borel cross section. Nonetheless, we prove here that wavelet representations correspond to equivalence classes of irreducible representations of $\mathcal{O}_N$, and they are effectively labeled by elements of the loop group, i.e., the group of measurable functions $A : \mathbb{T} \to \mathrm{U}_N(\mathbb{C})$. These representations of $\mathcal{O}_N$ are constructed here from an orbit picture analysis of the infinite-dimensional loop group.

1. Introduction

Recall the Cuntz algebra with N generators $s_0, \dots, s_{N-1}$ is the C^*-algebra $\mathcal{O}_N$ on the relations

$$s_i^* s_j = \delta_{ij} 1, \qquad \sum_{i=0}^{N-1} s_i s_i^* = 1. \tag{1.1}$$

Cuntz [**Cun77**] showed that it is simple and infinite. (We will consider N finite only, $N = 2, 3, \dots$.) By a theorem of [**Gli60**] there does not exist a Borel section parameterizing the irreducible representations of $\mathcal{O}_N$. Hence we shall restrict to special representations and consider the questions of irreducibility and decomposition. In particular we show that the elements in the loop group, i.e., all measurable maps

$$A : \mathbb{T} \longrightarrow \mathrm{U}_N(\mathbb{C}),$$

parameterize the equivalence classes of wavelet representations. The wavelets in turn correspond 1-to-1 to representations

$$\pi^{(A)} \in \mathrm{Rep}(\mathcal{O}_N, L^2(\mathbb{T}))$$

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where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the 1-torus. We review joint research papers [**BEJ00**], [**BrJo97**], [**BrJo98**], [**BrJo99**], [**BrJo00**] and the solo paper [**Jor00**]. See also [**Dau92**].

If A is given, we show that the operators

$$S_i^{(A)} = \pi^{(A)}(s_i)$$

are weighted shift operators on $L^2(\mathbb{T})$. Our papers [**BrJo97**] and [**BJKW00**] indicate generalizations to $\mathbb{T}^d$, $d > 1$, but we restrict to $d = 1$ here.

2. A Hilbert Module

Let (X, μ) be a measure space, and we assume that μ is a probability measure on X. Let $\sigma : X \to X$ be a measurable N-to-1 self map, and let N be given, and fixed.

It will further be assumed that there is a probability measure μ on X such that

$$\frac{1}{N}\sum_{i=0}^{N-1} \mu \circ \sigma_i^{-1} = \mu, \tag{2.1}$$

where $\sigma_i : X \to X$ is some chosen sections for σ, i.e., satisfying

$$\sigma \circ \sigma_i = \mathrm{id}_X, \qquad i = 0, \dots, N-1. \tag{2.2}$$

The analysis below refers to such a measure μ. If σ is expansive, μ is known to exist and be unique [**BJO99**]. If $X = \mathbb{T}$, and $\sigma(z) = z^N$, then μ is the usual Haar measure on $\mathbb{T}$.

Let

$$\mathcal{A}_\sigma := \{ f \circ \sigma \,;\, f \in L^\infty(X) \}. \tag{2.3}$$

We will also assume that $L^\infty(X)$ and its subalgebras act by multiplication on the Hilbert space $\mathcal{H}_\mu = L^2(X, \mu)$.

Lemma 2.1. (a) *A system of measurable functions*

$$m_i : X \longrightarrow \mathbb{C}, \qquad i = 0, \dots, N-1,$$

forms an orthonormal basis for $\mathcal{H}_\mu$ as an $\mathcal{A}_\sigma$-Hilbert module if and only if there are sections $\sigma_i : X \to X$, i.e., $\sigma \circ \sigma_i = \mathrm{id}_X$ such that the $N \times N$ matrix

$$M_m := (m_i \circ \sigma_j)_{i,j=0}^{N-1} \qquad \textit{is unitary,} \tag{2.4}$$

i.e., defines

$$M_m : X \longrightarrow \mathrm{U}_N(\mathbb{C}).$$

(b) *If $m_0 \in L^\infty(X)$ is given such that*

$$\sum_{y:\sigma(y)=x} |m_0(y)|^2 \equiv 1 \ \textit{a.a. } x \in X, \tag{2.5}$$

then there is a measurable selection $m_1, \dots, m_{N-1}$ such that the combined system $m_0, \dots, m_{N-1}$ satisfies (2.4).

(c) *A function system $\{m_i\}_{i=0}^{N-1}$ satisfies the conditions in* (a) *if and only if the system of operators*

$$S_i^{(m)} f := \sqrt{N} m_i \, f \circ \sigma, \qquad f \in \mathcal{H}_\mu, \tag{2.6}$$

defines a representation of $\mathcal{O}_N$, i.e., $S^{(m)} \in \mathrm{Rep}(\mathcal{O}_N, \mathcal{H}_\mu)$. (Note the right-hand side in (2.6) *is the function $\sqrt{N} m_i(x) f(\sigma(x))$.)*

DEFINITION 2.2. The system $\{m_i\}_{i=0}^{N-1}$ is called a quadrature mirror filter system (QMF) by analogy to the example $N = 2$, in which case $m_0 \sim (a_k)$ serves as the low pass filter, and $m_1 \sim (b_k)$ as the high pass filter. It is the orthogonality relations (2.4) which motivate the name QMF, and the use of filters in wavelet theory is further fleshed out in [**Dau92**].

PROOF OF LEMMA 2.1. Most of the details are contained in the paper [**BrJo97**], and others will be in a later more detailed paper. But we sketch here the argument for orthogonality in the Hilbert module. Let the functions $\{m_i\}_{i=0}^{N-1}$ be as stated in the lemma. Orthogonality refers to

$$\langle \mathcal{A}_\sigma m_i \mid \mathcal{A}_\sigma m_j \rangle_{\mathcal{H}_\mu} = 0$$

whenever $i \neq j$. Hence we must calculate, for $f \in L^\infty(X)$:

$$\int_X \overline{m_i(x)} f(\sigma(x)) m_j(x)\, d\mu(x) = \frac{1}{N} \int_X f(x) \sum_{y:\sigma(y)=x} \overline{m_i(y)} m_j(y)\, d\mu(x),$$

using (2.1), and the sum under the integral vanishes by (2.4) if $i \neq j$.

It is important that the selection result mentioned in (b) is generally not possible in the category of continuous functions. (See [**Kad84**].)

Conversely, if $\{m_i\}_{i=0}^{N-1} \subset L^\infty(X)$ is given such that (2.6) defines a representation of $\mathcal{O}_N$ on $\mathcal{H}_\mu$, then the closed subspaces $S_i^{(m)}\mathcal{H}_\mu$ $(\subset \mathcal{H}_\mu)$ are the submodules in a corresponding orthogonal $\mathcal{A}_\sigma$-module decomposition, $\mathcal{H}_\mu = \sum_{i=0}^{N-1} [\mathcal{A}_\sigma m_i]$, i.e., the $L^2(\mu)$-closure of $m_i \mathcal{A}_\sigma$ is $S_i^{(m)}\mathcal{H}_\mu$ for each i. Specifically, from (1.1), we get the identity $\mathcal{H}_\mu \to f = \sum_i S_i^{(m)} S_i^{(m)*} f = \sum_i m_i k_i \circ \sigma$, where $k_i = \sqrt{N} S_i^{(m)*} f$. $\square$

3. Comparing Representations

LEMMA 3.1. *Let $G_N(X)$ be the group of measurable maps*

$$A\colon X \longrightarrow \mathrm{U}_N(\mathbb{C}).$$

Then $G_N(X)$ acts transitively on the systems $\{m_i\}_{i=0}^{N-1}$ of functions from Lemma 2.1.

PROOF. Let $\{m_i\}$ be given as in Lemma 2.1, and let $A \in G_N(X)$. Set

$$n_i(x) := \sum_{j=0}^{N-1} A_{i,j}(\sigma(x))\, m_j(x). \tag{3.1}$$

Then $\{n_i\}$ satisfies the same orthogonality relations. For

$$\begin{aligned}\sum_k \overline{n_i(\sigma_k(x))} n_j(\sigma_k(x)) &= \sum_k \sum_{l,l'} \bar{A}_{i,l}(x) \bar{m}_l(\sigma_k(x))\, A_{j,l'}(x)\, m_{l'}(\sigma_k(x)) \\ &= \sum_{l,l'} \delta_{l,l'} \bar{A}_{i,l}(x)\, A_{j,l'}(x) = \sum_l \bar{A}_{i,l}(x)\, A_{j,l}(x) = \delta_{i,j},\end{aligned}$$

proving the assertion. We used the fact that $\sigma(\sigma_k(x)) = x$, see (2.2).

Conversely, if $S_i^{(m)}$ and $S_i^{(n)}$ are given representations as described in (2.6) of Lemma 2.1, then

$$\left(S_i^{(n)*} S_j^{(m)}\right) \in M_N(L^\infty(X)) \qquad (= M_N(\mathbb{C}) \otimes L^\infty(X)). \tag{3.2}$$

For it follows from the Cuntz relations (1.1) that the matrix in (3.2) is unitary, and a computation shows that its matrix entries are multiplication operators. Indeed,

$$\left(S_i^{(m)}\right)^* f(x) = \frac{1}{\sqrt{N}} \sum_{y\colon \sigma(y)=x} \overline{m_i(y)} f(y), \qquad \text{for } x \in X \text{ and for } f \in \mathcal{H}_\mu.$$

Hence

$$\left(S_i^{(n)}\right)^* S_j^{(m)} f(x) = \sum_{y\colon \sigma(y)=x} \overline{n_i(y)} m_j(y) f(x),$$

and so

$$A_{i,j}(x) := \sum_{y\colon \sigma(y)=x} \overline{n_i(y)} m_j(y)$$

defines an element of $G_N(X)$, and it satisfies (3.1) by its very construction. This proves transitivity. $\square$

When the lemma is applied to the example $X = \mathbb{T}$, $\sigma(x) = z^N$, $z \in \mathbb{T}$, and $\sigma_k(x) = e^{i\frac{2\pi k}{N}} \sqrt[N]{z}$, $k = 0, \dots, N-1$, we conclude that the loop group (see Section 1) of measurable $A\colon \mathbb{T} \to \mathrm{U}_N(\mathbb{C})$ acts transitively on the wavelet representations. Let $n_k(z) = \frac{1}{\sqrt{N}} z^k$, $k = 0, \dots, N-1$, and let $\{m_k\}_{k=0}^{N-1}$ be an arbitrary m-system as in Lemma 2.1.

Then we have

Corollary 3.2. *There is a* 1*-to-*1 *bijective correspondence between the loops A and the m-systems of Lemma* 2.1 *given as follows:*

$$m_i(z) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} A_{i,j}\left(z^N\right) z^j$$

and

$$A_{i,j}(z) = \frac{1}{\sqrt{N}} \sum_{w\in\mathbb{T}\colon w^N = z} m_i(w)\, w^{-j}.$$

4. Wavelets

Let $\varphi \in L^2(\mathbb{R})$ be the compactly supported scaling function, i.e., a solution to the scaling identity

$$\varphi(x) = \sum_{k=0}^{Ng-1} a_k \varphi(Nx - k). \tag{4.1}$$

Then the wavelet generators $\psi_1, \dots, \psi_{N-1} \in L^2(\mathbb{R})$ are constructed from φ by use of Lemma 2.1(b) above, and standard wavelet tools from [**Dau92**]. The generators make the system

$$\{N^{j/2}\psi_i(N^j x - k)\,;\, i = 0, 1, \dots, N-1,\ j, k \in \mathbb{Z}\}$$

into an orthonormal basis for $L^2(\mathbb{R})$; except for a smaller variety of cases when the system is only a tight frame. The coefficients $\{a_k\}$ represent a wavelet filter, and

$$m_0(z) = \sum_k a_k z^k. \tag{4.2}$$

Then define the operator

$$W : \ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{R}) \tag{4.3}$$

by

$$W\xi = \sum_k \xi_k \varphi(x-k). \tag{4.4}$$

The conditions on the wavelet filter $\{a_k\}$ may now be restated in terms of $m_0(z)$ in (4.2) as follows:

$$\sum_{k=0}^{N-1} \left| m_0(ze^{i\frac{k2\pi}{N}}) \right|^2 = 1, \tag{4.5}$$

and

$$m_0(1) = 1, \qquad \text{the low pass property.} \tag{4.6}$$

Then W in (4.4) maps $\ell^2(\mathbb{Z})$ onto the resolution subspace $\mathcal{V}_0$ $(\subset L^2(\mathbb{R}))$, and we note that

$$U_N W = W S_0, \tag{4.7}$$

where

$$U_N f(x) = N^{-\frac{1}{2}} f\left(\frac{x}{N}\right), \qquad f \in L^2(\mathbb{R}),\ x \in \mathbb{R}. \tag{4.8}$$

We showed in [**BrJo00**] that there are functions $m_1, \dots, m_{N-1}$ such that the N-by-N complex matrix

$$\left(m_j(e^{i\frac{k2\pi}{N}}z)\right)_{j,k=0}^{N-1} \tag{4.9}$$

is unitary for all $z \in \mathbb{T}$. (See Lemma 2.1(b).) We define

$$S_j f(z) = \sqrt{N} m_j(z) f(z^N), \qquad f \in L^2(\mathbb{T}),\ z \in \mathbb{T}. \tag{4.10}$$

The main result will be stated in the present section, but without proof. Instead the reader is referred to [**Jor00**] for the full proof, and for a detailed discussion of its implications. We noted above that the representation (4.10) given from a QMF system $m_j = m_j^{(A)}$, via

$$m_j^{(A)}(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} A_{j,k}(z^N) z^k, \tag{4.11}$$

$$A_{j,k}(z) = \frac{1}{\sqrt{N}} \sum_{w^N = z} w^{-k} m_j(w) \tag{4.12}$$

is irreducible if and only if the subbands are optimal, in that they do not admit further reduction into a refined system of closed subspaces of $L^2(\mathbb{R})$.

THEOREM 4.1. *The representation*

$$S_j^{(A)} f(z) = \sqrt{N} m_j^{(A)}(z) f\left(z^N\right), \qquad f \in L^2(\mathbb{T}),\ z \in \mathbb{T},$$

is an irreducible representation of $\mathcal{O}_N$ *on* $L^2(\mathbb{T})$ *if and only if* $A\colon \mathbb{T} \to \mathrm{U}_N(\mathbb{C})$ *does not admit a matrix corner of the form*

$$V \begin{pmatrix} z^{n_0} & & & 0 \\ & z^{n_1} & & \\ & & \ddots & \\ 0 & & & z^{n_{M-1}} \end{pmatrix}, \tag{4.13}$$

for some $V \in \mathrm{U}_M(\mathbb{C})$*, and where* $n_0, n_1, \dots, n_{M-1} \in \{0, 1, 2, \dots\}$*. Moreover, two representations* $\pi^{(A)}$ *and* $\pi^{(B)}$ *defined from different loops* A*,* B *are unitarily inequivalent unless* $A \equiv B$ *modulo a matrix corner of type* (4.13).

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References

[BEJ00] O. Bratteli, D.E. Evans, and P.E.T. Jorgensen, *Compactly supported wavelets and representations of the Cuntz relations*, Appl. Comput. Harmon. Anal. **8** (2000), 166–196.

[BrJo97] O. Bratteli and P.E.T. Jorgensen, *Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale* N, Integral Equations Operator Theory **28** (1997), 382–443.

[BrJo98] O. Bratteli and P.E.T. Jorgensen, *A connection between multiresolution wavelet theory of scale* N *and representations of the Cuntz algebra* $\mathcal{O}_N$, Operator Algebras and Quantum Field Theory (Rome, 1996) (S. Doplicher, R. Longo, J.E. Roberts, and L. Zsido, eds.), International Press, 1998, pp. 151–155.

[BrJo99] O. Bratteli and P.E.T. Jorgensen, *Convergence of the cascade algorithm at irregular scaling functions*, The Functional and Harmonic Analysis of Wavelets and Frames (San Antonio, 1999) (L.W. Baggett and D.R. Larson, eds.), Contemp. Math., vol. 247, American Mathematical Society, Providence, 1999, pp. 93–130.

[BrJo00] O. Bratteli and P.E.T. Jorgensen, *Wavelet filters and infinite-dimensional unitary groups*, preprint, 2000, http://arXiv.org/abs/math.FA/0001171 .

[BJKW00] O. Bratteli, P.E.T. Jorgensen, A. Kishimoto, and R. Werner, *Pure states on* $\mathcal{O}_d$, J. Operator Theory **43** (2000), 97–143.

[BJO99] O. Bratteli, P.E.T. Jorgensen, and V. Ostrovs'kyĭ, *Representation theory and numerical AF-invariants: The representations and centralizers of certain states on* $\mathcal{O}_d$, submitted to Mem. Amer. Math. Soc.

[Cun77] J. Cuntz, *Simple* C^**-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[Dau92] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[Gli60] J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.

[Jor00] P.E.T. Jorgensen, *Minimality of the data in wavelet filters*, preprint, 2000, The University of Iowa; submitted to Advances in Mathematics, http://arXiv.org/abs/math.FA/0004098 .

[Kad84] Richard V. Kadison, *Diagonalizing matrices*, Amer. J. Math. **106** (1984), 1451–1468.

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