Quantum diffusion and generalized fractal dimensions: The continuous case $L^2(\mathbb{R}^d)$.

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June 2000, revised version September 2000

Abstract

We estimate the spreading of the solution of the Schrödinger equation asymptotically in time, in term of the fractal properties of the associated spectral measures. For this, we exhibit a lower bound for the moments of order p at time T, for the state ψ, defined by

$$\frac{1}{T}\int_0^T \| \, |X|^{p/2} \mathrm{e}^{-itH}\psi\|^2 \, \mathrm{d}t \ .$$

We show that this lower bound can be expressed in term of the generalized fractal dimensions of the spectral measure μ_ψ associated to the Hamiltonian H and the state ψ. We especially focus on continuous models.

Keywords: Schrödinger operators, quantum dynamics, spectral measures, fractal dimensions.
1991 Mathematics Subject Classification. Primary: 81Q10, 28A80; Secondary: 35J10.

1 Introduction and brief review

The dynamical properties of quantum electrons in solid media are given by the spreading of the wave packet ψ_t, solution at time t of the Schrödinger equation $i\partial_t\psi_t = H\psi_t$ with initial

*J.M.-B. gratefully acknowledged GDR 1151 du CNRS and the M.E.N.R.T. through the project ACI Blanche for financial support.

condition $\psi_{t=0} = \psi$. The Hamiltonian H of the system is a self-adjoint operator acting on a separable Hilbert space $\mathcal{H}$, and ψ is fixed in $\mathcal{H}$. In our case, we consider $\mathcal{H}$ to be equal either to $L^2(\mathbb{R}^d)$ for continuous models or $\ell^2(\mathbb{Z}^d)$ for discrete one (tight-binding approximation).

A good estimate of the spreading of ψ_t is given by the averaged moments of order p, $(p > 0)$

$$\langle\langle |X|^p \rangle_\psi\rangle(T) := \frac{1}{T}\int_0^T \| \, |X|^{p/2} \mathrm{e}^{-itH}\psi\|^2 \, \mathrm{d}t \ , \tag{1}$$

and its increasing exponents, the so-called lower and upper diffusion exponents

$$\beta_p^-(\psi) := \liminf_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle_\psi\rangle(T)}{\log T} \quad \text{and} \quad \beta_p^+(\psi) := \limsup_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle_\psi\rangle(T)}{\log T} \ .$$

In the particular case $p = 2$, a behaviour $\langle\langle |X|^p\rangle_\psi\rangle(T) \sim T^2$ is representative of a ballistic motion, i.e. the motion of a free electron; if $\langle\langle |X|^p\rangle_\psi\rangle(T) \sim T$, we have a diffusive dynamics, whereas $\sup_{T>0}\langle\langle |X|^p\rangle_\psi\rangle(T) < \infty$ is characteristic of the dynamical localization.

The first rigorous result connecting the moments of order p and the fine properties of spectral measures is due to Guarneri ([11]) for discrete models and was extended by Combes [7] to continuous one. In the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, the result can easily be stated.

Guarneri-Combes Theorem [11, 7]. *Let $H = H^*$ be a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. Let $\psi \in \ell^2(\mathbb{Z}^d)$ be the initial state, $\|\psi\| = 1$. If the spectral measure μ_ψ associated to ψ and H verifies*

$$\text{There exists } \alpha \in [0,1] \text{ such that } \sup_{x\in\mathbb{R}} \left(\sup_{\varepsilon\in(0,1)} \left(\frac{\mu_\psi(x-\varepsilon, x+\varepsilon)}{\varepsilon^\alpha} \right) \right) < \infty \ , \tag{2}$$

then

$$\beta_p^-(\psi) \geq \alpha\frac{p}{d} \ . \tag{3}$$

The proof of this result is simple. We sketch it here since more recent results, including those presented in this paper, use similar approaches. It is based on a Strichartz estimate ([17]) connecting the local properties (2) of the spectral measure μ_ψ with the Fourier transform of measures absolutely continuous with respect to μ_ψ, with density in $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Namely, if μ_ψ verifies (2), then there exists $C < \infty$ such that for any $f \in L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, $\|f\|_2 \leq 1$, and for all $T > 1$

$$\frac{1}{T}\int_0^T |\widehat{f\mathrm{d}\mu_\psi}|^2(t)\,\mathrm{d}t := \frac{1}{T}\int_0^T \left| \int_{\mathbb{R}} \mathrm{e}^{-itx} f(x)\,\mathrm{d}\mu_\psi(x) \right|^2 \mathrm{d}t \leq CT^{-\alpha} \ . \tag{4}$$

A very short proof of this result can be found in [15].

For a given $N > 1$, one now estimates the time averaged probability $B_\psi(T,N)$ of finding the electron at time T in the ball centered at the origin and of radius N

$$B_\psi(T,N) := \frac{1}{T}\int_0^T \sum_{|n|\leq N} |\langle \psi_t, \, e_n\rangle|^2 \, \mathrm{d}t \ ,$$

where $(e_n)_{n\in\mathbb{Z}^d}$ is the canonical basis of $\ell^2(\mathbb{Z}^d)$ (remark that for technical reasons, $B_\psi(T,N)$ is defined slightly differently in (29) of Section 4). Noting that for all $n \in \mathbb{Z}^d$ there exists $f_n \in L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, $\|f_n\|_2 \leq 1$, such that ([1]) $\langle \psi_t, e_n\rangle = \int_{\mathbb{R}} \mathrm{e}^{-itx} f_n(x)\,\mathrm{d}\mu_\psi(x)$, we get by using (4) the following rough estimate

$$B_\psi(T,N) \leq 2^d C N^d T^{-\alpha} \ . \tag{5}$$

Now, the idea is to pick N not too large so that $B_\psi(T,N)$ remains "small enough". Therefore, a "sufficiently large" part of the wave packet ψ_t "lives" at a distance at least N of the origin, and contributes significantly to $\langle\langle |X|^p\rangle_\psi\rangle(T)$. More precisely

$$\begin{aligned}\langle\langle |X|^p\rangle_\psi\rangle(T) &\overset{\ell^2(\mathbb{Z}^d)}{=} \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} |n|^p |\langle \psi_t, e_n\rangle|^2\,\mathrm{d}t \\ &\geq \frac{1}{T}\int_0^T N^p \sum_{|n|>N} |\langle \psi_t, e_n\rangle|^2\,\mathrm{d}t = N^p\left(1 - B_\psi(T,N)\right) \ .\end{aligned} \tag{6}$$

Choosing $N = (T^\alpha/2^{d+1}C)^{1/d}$ in (5) gives $1 - B_\psi(T,N) \geq 1/2$, therefore, from (6), we obtain $\langle\langle |X|^p\rangle_\psi\rangle(T) \geq cT^{\alpha p/d}$ and thus (3) follows.

The first major improvement of this result is due to Last ([15]) and leads under the same assumption as in the previous Theorem to

Last Theorem [15]. *For $H = H^*$ on $\ell^2(\mathbb{Z}^d)$, $\psi \in \ell^2(\mathbb{Z}^d)$, $\|\psi\| = 1$.*

$$\beta_p^-(\psi) \geq \dim_H(\mu_\psi)\frac{p}{d} \ , \tag{7}$$

where $\dim_H(\mu_\psi)$ is the Hausdorff dimension of the spectral measure μ_ψ ([10]

This theorem improves the previous one since the Hausdorff dimension of μ_ψ is not smaller than the largest α such that (2) holds true. A statement for continuous models is also provided in [15].

The basic strategy to prove (7) remains essentially the same as above, especially the "trick" (6). The main improvement comes from the estimate of $1 - B_\psi(T,N)$; dividing spectraly ψ into two orthogonal states φ and χ, i.e. $\varphi = E(I)\psi$, $\chi = E(\mathbb{R}\setminus I)\psi$, for $E(\,.\,)$ spectral family of H and I some Borel set, one estimates $1 - B_\psi(T,N)$ by $1 - B_\varphi(T,N)$. In other words, the spreading of ψ is roughly given by the one of a piece of ψ. The aim is thus to extract the most relevant spectral information of ψ for the dynamics, by fixing a state $\varphi = E(I)\psi$ such that it spreads "fast", i.e., such that μ_φ verifies (2) with a larger α than the one for μ_ψ.

This optimal choice for φ finally leads to an α equals to $\dim_H(\mu_\psi)$, which is a dimension that characterizes only the "most continuous" part of μ_ψ (the reader should refer to [15] or [2] for details).

There exists a similar result to (7) established by Guarneri and Schulz-Baldes concerning the upper oscillations of $\langle\langle |X|^p\rangle_\psi\rangle(T)$ stating that (see [12, 6])

$$\beta_p^+(\psi) \geq \dim_P(\mu_\psi)\frac{p}{d} \ , \tag{8}$$

where $\dim_P(\mu_\psi) \geq \dim_H(\mu_\psi)$ is the packing dimension of μ_ψ ([19]).

The bounds (7) and (8) ought to be very satisfactory since apparently, very refined properties of μ_ψ are involved. However, we are forced to conclude that in some cases, these bounds are far from being optimal. This remark is based upon two different observations. First it is known from examples in [9] and [2] that $\beta_p^+(\psi)$ and $\beta_p^-(\psi)$ can be strictly positive whereas μ_ψ is pure point and thus its packing and Hausdorff dimensions equal zero. Second, some numerics performed by Mantica in [16] for one dimensional models of Julia matrices suggest that the variation of the exponents $\beta_p^\pm(\psi)$ may increase in p faster than a linear function, whereas the given bounds (7) and (8) are both linear in p. This nonlinear behaviour is sometimes called "intermittency".

The lower bounds we have established for $\langle\langle |X|^p\rangle_\psi\rangle(T)$ in [3] for discrete models, and that we present here in details for continuous one, actually appear to be "intermittent". These bounds reads as follow (see Theorem 1 and Theorem 2 for a precise statement)

$$\beta_p^-(\psi) \geq D_{\mu_\psi}^-\left(\frac{1}{1+p/d}\right)\frac{p}{d} \quad \text{and} \quad \beta_p^+(\psi) \geq D_{\mu_\psi}^+\left(\frac{1}{1+p/d}\right)\frac{p}{d}\ , \tag{9}$$

where $D_{\mu_\psi}^\pm$ are the so called *generalized Rényi dimensions* of the measure μ_ψ. The monotonicity property $D_{\mu_\psi}^\pm(q) \geq D_{\mu_\psi}^\pm(q')$, if $q < q'$, implies that the lower bounds (9) are intermittent. Furthermore since for all $q \in (0,1)$, $D_{\mu_\psi}^-(q) \geq \dim_H(\mu_\psi)$ and $D_{\mu_\psi}^+(q) \geq \dim_P(\mu_\psi)$, the result (9) improves notably (7) and (8). In particular it is possible to construct a self-adjoint operator H on $\ell^2(\mathbb{Z})$ and a state $\psi \in \ell^2(\mathbb{Z})$ such that μ_ψ is pure point and $D_{\mu_\psi}^\pm(\frac{1}{1+p/d}) > 0$ for all $p > 1$, giving rise to a nontrivial lower bound in (9).

The proof is essentially performed in two steps. We sketch it here in the discrete case; in the continuous case it is basically the same and it is detailed in the next sections.

The first step (Lemma 1 and 2) consists in estimating sharply $1 - B_\psi(T,N)$ by an integral expression involving μ_ψ and μ_φ, where $\varphi = E(I)\psi$ for some Borel set I. The estimate is roughly

$$1 - B_\psi(T,N) \geq \|\varphi\|^2 - CN^{d/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2T^2/4}\,\mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\psi(y)\right)^{1/2}\ .$$

Actually, this is a more complicated function than the gaussian $\mathrm{e}^{-(x-y)^2T^2/4}$ that enters the game, due to the fact that the time averaged in (1) is between 0 and T instead of $-T$ and T. Picking the largest N so that $1 - B_\psi(T,N) \geq \|\varphi\|^2/2$, using the "trick" given in (6) and optimizing the obtained result *at fixed* T over all possible φ, leads to the lower bound stated in Lemma 2

$$\langle\langle |X|^p\rangle_\psi\rangle(T) \geq CL_\psi(T)\ ,$$

where $L_\psi(T)$ is a function depending on p, d, ψ and T and involving local properties of the measure μ_ψ (see (26) for the exact definition).

Compared to above methods, the new idea in this step is to optimize in φ at a given time T as in [6], instead of optimizing in φ uniformly in T as done in [15, 2].

The second step is the most intricated one. It amounts to prove that the increasing exponents of $L_\psi(T)$ are $D_{\mu_\psi}^\pm(\frac{1}{1+p/d})\,\frac{p}{d}$. This is done by performing a fine analysis of $L_\psi(T)$ and with a good understanding of what part of the measure μ_ψ contributes essentially to the dimensions $D_{\mu_\psi}^\pm(q)$ for $q \in (0,1)$ (see Lemma 3 and 4).

The paper is organized as follows: in Section 2 we define the generalized Rényi dimensions $D_\mu^\pm(q)$ of a measure μ and state our two main results. The first one is an extension of Theorem 2.1 in [3] in the case of spectral measures whose support is not necessarily compact and when $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. Theorem 2 is a statement of Theorem 2.1 in [3] for continuous Schrödinger operators on $L^2(\mathbb{R}^d)$. In this case, the proof given in [3] needs some modifications presented here. In Section 3, we give some general properties for the generalized Rényi dimensions that will be used in the proof of the main results. Most of them are not proven here since details can be found in [3] and [4]. The last section is devoted to technical but crucial lemmas needed to prove the theorems of Section 2.

2 Main results

We start with the definition of the main object we need to state our theorems, namely the generalized Rényi dimensions.

Definition.
Let μ be a (positive) Borel probability measure. Let $q \in (0,1)$ and $\varepsilon \in (0,1)$. We consider the following function with values in $(0,\infty]$

$$I_\mu(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mu([x-\varepsilon, x+\varepsilon))^{q-1} \mathrm{d}\mu(x).$$

The lower and upper generalized Rényi dimensions of μ are respectively defined as

$$D_\mu^-(q) = \frac{1}{1-q} \liminf_{\varepsilon \to 0} \frac{\log I_\mu(q,\varepsilon)}{-\log \varepsilon} \quad \text{and} \quad D_\mu^+(q) = \frac{1}{1-q} \limsup_{\varepsilon \to 0} \frac{\log I_\mu(q,\varepsilon)}{-\log \varepsilon}. \tag{10}$$

with the understanding that both are $+\infty$, if for some $\varepsilon > 0$, I_μ takes the value $+\infty$.

Remark.
i) Strictly speaking, the Rényi dimensions are defined with a discrete sum instead of an integral for $I_\mu(q,\varepsilon)$. However, this leads to the same result since $q > 0$ (see [4]).
ii) Actually, these dimensions can be defined for all $q \in \mathbb{R}\backslash\{1\}$. For our purpose, it is sufficient to discuss the case $q \in (0,1)$.

Theorem 1. *Let H be a self-adjoint operator acting on $\ell^2(\mathbb{Z}^d)$. Let $\psi \in \ell^2(\mathbb{Z}^d)$, $\|\psi\| = 1$, and let $\tilde{q}_\mu := \inf\{q > 0 \,|\, D_\mu^+(q) < +\infty\}$. Then, for all $p \in (0, d(1-\tilde{q}_\mu)/\tilde{q}_\mu)$, we have*

$$\liminf_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \frac{p}{d} D_{\mu_\psi}^- \left(\frac{1}{1+p/d}\right),$$

and

$$\limsup_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \frac{p}{d} D_{\mu_\psi}^+ \left(\frac{1}{1+p/d}\right).$$

Remark. If μ_ψ has compact support, then $\tilde{q}_\mu = 0$ and the above result is thus valid for all $p > 0$ (Proposition 1, iv)).

In the next theorem, we consider continuous models. In this case, a similar result as Theorem 1 holds if we restrict ourselves to Schrödinger operators $H = -\Delta + V$ and under some assumptions on the potential V.

Remind that a real-valued measurable function W on $\mathbb{R}^d$ is said to lie in the Kato class K_d if and only if ([18])

$$\begin{array}{ll} \text{If } d \geq 3 & \lim_{\alpha \searrow 0} \left[\sup_{X \in \mathbb{R}^d} \int_{|X-Y| \leq \alpha} |X-Y|^{-(d-2)} |W(Y)| \, dY \right] = 0 \ , \\ \text{if } d = 2 & \lim_{\alpha \searrow 0} \left[\sup_{X \in \mathbb{R}^2} \int_{|X-Y| \leq \alpha} \ln(|X-Y|^{-1}) |W(Y)| \, dY \right] = 0 \ , \\ \text{if } d = 1 & \sup_{X \in \mathbb{R}} \int_{|X-Y| \leq 1} |W(Y)| \, dY < \infty \ . \end{array}$$

We say that W is in K_d^{loc} if an only if $W \chi_l$ is in K_d for all $l > 0$, where χ_l is the characteristic function of the ball in $\mathbb{R}^d$ of radius l centered at the origin .

The theorem is as follows

Theorem 2. *Let $H = -\Delta + V$ be a self-adjoint Schrödinger operator acting on $L^2(\mathbb{R}^d)$. We assume that the positive part and the negative part of the potential V verify $V_+ \in K_d^{\mathrm{loc}}$ and $V_- \in K_d$. Pick a state $\psi \in L^2(\mathbb{R}^d)$, $\|\psi\| = 1$, such that for some bounded interval $I \subset \mathbb{R}$, $E(I)\psi = \psi$ (where $E(\,.\,)$ is the spectral family of H), or equivalently μ_ψ is compactly supported. Then, for all $p > 0$, we have*

$$\liminf_{T \to \infty} \frac{\log \langle \langle |X|^p \rangle_\psi \rangle (T)}{\log T} \geq \frac{p}{d} D_{\mu_\psi}^{-} \left(\frac{1}{1 + p/d} \right) , \tag{11}$$

and

$$\limsup_{T \to \infty} \frac{\log \langle \langle |X|^p \rangle_\psi \rangle (T)}{\log T} \geq \frac{p}{d} D_{\mu_\psi}^{+} \left(\frac{1}{1 + p/d} \right) . \tag{12}$$

Remark.

i) The assumption on bounded support for the spectral measure μ_ψ is crucial in the continuous case $\mathcal{H} = L^2(\mathbb{R}^d)$ since we use the Lemma 2 in the proof of Theorem 2. Therefore, we need the Hilbert-Schmidt norm estimate (31). Very likely, this assumption can be weakened as soon as a similar relation to (31) holds (see e.g. [7] for other assumptions). In the other lemmas (1, 3 and 4), the compact support assumption is not necessary.

ii) There are not many time-independent Schrödinger operators for which we know that the diffusion exponents $\beta_p^{\pm}$ are not trivial (i.e. different from 0, 1 and 2). To our best knowledge, there exist only two.

The first is a model of Julia matrix H on $\ell^2(\mathbb{N})$ treated in [5]. For this model, the upper and lower fractal dimensions $D_\mu^{\pm}(q)$ are equal ($:=D_\mu(q)$), and continuous for $q \in (0,1)$. Theorem 1 of [5] states that there exists $p_c > 2$ such that for all $0 < p < p_c$, $\beta_p^{+}(\delta_0) \leq D_{\mu_{\delta_0}}(1-p)$, where δ_0 is the state in $\ell^2(\mathbb{N})$ located at $n = 0$. Furthermore, we have $D_\mu(1/(1+p)) = D_\mu(1-p) + o(p)$ (see [5] and references therein). Therefore, putting together the above Theorem 1 and Theorem 1 of [5], we get, for the exponents of any moment of order $0 < p < p_c$ and for the initial state δ_0, that

$$D_{\mu_{\delta_0}} \left(\frac{1}{1+p} \right) \leq \beta_p^{-}(\delta_0) \leq \beta_p^{+}(\delta_0) \leq D_{\mu_{\delta_0}}(1-p) \ .$$

and thus for small $p > 0$,

$$D_{\mu_{\delta_0}}(1-p) + o(p) \le \beta_p^-(\delta_0) \le \beta_p^+(\delta_0) \le D_{\mu_{\delta_0}}(1-p) \ .$$

This is the first model treated rigorously for which intermittency is now established, at least for small p.

The second models form a class of discrete Schrödinger operators of the form $H = -\Delta_d + V$ acting on $\ell^2(\mathbb{N} \setminus \{0\})$, where Δ_d stands for the discrete Laplacian and V is a sparse barrier potential. Roughly speaking, sparse potentials are positive potentials equal zero on very large regions and are very high in between these regions. The interplay between the distribution and width of "zero regions" and the height of V on "nonzero regions", give rise in some cases to non trivial spectra.

The fine spectral properties of such models where analyzed by Last and Jitomirskaya in [14] who proved that for all $\alpha \in (0,1)$, one can construct a sparse potential V and an initial state ψ in $\ell^2(\mathbb{N} \setminus \{0\})$ such that $\dim_H(\mu_\psi) = \alpha$. Recent results due to Combes and Mantica pushed further the analysis. They established that $\dim_P(\mu_\psi) = 1$, and a direct estimate of the lower diffusion exponents: For all $p \in (0,2]$,

$$\beta_p^-(\psi)/p \le \alpha \frac{p+1}{\alpha p + 1} \ .$$

According to these results and the above theorems it turns out that $\alpha \le \beta_p^-(\psi)/p < \beta_p^+(\psi)/p = 1$, and if p is small enough (α being fixed), we are sure that $\alpha + \mathcal{O}(p) = \beta_p^-(\psi)/p$. Furthermore if one consider e.g. $p = 2$, we get $\beta_2^-(\psi) \in (2\alpha, 6\alpha/(2\alpha+1)) \subset (0, 1/2)$ if $\alpha < 1/4$, giving rise, for the lower oscillations of $\langle\langle |X|^2 \rangle_\psi\rangle(T)$, to subdiffusive transport, and therefore to anomalous transport.

iii) A result similar to Theorem 1 was proved simultaneously and independently in [13] under stronger assumptions. The approach is also interesting since in this case, the main tool is the so-called function f_{μ_ψ}, singularity spectrum of μ_ψ. However, due to their assumptions, the result allows only to treat a restricted class of Hamiltonian H, and is presented in the specific case $\mathcal{H} = \ell^2(\mathbb{N})$.

Proof. The proof of Theorem 1 can be found in [3], and can also be easily recovered from the one of Theorem 2 which is as follows: Let p be in $(0, \infty)$. We fix the function h to be equal to $h(x) = a\chi_{]0,1[}\mathrm{e}^{-1/(x(x-1))^2}$, with $a = (\int_0^1 \mathrm{e}^{-1/(x(x-1))^2}\,\mathrm{d}x)^{-1}$, so that $h \in C_0^\infty([0,1])$ and $\int_0^1 h(z)\mathrm{d}z = 1$. Under the assumptions on H and ψ, we know from Lemma 2 that there exists a strictly positive constant $C = C(p,h,d)$ such that for all $T > 0$

$$\langle\langle |X|^p \rangle_\psi\rangle(T) \ge C L_\psi(T) \ . \tag{13}$$

(Note that $L_\psi(T)$ depends on a function R defined by using h as in (27). Now remind that ψ is such that $E(I)\psi = \psi$, for I bounded interval. Therefore, the spectral measure μ_ψ associated to ψ and H is compactly supported, and thus $\tilde{q}_\mu := \inf\{q \in (0,1) \,|\, D_\mu^+(q) < \infty\} = 0$ (see iv) of Proposition 1), and Lemma 4 holds for all $p \in (0,\infty)$, with R defined by (27). This implies the existence of $c' = c'(\psi, p, R) > 0$ such that for all $\varepsilon \in (0,1)$

$$L_\psi(\varepsilon^{-1}) \ge \frac{c'}{|\log\varepsilon|^{1+p/d}} K_{\mu_\psi}^{(R)}\left(\frac{1}{1+p/d}, \varepsilon\right)^{1+p/d} \ , \tag{14}$$

where $K^{(R)}_{\mu_\psi}$ is defined by (16) in Proposition 2. We conclude by using Proposition 2 giving an equivalent definition of the generalized Rényi dimensions by help of the function $K^{(R)}_{\mu_\psi}$; more precisely, we get for some constant $c(q) > 0$ and for all $T > 0$

$$K^{(R)}_{\mu_\psi}\left(\frac{1}{1+p/d}, \varepsilon\right) \geq c(q)^{-1} I_{\mu_\psi}\left(\frac{1}{1+p/d}, \varepsilon\right) \ . \tag{15}$$

Inequalities (13)-(15) together with the definition (10) of $D^{\pm}_{\mu_\psi}(q)$ with $q = \frac{1}{1+p/d}$ gives (11) and (12).

3 Properties of generalized Rényi dimensions

We first state some basic properties of the generalized Rényi dimensions. As already mentionned in the introduction, properties ii) and iii) show that our results in Section 1 are improvement of previous one ([11, 15, 2, 12]). Property i) emphasizes the possible nonlinear variation in p of the diffusion exponents $\beta^{\pm}_p$.

Proposition 1. *Let μ be a Borel probability measure.*
i) $D^-_\mu(q)$ and $D^+_\mu(q)$ are nonincreasing continuous functions of $q \in (0,1)$.
ii) For all $q \in (0,1)$, $D^-_\mu(q) \geq d_H(\mu)$.
iii) For all $q \in (0,1)$, $D^+_\mu(q) \geq d_P(\mu)$.
iv) If μ has a bounded support, then for all $q \in (0,1)$, $0 \leq D^-_\mu(q) \leq D^+_\mu(q) \leq 1$.

The next propostion gives an equivalent definition of the dimension $D^{\pm}_\mu(q)$, by replacing the characterictic function $\chi_{[-1,1]}$ in $\mu(x-\varepsilon, x+\varepsilon) = \int_{\mathbb{R}} \chi_{[-1,1]}((x-y)/\varepsilon)\, \mathrm{d}\mu(y)$ by a function R constant on $[-1,1]$ and decaying fast at infinity. This result allows us to deal with time-averaged between 0 ans T in the definition of $\langle\langle |X|^p\rangle_\psi\rangle(T)$, instead of time averaged between $-T$ and T as e.g. in [13].

Proposition 2. *Let R be a real function on $\mathbb{R}$ such that for all $x \in [-1,1]$, $R(x) = 1$, $\sup_{x\in\mathbb{R}} |R(x)| \leq 1$, and R decays faster than any polynomials at infinity. We define the following function*

$$K^{(R)}_\mu(q,\varepsilon) = \int_{\mathrm{supp}\mu} \left(\int_{\mathbb{R}} R((x-y)\varepsilon^{-1}) \mathrm{d}\mu(y)\right)^{q-1} \mathrm{d}\mu(x) \ , \tag{16}$$

then there exists a constant $c(q)$ such that for all $\varepsilon \in (0,1)$, we have

$$c(q)^{-1} K^{(R)}_\mu(q,\varepsilon) \leq \int_{\mathrm{supp}\mu} \mu([x-\varepsilon, x+\varepsilon))^{q-1} \mathrm{d}\mu(x) \leq c(q) K^{(R)}_\mu(q,\varepsilon) \ .$$

The proof of this result is based on an equivalent definition of (10) proven in [4] using the function $\sum_{k\in\mathbb{Z}} \mu([k\varepsilon,\ (k+1)\varepsilon))^q$ instead of $I_\mu(q,\varepsilon)$. See also [3] for an alternative proof in the case of measure μ with compact support.

The last proposition is a technical one used in the proof of Lemma 3 and Lemma 4.

Proposition 3. *Given a probability measure μ, fix a real $q \in (0,1)$ and a function R with the same properties as in Proposition 2. Denote by*

$$b(x,\varepsilon) := \int_{\mathrm{supp}\mu} R((x-y)\varepsilon^{-1}) \mathrm{d}\mu(y) \ .$$

Assume that there exists $q' < q$ such that $D_\mu^+(q') < \infty$; then there exists two constants $\alpha = \alpha(q,q',R) > 0$ and $C = C(q,q',R) < \infty$ dependins only on q, q' and R, such that for all $\varepsilon \in (0,1)$

$$\int_{\{x \in \mathrm{supp}\mu \,|\, b(x,\varepsilon) \le \varepsilon^\alpha\}} b(x,\varepsilon)^{q-1} \mathrm{d}\mu(x) \le C\varepsilon \ . \tag{17}$$

Proof of proposition 3. By definition of $D_\mu^+(q')$ and with Proposition 2, for all ε small enough, we have

$$\int_{\mathrm{supp}\mu} b(x,\varepsilon)^{q'-1} \mathrm{d}\mu(x) \le \varepsilon^{\frac{3}{2}(q'-1)D_\mu^+} \ .$$

Therefore, for all ε small enough

$$\begin{aligned} \int_{\{x \in \mathrm{supp}\mu \,|\, b(x,\varepsilon) \le \varepsilon^\alpha\}} b(x,\varepsilon)^{q-1} \mathrm{d}\mu(x) &= \int_{\{x \in \mathrm{supp}\mu \,|\, b(x,\varepsilon) \le \varepsilon^\alpha\}} b(x,\varepsilon)^{q'-1} b(x,\varepsilon)^{q-q'} \mathrm{d}\mu(x) \\ &\le \varepsilon^{(q-q')\alpha + \frac{3}{2}(q'-1)D_\mu^+(q')} \le \varepsilon \ , \end{aligned}$$

for $\alpha = (1 - 3/2(q'-1)D_\mu^+(q'))(q-q')$.

4 Technicalities

This section is devoted to technical lemmas needed in the proof of Theorem 1 and Theorem 2. The first two lemmas deal with estimates for $\langle\langle |X|^p\rangle_\psi\rangle(T)$ and $B_\psi(T,N)$ in term of a function L_ψ involving local properties of the measure μ_ψ. Lemmas 3 and 4 show how this function L_ψ is connected with the generalized Rényi dimensions $D^\pm_{\mu_\psi}$.

Lemma 1. *Let H be a self-adjoint operator acting on a separable Hilbert space $\mathcal{H}$. Let A be a Hilbert-Schmidt operator in $\mathcal{H}$. For all given vectors ψ, φ in $\mathcal{H}$ and for all function h in $L^1(\mathbb{R})$, we define for $T > 0$ the two quantities*

$$D^{(h)}_{\varphi,\psi}(T) = \frac{1}{T}\int_{\mathbb{R}} \langle A\mathrm{e}^{-itH}\varphi, \mathrm{e}^{-itH}\psi\rangle h(t/T) \mathrm{d}t \ , \tag{18}$$

$$U^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{h}((x-y)T)|^2 \, \mathrm{d}\mu_\varphi(x) \mathrm{d}\mu_\psi(y). \tag{19}$$

Then, for all $T > 0$, we have

$$|D^{(h)}_{\varphi,\psi}(T)| \le \|A\|_2 \left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}} \ ,$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of A.

Proof of Lemma 1. Since A is Hilbert-Schmidt, there exist two orthonormal bases $\{\zeta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and a decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$, $E_n \geq 0$, such that $\sum_{n=1}^{\infty} E_n^2 = \|A\|_2^2 < +\infty$ and $A = \sum_{n=1}^{\infty} E_n \langle\cdot, \zeta_n\rangle \xi_n$. Therefore,

$$D_{\varphi,\psi}^{(h)}(T) = \frac{1}{T}\int_{-\infty}^{+\infty} \sum_{n=1}^{\infty} E_n \langle \mathrm{e}^{-itH}\varphi, \zeta_n\rangle \langle \xi_n, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{d}t. \tag{20}$$

Furthermore, there exists $f_n \in L^2(\mathbb{R}, \mathrm{d}\mu_\varphi)$ and $g_n \in L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ such that ([1])

$$\langle \mathrm{e}^{-itH}\varphi, \zeta_n\rangle = \int_{\mathbb{R}} \mathrm{e}^{-itx}\overline{f_n(x)}\,\mathrm{d}\mu_\varphi(x)\ , \text{ and } \langle \xi_n, \mathrm{e}^{-itH}\psi\rangle = \int_{\mathbb{R}} \mathrm{e}^{itx} g_n(x)\mathrm{d}\mu_\psi(x)\ . \tag{21}$$

Thus

$$D_{\varphi,\psi}^{(h)}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{h}((x-y)T)S(x,y)\,\mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)\ , \tag{22}$$

where

$$S(x,y) = \sum_{n=1}^{\infty} E_n \overline{f_n(x)} g_n(y).$$

The sum converges in $L^2(\mathbb{R}^2, \mu_\varphi \times \mu_\psi)$. Applying the Cauchy-Schwarz inequality to (22), one gets

$$|D_{\varphi,\psi}^{(h)}(T)|^2 \leq U_{\varphi,\psi}^{(h)}(T)\|S\|^2_{L^2(\mathbb{R}^2,\,\mathrm{d}\mu_\varphi\times\mathrm{d}\mu_\psi)}\ . \tag{23}$$

One has

$$\|S\|^2_{L^2(\mathbb{R}^2,\mathrm{d}\mu_\varphi\times\mathrm{d}\mu_\psi)} = \sum_{n,k=1}^{\infty} E_n E_k a_{nk}\overline{b_{nk}}\ ,$$

where

$$a_{nk} = \int_{\mathbb{R}} f_k(x)\overline{f_n(x)}\,\mathrm{d}\mu_\varphi(x) = \langle P_\varphi \zeta_k, \zeta_n\rangle_{\mathcal{H}}\ , \tag{24}$$

where P_φ is the projection onto the cyclic subspace $\mathcal{H}_\psi$ spanned by H and φ; similarly, we have

$$b_{nk} = \langle P_\psi \xi_k, P_\psi \xi_n\rangle_{\mathcal{H}} = \langle \xi_k, P_\psi \xi_n\rangle_{\mathcal{H}}.$$

We have also used the fact that both P_φ and P_ψ are orthogonal projections. By Parseval equality,

$$\sum_{n=1}^{\infty} |a_{nk}|^2 = \|P_\varphi \zeta_k\|^2 \quad \text{and} \quad \sum_{k=1}^{\infty} |b_{nk}|^2 = \|P_\psi \xi_n\|^2\ .$$

Therefore, as $\|\zeta_k\| = \|\xi_n\| = 1$ for all k, n,

$$\|S\|^4_{L^2(\mathbb{R}^2,\mathrm{d}\mu_\varphi\times\mathrm{d}\mu_\psi)} \leq \sum_{k=1}^{\infty} E_k^2 \|P_\varphi\zeta_k\|^2 \sum_{n=1}^{\infty} E_n^2\|P_\psi \xi_n\|^2 \leq \left(\sum_{n=1}^{\infty} E_n^2\right)^2 = \|A\|_2^4\ . \tag{25}$$

The statement of the Lemma follows from (23) and (25).

For R being a function as in Proposition 2, and for $p > 0$ we define the function $L_\psi(T)$ as

$$L_\psi(T) := \sup\left\{L_\psi(\varphi, T)\,|\,\varphi \in \mathcal{H}_\psi,\ \langle\psi,\,\varphi\rangle \neq 0\right\}\ ,$$

with

$$L_\psi(\varphi, T) := \frac{|\langle\varphi,\, \psi\rangle|^{2+2p/d}}{\|\varphi\|^2 \tilde{U}^{(R)}_{\varphi,\psi}(T)^{p/d}}$$

and

$$\tilde{U}^{(R)}_{\varphi,\psi}(T) := \int_{\mathbb{R}}\int_{\mathbb{R}} R((x-y)T)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \ , \tag{26}$$

and where $\mathcal{H}_\psi$ is the cyclic subspace spanned by ψ and H

Lemma 2. *Let H be a self-adjoint Schrödinger operator defined on $\mathcal{H} = L^2(\mathbb{R}^d)$ or $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, and let ψ be a vector in $\mathcal{H}$, $\|\psi\| = 1$. If $\mathcal{H} = L^2(\mathbb{R}^d)$ we further assume that $H = -\Delta + V$, where V fulfil the same assumptions as in Theorem 2, and we assume that there exists a bounded interval I such that $E(I)\psi = \psi$, with $E(\,.\,)$ being the spectral family of H. The function $\langle\langle |X|^p\rangle_\psi\rangle(T)$ is defined by (1). Let $h \in C_0^\infty([0,1])$ be a given positive function such that $\int_0^1 h(z)dz = 1$. We suppose that R verifies*

$$R(w) = \begin{cases} 1 & \text{if } |w| \leq 1 \\ |\hat{h}(w)|^2 & \text{if } |w| > 1, \end{cases} \tag{27}$$

where $\hat{h}$ stands for the Fourier transform of h. Then there exists a strictly positive constant $C(\psi, p, h)$ such that for all $T > 1$

$$\langle\langle |X|^p\rangle\rangle_\psi(T) \geq C\, L_\psi(T) \ .$$

Proof of Lemma 2. Pick a positive function $h \in C_0^\infty([0,1])$ such that $\int_0^1 h(z)\mathrm{d}z = 1$. The role of h is to supply a fast decaying function $|\hat{h}(w)|^2$. Note that one trivially has, for any $z \in [0,1]$, $h(z) \leq \|h\|_\infty \chi_{[0,1]}(z)$. Furthermore, denoting by $F_{\leq N}$ (respectively $F_{>N}$) the multiplication operator by the characteristic function of the closed ball of center 0 and radius N (resp. of the complementary of the closed ball of center 0 and radius N), and noting that for all vector $\phi \in \mathcal{H}$ we have $\|\phi\|^2 = \|F_{\leq N}\phi\|^2 + \|F_{>N}\phi\|^2$, we get

$$\begin{aligned}
\langle\langle X^p\rangle_\psi\rangle(T) &\geq \frac{1}{\|h\|_\infty}\int_{\mathbb{R}} \|\,|X|^{p/2}\mathrm{e}^{-itH}\psi\|^2 h\left(\frac{t}{T}\right)\frac{\mathrm{d}t}{T} \\
&\geq \frac{1}{\|h\|_\infty}\int_{\mathbb{R}} \|\,F_{>N}|X|^{p/2}\mathrm{e}^{-itH}\psi\|^2 h\left(\frac{t}{T}\right)\frac{\mathrm{d}t}{T} \\
&\geq \frac{N^p}{\|h\|_\infty}\int_{\mathbb{R}} \|\,F_{>N}\mathrm{e}^{-itH}\psi\|^2 h\left(\frac{t}{T}\right)\frac{\mathrm{d}t}{T} \\
&= \frac{N^p}{\|h\|_\infty}\left(\|\psi\|^2 - B_\psi(T,N)\right) \ ,
\end{aligned} \tag{28}$$

with $B_\psi(T,N)$ defined by

$$B_\psi(T,N) := \frac{1}{T}\int_{\mathbb{R}} h(t/T)\|\,F_{\leq N}\mathrm{e}^{-itH}\psi\|^2\mathrm{d}t \ . \tag{29}$$

We now need to control this quantity $B_\psi(T,N)$ which represents the time-averaged behaviour of the wave-packet in a ball of radius N. Decompose the vector ψ into $\varphi+\chi$, with $\langle\varphi,\chi\rangle=0$ and $\varphi\neq 0$. Thus

$$B_\psi(T,N)=-B_\varphi(T,N)+B_\chi(T,N)+\frac{2}{T}\operatorname{Re}\int_{\mathbb{R}}\langle F_{\le N}\mathrm{e}^{-itH}\varphi,\,F_{\le N}\mathrm{e}^{-itH}\psi\rangle h\left(\frac{t}{T}\right)\mathrm{d}t\ .$$

Taking into account that $1/T\int_0^\infty h(t/T)\mathrm{d}t=1$ and $h(z)\ge 0$, we have $B_\chi(T,N)\le\|\chi\|^2=\|\psi\|^2-\|\varphi\|^2$. Then, since $-B_\varphi(T,N)<0$,

$$B_\psi(T,N)\le\|\psi\|^2-\|\varphi\|^2+2\mathrm{Re}D^{(h)}_{\varphi,\psi}(T,N)\ ,$$

where $D^{(h)}_{\varphi,\psi}(T,N)$ is defined as in Lemma 1, with $A=F_{\le N}$ if $\mathcal{H}=\ell^2(\mathbb{Z}^d)$, and $A=E(I)F_{\le N}$ if $\mathcal{H}=L^2(\mathbb{R}^d)$. The statement of Lemma 1 gives immediately

$$B_\psi(T,N)\le\|\psi\|^2-\|\varphi\|^2+CN^{\frac{d}{2}}\left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}}\ , \tag{30}$$

where $U^{(h)}_{\varphi,\psi}(T)$ is also defined in Lemma 1.

In inequality (30) we have used the important fact that there exists a constant C such that for all $N\in\mathbb{N}$,

$$\|A\|_2\le CN^{d/2}\ , \tag{31}$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of the operator A. This estimate is easy to check in the case $\mathcal{H}=\ell^2(\mathbb{Z}^d)$, and is valid in the continuous case $\mathcal{H}=L^2(\mathbb{R}^d)$ due to the assumptions we made on the potential V and the fact that $E(I)\psi=\psi$ (see [15, Lemma 6.1] and [18] for details).

As $|\hat{h}(w)|\le 1$ for all w and by definition (27) of R, we clearly have $R(w)\ge|\hat{h}(w)|^2$ for all w. Therefore $U^{(h)}_{\varphi,\psi}(T)\le\tilde{U}^{(R)}_{\varphi,\psi}(T):=\int_{\mathbb{R}}\int_{\mathbb{R}}R((x-y)T)\,\mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)$, and the inequality (30) is valid if we replace $U^{(h)}_{\varphi,\psi}(T)=$ by $\tilde{U}^{(R)}_{\varphi,\psi}(T)$, that is with the function R instead of $|\hat{h}|^2$.

We are now in position to finish the proof. Recall that T is fixed. The basic strategy is standard: let N be the largest integer such that $CN^{d/2}\tilde{U}^{(R)}_{\varphi,\psi}(T)^{\frac{1}{2}}\le\|\varphi\|^2/2$, it yields

$$B_\psi(T,N)\le\|\psi\|^2-\frac{\|\varphi\|^2}{2}\ . \tag{32}$$

The inequalities (32) and (28) yield with some positive constant $C(\psi,p,h)$

$$\langle\langle|X|^p\rangle\rangle_{\psi,T}\ge C(\psi,p,h)\frac{\|\varphi\|^{2+4p/d}}{\tilde{U}^{(R)}_{\varphi,\psi}(T)^{p/d}}\ . \tag{33}$$

In the more general case (i.e. φ is any function of $\mathcal{H}$ with $\langle\psi,\varphi\rangle\neq 0$), one gets the bound with

$$L_\psi(\varphi,T):=\frac{|\langle\varphi,\psi\rangle|^{2+2p/d}}{\|\varphi\|^2\tilde{U}^{(R)}_{\varphi,\psi}(T)^{p/d}}\ . \tag{34}$$

Indeed take such a φ, and then define $\tilde{\varphi}=(\langle\varphi,\psi\rangle\|\varphi\|^{-2})\varphi$. One checks that if $\chi=\psi-\tilde{\varphi}$, then $\langle\tilde{\varphi},\chi\rangle=0$ and thus (33) is valid with $\tilde{\varphi}$ instead of φ. Taking into account that $\|\tilde{\varphi}\|=$

$|\langle\varphi,\psi\rangle|\|\varphi\|^{-1}$ and that $\tilde U^{(R)}_{\tilde\varphi,\psi}(T)=|\langle\varphi,\psi\rangle|^2\|\varphi\|^{-4}\tilde U^{(R)}_{\varphi,\psi}(T)$, one finds the announced expression (34). To optimize the lower bound, we take the supremum of $L_\psi(\varphi,T)$ for a given T over all possible φ, and this ends the proof.

Lemma 3 *Let μ be a probability Borel measure. Fix $q\in(0,1)$ and a function R as in Proposition 2. Let N be in $\mathbb{N}^*$. We assume that for some $q'<q$, we have $D_\mu(q')<\infty$; then there exists an r_0, a set $\Omega(r_0):=\{x\in\mathrm{supp}\mu\,|\,\varepsilon^{r_0+\alpha/N}<b(x,\varepsilon)\le\varepsilon^{r_0}\}$, where α is the constant given by Proposition 3, and an $\varepsilon_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_0)$*

$$\frac{1}{2N}\int_{\mathrm{supp}\mu} b(x,\varepsilon)^{q-1}d_\mu(x)\le\int_{\Omega(r_0)} b(x,\varepsilon)^{q-1}\,\mathrm{d}\mu(x) \tag{35}$$

Proof of Lemma 3. Denote by $B_\alpha:=\{x\in\mathrm{supp}\mu\,|\,b(x,\varepsilon)\le\varepsilon^\alpha\}$ and $B^\alpha:=\{x\in\mathrm{supp}\mu\,|\,b(x,\varepsilon)>\varepsilon^\alpha\}$. Then using Proposition 3 we get

$$\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)=\int_{B_\alpha}+\int_{B^\alpha} b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)\le\int_{B^\alpha} b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x)+C\varepsilon\ , \tag{36}$$

where C is the constant given by Propositon 3. Now, if we consider the partition of B^α into N sets $\Omega(k\alpha/N):=\{x\in\mathrm{supp}\mu\,|\,\varepsilon^{(k+1)\alpha/N}<b(x,\varepsilon)\le\varepsilon^{k\alpha/N}\}$ ($k\in\{0,1,\ldots,N\}$), we can pick a $k_0\in\{0,1,\ldots,N\}$ such that

$$\int_{\Omega(k_0\alpha/N)} b(x,\varepsilon)^{q-1}d_\mu(x)\ge\frac{1}{N}\int_{B^\alpha} b(x,\varepsilon)^{q-1}d_\mu(x)\ . \tag{37}$$

Since $b(x,\varepsilon)\le1$ and $q\in(0,1)$, we have $\int b(x,\varepsilon)^{q-1}d_\mu(x)\ge1$. Therefore, from (37) and (36), we get inequality (35) for all $0\le\varepsilon\le\varepsilon_0:=1/(2C)$, with $r_0=k_0\alpha/N$.

Lemma 4. *Let H be a self-adjoint operator on the separable Hilbert space $\mathcal{H}=L^2(\mathbb{R}^d)$ or $\mathcal{H}=\ell^2(\mathbb{Z}^d)$. Pick a normalized state $\psi\in\mathcal{H}$ and denote by μ_ψ the spectral measure associated to H and ψ. Let R be a given function as in Proposition 2. Then for all $p\in(0,\infty)$ such that $\tilde q_\mu:=\inf\{q>0\,|\,D^+_\mu(q)<+\infty\}<\frac{1}{1+p/d}$, there exists a constant $c'>0$ such that for all $\varepsilon\in(0,1)$,*

$$\frac{c'}{|\log\varepsilon|^{1+p/d}}K^{(R)}_{\mu_\psi}\left(\frac{1}{1+p/d},\varepsilon\right)^{1+p/d}\le L_\psi(\varepsilon^{-1})\ ,$$

where the function L_ψ is defined in Lemma 2 by (26), and $K^{(R)}_{\mu_\psi}$ is defined in Proposition 2

Proof of Lemma 4. Let $\alpha(q',q,R)$ be defined as in Proposition 3, $N:=[-\log\varepsilon]$ be the integer part of $-\log\varepsilon$, $R(z):=|\hat h(z)|^2$, q be such that $q=\frac{1}{1+p/d}$, and $\varphi:=\chi_{\Omega(r_0)}(H)\psi$ (r_0 given in Lemma 3). We have, by definition of $\tilde U^{(R)}_{\varphi,\psi}$ given in Lemma 3

$$\tilde U^{(R)}_{\varphi,\psi}(\varepsilon^{-1})=\int_{\Omega(r_0)} b(x,\varepsilon)\mathrm{d}\mu_\psi(x)\le\varepsilon^{r_0}\mu_\psi(\Omega(r_0))\ .$$

Therefore we have

$$L_\psi(\varepsilon^{-1})\quad\ge\quad\frac{|\langle\varphi,\,\psi\rangle|^{2+2p/d}}{\|\varphi\|^2U_{\varphi,\,\psi}(\varepsilon^{-1})^{p/d}}\ge\mu(\Omega(r_0))^{1+p/d}\varepsilon^{-r_0p/d}$$

$$
\begin{aligned}
&\geq \quad \varepsilon^{-r_0(1+2p/d)} \left(\int_{\Omega(r_0)} b(x,\varepsilon) \mathrm{d}\mu_\psi(x) \right)^{1+p/d} \\
&\geq \quad \varepsilon^{-r_0(1+2p/d)} \left(\int_{\Omega(r_0)} b(x,\varepsilon)^{\frac{-p/d}{1+p/d}} b(x,\varepsilon)^{\frac{1+2p/d}{1+p/d}} \mathrm{d}\mu_\psi(x) \right)^{1+p/d} \\
&\geq \quad \varepsilon^{-r_0(1+2p/d)+r_0(1+2p/d)+(1+2p/d)\alpha/N} \left(b(x,\varepsilon)^{\frac{-p/d}{1+p/d}} \right)^{1+p/d} \\
&\geq \quad \varepsilon^{(1+2p/d)\alpha/N} K^{(R)}_{\mu_\psi}\left(\frac{1}{1+p/d}, \varepsilon \right) \geq \frac{c'}{|\log \varepsilon|} K^{(R)}_{\mu_\psi}\left(\frac{1}{1+p/d}, \varepsilon \right) \quad ,
\end{aligned}
$$

where in the last inequality we used Lemma 3 with N being the integer part of $-\log \varepsilon$.

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