Renormalization Group Flow of the Two-Dimensional Hierarchical Coulomb Gas

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Abstract

We consider a quasilinear parabolic differential equation associated with the renormalization group transformation of the two–dimensional hierarchical Coulomb system in the limit as the size of the block $L \downarrow 1$. We show that the initial value problem is well defined in a suitable function space and the solution converges, as $t \to \infty$, to one of the countably infinite equilibrium solutions. The j–th nontrivial equilibrium solution bifurcates from the trivial one at $\beta_j = 8\pi/j^2$, $j = 1, 2, \ldots$. These solutions are fully described and we provide a complete analysis of their local and global stability for all values of inverse temperature $\beta > 0$. Gallavotti and Nicoló's conjecture on infinite sequence of "phases transitions" is also addressed. Our results rule out an intermediate phase between the plasma and the Kosterlitz–Thouless phases, at least in the hierarchical model we consider.

1 Introduction

We consider, for each $\beta > 0$, the partial differential equation

$$u_t - \frac{\beta}{4\pi}(u_{xx} - u_x^2) - 2u = 0 \tag{1.1}$$

on $\mathbb{R}_+ \times (-\pi, \pi)$ with periodic boundary condition, $u(t, -\pi) = u(t, \pi)$ and $u_x(t, -\pi) = u_x(t, \pi)$, in the space of even functions, satisfying an additional condition $u(t, 0) = 0$[1]. We show that the initial value problem is well defined in an appropriate function space $\mathcal{B}$ and the solution exists and is unique for all $t > 0$. Furthermore, as $t \to \infty$, the solution converges in $\mathcal{B}$ to one of the

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[1]This is assured by a Lagrange multiplier (see Remark 3.1).

(equilibrium) solutions ϕ of

$$\frac{\beta}{4\pi}\left(\phi''-(\phi')^2\right)+2\phi=0\,, \tag{1.2}$$

with $\phi(-\pi)=\phi(\pi)$ and $\phi'(-\pi)=\phi'(\pi)$. For $\beta>8\pi$, $\phi_0\equiv 0$ is the (globally) asymptotically stable solution of (1.1). For $\beta<8\pi$ such that $8\pi/(k+1)^2\le\beta<8\pi/k^2$ holds for some $k\in\mathbb{N}_+$, ϕ_0 is unstable and there exist $2k$ non–trivial equilibria solutions $\phi_1^{\pm},\ldots,\phi_k^{\pm}$ of (1.2) among which $\phi_1^{\pm}$ are the only asymptotically stable ones.

The aim of the present work is to show that, for $j\ge 1$, $\phi_j^{\pm}$ have a $(j-1)$–dimensional unstable manifold $\mathcal{M}_j\subset\mathcal{B}$ so $\phi_j^{\pm}$ are more stable than $\phi_{j'}^{\pm}$ if $j<j'$. As a consequence, there exists a dense open set of initial conditions in $\mathcal{B}$ such that ϕ_1^{+} (ϕ_1^{-} is not physically admissible) is the non–trivial stable solution for all $\beta<8\pi$.

Our description of equation (1.1) is motivated by two distinct goals. Firstly, it provides a new example of nonlinear parabolic differential equation by which a geometric theory can be carried out (see e.g. Henry [H]). According to this theory, the above scenario can be stated as follows: there exist a sufficient large ball $\mathcal{B}_0\subset\mathcal{B}$ about the origin such that, if $u(t,\mathcal{B}_0)$ denotes the set of points reached at time t starting from any initial function in $\mathcal{B}_0$, then the invariant set $\bigcap_{t\ge 0}u(t,\mathcal{B}_0)$ coincides with the k–dimensional unstable manifold $\mathcal{K}_k=\bigcup_{0\le j\le k}\mathcal{M}_j=\overline{\mathcal{M}_0}$ provided $8\pi/(k+1)^2\le\beta<8\pi/k^2$.

Secondly, the solution of the initial value problem (1.1) describes the renormalization group (RG) flow of the effective potential in the two–dimensional hierarchical Coulomb system and the stationary solutions $\{\phi_j^{+}\}$, the fixed points of RG, contain informations on its critical phenomena.

The analysis of equation (1.1) presented here can hopefully bring some light to a question raised by Gallavotti and Nicoló [GN] on the "screening phase transitions" in two–dimensional Coulomb systems. The existence of infinitely many thresholds of "instabilities" found in the Mayer series at inverse temperature $\beta_n=8\pi(1-1/(2n))$, $n\in\mathbb{N}_+$, indicates, according to the authors, a sequence of "intermediate" phase transitions from the plasma phase ($\beta\le\beta_1=4\pi$) to the multipole phase ($\beta\ge\beta_\infty=8\pi$). They conjectured that some partial screening takes place when the inverse temperature decreases from 8π to 4π, which prevents the formation of neutral multipole of order larger than $2n$ where n is the integer part of $1/(2-\beta/4\pi)$ (dipoles are the last to be prevented at 4π).

The Kosterlitz–Thouless phase (multipole phase) was established by Fröhlich–Spencer [FS] and extended up to 8π by one of the present authors and A. Klein [MK]. Debye screening (plasma phase) was only proved for sufficiently small $\beta<<4\pi$ [BF]. Study of the region $[4\pi,8\pi]$ began with the work by Benfatto, Gallavotti and Nicoló [BGN] on the ultraviolet collapses of neutral clusters in the Yukawa gas which served as a base for the results in [GN]. It seems improbable, on the light of the present knowledge, that a conclusive answer to the Gallavotti–Nicoló conjecture will come up soon. It may be noted, however, that the scenario of an intermediate phase, which has challenged the conventional picture due to Jose *et al* [JKKN], has been contested by Fisher *et al* [FLL] based on Debye–Hückel–Bjerrum theory and by Dimock and Hurd [DH] who have reinterpreted the ultraviolet collapses in the Yukawa gas.

The Kosterlitz–Thouless phase is manifested in the hierarchical model as a bifurcation from the trivial solution [MP]. Our results rule out the existence of further phase transitions since no other bifurcation arises from the stable solution (see Theorem 5.1 on the stability of ϕ_1^{+}).

Even though the existence of the invariant unstable manifold $\mathcal{K}_k$ may provide a suitable explanation to the appearance of Gallavotti–Nicoló's thresholds, the nature (and location) of the instabilities in the hierarchical Coulomb gas differs substantially from the one we have just described, because neutral multipoles cannot be formed in the hierarchical model. We believe, however, our investigation may be helpful for the plasma phase. Numerical analysis shows the stable solution ϕ_1^+ looks like the Debye–Hückel potential $\phi_{DH} = (2\pi/\beta)\, x^2$ in $(-\pi, \pi)$ right after the transition takes place (see Remark 4.6).

As in [F], the renormalization group (GR) flow (1.1) may be derived from the block–spin RG transformation of a two–dimensional hierarchical Coulomb system in the limit as the block size $L \downarrow 1$. This procedure, called *local potential approximation*, has been discussed by Felder [F] in the context of Dyson's hierarchical model, whose partial differential equation,

$$u_t - \frac{1}{2} u_{xx} + \frac{d-2}{2} x\, u_x - d\, u + \frac{1}{2} u_x^2 = 0\,, \tag{1.3}$$

coincides with (1.1) when his dimensional parameter $d = 2$ if β is equal to 2π (without boundary conditions). Felder showed that (1.3) has global stationary solutions u_{2n}^* on $\mathbb{R}$ for $2 < d < d_n$ with $u_{2n}^*(x) \to 0$ as $d \uparrow d_n$ and calculated their profile. Here, $d_n = 2 + 2/(n-1)$, $n = 2, 3, \ldots$, is the sequence of thresholds where nontrivial fixed points are expected to appear as a bifurcation from the trivial solution. We mean by global solution one which doesn't blow up at finite x.

The present paper begins with a derivation of equation (1.1) in Section 2. The existence, uniqueness and continuous dependence on the initial value are presented in Section 3 and the precise statements are given in Theorems 3.2 and 3.4. We describe all global solutions of (1.2) completely in Section 4. Due to smoothness and the periodic condition, blow–up of an admissible stationary solution is impossible. We show that the non–trivial stationary solution for $\beta < 8\pi$ is *unique* modulo solutions with period $2\pi/j$, $j = 2, 3, \ldots$, which are responsible for the existence of the unstable manifold (see Theorem 4.1). Finally, we analyze in detail the local and global stability of equilibrium solutions of (1.1) in Section 5. The main results are stated in Theorems 5.1 and 5.14.

2 The Flow Equation

This section is devoted to the derivation of (1.1) from the RG transformation of two–dimensional hierarchical Coulomb system. We begin with a brief review of this model.

A Coulomb system is an ensemble of two species (for simplicity) of charged particles, interacting via a two–body Coulomb potential V. In the grand canonical ensemble the total number of particles fluctuates around a mean value determined by the particle activity z. It will become clear that the charge ensemble, rather than the particle ensemble, is more appropriate for RG transformation.

A configuration q of this system is a function $q : \Lambda \subset \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ which associates to each site x of the lattice Λ the total charge $q(x)$ at this position.

To each configuration we introduce two functionals: the total energy $E : \mathbb{Z}^\Lambda \longrightarrow \mathbb{R}_+$,

$$E(q) = \frac{1}{2} \sum_{x,y\in\Lambda} q(x)\, V(x,y)\, q(y) \tag{2.1}$$

(self–energy is included) and an "a priori" weight $F : \mathbb{Z}^\Lambda \longrightarrow \mathbb{R}_+$,

$$F(q) = \prod_{x\in\Lambda} \lambda(q(x)) \tag{2.2}$$

defined for positive real valued functions λ.

The equilibrium Gibbs measure $\mu_\Lambda : \mathbb{Z}^\Lambda \longrightarrow \mathbb{R}_+$ is thus given by

$$\mu_\Lambda(q) := \frac{1}{\Xi_\Lambda} F(q)\, e^{-\beta\, E(q)} \tag{2.3}$$

where β is the inverse temperature and

$$\Xi_\Lambda = \sum_{q\in\mathbb{Z}^\Lambda} F(q)\, e^{-\beta\, E(q)} \tag{2.4}$$

is the grand partition function.

It has been shown (see e.g. [FS]) that the standard Coulomb system in the grand canonical ensemble with particle activity z has charge activity given by $\lambda(q) = I_q(2z)$, where I_q is the q–th modified Bessel function. If $\lambda(q) = \delta_{q,0} + z\,(\delta_{q,1} + \delta_{q,-1})$, Ξ_Λ is the grand canonical ensemble of charged particles with hard core.

Let us introduce our hierarchical model as proposed in ref. [MP]. The potential V in (2.1) is replaced by a function

$$V_h(x,y) = -\frac{1}{2\pi} \ln d_h(x,y)\,,$$

given by the asymptotic behavior of the two–dimensional Coulomb potential with the Euclidean distance $|x-y|$ replaced by hierarchical distance

$$d_h(x,y) := L^{N(x,y)}\,, \tag{2.5}$$

defined for an integer $L > 1$, where

$$N(x,y) := \inf\left\{N \in \mathbb{N}_+ : \left[\frac{x}{L^N}\right] = \left[\frac{y}{L^N}\right]\right\} \tag{2.6}$$

and $[z] \in \mathbb{Z}^2$ has components the integer part of the components of $z \in \mathbb{R}^2$. Notice that d_h is not invariant by translations.

Now, given an integer number $N > 1$, let $\Lambda = \Lambda_N = [-L^N, L^N - L^{N-1}]^2 \cap \mathbb{Z}^2$ and define, for each configuration $q \in \mathbb{Z}^\Lambda$, the block configuration $q^1 : \Lambda_{N-1} \longrightarrow \mathbb{Z}$,

$$q^1(x) = \sum_{\substack{0\le y_i<L \\ i=1,2}} q(Lx+y)\,. \tag{2.7}$$

The renormalization group transformation $\mathcal{R}$ acting on the space of Gibbs measures (2.3),

$$\mu^1_{\Lambda_{N-1}}(q^1) = [\mathcal{R}\mu_{\Lambda_N}](q^1) := \sum_{\substack{q\in\mathbb{Z}^{\Lambda_N}: \\ q^1\,\text{fixed}}} \mu_{\Lambda_N}(q)\,, \tag{2.8}$$

involves an integration over the fluctuations about q^1 following by a rescaling back to the original lattice.

As it has been shown in [MP], the RG transformation $\mathcal{R}$ preserves the form of the Gibbs measure in the grand canonical ensemble of charges. The measure $\mu^1_{\Lambda_{N-1}}$ is thus given by (2.3) with the "a priori weight" F replaced by

$$F^1(q^1) = \prod_{x\in\Lambda_{N-1}} \lambda^1(q^1(x)) \tag{2.9}$$

where

$$\lambda^1(p) = L^{-\beta p^2/(4\pi)}(\underbrace{\lambda\star\lambda\star\cdots\star\lambda}_{L^2-\text{times}})(p) \tag{2.10}$$

with $(\lambda\star\varrho)(p) = \sum_{q\in\mathbb{Z}} \lambda(p-q)\,\varrho(q)$. Note that $\Xi_{\Lambda_N}(\lambda) = \Xi_{\Lambda_{N-1}}(\lambda^1)$.

Remark 2.1 *A peculiar feature of hierarchical models is the reduction of the measure space where $\mathcal{R}$ acts to local functions. The RG transformation (2.8) induces a transformation $\lambda^1 = r\lambda$ given by (2.10) on the space of infinite sequences. Note that the space $\ell_1(\mathbb{Z})$ of summable sequences is closed by the r transformation: $(\lambda\star\lambda)\in\ell_1(\mathbb{Z})$ if $\lambda\in\ell_1(\mathbb{Z})$ by the Hausdorff-Young inequality.*

In order to take $L\downarrow 1$ limit of the RG transformation r it is convenient to write the system in the *sine–Gordon representation.* Fourier transforming (2.10),

$$\widehat{\lambda}(\varphi) = \sum_{q\in\mathbb{Z}} \lambda(q)\,e^{iq\varphi}\,,$$

and using the convolution theorem, yields

$$\widehat{\lambda^1}(\varphi) = \widehat{r\lambda}(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \vartheta(\varphi-\tau)\;\widehat{\lambda}^{L^2}(\tau)\,d\tau \tag{2.11}$$

where

$$\begin{aligned}\vartheta(\varphi) &= \sum_{q\in\mathbb{Z}} L^{-\beta q^2/(4\pi)}\,e^{iq\varphi} \\ &= \frac{1}{(\beta\ln L)^{1/2}}\sum_{n\in\mathbb{Z}} e^{-\pi(\varphi+2\pi n)^2/(\beta\ln L)}\end{aligned} \tag{2.12}$$

by the Poisson formula.

Plugging (2.12) into (2.11) and changing the variable $\zeta = \tau + 2\pi n$, equation (2.11) can be written as

$$\widehat{r\lambda}(\varphi) = \left(\nu * \widehat{\lambda}^{L^2}\right)(\varphi) \tag{2.13}$$

where $\nu*$ means convolution by a Gaussian measure with mean zero and variance $\beta \ln L/(2\pi)$:

$$\begin{aligned}(\nu * f)(\varphi) &= (\beta \ln L)^{-1/2} \int_{-\infty}^{\infty} d\zeta\; e^{-\pi(\varphi-\zeta)^2/(\beta \ln L)}\, f(\zeta) \\ &= e^{(\beta \ln L/4\pi)\left(d^2/d\varphi^2\right)}\, f(\varphi)\;,\end{aligned} \tag{2.14}$$

where in the second form of the Gaussian convolution we have used Wick's theorem.

Note that (2.13) is precisely the RG transformation derived by Gallavotti who has started directly from the sine-Gordon representation.

In order to let the block size L to 1, we introduce a variable $t := n \ln L$ which keeps track of the number of times the RG transformation (2.8) has to be iterated in order to bring two sites at hierarchical distance L^n to $\mathcal{O}(1)$ distance. We shall take the limit $L \downarrow 1$ together with $n \to \infty$ maintaining t fixed.

Define

$$u(t,x) = -\ln \widehat{\lambda^n}(x) \tag{2.15}$$

where $\widehat{\lambda^n} = \widehat{r^n \lambda}$ denotes the n –th iteration of the transformation (2.13). If one writes $t' = (n+1)\ln L$ then, by taking the logarithm and using (2.15), equation (2.13) reads

$$\begin{aligned}u(t',x) &= -\ln\left\{\exp\left(\frac{\beta t}{4\pi n}\frac{d^2}{dx^2}\right)\exp\left(-e^{2t/n}u(t,x)\right)\right\} \\ &= u(t,x) - \ln\left\{1 + \frac{t}{n}\left(\frac{\beta}{4\pi}\left(u_x^2(t,x) - u_{xx}(t,x)\right) - 2u(t,x)\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right\} \\ &= u(t,x) + \frac{t}{n}\left(\frac{\beta}{4\pi}\left(u_{xx}(t,x) - u_x^2(t,x)\right) + 2u(t,x)\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\end{aligned} \tag{2.16}$$

which, combined with

$$\begin{aligned}u_t(t,x) &= \lim_{t'\downarrow t} \frac{u(t',x) - u(t,x)}{t'-t} \\ &= \lim_{n\to\infty} \frac{n}{t}\left(u(t',x) - u(t,x)\right)\;,\end{aligned} \tag{2.17}$$

yields equation (1.1).

3 Existence, Uniqueness and Continuous Dependence

In this section the existence, uniqueness and continuous dependence on the initial value of equation (1.1) will be established by Picard's theorem for Banach spaces.

To avoid the appearance of zero modes upon linearization, we differentiate (1.1) with respect to x and consider the equation for $v = u_x$,

$$v_t - \frac{\beta}{4\pi}\left(v_{xx} - 2v\, v_x\right) - 2v = 0\,, \tag{3.1}$$

with $v\,(t,-\pi) = v\,(t,\pi)$ and $v_x\,(t,-\pi) = v_x\,(t,\pi)$, in the subspace of odd functions and initial value $v(0,\cdot) = v_0$. Note that the operator defined by the l. h. s. of (3.1) preserves this subspace.

Before we proceed, we have the following

Remark 3.1 *The "a priori weight" $\lambda(t,q) := \lambda^n(q)$ at scale $t = n \ln L$, is a positive* ***symmetric****, $\lambda(t,q) = \lambda(t,-q)$, sequence of real numbers and has to be normalized at all scales. In [MP] equation (2.10) was redefined so that $\lambda^n(0) = 1$ holds for all n. Here, the appropriated normalization is given by*

$$\sum_{q\in\mathbb{Z}} \lambda(t,q) = 1\,,$$

since, in view of equation (2.15), this leads to the condition $\widetilde{u}\,(t,0) = 0$, which is already imposed for all t if

$$\widetilde{u}(t,x) = \int_0^x v(t,y)\,dy \tag{3.2}$$

with $v(s,x)$ an odd solution of (3.1). From (3.2), we have

$$\begin{aligned} \widetilde{u}_t &= \int_0^x v_t(t,y)\,dy \\ &= \int_0^x \left[\alpha\left(\widetilde{u}_{xx} - \widetilde{u}_x^2\right) + 2\widetilde{u}\right]_x\,dy \\ &= \alpha\left(\widetilde{u}_{xx} - \widetilde{u}_x^2\right) + 2\widetilde{u} - \alpha\widetilde{u}_{xx}(t,0) \end{aligned} \tag{3.3}$$

where $\widetilde{u}_x(t,0) = v(t,0) = 0$ by parity. Note that $\widetilde{u}(t,x) = -\ln\widetilde{\lambda^n}(x) + \ln\widetilde{\lambda^n}(0)$ also satisfies (3.3) by equations (2.16) and (2.17). Moreover, note that there is a one–to–one correspondence between the solution of (1.1) and the solution of (3.3), with the same initial value u_0, given by

$$\widetilde{u}(t,x) = u(t,x) - u(t,0) \tag{3.4}$$

and

$$u(t,x) = \widetilde{u}(t,x) + \alpha\int_0^t e^{2(t-s)}\widetilde{u}_{xx}(s,0)\,ds\;, \tag{3.5}$$

where $\alpha\widetilde{u}_{xx}(t,0)$ is the required Lagrange multiplier introduced in (3.3) to assure that $\widetilde{u}(t,0) = 0$ (see comments after equation (1.1′) in ref. [F]). This correspondence will be useful in Section 5.

Because the standard initial condition $u_0(x) = z\,(1-\cos x)$ satisfies $u_0'(0) = u_0'(\pi) = 0$, equation (3.1) may equivalently be considered on $(0,\pi)$ with Dirichlet boundary conditions $v\,(t,0) = v\,(t,\pi) = 0$.

Another reason for considering (3.1) instead of (1.1) is the fact that the nonlinearity $2v\,v_x$ is more suitable than u_x^2 for the analysis of equilibrium solutions and corresponding stabilities given in the next sections.

The boundary and initial value problem (3.1) may be written as an ordinary differential equation

$$\frac{dz}{dt} + Az = F(z) \tag{3.6}$$

in a conveniently defined Banach space $\mathcal{B}$ where

$$Az = -\alpha z'' - 2z \qquad \text{and} \qquad F(z) = -2\alpha z' z\,, \tag{3.7}$$

with $\alpha = \beta/(4\pi)$ and initial value $z(0) = z_0$.

The linear operator A is defined on the space $C^2_{\mathrm{o,p}}$ of smooth odd and periodic real–valued functions in $[-\pi, \pi]$,[2] with inner product $(f, g) := \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$. Because of $(f, Ag) = (Af, g)$, A may be extended to a self–adjoint operator in $L^2_{\mathrm{o,p}}(-\pi, \pi)$. The domain $D(A)$ of A is

$$D(A) = \left\{ f \in L^2_{\mathrm{o,p}}(-\pi, \pi) : Af \in L^2_{\mathrm{o,p}}(-\pi, \pi) \right\}$$

and the spectrum of A,

$$\sigma(A) = \left\{ \lambda_n = \alpha n^2 - 2,\ n \in \mathbb{N}_+ \right\}\,, \tag{3.8}$$

consists of simple eigenvalues with corresponding eigenfunctions $\phi_n(x) = (1/\pi)^{1/2} \sin nx$.

Let A_1 denote a positive definite linear operator given by A if $\alpha > 2$ and $A + aI$ for some $a > 2 - \alpha$, otherwise. The following properties also hold for A given by the closure in $L^q_{\mathrm{o,p}}(-\pi, \pi)$, $1 \le q < \infty$, of the operator $(-\alpha\, d^2/dx^2 - 2)|_{\mathcal{C}^2_{\mathrm{o,p}}}$.

1. The operator A generates an analytic semi–group $T(t) = e^{-tA}$ given by the formula

$$T(t) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda + A}\, e^{\lambda t}\, d\lambda$$

where Γ is a contour in the resolvent set of A with $\arg \lambda \longrightarrow \pm\theta$, $\pi/2 < \theta < \pi$, as $|\lambda| \to \infty$. From this, we have

$$\left\| e^{-tA} \right\| \le C\, e^{-ct} \qquad \text{and} \qquad \left\| A e^{-tA} \right\| \le \frac{C}{t}\, e^{-ct} \tag{3.9}$$

for $t > 0$, $c < \inf_\lambda \sigma(A)$ and $C < \infty$.

2. Given $\gamma \ge 0$, let the fractional power of A_1 be given by

$$A_1^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma - 1}\, e^{-A_1 t}\, dt$$

and define $A_1^\gamma = \left(A_1^{-\gamma}\right)^{-1}$. $A_1^{-\gamma}$ is a bounded operator (compact if $\gamma > 0$) with $A_1^{-1/2}(d/dx)$ and $(d/dx)\, A_1^{-1/2}$ bounded in the $L^2_{\mathrm{o,p}}(-\pi, \pi)$ norm. In addition, for $\gamma > 0$, A_1^γ is closely defined with the inclusion $D(A_1^\gamma) \subset D(A_1^\tau)$ if $\gamma > \tau$.

[2] From here on, the subindexes in $C^2_{\mathrm{o,p}}$, $L^2_{\mathrm{o,p}}$, $L^2_{\mathrm{e,p}}$, $H^1_{\mathrm{e,p}}$ and *etc.*, indicate spaces of odd and periodic (o,p) or even and periodic (e,p) functions.

It thus follows from 1. and 2. (see e.g. [H])

$$\left\| A_1^{\gamma} e^{-tA_1} \right\| \leq \frac{C_\gamma}{t^\gamma}\, e^{-ct} \tag{3.10}$$

holds for $0 < \gamma < 1$, $t > 0$. Here C_γ is bounded in any compact interval of $(0,1)$ and also bounded as $\gamma \searrow 0$. Note that, if the operator norm is induced by the L^2–norm, equation (3.10) hold with

$$C_\gamma = \sup_{n \in \mathbb{N}_+} \left| (t\lambda_n)^\gamma\, e^{-t\lambda_n} \right| \leq \sup_{r > tc} \left| r^\gamma\, e^{-r} \right| \leq \left(\frac{\gamma}{e} \right)^\gamma , \tag{3.11}$$

uniformly in $\gamma, t \geq 0$.

Following Picard's method, let us replace F in (3.6) by a locally Hölder continuous function $f : [0,T] \longrightarrow \mathcal{B}$:

$$\| f(r) - f(s) \| \leq C \left| r - s \right|^\theta$$

for $0 \leq r \leq s < T$ and $\theta > 0$. In this case, a solution to (3.6) is given by the variation of constants formula

$$z(t) = e^{-tA} z_0 + \int_0^t e^{-(t-s)A} f(s)\, ds \, . \tag{3.12}$$

Note that $z : [0,T) \longrightarrow \mathcal{B}$ is continuously differentiable with $z \in D(A)$ satisfying the differential equation (3.6). Moreover, $z(t)$ is the unique solution with $z(0) = z_0$ provided f is such that $\lim_{\rho \to 0} \int_0^\rho \| f(s) \|\, ds = 0$.

Now, substituting $f(s) = F\left(z(s)\right)$ into (3.12) leads to an integral equation

$$z(t) = e^{-tA} z_0 + \int_0^t e^{-(t-s)A}\, F\left(z(s)\right)\, ds \tag{3.13}$$

whose solution, whether it exists, also solves the initial value problem (3.1) provided $F\left(z(s)\right)$ is shown to be locally Hölder continuous on the interval $0 \leq t < T$.

To formulate the necessary condition on F and state our results, let $\mathcal{B}^\gamma = D(A^\gamma)$, $\gamma \geq 0$, denote the Banach space with the graph norm

$$\| f \|_\gamma := \| A^\gamma f \| \, .$$

$F : \mathcal{B}^\gamma \longrightarrow L^2_{\mathrm{p,o}}\left(-\pi, \pi\right)$ is said to be locally Lipschtzian if there exist $U \subset \mathcal{B}^\gamma$ and a finite constant L such that

$$\| F(z_1) - F(z_2) \| \leq L \, \| z_1 - z_2 \|_\gamma \tag{3.14}$$

holds for any z_1, $z_2 \in U$.

Theorem 3.2 *The initial value problem (3.6) has a unique solution $z(t)$ for all $t \in \mathbb{R}_+$ with $z(0) = z_0 \in \mathcal{B}^{1/2}$. In addition, if $\| z(t) \|_{1/2}$ is bounded as $t \to \infty$, the trajectories $\{ z(t) \}_{t \geq 0}$ lie on a compact set in $\mathcal{B}^{1/2}$.*

Proof. The proof of Theorem 3.2 will be divided into four parts. Firstly, $F(z(t))$ will be shown to be Hölder continuous under the Lipschtzian condition (3.14), which establishes the equivalence between the integral equation (3.13) and the initial problem (3.6). Secondly, the Banach fixed point theorem will be used to show the existence of a unique solution $z(t)$ of (3.13) for $0 \le t \le T$. Hence, by a compactness argument, the solution $z(t)$ will be extended to all $t \in \mathbb{R}_+$. Finally, assuming that $\|z(t)\|_{1/2}$ stays bounded for all $t > 0$, we conclude the proof. We have to wait till Section 5 for the boundedness hypothesis to be established.

Part I: Continuity. Let us show that $F : D(A^{1/2}) \longrightarrow L^2_{\mathrm{o,p}}(-\pi,\pi)$ given by $F(z) = -2\alpha z\, z'$ is locally Lipschitz. We note that $D(A^{1/2}) = H^1_{\mathrm{o,p}}(-\pi,\pi)$ where $H^k_{\mathrm{o,p}}(-\pi,\pi)$ is the Sobolev space of odd periodic functions which have distributional derivatives up to order k. It thus follows that, if $z \in H^1_{\mathrm{o,p}}$, then $z(x) = \int_0^x z'(\xi)\, d\xi$ is absolutely continuous with

$$\sup_{x\in[-\pi,\pi]} |z(x)| \le \sqrt{2\pi}\, \|z\|_{1/2}\ ,$$

by the Schwarz inequality. Moreover, using (3.10), we have

$$\begin{aligned} \|F(z_1) - F(z_2)\| &\le 2\alpha \left\{ \|z_1(z_1' - z_2')\| + \|(z_1 - z_2)z_2'\| \right\} \\ &\le 2\alpha\sqrt{2\pi} \left\{ \|z_1\|\, \|z_1 - z_2\|_{1/2} + \|z_1 - z_2\|\, \|z_2\|_{1/2} \right\} \end{aligned} \tag{3.15}$$

which satisfies (3.14) with $\gamma = 1/2$ and $L = 2\alpha\sqrt{2\pi}\left(\|z_1\|_{1/2} + \|z_2\|_{1/2} \right)$.

Suppose that $z : (0,T) \longrightarrow \mathcal{B}^{1/2}$ is a continuous solution of (3.13). From the estimate (3.10), we have

$$\begin{aligned} \left\| \left(e^{-hA} - I\right) e^{-\tau A} w \right\|_{1/2} &\le \int_0^h \left\| A\, e^{-(s+\tau)A} w \right\|_{1/2}\, ds \\ &= \int_0^h \left\| A^{1-\delta}\, e^{-sA} \right\| ds\, \left\| A^{\delta} e^{-\tau A} w \right\|_{1/2} \\ &\le C_{1-\delta} \int_0^h \frac{1}{s^{1-\delta}} ds\, \left\| A^{\delta} e^{-\tau A} w \right\|_{1/2} \\ &\le \frac{C_{1-\delta}}{\delta} h^{\delta} C_{\delta+1/2}\, \frac{e^{-c\tau}}{\tau^{\delta+1/2}}\, \|w\| \end{aligned} \tag{3.16}$$

for $0 < \delta < 1/2$ which can be used in the equation (3.13) along with (3.14), to get

$$\begin{aligned} \|z(t+h) - z(t)\|_{1/2} \le\ & \left\| \left(e^{-hA} - I\right) e^{-tA} z_0 \right\|_{1/2} + \textstyle\int_0^t \left\| \left(e^{-hA} - I\right) e^{-(t-s)A} F(z(s)) \right\|_{1/2} ds \\ & + \textstyle\int_t^{t+h} \left\| e^{-(t+h-s)A} F(z(s)) \right\|_{1/2} ds \le K\, h^{\delta} \end{aligned} \tag{3.17}$$

for some constant $K < \infty$ in the open interval $(0,T)$. Combined with (3.14), this implies the Hölder continuity of $f(t) = F\left(z(t)\right)$ and the equivalence between the equations (3.6) and (3.13).

Part II: Local existence. Let $V = \left\{z \in \mathcal{B}^{1/2} : \|z - z_0\| \leq \varepsilon\right\}$ be an ε–neighborhood and let L be the Lipschitz constant of F on V. We set $B = \|F(z_0)\|$ and let T be a positive number such that

$$\left\|\left(e^{-hA} - I\right)\, z_0\right\|_{1/2} \leq \frac{\varepsilon}{2} \tag{3.18}$$

with $0 \leq h \leq T$ and

$$C_{1/2}\,(B + L\varepsilon) \int_0^T s^{-1/2}\, e^{-cs}\, ds \leq \frac{\varepsilon}{2} \tag{3.19}$$

hold.

Let $\mathcal{S}$ denote the set of continuous functions $y : [t_0, t_0 + T] \longrightarrow \mathcal{B}^{1/2}$ such that $\|y(t) - z_0\| \leq \varepsilon$. Equipped with the sup–norm

$$\|y\|_T := \sup_{t_0 \leq t \leq t_0 + T} \|y(t)\|_{1/2}$$

$\mathcal{S}$ is a complete metric space.

Defining $\Phi[y] : [t_0, t_0 + T] \longrightarrow \mathcal{B}^{1/2}$ for each $y \in \mathcal{S}$ by

$$\Phi[y](t) = e^{-(t-t_0)A} z_0 + \int_{t_0}^{t} e^{-(t-s)A}\, F\left(y(s)\right)\, ds\,,$$

we now show that, under the conditions (3.18) and (3.19), $\Phi : \mathcal{S} \longrightarrow \mathcal{S}$ is a strict contraction. Using

$$\|F(y(t))\| \leq \|F(y(t)) - F(z_0)\| + \|F(z_0)\| \leq L\,\|y(t) - z_0\|_{1/2} + B \leq L\varepsilon + B$$

and (3.10), we have

$$\begin{aligned}
\|\Phi[y](t) - z_0\|_{1/2} &\leq \left\|\left(e^{-(t-t_0)A} - I\right) e^{-tA} z_0\right\|_{1/2} + \int_{t_0}^{t_0+T} \left\|A^{1/2} e^{-(t-s)A}\right\|\, \|F(y(s))\|\, ds \\
&\leq \frac{\varepsilon}{2} + C_{1/2}\,(B + L\varepsilon) \int_0^T s^{-1/2}\, e^{-cs}\, ds \leq \varepsilon
\end{aligned}$$

and since $\Phi[y]$ is continuous by an estimate analogous to (3.17), $\Phi[y] \in \mathcal{S}$.

Analogously, from (3.14) and (3.19), for any $y, w \in \mathcal{S}$

$$\begin{aligned}
\|\Phi[y](t) - \Phi[w](t)\|_{1/2} &\leq \int_{t_0}^{t_0+T} \left\|A^{1/2} e^{-(t-s)A}\right\|\, \|F(y(s)) - F(w(s))\|\, ds \\
&\leq C_{1/2} L \int_0^T s^{-1/2}\, e^{-cs}\, ds\, \|y - w\|_T \leq \frac{1}{2}\, \|y - w\|_T
\end{aligned}$$

holds uniformly in $t \in [t_0, t_0 + T]$ concluding our claim.

By the contraction mapping theorem, Φ has a **unique** fixed point z in $\mathcal{S}$ which is the continuous solution of the integral equation (3.13) on (t_0, t_0+T) and, by *Part I*, is the solution of (3.6) in the same interval with $z(t_0) = z_0 \in \mathcal{B}^{1/2}$.

Part III: Global existence. As the set U where (3.14) holds is compact, the same T can be chosen in *Part II* for any initial condition $z_0 \in U$. Moreover, if $I_1 = (t_1, t_1+T)$ and $I_2 = (t_2, t_2+T)$ are two intervals containing t_0, then there exist $z_{0,1}, z_{0,2} \in U$ such that the two solutions $z_1(t)$ and $z_2(t)$ of equation (3.6) on I_1 with $z_1(t_1) = z_{0,1}$ and on I_2 with $z_2(t_2) = z_{0,2}$, respectively, coincide in the open interval $I_1 \cap I_2$. As a consequence, one can define an **open maximal interval** $I_{\max} = (t_-, t_+)$ (containing the origin), where the solution $z(t)$ of (3.6) is uniquely given by patching together the solutions $z_j(t)$ on intervals I_j with $z_j(t_j) = z_{0,j}$. By construction, there is no solution to (3.6) on (t_0, t') if $t' > t_+$. Therefore, either $t_+ = \infty$, or else there exist a sequence $\{t_n\}_{n\in\mathbb{N}_+}$, with $t_n \to t_+$ as $n \to \infty$ such that $z(t_n)$ tend to the boundary ∂U of the compact set U .

It thus follows that, if t_+ is finite, the solution $z(t)$ blows–up at finite time. In what follows we show that $\|z(t)\|_{1/2}$ remains finite for all $t > t_0$ and this implies global existence of $z(t)$. Let us start with the following generalization of the Gronwall inequality.

Lemma 3.3 (Gronwall) *Let ξ and γ be numbers and let θ and ζ be non–negative continuous functions defined in a interval $I = (0,T)$ such that $\xi \geq 0$, $\gamma > 0$ and*

$$\zeta(t) \leq \theta(t) + \xi \int_0^t (t-\tau)^{\gamma-1}\, \zeta(\tau)\, d\tau\,. \tag{3.20}$$

Then

$$\zeta(t) \leq \theta(t) + \int_0^t E'_\gamma(t-\tau)\, \theta(\tau)\, d\tau \tag{3.21}$$

holds for $t \in I$, where $E'_\gamma = dE_\gamma/dt$,

$$E_\gamma(t) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\gamma+1)} \left(\xi\Gamma(\gamma)\, t^\gamma\right)^n \tag{3.22}$$

and $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function. In addition, if $\theta(t) \leq K$ for all $t \in I$, then

$$\zeta(t) \leq K\, E_\gamma(t)\ \leq K'\, e^{\xi\Gamma(\gamma)T} \tag{3.23}$$

holds for some finite constant K'.

Proof. If $\mathcal{T}$ is an integral operator given by the convolution

$$\mathcal{T}\zeta(t) = \xi \int_0^t (t-\tau)^{\gamma-1}\, \zeta(\tau)\, d\tau\,, \tag{3.24}$$

then the inequality (3.20) can be formally solved by

$$\zeta(t) = \theta(t) + \sum_{n=1}^{\infty} \mathcal{T}^n \theta(t)$$

where $\mathcal{T}^n$ is also an convolution integral operator which can be explicitly evaluated by the Laplace transform,

$$\begin{aligned}\mathcal{T}^n\theta(t) &= \frac{1}{\Gamma(n\gamma)}(\xi\Gamma(\gamma))^n\int_0^t (t-\tau)^{n\gamma-1}\theta(\tau)\,d\tau \\ &= \frac{1}{\Gamma(n\gamma+1)}(\xi\Gamma(\gamma))^n\int_0^t \frac{d}{dt}(t-\tau)^{n\gamma}\theta(\tau)\,d\tau \equiv (f_n'*\theta)(t)\,,\end{aligned}$$

with $f_n(t) = (\xi\Gamma(\gamma)\,t^\gamma)^n/\Gamma(n\gamma+1)$.

Equation (3.21) (and (3.23) by the fundamental theorem of calculus) thus follows by setting $E_\gamma(t) = \sum_{n\in\mathbb{N}} f_n(t)$. Note that this series is absolutely and uniformly convergent in $t \in I$, with $E_\gamma(0) = 1$, and it cannot grow faster than exponential

$$E_\gamma(T) \sim \frac{1}{\gamma}e^{\xi\Gamma(\gamma)T} \tag{3.25}$$

as $T \to \infty$ (see Lemma 7.1.1 in [H]). This concludes the proof of Lemma 3.3.

□

Taking the graph norm of (3.13), we have in view of (3.9), (3.10) and (3.25)

$$\begin{aligned}\|z(t)\|_{1/2} &\le \left\|e^{-(t-t_0)A}z_0\right\|_{1/2} + L\int_{t_0}^t \left\|A^{1/2}e^{-(t-s)A}\right\|\,\|z(s)\|_{1/2}\,ds \\ &\le C\,\|z_0\|_{1/2} + L\int_{t_0}^t (t-s)^{-1/2}\,\|z(s)\|_{1/2}\,ds \\ &\le C\,\exp\left(LC_{1/2}\sqrt{\pi}t\right)\,\|z_0\|_{1/2}\,,\end{aligned} \tag{3.26}$$

which is finite for any $t \in \mathbb{R}_+$.

Part IV: Compact trajectories. Since $\mathcal{B}^\gamma \subset \mathcal{B}^{1/2}$ has compact inclusion if $1/2 < \gamma < 1$ [H], it suffices to show that $\|z(t)\|_\gamma$ remains bounded as $t \to \infty$. The hypothesis $\|z(t)\|_{1/2} < \infty$ combined with (3.15) implies the existence of $C' < \infty$ such that, analogously as in (3.26),

$$\begin{aligned}\|z(t)\|_\gamma &\le \left\|e^{-tA}z_0\right\|_\gamma + \int_0^t \left\|A^\gamma e^{-(t-s)A}\right\|\,\|F(z(s))\|\,ds \\ &\le C_{\gamma-1/2}\,t^{1/2-\gamma}\,e^{-ct}\,\|z_0\|_{1/2} + C'\,C_\gamma\int_0^t (t-s)^{-\gamma}\,e^{-c(t-s)}\,ds\,,\end{aligned}$$

which is bounded for $t > 0$ provided $c > 0$ (i.e. $\inf_\lambda \sigma(A) > 0$). Although the spectrum of A is not positive if $\beta \le 8\pi$, we shall see in Section 5 that A in the integral equation (3.13) can be replaced by a positive linear operator L (see Theorems 5.2 and 5.3).

This concludes the proof of Theorem 3.2.

□

It follows by analogous procedure that if z_1 and z_2 are solutions of (3.6) differing by their initial value in $\mathcal{B}^{1/2}$, then

$$
\begin{aligned}
\|z_1(t) - z_2(t)\|_{1/2} &\le \left\|e^{-tA}\left(z_{0,1} - z_{0,2}\right)\right\|_{1/2} + \int_0^t \left\|A^{1/2} e^{-(t-s)A}\right\| \, \|F(z_1(s)) - F(z_2(s))\| \, ds \\
&\le \left\|e^{-tA}\left(z_{0,1} - z_{0,2}\right)\right\|_{1/2} + C_{1/2} L \int_0^t (t-s)^{-1/2} \, e^{-cs} \, ds \, \|z_1(s) - z_2(s)\|_{1/2}
\end{aligned}
$$

which implies, by the Gronwall inequality, the continuous dependence of $z(t)$ with respect to its initial condition.

We may also consider the dependence of z with respect to the parameter $\alpha = \beta/(4\pi)$. The next statement is a corollary of the above analysis.

Theorem 3.4 *The solution $z(t) : \mathbb{R}_+ \times \mathcal{B}^{1/2} \longrightarrow \mathcal{B}^{1/2}$ to the initial value problem (3.6) as a function of the bifurcation parameter α and the initial value z_0 is continuous.*

Remark 3.5 *It can be shown (see [H]) that for any initial value $z_0 \in \mathcal{B}^\gamma$, $0 < \gamma < 1$, the solution is actually in $D(A)$ at any later time. Moreover, since $F : \mathcal{B}^{1/2} \longrightarrow L^2_{\mathrm{o,p}}(-\pi, \pi)$ is $\mathcal{C}^\infty$ (has Fréchet derivatives of all orders), it can also be shown that $(\alpha, z_0) \in \mathbb{R}_+ \times \mathcal{B}^{1/2} \longrightarrow z(t; \alpha, z_0)$ is $\mathcal{C}^\infty$ for all $t > 0$.*

Remark 3.6 *Under minor modifications, one can show existence, uniqueness and continuous dependence of (3.1) in Sobolev space $H^1_{\mathrm{o,p}}(-\pi, \pi)$ with norm $\|z\|_1 = \|z'\|_{L^2_{\mathrm{o,p}}}$ (just include the linear term of (3.1) in the definition of F). The same results hold for equation (1.1) in the Sobolev space of even and periodic function $H^1_{\mathrm{e,p}}(-\pi, \pi)$ with both norms $\|\cdot\|_1$ and $\|\cdot\|_{1/2}$. Note from item 2. after (3.9) and (3.7) that $\alpha \|z\|_1 = \|z\|_{1/2} + 2\|z\|_{L^2_{\mathrm{o,p}}}$ so, both norms are equivalent.*

4 Equilibrium Solutions

Time independent (equilibrium) solutions of (3.1) are odd solutions of the ordinary differential equation

$$
\alpha\left(\psi'' - 2\psi\psi'\right) + 2\psi = 0\,, \tag{4.1}
$$

with periodic conditions $\psi(-\pi) = \psi(\pi)$ and $\psi'(-\pi) = \psi'(\pi)$, $\alpha = \beta/(4\pi) \ge 0$, which can be written as

$$
\left\{
\begin{array}{rcl}
w' &=& 2p\left(w - \alpha^{-1}\right) \\
p' &=& w\,,
\end{array}
\right. \tag{4.2}
$$

by setting $p = \psi$ and $w = \psi'$.

In this section we give a qualitative and quantitative description of the solutions of (4.2) in the phase space $\mathbb{R}^2$ and study their implications for the equilibrium solutions of (3.1). Our results are summarized as follows.

Theorem 4.1 *The stationary equation (4.1) has two distinct regimes separated by* $\alpha = 2$ $(\beta = 8\pi)$*. For* $\alpha \geq 2$*,* $\psi_0 \equiv 0$ *is the unique solution. For* $\alpha < 2$ *such that* $2/(k+1)^2 \leq \alpha < 2/k^2$ *holds for some* $k \in \mathbb{N}_+$*, there exist* $2k$ *non–trivial solutions* ψ_j^+, ψ_j^- *,* $j = 1, \ldots, k$*, with fundamental period* $2\pi/j$*,* $\psi_j^\pm(-x) = -\psi_j^\pm(x)$ *and* $\psi_j^-(x) = \psi_j^+(x+\pi)$*. Moreover, each pair of non–trivial solutions bifurcate from the trivial solution* ψ_0 *at* $\alpha_j = 2/j^2$ $(\beta_j = 8\pi/j^2)$ *with* $\lim\limits_{\alpha \uparrow \alpha_j} \psi_j^\pm = 0$*.*

In the phase space, these solutions (ψ_j', ψ_j)*, are closed orbits around* $(0,0)$ *whose distance from the origin increases monotonically as* α *decreases. Numerical computations indicate that these orbits approach rapidly to the open orbit* $\{(\alpha^{-1}, \alpha^{-1}x)\,,\ x \in \mathbb{R}\}$ *from the left as* $\alpha \to 0$*.*

Let us begin by stating the general properties derived by the same tools used in the analysis performed in Section 3.

The vector field $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$(w,p) \longrightarrow f(w,p) = \left(2p(w - \alpha^{-1}), w\right)\ ,$$

in the right hand side of (4.2), defines a smooth autonomous dynamical system. It thus follows from Piccard's theorem (see e.g. [CL]) that there exist a unique solution $(w(x), p(x))$ of this system, globally defined in $\mathbb{R}^2$, with $(w(0), p(0)) = (w_0, p_0)$. As we have seen in Section 3, the existence of a global solution and its continuous dependence on the value (w_0, p_0), and on the parameter α, follow from Gronwall's lemma, which holds here in its standard form. As a consequence, the phase space $\mathbb{R}^2$ is foliated by non–overlapping orbits

$$\gamma_P = \{(w(x), h(x)) : x \in \mathbb{R} \text{ and } P = (w(0), p(0))\}$$

which passes by $P = (w_0, p_0) \in \mathbb{R}^2$ at $x = 0$. Note that, by varying continuously P and α, the orbit γ_P varies continuously in the phase space.

We shall now determine the values (P, α) by which the solution of (4.2) defines closed orbits. Note that the orbits are symmetric with respect to the w–axis, $L = \{(w,0) : w \in \mathbb{R}\}$, since the system of equations (4.2) remains invariant if the sign of both, x and p, are reversed. As we shall see, there is no loss of generality if the initial value $(w(0), p(0)) = P$ belongs to L. We write $\gamma_P = \gamma_{w_0}$.

Proposition 4.2 *Every orbit* γ_P *is determined by a single value* P *in the positive semi–axis* $L^+ = \{(w_0, 0) : w_0 \geq 0\}$*. For* $w_0 > 0$*, the orbit* γ_{w_0} *is either closed or unbounded depending on whether* $\alpha\, w_0 < 1$ *or* $\alpha\, w_0 \geq 1$*, respectively. The orbit* $\gamma_{\alpha^{-1}} = \{(\alpha^{-1}, \alpha^{-1}x) : x \in \mathbb{R}\}$ *separates the phase space* $\mathbb{R}^2$ *in such way that* γ_P *is closed if* P *is on the left of* $\gamma_{\alpha^{-1}}$ *and unbounded otherwise. In addition, if* $w_0 = 0$*, then* $\gamma_0 = \{(0,0)\}$*, and the origin is enclosed by every closed orbit.*

Proof. The proof of Proposition 4.2 follows from an explicit computation. By the chain rule, equation (4.2) can be written as

$$\frac{dp}{dw} = \frac{w}{2p\,(w - \alpha^{-1})} \tag{4.3}$$

provided $\alpha w \neq 1$. The trajectories γ_{w_0}, obtained by integrating $2p\,dp = w\,dw/\left(w-\alpha^{-1}\right)$ with initial point $P=(w_0,0)$,

$$p^2 = w - w_0 + \alpha^{-1}\ln\left(\frac{1-\alpha w}{1-\alpha w_0}\right), \tag{4.4}$$

are portrayed in Figure 1.

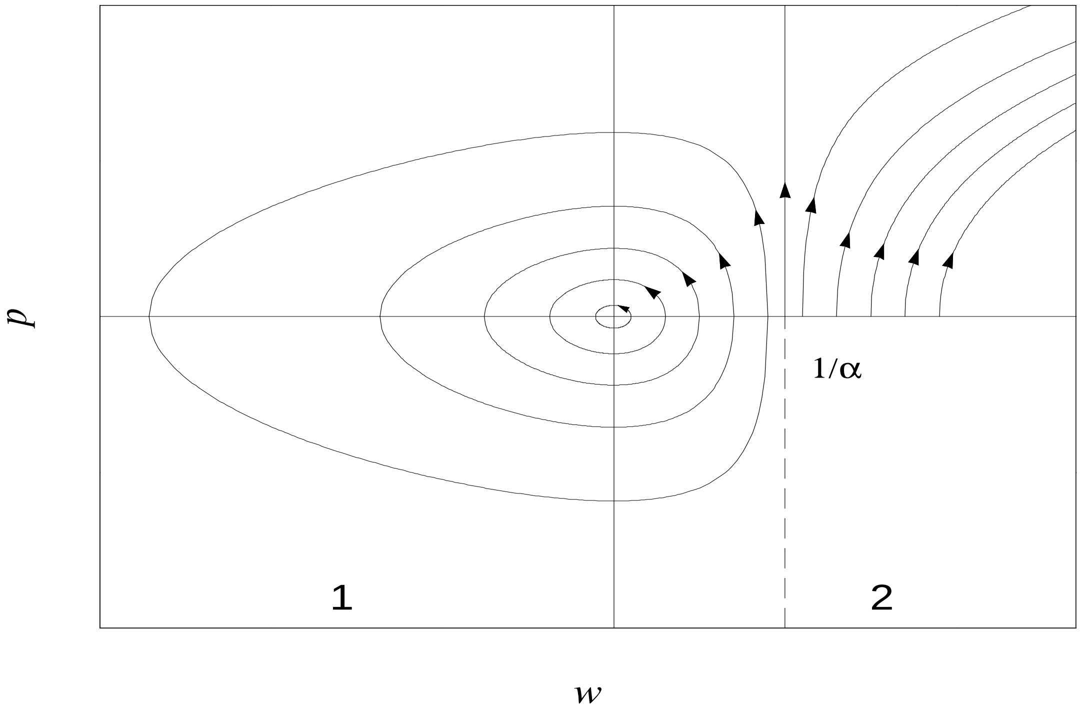

Figure 1: Trajectories of the dynamical system (4.2).

We note that $P=(0,0)$ is the only critical point of (4.2) which is a center for all $\alpha>0$ since, by linearizing $f(w,p)$ around $P=(0,0)$ gives a matrix whose eigenvalues are $\lambda_\pm = \pm i\sqrt{2\alpha^{-1}}$. This implies that $\gamma_0=\{(0,0)\}$ and the orbits γ_{w_0} with w_0 sufficiently closed to 0 are, in view of (4.4), ellipses defined by the equation $2\alpha^{-1}p^2+w^2=C$.

When $\alpha w_0 = 1$, using mathematical induction and equations (4.2) with $(w(0),p(0))=(w_0,0)$, we have

$$\frac{d^n w}{dx^n}(0) = 0\,,$$

for all $n\geq 1$, which leads

$$\gamma_{\alpha^{-1}} = \left\{\left(\alpha^{-1},\alpha^{-1}x\right) : x\in\mathbb{R}\right\}\ .$$

Hence, if $\omega=\omega(P)$ denotes the set of limit points (the ω –limit set) given by

$$\omega(P) = \left\{(w^*,h^*)\in\mathbb{R}^2 : \lim_{n\to\infty}\left(w(x_n),h(x_n)\right) = (w^*,h^*)\right\} \tag{4.5}$$

for some sequence of points $\{x_n\}$ such that $x_n \to \infty$, as $n \to \infty$, $\gamma_{\alpha^{-1}}$ separates two different type of orbits: $\omega(P) = \gamma_P$ or $\omega(P) = \{\infty\}$ depending on whether the point P is at the left or at the right of $\gamma_{\alpha^{-1}}$.

□

Proof of Theorem 4.1. The stationary solutions satisfy (4.2) with periodic conditions $w(0) = w(2\pi)$ and $p(0) = p(2\pi)$. By fixing the period T of an orbit γ_{w_0} in 2π, the label w_0 becomes implicitly dependent on the parameter α. In view of Proposition 4.2, Theorem 4.1 follows if for $\alpha \geq 2$, except by the orbit $\gamma_0 = \{(0,0)\}$, no (non–trivial) solution has period $T = 2\pi$ and for $\alpha < 2$ there is a one–to–one correspondence between w_0 and α for T fixed at any value $2\pi/k$, $k = 1, \ldots, \left[\sqrt{2/\alpha}\right]$.

More precisely, let $T = T(\alpha, w_0)$ denote the period of the dynamical system (4.2) with initial value $(w(0), p(0)) = (w_0, 0)$:

$$T = \int_{\gamma_{w_0}} dx = 2 \int \frac{dp}{w} , \tag{4.6}$$

where, by symmetry, the second integration is over the semi–orbit above the w–axis. For $\mathcal{D} = \{(\alpha, w_0) \in \mathbb{R}_+ \times \mathbb{R}_+ : \alpha w_0 \leq 1\}$, we set

$$G_j = T - \frac{2\pi}{j}$$

and note that $G_j : \mathcal{D} \longrightarrow \mathbb{R}$ is a continuous function of both variables satisfying

$$G_j\left(2/j^2, 0\right) = 0. \tag{4.7}$$

To see (4.7), we compute the period T_L of an elliptic orbit, e.g. $\{(2/\alpha)\, p^2 + w^2 = 1\}$, of (4.2) linearized at the origin ($f(w,p)$ replaced by $(2\alpha^{-1}p, w)$),

$$T_L = 4 \int_0^{\sqrt{\alpha/2}} \frac{dp}{\sqrt{1 - (2/\alpha)\, p^2}} = 2\pi \sqrt{\frac{\alpha}{2}} , \tag{4.8}$$

and note that $\lim_{w_0 \to 0} T(\alpha, w_0) = T_L$. Continuity follows from the general properties stated previously.

Hence, provided

$$\frac{\partial T}{\partial w_0} > 0 \tag{4.9}$$

holds for all $(\alpha, w_0) \in \mathcal{D}$, by the implicit function theorem, there exists a **unique** (strictly) monotone decreasing function $\widehat{w}_j : [0, 2/j^2] \longrightarrow \mathbb{R}_+$ with $\widehat{w}_j(2/j^2) = 0$ such that $G_j(\alpha, \widehat{w}_j(\alpha)) = 0$. Note that (4.9) and

$$T(\alpha, w_0) = \sqrt{\alpha}\, T(1, \alpha w_0) \tag{4.10}$$

imply that T is an increasing function of both α and w_0, independently. This fact, which can be seen by rescaling (4.2) by $x \to \overline{x} = x/\sqrt{\alpha}$, $w \to \overline{w} = \alpha w$ and $p \to \overline{p} = \sqrt{\alpha} p$, explains the monotone behavior of $\widehat{w}_j$.

It thus follows that, if $\alpha < 2$, for each $j = 1, \ldots, k$ such that $2/(k+1)^2 \le \alpha < 2/k^2$ holds, a unique function $\widehat{w}_j$ such that $\widehat{w}_j(2/j^2) = 0$ exists. The non–trivial solutions $\psi_1^{\pm}, \ldots, \psi_k^{\pm}$ of (4.1) are the p–component of $\gamma_{\widehat{w}_j}$, $j = 1, \ldots, k$, which winds around the origin j–times: ψ_j^+ is 2π–periodic odd function with fundamental period $2\pi/j$, $\left(\psi_j^+\right)'(0) > 0$ and satisfies $\psi_j^+(x+\pi) = \psi_j^-(x)$. If $\alpha \ge 2$, because $T(\alpha, w_0)$ is a strictly increasing function of w_0 and $T(\alpha, 0) \ge 2\pi$ (see eq. (4.8)), there is no solution of $G_j(\alpha, w_0) = 0$ besides $\widehat{w}_j(\alpha) = 0$ for $j = 1$. This reduces the proof of Theorem 4.1 to the proof of inequality (4.9).

To prove (4.9), it is convenient to change variables. Let

$$q = \ln(1 - \alpha\, w) \tag{4.11}$$

be defined for $\alpha w < 1$. From (4.10), there is no loss of generality in taking $\alpha = 1$. The system of equations (4.2) under this condition is thus equivalent to the following Hamiltonian system[3]

$$\begin{cases} q' &= 2p \\ p' &= 1 - e^q, \end{cases} \tag{4.12}$$

whose energy function is given by

$$H(q,p) = p^2 + e^q - q - 1\,. \tag{4.13}$$

The trajectory equation (4.4), when written in terms of the q–variable, gives exactly the energy level equation $H(q,p) = E$ with

$$E = -w_0 - \ln(1 - w_0)\,. \tag{4.14}$$

We denote by γ_E the orbits of (4.12) and note that, in view of the fact

$$\frac{dE}{dw_0} = \frac{w_0}{1 - w_0} > 0,$$

there is a one–to–one correspondence between the two families of closed orbits $\{\gamma_{w_0},\, 0 \le w_0 < 1\}$ and $\{\gamma_E,\, 0 \le E < \infty\}$.

Now, let $\widetilde{T} = \widetilde{T}(E)$ be the period of an orbit γ_E,

$$\widetilde{T} = \int_{\gamma_E} dx = \int_{q_-}^{q_+} \frac{dq}{p}\,. \tag{4.15}$$

Using the energy conservation law, we have

$$p = p(q,E) = \sqrt{E - v(q)}\,, \tag{4.16}$$

where the potential energy is given by

$$v(q) = e^q - q - 1\,, \tag{4.17}$$

and $q_{\pm} = q_{\pm}(E)$ are the positive and negative roots of equation $v(q) = E$.

Equation (4.9) holds if and only if $\dfrac{d\widetilde{T}}{dE} > 0$ holds uniformly in $E \in \mathbb{R}_+$. But this follows from the monotonicity criterion given by C. Chicone [C] (see also [CG]):

[3] We thank G. Benfatto for explaining this tranformation and for pointing us equation (4.4) in a footnote of [F].

Lemma 4.3 *Let $v \in \mathcal{C}^3(\mathbb{R})$ be a three–times differentiable function and let $f(q) = -v'(q)$ be the force acting at q. If v/f^2 is a convex function with*

$$\left(\frac{v}{f^2}\right)'' = \frac{6v\,(v'')^2 - 3\,(v')^2\,v'' - 2vv'v'''}{(v')^4} > 0\,, \qquad q \neq 0\,, \tag{4.18}$$

then the period $\widetilde{T}$ is a monotone (strictly) increasing function of E.

Proof. It follows from (4.16) two basic facts:

$$\frac{\partial p}{\partial q} = \frac{f}{2p} \qquad \text{and} \qquad p(q_\pm, E) = 0\,. \tag{4.19}$$

These will be used for deriving an appropriated integral representation of $d\widetilde{T}/dE$.

Let

$$K := \frac{1}{3}\int_{q_-}^{q_+} p^3 \left(\frac{v}{f^2}\right)'' dq\,. \tag{4.20}$$

Integrating twice by parts, gives

$$\begin{aligned} K &= \frac{p^3}{3}\left(\frac{v}{f^2}\right)'\Bigg|_{q_-}^{q_+} - \frac{pv}{2f}\Bigg|_{q_-}^{q_+} + \int_{q_-}^{q_+} (pf)'\,\frac{v}{f^2}\,dq \\ &= \frac{1}{2}\int_{q_-}^{q_+}\left(\frac{v}{2p} + vp\frac{f'}{f^2}\right) dq \end{aligned}$$

in view of (4.19). Note that $f(q_\pm) \neq 0$ since

$$v'(q_\pm) = v(q_\pm) - q_\pm = E - q_\pm$$

vanishes only at $E = 0$. This follows from the fact that v is a convex positive function with $v(0) = 0$ and asymptotic behavior $v(q) \sim q - 1$ and $\sim e^{\alpha q}$, as q goes to $-\infty$ and ∞.

Now, using $(v/f)' = v'/f - vf'/f^2 = -1 - vf'/f^2$, and integrating by parts, we continue

$$\begin{aligned} K &= \frac{1}{2}\int_{q_-}^{q_+}\left(\frac{v}{2p} - p\left(\frac{v}{f}\right)' - p\right) dq \\ &= \frac{1}{2}\int_{q_-}^{q_+}\left(\frac{v}{p} - p\right) dq - \frac{1}{2}p\left(\frac{v}{f}\right)\Bigg|_{q_-}^{q_+} \\ &= \frac{1}{2}\int_{q_-}^{q_+}\left(\frac{E}{p} - 2p\right) dq \end{aligned} \tag{4.21}$$

where in the last equation we have used $v = E - p^2$.

From (4.15), (4.20) and (4.21), we have

$$E\widetilde{T} = 2\int_{q_-}^{q_+} p\, dq + \frac{2}{3}\int_{q_-}^{q_+} p^3 \left(\frac{v}{f^2}\right)'' dq\,.$$

Differentiating this with respect to E and using (4.19), gives

$$\widetilde{T} + E\,\frac{d\widetilde{T}}{dE} = \int_{q_-}^{q_+} \frac{dq}{p} + \int_{q_-}^{q_+} p \left(\frac{v}{f^2}\right)'' dq$$

which, in view of (4.15) and the assumption of Lemma 4.18, implies

$$\frac{d\widetilde{T}}{dE} = \frac{1}{E}\int_{q_-}^{q_+} p \left(\frac{v}{f^2}\right)'' dq > 0\,.$$

□

It remains to verify (4.18) for v given by (4.17). By an explicit computation (see Chicone [C])

$$\left(\frac{v}{f^2}\right)'' (v')^4 = e^q\, g(q)$$

where

$$g(q) := e^{2q} + 4\,(1-q)\,e^q - 2q - 5$$

is such that $g(0) = g'(0) = 0$ and $g''(q) = 4e^q v(q) \geq 0$. This implies $g(q) \geq 0$ ($g(q) = 0$ only if $q = 0$), the hypothesis of Lemma 4.3 and concludes the proof of Theorem 4.1.

□

Turning back to the Coulomb system problem, some remarks are now in order.

Remark 4.4 *Recalling $v(t,x) = u_x(t,x)$ and denoting $\lambda^* = \lim_{n\to\infty} \lambda^n$ the charge activity at the fixed point, we have from (2.15)*

$$\psi(0) = -i\sum_{q\in\mathbb{Z}} q\,\lambda^*(q) \Big/ \sum_{q\in\mathbb{Z}} \lambda^*(q) = 0$$

and

$$\psi'(0) = \sum_{q\in\mathbb{Z}} q^2\,\lambda^*(q) \Big/ \sum_{q\in\mathbb{Z}} \lambda^*(q) \geq 0\,.$$

These boundary conditions select ψ_j^+, $j = 1, \dots, k$, as being the only physically meaningful stationary solutions and implies $\phi^+(x) = \int_0^x \psi^+(y)\,dy \geq 0$ on $(-\pi,\pi)$.

Remark 4.5 *The value $\alpha = 2$ is a bifurcation point as one can see by linearizing (4.1) about $\psi \equiv 0$. The linear operator $L[0] = A$ given by (3.7) in the subspace of odd 2π–periodic functions has eigenvalues and associate eigenfunctions given by (3.8). Hence, if $\alpha > 2$, the eigenvalues are all positive and $\psi \equiv 0$ is locally stable. When $\alpha < 2$ (but close to 2) a single eigenvalue becomes negative and one can apply the Crandall–Rabinowitz bifurcation theory [C] to locally describe the stable solution which bifurcates from the trivial one. Note that Crandall–Rabinowitz theory can also be applied in the neighborhood of $\alpha_j = 2/j^2$, $j > 1$, in the orthogonal complement of the span $\left\{\pi^{-1/2}\sin mx,\ m = 1, ..., j-1\right\}$ corresponding to the odd functions with fundamental period $T = 2\pi/j$. These points were referred to in the introduction as a sequence of instability thresholds.*

In Theorem 4.1 we have given a global characterization of the non–trivial stationary solutions.

Remark 4.6 *In the sine–Gordon representation, the effective potential $\phi(x) = \int_0^x \psi(y)\,dy = x^2/(2\alpha)$ at $\gamma_{\alpha^{-1}}$ corresponds the Debye–Hückel regime with Debye length α. Although this regime is not reached for all $\beta > 0$, it gets closed quite fast as $\beta = 4\pi\alpha$ approaches 0. Numerical calculation is shown in Figure 2. Note that at $\alpha = 1$ ($\beta = 4\pi$), $\widehat{w}_1$ cannot be distinguished from α^{-1} (numerical error is in the sixth decimal order).*

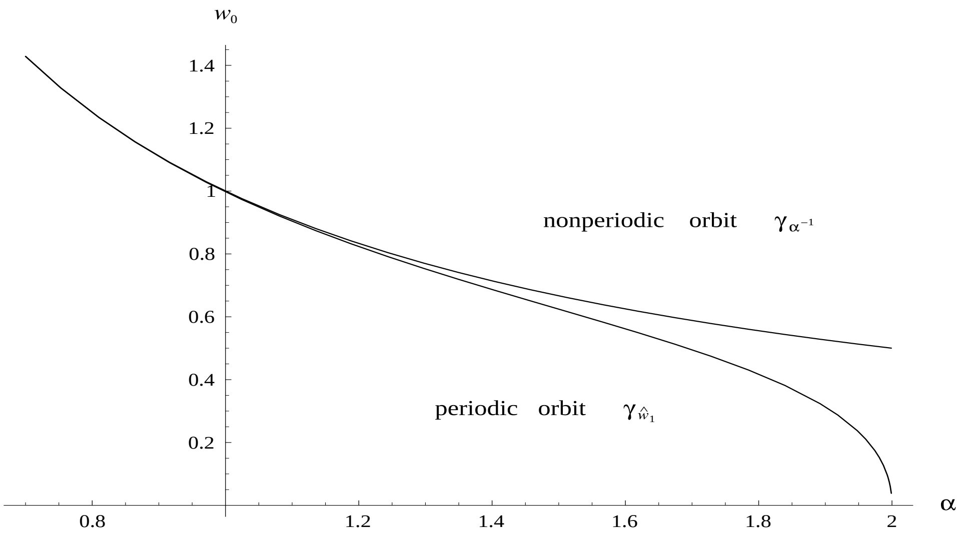

Figure 2: Comparison between the initial value function for the periodic orbit of period 2π, $\widehat{w}_1 = \widehat{w}_1(\alpha)$, and for the nonperiodic Debye–Hückel orbit, $\widehat{w}_{\rm DH}(\alpha) = \alpha^{-1}$.

Remark 4.7 *The derivative of (4.6) with respect to w_0, computed from equation (4.4),*

$$\frac{\partial T}{\partial w_0} = \frac{2\alpha w_0}{1-\alpha w_0}\int \operatorname{sign}(w)\,\frac{2\,(1-\alpha w)\,p^2/\alpha}{\left(1+2\,(1-\alpha w)\,p^2/\alpha\right)}dp\,,$$

indicates that an estimate from below can be very delicate to obtain. Note $\operatorname{sign}(w)$ *changes along the orbit γ_{w_0}. This shows how amusing Chicone's monotonicity result is for the problem at hand.*

5 Stability

Let $z(t; z_0)$ denote the solution of the initial value problem (3.6) – (3.7). It follows from the analysis in Section 3 that

$$S(t) z_0 = z(t; z_0) \tag{5.1}$$

defines a dynamical system on a closed subset $\mathcal{V} \subset D(A)$ of $\mathcal{B}^{1/2}$ with the topology induced by the graph norm $\|\cdot\|_{1/2}$. Note that $z(t; z_0)$ is continuous in both t and z_0 with $z(0; z_0) = z_0$ and satisfies the (nonlinear) semi–group property $S(t+\tau) z_0 = z(t; z(\tau; z_0)) = S(t) S(\tau) z_0$.

This section is devoted to the stability analysis of the equilibrium solutions described in Section 4. By local stability it is meant that $z(t; z_0)$ is uniformly continuous in $\mathcal{V}$ for all $t \geq 0$: given $\varepsilon > 0$, $\|z(t; z_0) - z(t; z_1)\|_{1/2} < \varepsilon$ for all $t \geq 0$ and $z_1 \in \mathcal{V}$ such $\|z_1 - z_0\|_{1/2} < \delta$ for some $\delta = \delta(\varepsilon) > 0$. It is uniformly asymptotically stable if, in addition, $\lim\limits_{t\to\infty} \|z(t; z_0) - z(t; z_1)\|_{1/2} = 0$.

The Liapunov (global) stability analysis as developed by LaSalle and applied to semilinear parabolic differential equations by Chafee and Infante [CI] (see also [H]) will also be discussed and extended in this section.

Let us begin with the local analysis.

Theorem 5.1 (Local Stability) *There exists a neighborhood $\mathcal{U} \subset \mathcal{B}^{1/2}$ of the origin such that, if $\alpha > 2$ and z_0 in $\mathcal{U}$, then $\psi_0 \equiv 0$ is stable, i.e., $\lim\limits_{t\to\infty} \|z(t; z_0)\|_{1/2} = 0$. If $\alpha < 2$ is such that $2/(k+1)^2 \leq \alpha < 2/k^2$ holds, among all equilibrium solutions of (4.1), $\psi_0, \psi_j^{\pm}$, $j = 1, \ldots, k$, $\psi_1^{\pm}$ are the only asymptotically stables. So, there exists $\rho > 0$ such that if $\|z_0 - \psi\|_{1/2} \leq \rho$, then $\lim\limits_{t\to\infty} \|z(t; z_0) - \psi\|_{1/2} = 0$ for $\psi = \psi_1^{\pm}$ and, for any sequence $\{z_n\}_{n\geq 1}$ with $\lim\limits_{n\to\infty} \|z_n - \psi\| = 0$, we have $\sup\limits_{t>0} \|z(t; z_n) - \psi\|_{1/2} \geq \varepsilon > 0$ for all n and $\psi = \psi_j^{\pm}$, $j \neq 1$.*

It is convenient to consider the equation

$$\frac{d\zeta}{dt} + L\zeta = F(\zeta) \tag{5.2}$$

for $\zeta = z - \psi$ where ψ is a solution of (4.1). Here

$$L\zeta = L[\psi]\,\zeta = -\alpha\zeta'' + 2\alpha\psi\zeta' - 2(1 - \alpha\psi')\,\zeta \tag{5.3}$$

is the linearization of the differential operator (3.1) around ψ and F is as in (3.7). Note $L = A$ and (5.2) reduces to (3.6) if $\psi = \psi_0 = 0$.

Proof. The proof of the Theorem 5.1 follows from the next two theorems.

Theorem 5.2 *If the spectrum $\sigma(L)$ of (5.3) lies in $\{\lambda \in \mathbb{R} : \lambda \geq c\}$ for some $c > 0$, then $\zeta = 0$ is the unique uniformly asymptotically stable solution of (5.2). On the other hand, if $\sigma(L) \cap \{\lambda \in \mathbb{R} : \lambda < 0\} \neq \emptyset$, then $\zeta = 0$ is unstable.*

Theorem 5.3 *Let $L = L[\psi]$ be given by (5.3). Then $\sigma(L) > 0$ whenever $\psi = \psi_0$ and $\alpha > 2$ or $\psi = \psi_1^{\pm}$ and $\alpha < 2$. If α is such that $2/(k+1)^2 \leq \alpha < 2/k^2$ holds for some $k \in \mathbb{N}_+$, then $\sigma(L) \cap \{\lambda \in \mathbb{R} : \lambda < 0\} \neq \emptyset$ for $\psi = \psi_0$ and $\psi = \psi_j^{\pm}$, $j = 2, \ldots, k$.*

Proof of Theorem 5.2. We shall prove only the first part of Theorem 5.2 and refer to Theorem 5.1.3 of Henry's book [H] for the instability part.

It follows from (3.13), (3.10), (3.15) and the hypothesis on $\sigma(L)$ that

$$\|\zeta(t)\|_{1/2} \leq C_{1/2}\, e^{-ct}\, \|\zeta_0\|_{1/2} + \xi \int_0^t (t-s)^{-1/2}\, e^{-c(t-s)}\, \|\,\zeta(s)\|_{1/2}^2\; ds\,, \tag{5.4}$$

with $c > 0$, $C_{1/2} = 1/\sqrt{2e}$ and $\xi = 2\sqrt{2\pi}\alpha$.

Let us assume that $\|\zeta(s)\|_{1/2} \leq \rho$ on a interval $(0, t)$ for some ρ satisfying

$$\xi \int_0^\infty t^{-1/2}\, e^{-ct}\, dt = \xi \sqrt{\frac{\pi}{c}} < \frac{1}{2\rho}, \tag{5.5}$$

i. e., $\rho < \dfrac{1}{4\pi\alpha}\sqrt{\dfrac{c}{2}}$. If $\|\zeta_0\|_{1/2} \leq \rho\sqrt{\dfrac{e}{2}}$, then equation (5.4) can be bounded as

$$\|\zeta(t)\|_{1/2} \leq \frac{\rho}{2} + \rho^2 \xi \int_0^t (t-s)^{-1/2}\; e^{-c(t-s)} < \rho \tag{5.6}$$

and this implies the existence of a unique solution of (5.2) with $\|\zeta(t)\|_{1/2} \leq \rho$ for all $t > 0$. Note that $\|\zeta_0\|_{1/2} < \rho$ and if t_1 is the maximum value under which $\|\zeta(t)\|_{1/2} < \rho$ for all $0 < t < t_1$, then either $\|\zeta(t_1)\|_{1/2} = \rho$ or $t_1 = \infty$. But the first case is impossible by (5.6).

Going back to (5.4), using $\|\,\zeta(s)\|_{1/2} < \rho$ and a slightly modification of Gronwall inequality (3.3) with $E_{1/2}(t) = \sum_{n=0}^{\infty} \left(\rho\xi\sqrt{\pi}t^{1/2}\right)^n / \Gamma(n/2+1)$, we have

$$\begin{aligned}
\|\zeta(t)\|_{1/2} &\leq C_{1/2}\, \|\zeta_0\|_{1/2}\, E_{1/2}(t)\; e^{-ct} \\
&\leq C_{1/2}\, \|\zeta_0\|_{1/2} \left(1 + \rho\xi t^{1/2}\right) e^{-\left(c - \rho^2\xi^2\pi\right)t} \\
&\leq \frac{1}{2e}\, \|\zeta_0\|_{1/2} \left(1 + \frac{1}{2}\sqrt{\frac{ct}{\pi}}\right)\; e^{-3ct/4}\,,
\end{aligned}$$

in view of (5.5). This proves the stability statement of Theorem 5.2, since (5.2) defines a dynamical system in a closed subset $\mathcal{V}_\rho = \left\{\zeta \in \mathcal{B}^{1/2} : \|\zeta\|_{1/2} \leq \rho\right\}$ with $\lim_{t\to\infty} \|\zeta(t)\|_{1/2} = 0$ if $\|\zeta_0\|_{1/2} = \|z_0 - \psi\|_{1/2} \leq \rho\sqrt{\dfrac{e}{2}}$.

□

Remark 5.4 *One can actually show that if* $c = \inf_\lambda \sigma(L)$ *then* $\zeta(t;\zeta_0) = z(t;z_0) - \psi$ *decays exponentially fast to* 0 *as*

$$\zeta(t;\zeta_0) = \kappa(\zeta_0)\, e^{-ct} + \varepsilon(t;\zeta_0)$$

where $\|\varepsilon(t;\zeta_0)\|_{1/2} \le C\, \|\zeta_0\|_{1/2}\, e^{-c't}$ *with* $0 < c < c'$ *and* $\kappa : \mathcal{V}_\rho \longrightarrow \mathcal{N}\,(L - cI)$ *is continuous and such that* $\kappa(0) = 0$*, where* $\mathcal{N}\,(L - cI)$ *is the one–dimensional span of the eigenfunction of* L *associated to* c*.*

Proof of Theorem 5.3. Since $L[\psi_0] = A$, Theorem 5.3 for $\psi = \psi_0$ with $\alpha \ge 0$ follows from the spectral computation in (3.8).

Now, let ψ be a nontrivial solution of the equilibrium equation (4.1) and note that $\psi(0) = \psi(\pi) = 0$ by parity. According to Theorem 5.2, ψ is asymptotically stable if $\sigma(L) > 0$ and unstable if $\sigma(L) \cap \{\lambda < 0\} \neq \emptyset$.

Let φ be the solution of

$$L[\psi]\varphi = 0 \tag{5.7}$$

in the domain $0 < x < \pi$ satisfying

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi'\,(0) = 1. \tag{5.8}$$

As in [H], we shall use the comparison theorem to establish that ψ is asymptotically stable if $\varphi(x) > 0$ on $0 < x \le \pi$ and unstable if $\varphi(x) < 0$ somewhere in $0 < x < \pi$.

To apply the comparison theorem and complete the proof of Theorem 5.3, let

$$p(x) := e^{-2\int_0^x \psi(y)\,dy} \tag{5.9}$$

be the weight which makes L a self–adjoint operator:

$$p\,L[\psi]\zeta = -\alpha\,(p\,\zeta')' - 2p\,(1 - \alpha\psi')\,\zeta\,. \tag{5.10}$$

Note that $(L\zeta,\eta)_p = (\zeta, L\eta)_p$ for any odd periodic functions ζ and η of period 2π were $(f,g)_p := \displaystyle\int_{-\pi}^{\pi} f(x)\,g(x)\,p(x)\,dx$.

Theorem 5.5 (Comparison) *Suppose* ζ_1 *and* ζ_2 *are two real solutions on the domain* $(0,\pi)$ *of*

$$p\,L[\psi]\zeta = f_i\,, \qquad i = 1,2\,,$$

respectively, with $\zeta_1(0) = \zeta_1(\pi) = 0$*,* $\zeta_1'(0) > 0$ *and* $\zeta_2(0) = 0$*,* $\zeta_2'(0) > 0$*. If* $\zeta_1 > 0$ *and* $f_i = f_i(\zeta; x)$ *is such that*

$$f_2 > f_1 \tag{5.11}$$

on $(0,\pi)$*, then* ζ_2 *must vanish at some point of this domain.*

Proof. Let assume that $\zeta_2 > 0$ on $(0,\pi)$. Then, from (5.10) and the hypotheses of Theorem 5.5, we have

$$
\begin{aligned}
2\int_0^\pi (f_2 - f_1)\, dx &= (\zeta_1, L\zeta_2)_p - (L\zeta_1, \zeta_2)_p \\
&= 2\alpha \int_0^\pi \left[(p\,\zeta_1')'\,\zeta_2 - \zeta_1\,(p\,\zeta_2')'\right] dx \\
&= 2\alpha \int_0^\pi \left[p\,(\zeta_1'\zeta_2 - \zeta_1\,\zeta_2')\right]' dx
\end{aligned}
$$

which, in view of the boundary conditions and (5.11), implies a contradiction

$$p(\pi)\,\zeta_1'(\pi)\,\zeta_2(\pi) > 0\,.$$

Note that $\zeta_1'(\pi) < 0$ since $\zeta_1 > 0$ on $(0,\pi)$ and $\zeta_1(\pi) = 0$. So, there must exist $\overline{x} \in (0,\pi)$ such that $\zeta_2(\overline{x}) = 0$.

□

If we consider the eigenvalue equation

$$L[\psi]\theta = \lambda\theta \tag{5.12}$$

on $(0,\pi)$ for the smallest eigenvalue λ in the space of odd periodic function, θ satisfies the conditions of ζ_1 in Theorem 5.5 with $f_1 = \lambda p\zeta$. Note the eigenfunction associated to the smallest eigenvalue may be chosen to be positive in the domain $(0,\pi)$.

Applying Theorem 5.5 for (5.7) and (5.12) we arrive to the following stability criterium:

Criterium 5.6 *The smallest eigenvalue λ of $L[\psi]$ is positive if $\varphi > 0$ on $(0,\pi)$ and negative if there exist $\overline{x} \in (0,\pi)$ such that $\varphi(\overline{x}) = 0$, where φ is the solution of equations (5.7) and (5.8).*

Now, for a given non–trivial stationary solution ψ let

$$\chi = c\,(-\alpha\psi'' + 4\psi)\ , \tag{5.13}$$

where $c > 0$ is chosen so that $\chi'(0) = 1$. It follows from the equation $-\alpha\psi'' = 2\,(1 - \alpha\psi')\,\psi$ (see (4.1)), that

$$\chi(0) = 0 \qquad \text{and} \qquad \chi > 0$$

whenever $\psi > 0$ (recall $\psi(0) = 0$ and $1 - \alpha\psi' > 0$ for all closed orbits). Moreover, we have

Proposition 5.7

$$L[\psi]\chi = 8c\alpha^2\psi\,(\psi')^2 > 0 \tag{5.14}$$

on the same domain $(0,\overline{x})$ that $\psi > 0$.

Proof. Differentiating (4.1) twice,

$$-\alpha\left(\psi''\right)'' = -2\alpha\psi\left(\psi''\right)' + 2\left(1-3\alpha\psi'\right)\psi''\,,$$

and using (4.1) again, gives

$$\begin{aligned} L[\psi]\psi'' &= -\alpha\left(\psi''\right)'' + 2\alpha\psi\left(\psi''\right)' - 2\left(1-\alpha\psi'\right)\psi'' \\ &= -4\alpha\psi'\psi'' \\ &= 8\left(1-\alpha\psi'\right)\psi\,\psi' \end{aligned}$$

In addition, we have

$$\begin{aligned} L[\psi]\psi &= -\alpha\psi'' + 2\alpha\psi\psi' - 2\left(1-\alpha\psi'\right)\psi \\ &= 2\alpha\psi\psi' \end{aligned}$$

which combined with the above equation, gives the equality in Proposition 5.7.

□

Completion of the proof of Theorem 5.3. We are in position to prove Theorem 5.3 for non–trivial equilibrium solutions. Let χ be given by (5.13) with $\psi = \psi_1^+$. Then $\chi > 0$ on $(0,\pi)$ and Theorem 5.5 can be used to compare equation (5.14) with (5.7). This yields $\varphi > \chi \geq 0$ on $(0,\pi]$ which implies the stability of ψ_1^+ by Criterium 5.6.

For instability, we observe that ψ' satisfies

$$\begin{aligned} L[\psi]\psi' &= -\alpha\psi''' + 2\alpha\psi\psi'' - 2\left(1-\alpha\psi'\right)\psi' \\ &= \left(-\alpha\psi'' + 2\alpha\psi\psi' - 2\psi\right)' = 0\,, \end{aligned}$$

in view of equation (4.1). Recall that $\psi = \psi_j^+$ with $j \geq 2$, has fundamental period $2\pi/j$ and satisfies $\psi(\pi/j) = \psi''(\pi/j) = 0$ by the odd parity and equation (4.1). Since $\psi'(0) > 0$, this implies $\psi < 0$ on $(\pi/j, 2\pi/j)$ and the minimum of ψ is attained at $\underline{x} = \dfrac{3\pi}{2j}$. Since ψ' and φ satisfies the same self–adjoint equation $pL[\psi]\zeta = 0$, their Wronskian

$$\begin{aligned} W\left(\varphi,\psi';x\right) &= \begin{vmatrix} \varphi & \psi' \\ -\alpha p\varphi' & -\alpha p\psi'' \end{vmatrix} \\ &= \alpha p\left(\varphi'\psi' - \varphi\psi''\right) = \alpha\psi'(0) > 0 \end{aligned}$$

is a non–vanishing constant (recall $p(0) = 1$, $\varphi(0) = 0$ and $\left(\psi_j^+\right)'(0) > 0$). As a consequence

$$W\left(\varphi,\psi';\pi/j\right) = -\alpha p(\underline{x})\varphi\left(\underline{x}\right)\psi''\left(\underline{x}\right) > 0$$

implies $\varphi\left(\underline{x}\right) < 0$ because $\psi''\left(\underline{x}\right) > 0$. It thus follows from the stability criterium that ψ_j^+, $j = 2,\ldots,k$, are unstable since $\underline{x} \in (0,\pi)$ provided $j \geq 2$ and there exist $\overline{x} \in (0,\pi)$, $\overline{x} < \underline{x}$, such that $\varphi(\overline{x}) = 0$.

By a slight modification of these arguments, one may conclude the stability of ψ_1^- and instability of ψ_j^-, $j = 2, \ldots, k$, as well. This concludes the proof of Theorem 5.3 and, consequently, the proof of Theorem 5.1.

$\square$

Now, we turn to the Liapunov stability analysis with a proof of global stability of the trivial solution $\phi_0 \equiv 0$.

Let V be a real–valued functional on the subspace of absolutely continuous function of $D(A)$ given by

$$V(v) = \int_{-\pi}^{\pi} \left\{ \left(\alpha^{-1} - v'\right) \ln\left(1 - \alpha v'\right) + v' - v^2 \right\} \, dx \tag{5.15}$$

and notice that $V(0) = 0$ and $V(\eta) = W(\eta) + o\left(\|\eta\|^2\right)$, as $\|\eta\| \to 0$, where

$$W(v) = \frac{1}{2} \int_{-\pi}^{\pi} \left(\alpha v'^2 - 2v^2\right) \, dx \,,$$

by Taylor expanding $g(w) = (\alpha^{-1} - w) \ln(1 - \alpha w) - w$ around $w = 0$. Observe that $W(v) = (1/2) \|v\|_{1/2}^2$ if $\alpha > 2$ and since $g(w) - (\alpha/2) w^2 \geq 0$ if $\alpha w < 1$, $V(v) \leq W(v)$ holds on the space

$$\mathcal{V} = \left\{ v \in H^1_{\mathrm{o,p}} \cap H^2_{\mathrm{o,p}} : \alpha v' < 1 \right\} \,,$$

of odd, positive and 2π–periodic functions with distributional derivative up to second order.

A Liapunov function V of a dynamical system $\{S(t), t \geq 0\}$ satisfies

$$\dot{V}(v) = \overline{\lim_{t \downarrow 0}} \frac{1}{t} \left(V\left(S(t)v\right) - V(v) \right) \leq 0 \tag{5.16}$$

for all $v \in \mathcal{V}$. We now show that (5.16) holds if $S(t)$ is given by equation (3.1). More precisely,

Proposition 5.8 *Let $S(t)v_0 = v(t; v_0)$ be the dynamical system in $\mathcal{V}$ given by (5.1). Then, the pair of functions*

$$\rho(w) = \frac{1}{1 - \alpha w} \tag{5.17}$$

and

$$\Phi(p, w) = \left(\alpha^{-1} - w\right) \ln\left(1 - \alpha w\right) + w - p^2 \tag{5.18}$$

generate the Liapunov function given by (5.15):

$$V(v) = \int_0^{\pi} \Phi(v, v_x) \, dx \qquad \text{with} \qquad \dot{V}(v) = \int_0^{\pi} \rho(v_x) v_t^2 \, dx \,. \tag{5.19}$$

Proof. Note that, from the parity of v the integral in (5.15) can be made over $[0,\pi]$. By the calculus of variations and equations (5.17) and (5.18) we have

$$\begin{aligned}\dot{V}\left(v\left(t,\cdot\right)\right) &= -\int_0^\pi \left(\frac{d}{dx}\frac{\partial \Phi}{\partial v_x} - \frac{\partial \Phi}{\partial v}\right) v_t\, dx + \left.\frac{\partial \Phi}{\partial v_x} v_t\right|_0^\pi \\ &= -\int_0^\pi \left(-\frac{d}{dx}\ln(1-\alpha v_x) + 2v\right) v_t\, dx \\ &= -\int_0^\pi \rho(v_x)\,\left(\alpha v_{xx} - 2\alpha v v_x + 2v\right) v_t\, dx\,, \qquad (5.20)\end{aligned}$$

where $v_t(t,0) = v_t(t,\pi) = 0$, $t \geq 0$, in view of the boundary conditions on $\mathcal{V}$. Since $\rho(w) \geq 0$ for $\alpha w < 1$, this with (3.1) concludes the proof of Proposition.

□

Remark 5.9 *We have used the construction method based in the Euler–Lagrange equation to find this Liapunov function (see e.g. Chap. 2 of Zelenyak, Lavrentiev and Vishnevskii [ZLV]). A sufficient condition for (5.19) hold leads to a first order partial differential equation for* ρ

$$w\rho_p - \frac{2}{\alpha}(1-\alpha w)p\rho_w = -2p\rho$$

whose characteristics are given by the orbits γ_{w_0} *described in Section 4 in the study of the equilibrium solutions of (3.1). Note that equation (5.18) is the Lagrangian associated with the Hamiltonian (4.13) (with* q *defined by (4.11)). Due to the requirement* $\alpha w < 1$*, our particular solution takes into account only the closed orbits. There may be other suitable choices which includes all orbits.*

The proof of global stability of ϕ_0 requires that a subspace of $\mathcal{V}$ be invariant under the flow equation (3.1). This is shown in the following by using the maximum principle.

Theorem 5.10 *If* $v(t,x)$ *is a classical solution of equation (3.1) with initial condition* $v(0,x) = v_0(x) \in \mathcal{V}$*, then* $\alpha v_x(t,x) < 1$ *and*

$$\alpha^{-1}\left(x-\pi\right) < v(t,x) < \alpha^{-1}x\,, \qquad (5.21)$$

hold for all $t \geq 0$ *and* $0 \leq x \leq \pi$.

Proof. Denoting

$$L[v] := F(v_{xx}, v_x, v) - v_t, \qquad (5.22)$$

where $F(a_1,a_2,a_3) = \alpha\,(a_1 - 2a_2\,a_3) + 2a_3$ is a continuous and differentiable function of its variables, the differential equation (3.1) can be written as

$$L[v] = 0. \qquad (5.23)$$

For v satisfying (5.23) with $v(t,0)=v(t,\pi)=0\,,\ 0\le t\le\tau$, and initial data $v(0,\cdot)=v_0$, let us suppose $z=z(t,x)$ and $Z=Z(t,x)$ are such that

$$L[Z]\le 0\le L[z] \tag{5.24}$$

for all (t,x) in $D=(0,\tau)\times(0,\pi)$ with $z(t,y)\le 0\le Z(t,y)$, $y=0,\pi$ and $0\le t\le\tau$, and

$$z(0,x)\le v_0(x)\le Z(0,x)\,,$$

for $0\le x\le\pi$. Then, by the maximum principle (see [PW], Theorem 12 in Chap. 3),

$$z(t,x)\le v(t,x)\le Z(t,x) \tag{5.25}$$

in $\overline{D}=[0,\pi]\times[0,\tau]$.

The lower limit function z is given by

$$z(x)=\theta\,(x-\pi)\ , \tag{5.26}$$

with $\theta\ge 0$. From (5.23),

$$L[z]=2\theta(\alpha\theta-1)(\pi-x)\}\,,$$

is always positive provided $\alpha\theta\ge 1$.

Analogously, the upper limit function Z is given by

$$Z(x)=\delta x\,, \tag{5.27}$$

from which

$$L[Z]=-2\delta(\alpha\delta-1)x$$

is always negative provided $\alpha\delta\ge 1$.

Since (5.24) holds uniformly in τ, equation (5.25) holds for all (t,x) in $\mathbb{R}_+\times[0,\pi]$. Note that v remains bounded irrespective of $\alpha v_x<1$. However, if this condition holds for $t=0$, it remains for all $t>0$. To see this, observe from the equation $v_t=v_{xx}+(1-\alpha v_x)v$ with $v_{xx}=0$, that the rate by which $|v|$ increases tends to zero when the inequality saturates. The inclusion of the Laplacian only smooths v and prevents, even more, v_x to increase beyond the threshold. The same argument justify the strict inequality (5.21).

This concludes proof of Theorem 5.10.

□

We pause to discuss some properties of the classical solutions of equations (1.1) and (3.3). Recall that $\widetilde{u}(t,x)=\displaystyle\int_0^x v(t,y)\,dy$ with v satisfying (3.1).

Remark 5.11 *Note that the cone* $\mathcal{C}=\left\{u\in H^1_{\mathrm{e,p}}\cap H^2_{\mathrm{e,p}}:u\ge 0,\ \alpha u_{xx}<1\right\}$ *is invariant under the unnormalized evolution (1.1). For this, let*

$$M[u]:=\alpha(u_{xx}-u_x^2)+2u.$$

If $u(t,x)$ is a classical solution of (1.1) with initial value $u_0 \in \mathcal{C}$, since $M[u] = 0$ for $u \equiv 0$, we have by Theorem 7 in Chap. 3 of [PW] (see also Remark (ii) after this) that $u(t,x) \geq 0$ for all $t > 0$. This, however, does not imply that $\widetilde{u}(t,x)$ remains positive (recall (3.5)). A proof of this assertion goes as follows.

Theorem 5.10 implies $\widetilde{u}(t,x)$ remains bounded, and $\widetilde{u}_{xx}(t,0)$ bounded from above, if $\widetilde{u}$ satisfies (3.3) with initial condition u_0 satisfying $\alpha^{-1}[(x-\pi)^2/2 - \pi^2/2] < u_0 < \alpha^{-1}x^2/2$ (by integrating (5.21)). The comparison principle applied directly to equation (3.3) leads to (5.24) with L replaced by M and 0 replaced by $\alpha\widetilde{u}_{xx}(t,0)$. An upper and lower solutions, z and Z, can be obtained from the solution of the equilibrium initial value problem (4.2):

$$\Phi^{\pm}(\alpha, w_0; x) = \int_0^x \Psi^{\pm}(\alpha, w_0; y)\, dy\,,$$

where $\Psi^{\pm}(\alpha, w_0; x)$ is the p–component of the closed orbit $\gamma_{\pm w_0}$ starting at $(\pm w_0, 0)$. Note that $\Phi^{\pm}$ is an even periodic function of period $T = T(\alpha, w_0)$ given by a monotonically increasing function of both w_0 and α with $T \to \infty$ as $\alpha w_0 \uparrow 1$ for Φ^+ and as $w_0 \to \infty$ for Φ^- (see proof of Theorem 4.1 for details). $\Phi^{+(-)}$ is also a monotone increasing (decreasing) function of x in $[0, T/2]$ and satisfies

$$M[\Phi^{\pm}] = \pm\alpha w_0\,,$$

with $\Phi^{\pm}(0) = \Phi^{\pm\prime}(0) = \Phi^{\pm\prime}(T/2) = 0$ and $\Phi^{\pm\prime\prime}(0) = w_0$. The lower limit function z is given by

$$z(x) = \theta\left(\Phi^{-}(\alpha, w_0; x + \widetilde{x}) + \frac{\alpha}{2}\widetilde{w}_0\right)\,, \tag{5.28}$$

where $\theta < 1$, $\widetilde{w}_0 \geq w_0$, $\widetilde{x} = \widetilde{x}(\alpha, w_0, \widetilde{w}_0)$ is such that $z(0) = 0$, with w_0 and $\widetilde{w}_0$ so that T is very large and $z'(\pi) = 0$ which can always be done in view of the properties of Φ^-. The upper limit function Z can be written also as (5.28) with Φ^- replaced by Φ^+ and the second term with minus sign. We have

$$M[W^{\pm}] = \alpha\theta\{(1-\theta)(\Phi^{\pm\prime})^2 \mp (\widetilde{w}_0 - w_0)\}\,,$$

with $W^+ = Z$ and $W^- = z$. In order inequality (5.25) holds uniformly in τ, $\widetilde{u}_{xx}(t,0)$ has to remain bounded from above and below. Since $\widetilde{u}_{xx}(t,0) < \alpha^{-1}$ by Theorem 5.10, one may choose θ arbitrarily small in (5.28) and take $\widetilde{w}_0$ and w_0 so large that $\theta(\widetilde{w}_0 - w_0) > \alpha^{-1}$. In the limit as $\theta \to 0$ we have $\widetilde{u}(t,x) \geq 0$ and $0 \leq \widetilde{u}_{xx}(t,0) < \alpha^{-1}$ for $0 \leq t \leq \tau$, uniformly in τ, implying $\widetilde{u}(t,x) \geq 0$ for all $t \geq 0$.

LaSalle's invariance principle allows us to apply Liapunov function techniques under milder assumptions. A subset $\mathcal{K} \subset \mathcal{V}$ of a complete metric space $\mathcal{V}$ is said to be *invariant* (positive invariant) if, for any $v_0 \in \mathcal{K}$, there exist a continuous curve $v : \mathbb{R} \longrightarrow \mathcal{K}$ with $v(0) = v_0$ and $S(t)v(\tau) = v(t+\tau)$ for all $t \geq 0$ and $\tau \in \mathbb{R}$ ($\mathbb{R}_+$). The following two theorems express the content of this principle.

Theorem 5.12 *Suppose $v_0 \in \mathcal{V}$ is such that the orbit $\gamma(v_0) = \{S(t)v_0, t \geq 0\}$ through v_0 lies in a compact set in $\mathcal{V}$ and let $\omega(v_0)$ denote its ω –limit set, i.e.,*

$$\omega(v_0) = \bigcap_{\tau \geq 0} \overline{\gamma\left(S(\tau)v_0\right)}$$

(see (4.5) for alternative definition). Then $\omega(v_0)$ is nonempty, compact, invariant, connected and $\operatorname{dist}\left(S(\tau)u_0, \omega(v_0)\right) \longrightarrow 0$ *as $t \to \infty$.*

Proof. We refer to Theorem 4.3.3 of [H] for details. Note that $\omega(v_0)$ is the intersection of a decreasing collection of nonempty compact sets. Note, in addition, that $\omega(v_0)$ is positive invariant by definition and is invariant by compactness argument.

□

Theorem 5.13 *Let V be a Liapunov function for $t \geq 0$ and, for*

$$\mathcal{E} := \left\{ v \in \mathcal{V} : \dot{V}(v) = 0 \right\}, \tag{5.29}$$

let $\mathcal{K}$ be the maximal invariant set in $\mathcal{E}$. If the orbit $\gamma(v_0)$ lies in a compact set in $\mathcal{V}$, then $S(t)v_0 \longrightarrow \mathcal{K}$ as $t \to \infty$.

Proof. By definition, $V(S(t)v_0)$ is a nonincreasing function of t and bounded from below, by hypothesis. So, $\lim\limits_{t\to\infty} V(S(t)v_0) = v$ exists. If $y \in \omega(v_0)$, then $V(y) = v$ and, in view of the fact that $S(t)y = y$, we have $V(S(t)y) = v$ which implies $\dot{V}(t) = 0$ and $\omega(v_0) \in \mathcal{K}$.

□

Now, we apply the invariance principle to the problem at our hand. As we will see, if $\mathcal{B}_0$ is a sufficient large ball around $\phi_0 = 0$ in the cone $\mathcal{C}$ (with the induced topology of $H^1_{\rm e,p}$), the invariant set $\mathcal{K}_k = \{\omega(u_0),\ u_0 \in \mathcal{B}_0\} \subset \mathcal{E}$ consists of the union of unstable manifolds for the equilibrium points $\phi_0, \phi_1, \ldots, \phi_k$, with $\phi_j(x) = \displaystyle\int_0^x \psi_j^+(y)\, dy$, provided α is such that $2/\left(k+1\right)^2 \leq \alpha < 2/k^2$ holds for some $k \in \mathbb{N}$. Note that the hypotheses of Theorems 5.12 and 5.13 hold since the orbits of $S(t)v_0$ are bounded in $H^1_{\rm o,p}$ by Theorem 5.10 and remain in a compact set of $H^1_{\rm o,p}$ in view of Theorem 3.2. For this the Sobolev embedding theorem is evoked: $W^{2,2}\left(-\pi,\pi\right) \subset C^{1+a}\left(-\pi,\pi\right)$ with continuous inclusion, so v has a continuous representative in $C^{1+a}_{\rm o,p}$ which belongs to $C^{2+a}_{\rm o,p}$ by Schauder estimates (see e.g [S] and references therein). Therefore, any solutions $\widetilde{u}(t,x) = \int_0^x v(t,y)\, dy$ of (3.3) in $\mathcal{C}$ has a continuously three–times differentiable representative.

We thus have

Theorem 5.14 *If $\alpha > 2$, $\phi_0 = 0$ is globally asymptotically stable solution of (3.3) in*

$$\widetilde{\mathcal{C}} = \left\{ u \in H^1_{\rm e,p} \cap H^2_{\rm e,p} : u(0) = 0,\ u \geq 0 \text{ and } \alpha u_{xx} < 1 \right\}.$$

If $\alpha < 2$, the origin is unstable in $\widetilde{\mathcal{C}}$ and there exits an open dense set $\mathcal{U} \subset \widetilde{\mathcal{C}}$ of initial conditions such that $\lim\limits_{t\to\infty} \widetilde{u}(t;u_0) \longrightarrow \phi_1^+$ for all $u_0 \in \mathcal{U}$.

Proof. It follows from Theorem 5.13 $v(t;\cdot) \longrightarrow \omega\left(v(0;\cdot)\right) \subset \left\{ \psi : \dot{V}\left(\psi\right) = 0 \right\}$ in $\mathcal{V}$ as $t \to \infty$. But, from (5.20), $\dot{V}\left(\psi\right) = 0$ iff

$$\alpha\psi'' - 2\alpha\,\psi\psi' + 2\psi = 0\,, \tag{5.30}$$

whose solutions are $\psi = \psi_0$ and ψ_j^+, $j = 1, \dots, k$, studied in Section 4. We note that $\phi_j^+(x) = \int_0^x \psi_j^+(y)\, dy \geq 0$ ($\phi_j^-(x) \leq 0$) for all $x \in [-\pi, \pi]$ and $j \geq 1$, since $\left(\psi_j^+\right)'(0) > 0$ ($\left(\psi_j^-\right)'(0) < 0$).

Multiplying (5.30) by ψ and integrating over $(-\pi, \pi)$, gives

$$\int_{-\pi}^{\pi} (\alpha\psi'' + 2\psi)\, \psi\, dx = -\left\|\psi\right\|_{1/2}^2 \leq 0$$

if $\alpha > 2$. The nonlinear term vanishes since, by integration by parts,

$$\int_{-\pi}^{\pi} \psi' \psi^2\, dx = -2 \int_{-\pi}^{\pi} \psi' \psi^2\, dx\,.$$

This implies $\psi \equiv 0$ and proves that $S(t)v_0 \longrightarrow 0$ as $t \to \infty$ in $\mathcal{V}$. We quote Theorem 4.3.5 in [H] for the instability assertion.

Since the spectrum $\sigma(L)$ of the linearized operator around the equilibrium points (see Theorem 5.3) lies on the real line, all equilibrium points are hyperbolic, $\mathcal{E}$ given in (5.29) is a discrete and finite set and

$$\mathcal{V} = \bigcup_{\psi \in \mathcal{E}} \mathcal{W}^s(\psi)$$

holds with $\mathcal{W}^s(\psi) = \{u_0 \in \mathcal{V} : S(t)v_0 \longrightarrow \psi \text{ as } t \to \infty\}$. It is proven in [H] that each stable manifold $\mathcal{W}^s(\psi)$ is a C^2 embedded submanifold of $\mathcal{V}$ ($\mathcal{W}^s(\phi)$ is C^3 submanifold of $\widetilde{\mathcal{C}}$) and, if ψ is locally unstable, than $\mathcal{W}^s(\psi)$ has codimension larger than or equal to 1. Therefore, $\mathcal{V}$, and consequently $\widetilde{\mathcal{C}}$, can be written as a *finite* union of open connected sets together with a closed nowhere–dense remainder.

□

Finally, we show that, for an open set $\mathcal{V}_0 \subset \mathcal{V}$ given as before, the maximal invariant set

$$\mathcal{K}_k = \bigcup_{\psi \in \mathcal{E}} \mathcal{W}^u(\psi) \tag{5.31}$$

where $\mathcal{W}^u(\psi) = \{v_0 \in \mathcal{V} : S(t)v_0 \longrightarrow \psi \text{ as } t \to -\infty\}$ is the unstable manifold of ψ. By Theorem 5.13, the orbit $v(t; u_0) = S(t)v_0$ exists and remains, by invariance, in $\mathcal{K}_k$ for all $t \in \mathbb{R}$. Therefore, $\lim_{t\to\infty} v(t, u_0) = \psi$ exists and $\psi \in \mathcal{K}_k$, so $\mathcal{K}_k \subset \bigcup_{\psi\in\mathcal{E}} \mathcal{W}^u(\psi)$. Since the converse is also true, the equality (5.31) thus holds.

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References

[BF] D. Brydges and P. Federbush, *Debye Screening*. Commun. Math. Phys. **73**, 197-245 (80).

[BGN] G. Benfatto, G. Gallavotti and F. Nicoló, *On the massive sine–Gordon equation in the first few regions of collapse*. Commun. Math. Phys. **85**, 387-410 (82).

[C] Carmen Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields*. J. Diff. Eqns. **69**, 310-321 (87).

[CG] W. A. Coppel and L. Gavrilov, *The period function of a Hamiltonian quadratic system.* Diff. and Int. Eqns. **6**, 1357-1365 (93).

[CI] N. Chafee and E. F. Infante, *A bifurcation problem for a nonlinear partial differential equation of parabolic type*. J. Aplicable Anal. **4**, 17-37 (74).

[CL] Earl A. Coddington and N. Levinson, *Theory of ordinary differential equations*. MacGraw-Hill Book Company, 1955.

[DH] J. Dimock and T. R. Hurd, *Construction of the two–dimensional sine–Gordon model for* $\beta < 8\pi$. Commun. Math. Phys. **156**, 547-580 (93).

[F] Giovanni Felder, *Renormalization group in the local potential approximation*, Commun. Math. Phys. **111**, 101-121 (87).

[FLL] Michael E. Fisher, Xiao–jun Li and Yan Levin, *On the absence of intermediate phases in the two–dimensional Coulomb gas*. J. Stat. Phys. **79**, 1-11 (95).

[FS] J. Fröhlich and T. Spencer, *The Kosterlitz–Thouless in two-dimensional abelian spin systems and Coulomb gas* . Commun. Math. Phys. **81**, 527-602 (81)

[GN] G. Gallavotti and F. Nicoló, *The "The screening phase transitions" in the two–dimensional Coulomb gas*. J. Stat. Phys. **39**, 133-156 (85).

[H] Daniel Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics **840**, Springer-Verlag, 1981.

[JKKN] J. V. José, L. P. Kadanoff, S. Kirkpatric and D. R. Nelson, *Renormalization, vortices, and symmetry–breaking perturbations in the two–dimensional planar model*. Phys. Rev. D **16** , 1217-1241 (77).

[MK] D. H. U. Marchetti and A. Klein, *Power–law falloff in two–dimensional Coulomb gases at inverse temperature* $\beta > 8\pi$. J. Stat. Phys. **64**, 135-162 (91).

[MP] D. H. U. Marchetti and J. F. Perez, *The Kosterlitz–Thouless phase transition in two–dimensinal hierarchical Coulomb gases*. J. Stat. Phys. **55**, 141-156 (89).

[PW] Murray H. Protter and Hans F. Weinberger, *Maximum principles in differential equations.* Prentice–Hall, Inc., New Jersey, 1964.

[S] D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems.* Indiana Univ. Math. J. **21**, 979-1000 (72).

[ZLV] T. I. Zelenyak, M. M. Lavrentiev Jr. and M. P. Vishnevskii, *Qualitative theory of parabolic equations.* VSP, Utrecht, 1997.