Lindstedt series for perturbations of isochronous systems. II. KAM theorem and stability of the upside-down pendulum

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Abstract. *We consider the planar pendulum with support point oscillating in the vertical direction, and we study its motion around the equilibrium point corresponding to the upside-down position. We prove that the equilibrium point is stable for the projection of the motion on the pendulum phase space (for a full measure subset of the stability region of the linearized system inside the two-dimensional space of parameters), by proving the persistence of invariant KAM tori for the two-dimensional Hamiltonian system describing the model.*

1. Introduction

1.1. *The state of the art.* The upside-down pendulum with the support point oscillating with a frequency ω large enough has been extensively studied in literature as a simple model exhibiting a quite nontrivial behaviour; see [BGG] and the references quoted therein.

The stability of the upside-down position can be proven by the averaging method (see for instance [PR], Ch. 9): the result is that if the support point oscillates fast enough then the position becomes stable. However such a kind of analysis is not completely rigorous both because no explicit control on the corrections can be obtained and because it can lead to un-correct results, as already pointed out in [BSG]. In fact the averaging approach can be followed also for studying the stability of the downward position, and the result one finds in doing so is that such a position is always stable – provided that ω is large enough to make possible to apply the averaging, say $\omega > \omega_1$, for some ω_1 –, a result obviously unacceptable as by varying ω above ω_1 one can lose also the linear stability (as it follows from Mathieu's equation theory).

A rigorous proof has been given for the linearized system in [Ac], where also the case of several pendula has been considered. In the latter case the linearized system can be written (by a diagonalization procedure) as a system of several un-coupled Mathieu's equations. So both for a single pendulum and for more than one pendulum, the theory of Mathieu's equation applies. In particular for a single pendulum the physical parameters describing the system have to be such that the parameters (a, q) appearing in Mathieu's equation $\ddot{x} + (a + 2q \cos 2\tau)x = 0$ lay inside the stability region corresponding to negative values of a (see [Ac]): if the amplitude of the oscillation of the support point is small enough (with respect to the length of the pendulum), then one has stability if the frequency ω is large enough above a threshold value ω_0, and the same value predicted by the averaging method is found for ω_0.

1.2. *Stability and KAM theory.* What is missing in the literature is a rigorous discussion of the full system (not only the linearized one). In this paper we achieve such a task, by studying the full system by means of the perturbation theory techniques. As the linearized system can be written as a two-dimensional integrable Hamiltonian system, the full system becomes a perturbation of an integrable system: so KAM theory applies. Then we can prove that a large quantity of invariant tori persist under the perturbations, and the nearer the initial data $(\theta(0), \dot{\theta}(0))$ are to the upside-down position $(\pi, 0)$ the nearer the curves obtained by projection of the tori on the pendulum phase space $(\theta, \dot{\theta})$ are to such a position: this proves the stability of the upside-down position for the pendulum.

Note that in applying the above argument it is a fundamental fact that the system is a two-dimensional one, so that the existence of the tori yield a topological obstruction for trajectories starting inside a torus to cross it and move far from it. So such a result can not be used in order to study the stability of the upside-down position in the case of more than one pendulum: as a matter of fact, in the latter case, it is even likely that Arnol'd diffusion can occur and, as a consequence, the position is not stable at all. See also comments in [BGG].

1.3. *Contents.* As far as the proof is concerned, in such a note we simply show that the system can be written in terms of action-angle variables, and gives rise to a perturbed isochronous system, by referring to [BG] for the proof of persistence of KAM invariant tori within the framework of the Lindstedt series recently developed in literature (see [BG] for references) starting from the pioneering work by Eliasson [E].

In Section 2 we consider preliminarily a very simple model just to illustrate the problem in a easier case: it plays the rôle of a propedeutic problem. Then in Section 3 we introduce the model which we are really interested in, and we show how the analysis of [BG] can be applied to it. Finally in Appendix A2 we show how the condition the oscillation frequency of the support point of the pendulum can be relaxed into a weaker one still using the Lindstedt series methods: from a technical point of view Appendix A2 represents the main interest of the present note. [Appendix A1 is simply a brief review on the canonical change of coordinates leading the Hamiltonian into action-angle variables.]

2. First model: a propedeutic example

2.1. *A simple model.* Consider the system described by the non-autonomous Hamiltonian

$$H_0 = \frac{1}{2}\left(y_1^2 + \mu^2 x_1^2\right) + f_0(y_1, x_1, t), \tag{2.1}$$

where $\mu \in (0,1)$, $(y_1, x_1) \in \mathbb{R}^2$ are conjugated variables, and f_0 is analytic in its arguments (in a suitable domain $\mathcal{B}$), is 2π-periodic in t and depends on terms of order at least 3 in x_1, y_1: therefore the origin $(0,0)$ is an elliptic equilibrium point for the motion of the variables (x_1, y_1). Then we can define

$$\begin{cases} y_1 = \varepsilon p_1, \\ x_1 = \varepsilon q_1, \end{cases} \tag{2.2}$$

with ε a small parameter. Then (2.1), divided by ε^2, becomes

$$H = \frac{1}{2}\left(p_1^2 + \mu^2 q_1^2\right) + \tilde{f}(p_1, q_1, t, \varepsilon), \tag{2.3}$$

where $\tilde{f}(p_1, q_1, t, \varepsilon) = \varepsilon^{-2} f_0(y_1, x_1, t)$ is such that $\tilde{f} = O(\varepsilon)$.

By passing to action-angle variables (as far as (q_1, p_1) is different from $(0,0)$)

$$\begin{cases} p_1 = \sqrt{2A_1\mu}\cos\alpha_1, \\ q_1 = \sqrt{2A_1/\mu}\sin\alpha_1, \end{cases} \tag{2.4}$$

we can write H as an autonoumous Hamiltonian by introducing an auxiliary variable A_2 conjugated to $\alpha_2 \equiv t$, *i.e.*

$$H = \boldsymbol{\omega} \cdot \mathbf{A} + f(\boldsymbol{\alpha}, A_1, \varepsilon), \tag{2.5}$$

where $\mathbf{A} = (A_1, A_2) \in \mathbb{R}^2$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{T}^2$ are conjugated variables, $\boldsymbol{\omega} = (\mu, 1)$, $\cdot$ denotes the inner product in $\mathbb{R}^2$ and f a function analytic in the variables A_1, $\boldsymbol{\alpha}$ and ε, in a domain

$$\{(\mathbf{A}, \boldsymbol{\alpha}, \varepsilon) \;:\; |\mathrm{Im}\,\alpha_j| < \kappa, j = 1, 2, \quad |\varepsilon| < \varepsilon_1, \quad A_1 \in \mathcal{A}\}, \tag{2.6}$$

where $\mathcal{A}$ is an open subset of $\mathbb{R} \setminus \{0\}$ (depending on $\mathcal{B}$ as well as κ and ε_1) and such that $f = O(\varepsilon)$. Note that the dependence of f on A_1 and ε is only through the product $\varepsilon A_1^{1/2}$; more explicitly, by denoting $\boldsymbol{\nu} = (\nu_1, \nu_2)$, one has

$$f(\boldsymbol{\alpha}, A_1, \varepsilon) = \sum_{p=3}^{\infty} f_p(\boldsymbol{\alpha}) A_1^{p/2} \varepsilon^{p-2}, \tag{2.7}$$

with

$$f_p(\boldsymbol{\alpha}) = \sum_{|\nu_1| \le p} \sum_{\nu_2 \in \mathbb{Z}} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{p,\boldsymbol{\nu}}, \tag{2.8}$$

a property which one can take advantage of in order to improve the conditions one has to impose on $\boldsymbol{\omega}$ to prove the persistence of invariant tori (see §2.3 below and Appendix A2).

We shall explicitly consider also the special case

$$f(\boldsymbol{\alpha}, A_1, \varepsilon) = F(\alpha_1, A_1, \varepsilon) \cos \alpha_2, \tag{2.9}$$

i.e. $f_0(y_1, x_1, t) = F_0(y_1, x_1) \cos t$ in (2.1), so that the coefficients $f_p(\boldsymbol{\alpha})$ in (2.7) can be written as

$$f_p(\boldsymbol{\alpha}) = \sum_{|\nu_1| \le p} \sum_{\nu_2 = \pm 1} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{p,\boldsymbol{\nu}} = \sum_{|\boldsymbol{\nu}| \le p+1} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{p,\boldsymbol{\nu}}, \tag{2.10}$$

where in the last formula one puts $f_{p,\boldsymbol{\nu}} = 0$ whenever $\boldsymbol{\nu}$ is not of the form $\boldsymbol{\nu} = (\nu_1, \pm 1)$, with $|\nu_1| \le p$.

The system (2.1) represents two harmonic oscillators interacting through a potential depending only on the angles and on the action variable A_1: the latter condition, in particular, implies that one oscillator is simply a clock, *i.e.* it rotates with fixed frequency.

The corresponding equations of motion are

$$\begin{cases} \dot\alpha_1 = \mu + \partial_{A_1} f, \\ \dot\alpha_2 = 1, \\ \dot A_1 = -\partial_{\alpha_1} f, \\ \dot A_2 = -\partial_{\alpha_2} f, \end{cases} \tag{2.11}$$

so that one sees that $\alpha_2(t) = t$.

2.2. *Stability of the origin.* We want to study the persistence of tori near the origin. Note that, for any value of $\mathbf{A}$, the origin is recovered by setting $\varepsilon = 0$ (see (2.2)). On the other hand for $\varepsilon = 0$ the scaled Hamiltonian (2.5) reduces to $\boldsymbol{\omega} \cdot \mathbf{A}$, so that it admits invariant tori for any value $\mathbf{A}_0 = (A_{01}, A_{02})$ of the action variable $\mathbf{A}$, all run with the same rotation vector $\boldsymbol{\omega}$: such tori are defined by

$$\mathcal{T} = \{\boldsymbol{\alpha}(t) = \boldsymbol{\omega} t, \mathbf{A}(t) = \mathbf{A}_0\}, \tag{2.12}$$

where $\boldsymbol{\omega}$ is fixed and $\mathbf{A}_0 \in \mathbb{R}^2$.

Note that the Hamiltonian (2.5) is of the form considered in [BG], so that we can apply the theorem 1.4 of [BG]. Then suppose $\boldsymbol{\omega} = (\mu, 1)$ to be Diophantine with Diophantine constants $C > 0$ and $\tau > 1$, *i.e.*

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > C \, |\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : \tag{2.13}$$

by the theorem 1.4 of [BG] we know that there exist two functions $\mathbf{h}(\boldsymbol{\psi}, \mathbf{A}, \varepsilon, \mu_0)$ and $\mathbf{H}(\boldsymbol{\psi}, \mathbf{A}, \varepsilon, \mu_0)$, called the *conjugating functions*, analytic in their first three arguments in a domain

$$\mathcal{D}_{\rho', \kappa', \varepsilon_0} = \{(\mathbf{A}, \boldsymbol{\alpha}, \varepsilon) \ : \ |A_1 - A_{01}| < \rho', \quad |\mathrm{Im}\, \alpha_j| < \kappa', j = 1, 2, \quad |\varepsilon| < \varepsilon_0\}, \tag{2.14}$$

for some $\rho' \in (0, \rho)$, $\kappa' \in (0, \kappa)$ and $0 < \varepsilon_0 \le \varepsilon_1$ small enough, and vanishing for $\varepsilon = 0$, such that the following holds: fixed any value $\mathbf{A}_0$ for which $B_\rho(\mathbf{A}_0) \subset \mathcal{A}$ for some $\rho > 0$, it is possible to find arbitrarily near to μ a value μ_0 with $\boldsymbol{\omega}_0 = (\mu_0, 1)$ Diophantine with Diophantine constants $C_0 = bC$ and τ, for some constant b, and, correspondingly, a value $\varepsilon = \varepsilon^*$ with $|\varepsilon^*| < \varepsilon_0$, such that

$$\begin{cases} \boldsymbol{\alpha}(t) = \boldsymbol{\omega}_0 t + \mathbf{h}(\boldsymbol{\omega}_0 t, \mathbf{A}_0, \varepsilon^*, \mu_0), \\ \mathbf{A}(t) = \mathbf{A}_0 + \mathbf{H}(\boldsymbol{\omega}_0 t, \mathbf{A}_0, \varepsilon^*, \mu_0) \end{cases} \tag{2.15}$$

is a solution of the equations of motion (2.11). As the value of $\varepsilon = \varepsilon^*$ is such that difference between μ and μ_0 is $O(\varepsilon)$, see [BG], the closer μ_0 is to μ, the smaller is the value of ε, *i.e.* , by (2.2), the closer is the motion to the origin.

Note that (2.15) defines an invariant torus $\mathcal{T}$ run with rotation vector $\boldsymbol{\omega}_0$. The projection of such a torus on the plane (x_1, y_1) gives a closed curve $\mathcal{C}$ with distance of order $O(\varepsilon)$ from the origin: in fact in (2.15) one has $|A_1(t) - A_{01}| = O(\varepsilon)$, by analyticity, while we can take $A_{01} = O(1)$ (in ε).

In the original variables, along the motion corresponding to the considered torus $\mathcal{T}$, one has

$$x_1^2 + y_1^2 = O(\varepsilon^2), \tag{2.16}$$

in the sense that, for initial data $(x_1(0), y_1(0))$ with $x_1^2(0)+y_1^2(0) = \varepsilon^2$, one has $|x_1^2(t)+y_1^2(t)-\varepsilon^2| < C\varepsilon^4$, for some constant C; in particular for ε small enough along the motion one has $A_1(t) \neq 0$, so that the trajectory never cross the origin (which can be outside the analyticity domain of the conjugating functions). So we have a closed curve surrounding the origin and which can be made arbitrarily near to it. As all trajectories starting from initial data contained inside the torus described by (2.15) have to remain inside, we have that the projections of them onto the plane (x_1, y_1) have to remain inside the closed curve $\mathcal{C}$. Therefore we obtain the stability of the origin for the motion of the variables (x_1, y_1).

2.3. *Extension to vectors satisfying only a non-resonance condition.* Note that the theorem 1.4 of [BG] requires $\boldsymbol{\omega}$ to be Diophantine. As it is well known (see for instance [HZ]) such a condition is not necessary and in fact it can be relaxed (by using that the perturbation depends on $\mathbf{A}$ and ε in a precise way, see (2.7)÷(2.10)): such extension of the proof is discussed in Appendix A2, within the formalism introduced in [BG], and gives the following result.

Given the Hamiltonian (2.1), if $\boldsymbol{\omega} = (\mu, 1)$ verifies the non-resonance condition

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \neq 0 \qquad \forall \boldsymbol{\nu} \in \{\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{Z}^2 \ : \ |\nu_1| \leq 3\}, \tag{2.17}$$

then the equilibrium position $(0, 0)$ for the motion of the variables (x_1, y_1) is stable, *i.e.* for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all initial data $(x_1(0), y_1(0))$ verifying

$$x_1^2(0) + y_1^2(0) \leq \delta^2, \tag{2.18}$$

one has

$$x_1^2(t) + y_1^2(t) \leq \varepsilon^2, \tag{2.19}$$

for all $t \in \mathbb{R}$. The stability follows from the existence of KAM invariant tori corresponding to rotation vectors $\boldsymbol{\omega}_0 = (\mu_0, 1)$ which can be made arbitrarily close to $\boldsymbol{\omega}$ and verify some suitable Diophantine condition (see Appendix A2 for details).

3. Second model: the vertically driven pendulum

3.1. *The upside-down pendulum.* Consider a pendulum with support point $P = (x_P, y_P)$ oscillating in the vertical direction with the law $y_P(t) = b \cos \omega t$ (and $x_P(t) \equiv 0$). The system is described by the Lagrangian (see, for instance [LL] or [PR])

$$\mathcal{L}_0 = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell\,(\cos\theta + 1) - m\ell b\omega^2\,(\cos\theta + 1)\cos\omega t, \qquad \dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}, \tag{3.1}$$

which, by the change of variable

$$\theta(t) = \pi + x(\tau), \qquad \tau = \omega t/2, \tag{3.2}$$

can be written as

$$\mathcal{L}_0 = \frac{1}{8} m\ell^2\omega^2\dot{x}^2 - mg\ell\,(\cos x - 1) + m\ell b\omega^2\,(\cos x - 1)\cos 2\tau, \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\tau}. \tag{3.3}$$

Note that with respect to [BGG] we have added to $\mathcal{L}_0$ a total derivative (which does not change the equations of motion).

The corresponding (non-autonomous) Hamiltonian is

$$H_0 = \frac{2y_1^2}{m\ell^2\omega^2} + \left(mg\ell - m\ell b\omega^2\cos 2\tau\right)(\cos x_1 - 1)\,, \tag{3.4}$$

where $x_1 = x$ and y_1 is the momentum conjugated to x_1. Again we are interested in the stability of the position $(\theta,\dot{\theta}) = (\pi, 0)$, *i.e.* $(x,\dot{x}) = (0,0)$, for ω large enough (see [PR] and [Ac]), so that we can define

$$\begin{cases} y_1 = \delta p_1, \\ x_1 = \delta q_1, \end{cases} \tag{3.5}$$

and write (3.4), divided by δ^2, as

$$\begin{aligned} H = {} & \frac{2p_1^2}{m\ell^2\omega^2} - \left(mg\ell - m\ell b\omega^2\cos 2\tau\right)\frac{q_1^2}{2} \\ & + \left(mg\ell - m\ell b\omega^2\cos 2\tau\right)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}q_1^{2p}\delta^{2(p-1)}, \end{aligned} \tag{3.6}$$

where the last sum is obtained by Taylor expanding the cosine function and disappears for $\delta = 0$.

As in §2.1 we can consider the autonomous Hamiltonian

$$\begin{aligned} H = {} & \frac{2p_1^2}{m\ell^2\omega^2} + I_2 - \left(mg\ell - m\ell b\omega^2\cos 2\alpha_2\right)\frac{q_1^2}{2} \\ & + \left(mg\ell - m\ell b\omega^2\cos 2\alpha_2\right)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}q_1^{2p}\delta^{2(p-1)}, \end{aligned} \tag{3.7}$$

with (I_2,α_2) conjugated variables.

By a (canonical) rescaling

$$p_1 \to \frac{2}{\sqrt{m\ell^2\omega^2}}p_1, \qquad q_1 \to \frac{\sqrt{m\ell^2\omega^2}}{2}q_1, \tag{3.8}$$

we can put (3.7) into the form

$$\begin{aligned} H = {} & \frac{p_1^2}{2} + I_2 + \left(-\frac{4g}{\ell\omega^2} + \frac{4b}{\ell}\cos 2\alpha_2\right)\frac{q_1^2}{2} \\ & + \left(-\frac{4g}{\ell\omega^2} + \frac{4b}{\ell}\cos 2\alpha_2\right)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}\left(\frac{4}{m\ell^2\omega^2}\right)^{p-1} q_1^{2p}\delta^{2(p-1)}, \end{aligned} \tag{3.9}$$

where, for notational simplicity, we still denote by (p_1, q_1) the new variables.

Let us consider the Hamiltonian obtained from (3.7) by putting $\delta = 0$: one easily realizes that the corresponding equation for x_1 is Mathieu's equation

$$\ddot{q}_1 + (a + 2q\cos 2\tau)\,q_1 = 0 \qquad a = -\frac{4g}{\ell\omega^2}, \qquad q = \frac{2b}{\ell}, \tag{3.10}$$

so that the solution is of the form

$$q_1(\tau) = \rho\,\mathrm{Re}\,u_1(\tau), \qquad u_1(\tau) = e^{i\mu\tau}p_0(\tau), \tag{3.11}$$

for some ρ (and for a particular choice of the initial phase) and with $p_0(\tau)$ a periodic function of period π; μ is a real number in $(0,1)$, for ω large enough (see [Ar], [JS] and [B]; see also [BGG] for the notations and for a review of some basic properties of Mathieu's equation which will be used below). In (3.11) we can choose ρ so that $q_1^2(0)+p_1^2(0)=1$ (*i.e.* $x_1^2(0)+y_1^2(0)=O(\delta^2)$).

It is possible to pass to action-angle variables through a canonical transformation

$$(p_1,I_2,q_1,\alpha_2)\to(\mathbf{A},\boldsymbol{\alpha})\equiv(A_1,A_2,\alpha_1,\alpha_2), \tag{3.12}$$

see [G] and [C] (see also Appendix A1 below), with α_2 the same in both sides. Then the Hamiltonian (3.9) at $\delta=0$ becomes

$$H=\boldsymbol{\omega}\cdot\mathbf{A},\qquad \boldsymbol{\omega}=(\mu,1), \tag{3.13}$$

which is an isochronous Hamiltonian, while the perturbation

$$\big(mg\ell-m\ell b\omega^2\cos 2\alpha_2\big)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}q_1^{2p}\delta^{2(p-1)}, \tag{3.14}$$

in (3.9) can be written in terms of the variables $(\mathbf{A},\boldsymbol{\alpha})$, and gives a function analytic in its arguments (in a suitable domain). By setting $\delta^2=\varepsilon$ and defining

$$G(\alpha_2)\equiv\big(mg\ell-m\ell b\omega^2\cos 2\alpha_2\big)=\sum_{|\nu_2|\le1}G_{\nu_2}\,e^{i2\nu_2\alpha_2}, \tag{3.15}$$

the Hamiltonian (3.9) becomes

$$H=\boldsymbol{\omega}\cdot\mathbf{A}+f(\boldsymbol{\alpha},A_1,\varepsilon), \tag{3.16}$$

where (see Appendix A1)

$$f(\boldsymbol{\alpha},A_1,\varepsilon)=\sum_{p=2}^{\infty}\varepsilon^{p-1}\Phi_p(\boldsymbol{\alpha},A_1)=\sum_{p=2}^{\infty}\varepsilon^{p-1}\sum_{|\nu_1|\le2p}\sum_{\nu_2\in\mathbb{Z}}\sum_{|\nu_2|\le1}\Phi_{p,\boldsymbol{\nu}}(A_1)\,e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}, \tag{3.17}$$

with

$$\Phi_p(\boldsymbol{\alpha},A_1)=G(\alpha_2)F_p(\boldsymbol{\alpha},A_1), \tag{3.18}$$

where the functions $F_p(\boldsymbol{\alpha},A_1)$ are analytic in $(\boldsymbol{\alpha},A_1)$ uniformly for all p.

3.2. *KAM results.* Then we can reason as in the previous section, and the same results follow. In particular for the motion in the plane $(\theta,\dot\theta)$, the equilibrium position $(\theta,\dot\theta)=(0,0)$ is stable (in the same sense as in §2.2). As for the model studied in §2, also for the upside-down pendulum we can relax the condition on the rotation vector $\boldsymbol{\omega}$; see Appendix A2.7 below.

Appendix A1. Action-angle variables for the upside-down pendulum

A1.1. *First change of coordinates.* Define $F_0(\varphi,\psi)=e^{i\varphi}p_0(\psi)$, so that $u_1(t)=F_0(\mu t,t)$. Consider the change of coordinates $\mathcal{C}_1:\mathbb{R}^3\times\mathbb{T}\setminus\{(0,0)\times\mathbb{R}\times\mathbb{T}\}\to\mathbb{R}_+\times\mathbb{T}\times\mathbb{R}\times\mathbb{T}$ defined as

$$\begin{aligned}&(q_1,p_1,I_2,\alpha_2)\to(\rho,\varphi,I_2,\psi),\\&q_1=q_1(\rho,\varphi,\psi)=\rho\,\mathrm{Re}\,(F_0(\varphi,\psi))\equiv\rho\,Q(\varphi,\psi),\\&p_1=p_1(\rho,\varphi,\psi)=\rho\,\mathrm{Re}\,(D\,F_0(\varphi,\psi))\equiv\rho\,P(\varphi,\psi),\\&\alpha_2=\psi,\end{aligned} \tag{A1.1}$$

where D is the differential operator

$$D=\mu\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi}. \tag{A1.2}$$

Define also

$$\kappa = \frac{i}{2}\left(u_1(t)\,\frac{\mathrm{d}\overline{u}_1(t)}{\mathrm{d}t} - \frac{\mathrm{d}u_1(t)}{\mathrm{d}t}\,\overline{u}_1(t) \right); \tag{A1.3}$$

with the notations of [BGG] one has $\kappa = c \neq 0$ (see (6.11) in [BGG]). One can easily show that the change of coordinates (A1.1) is analytic (see [C] and [G]).

A1.2. *Second change of coordinates.* Consider the change of coordinates $\mathcal{C}_2 \colon \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ defined as

$$\begin{aligned}
&(\rho, \varphi, I_2, \psi) \to (A_1, \alpha_1, A_2, \alpha_2),\\
&A_1 = \frac{1}{2}\kappa\rho^2,\\
&A_2 = I_2 + \frac{1}{2}\rho^2\left(\frac{\partial Q}{\partial\psi}\,P - Q\,\frac{\partial P}{\partial\psi}\right),\\
&\alpha_1 = \varphi,\\
&\alpha_2 = \psi.
\end{aligned} \tag{A1.4}$$

Then (A1.4) is an analytic change of coordinates, such that the change of coordinates $\mathcal{C}_2 \circ \mathcal{C}_1$ is analytic and canonical, and, in the new variables, the Hamiltonian (3.7) for $\delta = 0$ becomes

$$H = \boldsymbol{\omega} \cdot \mathbf{A} \equiv \mu A_1 + A_2. \tag{A1.5}$$

For the proof see [C] and [G], in particular the theorem 2 in Section 1.3 of [C].

Moreover the second line of (3.9) can be written as a function

$$f(\boldsymbol{\alpha}, A_1, \delta^2) = \sum_{p=2}^{\infty} \varepsilon^p f_p(\boldsymbol{\alpha}, A_1), \qquad \varepsilon = \delta^2, \tag{A1.6}$$

which does not depend on A_2, and such that the dependence of $f_p(\boldsymbol{\alpha}, A_1)$ on α_1 involves only harmonics with $|\nu_1| \leq 2p$ (simply combine the definitions of the two changes of coordinates $\mathcal{C}_1$ and $\mathcal{C}_2$).

Appendix A2. Non-resonance conditions on the rotation vectors

A2.1. *Non-resonant condition on the rotation vector.* The result in sect. 2 can be improved in the following way. Consider first the case (2.10); see the remark A2.6 below for the extension of the results to the case (2.8).

Let $\boldsymbol{\omega}$ be such that

$$\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0 \qquad \forall |\boldsymbol{\nu}| \leq 4. \tag{A2.1}$$

Then it is still possible to perform the analysis of [BG] with some minor changes. We refer to the formalism introduced in [BG]: we use the same notations used there, by assuming a full knowledge of that paper by the reader, with no further comments henceforth.

Assume that the rotation vector $\boldsymbol{\omega}$ satisfies the non-resonance condition (A2.1). This means that there exists a constant α such that

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > \alpha \qquad \forall |\boldsymbol{\nu}| \leq 4, \tag{A2.2}$$

so that, for ε small enough

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > \alpha/2 \qquad \forall |\boldsymbol{\nu}| \leq 4, \tag{A2.3}$$

provided that $|\boldsymbol{\omega} - \boldsymbol{\omega}_0| < b\varepsilon$ for some constant b. Defining

$$N_\alpha(\theta) = \left|\{\ell \in \Lambda(\theta) \;:\; |\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell| \leq \alpha/8\}\right|, \tag{A2.4}$$

then one can prove the following bound.

A2.2. Lemma. *For any tree $\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}$ one has*

$$N_\alpha(\theta) \le \frac{3k}{5}, \tag{A2.5}$$

where k is the order of θ.

A2.3. *Proof of the lemma A2.2.* We prove by induction on the number of nodes $V = V(\theta)$ that

$$N_\alpha(\theta) \le \max\{(3k-1)/5, 0\}. \tag{A2.6}$$

For $V = 1$ one has $N_\alpha(\theta) = 1$ if $N_\alpha(\theta) \neq 0$. On the other hand, if k is the order of the tree, one has $p = k+2$ in (2.7) so that $|\boldsymbol{\nu}| \le k+3$ by (2.10). Then by (A2.3) one can have $N_\alpha(\theta) \neq 0$ only for $k \ge 2$: as $3k-1 \ge 5$ for $k \ge 2$ the bound (A2.6) follows for $V = 1$.

In general note that $k = 1$ implies $V = 1$ so that the above argument yields $N_\alpha(\theta) = 0$ for all θ of order 1.

Given $V > 1$, assuming (A2.6) to hold for $V' < V$ we can show that it holds also for V. Let θ be a tree of order k with V nodes and let v_0 be the node which the root line $\ell_0 \equiv \ell_{v_0}$ of θ exits from. Call $\theta_1, \ldots, \theta_m$ the subtrees of order ≥ 2 entering the node v_0 and denote by $k_1, \ldots, k_m$ and $V_1, \ldots, V_m$, respectively, the orders and the numbers of nodes of such subtrees. Call also k' and V' the sum of the orders and of the numbers of nodes, respectively of the subtrees of order 1 entering v_0 (of course $k' = V'$). One has $m \ge 0$, $k_1 + \ldots + k_m + k_{v_0} + k' = k$ and $V_1 + \ldots + V_m + 1 + V' = V$.

If either $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| > \alpha/8$ or $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| \le \alpha/8$ and $k_{v_0} + k' \ge 2$, then the bound (A2.6) follows by the inductive hypothesis. In fact one has in the first case ($|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| > \alpha/8$)

$$\begin{aligned} N_\alpha(\theta) &= N_\alpha(\theta_1) + \ldots + N_\alpha(\theta_m) \\ &\le \frac{3k_1 - 1}{5} + \ldots \frac{3k_m - 1}{5} \le \frac{3(k-1) - m}{5} \le \frac{3k-1}{5}, \end{aligned} \tag{A2.7}$$

while in the second case ($|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| > \alpha/8$ and $k_{v_0} + k' \ge 2$)

$$\begin{aligned} N_\alpha(\theta) &= 1 + N_\alpha(\theta_1) + \ldots + N_\alpha(\theta_m) \\ &\le 1 + \frac{3k_1 - 1}{5} + \ldots \frac{3k_m - 1}{5} \le 1 + \frac{3(k-2) - m}{5} \\ &\le \frac{3k-1}{5} + \left(1 + \frac{1 - 6 - m}{5}\right) \le \frac{3k-1}{5}. \end{aligned} \tag{A2.8}$$

So consider the case $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| \le \alpha/8$ and $k_{v_0} + k' = 1$ (*i.e.* $k_{v_0} = 1$ so that $k' = 0$ and $m \ge 1$). One has

$$\begin{aligned} N_\alpha(\theta) &= 1 + N_\alpha(\theta_1) + \ldots + N_\alpha(\theta_m) \\ &\le 1 + \frac{3k_1 - 1}{5} + \ldots + \frac{3k_m - 1}{5} \le \frac{3k-1}{5} + \left(1 + \frac{1 - 3 - m}{5}\right), \end{aligned} \tag{A2.9}$$

so that, for $m \ge 3$, (A2.6) follows also for V.

If $m = 2$ one has

$$N_\alpha(\theta) = 1 + N_\alpha(\theta_1) + N_\alpha(\theta_2). \tag{A2.10}$$

In such a case the lines ℓ_1 and ℓ_2 entering v_0 can not be such that, simultaneously, $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_1}| \le \alpha/8$ and $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_2}| \le \alpha/8$: this would imply $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{v_0}| < \alpha/2$, which is not possible, by (A2.3), as $|\boldsymbol{\nu}_{v_0}| \le 4$ for $k_{v_0} = 1$. Therefore at least one line entering v_0, say ℓ_1, does not contribute to $N_\alpha(\theta)$, so that, if $\theta'_1, \ldots, \theta'_{m'}$ are the subtrees of orders ≥ 2 entering the node v_1 which the line ℓ_1 exits from, one has in (A2.10)

$$N_\alpha(\theta_1) = N_\alpha(\theta'_1) + \ldots + N_\alpha(\theta'_{m'}), \tag{A2.11}$$

and by the inductive hypothesis, by denoting by k'' the sum of the subtrees of order 1 entering v_1 and noting that $k'_1+\ldots+k'_{m'}=k_1-k_{v_1}-k''\le k_1-1$ and $k_1+k_2=k-1$, one has

$$\begin{aligned} N_\alpha(\theta) &= 1+N_\alpha(\theta'_1)+\ldots+N_\alpha(\theta'_{m'})+N_\alpha(\theta_2) \\ &\le 1+\frac{3(k_1-1)-m'}{5}+\frac{3k_2-1}{5} \\ &\le \frac{3k-1}{5}+\left(1-\frac{3+3+m'}{5}\right)\le\frac{3k-1}{5}. \end{aligned} \tag{A2.12}$$

Therefore (A2.6) is proved also in such a case.

If $m=1$ one has

$$N_\alpha(\theta)=1+N_\alpha(\theta_1), \tag{A2.13}$$

and the line ℓ_1 entering v_0 must have a momentum $\boldsymbol{\nu}_{\ell_1}$ such that $|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}_{\ell_1}|>\alpha/8$ (simply reason as above by using that $|\boldsymbol{\nu}_{v_0}|\le 4$), so that

$$N_\alpha(\theta)=N_\alpha(\theta'_1)+\ldots+N_\alpha(\theta'_{m'}), \tag{A2.14}$$

if $\theta'_1,\ldots,\theta'_{m'}$ are the subtrees of order ≥ 2 entering the node v_1 which the line ℓ_1 exits from; if $m'\ge 2$ the bound follows again by the inductive hypothesis. If $m'=1$ then one has only one subtree θ'_1 entering v_1, such that its order is $k'_1\le k-1-k_{v_1}\le k-2$ and

$$N_\alpha(\theta)=1+N_\alpha(\theta'_1), \tag{A2.15}$$

so that the inductive hypothesis gives

$$N_\alpha(\theta)\le 1+\frac{3k'_1-1}{5}\le 1+\frac{3(k-2)-1}{5}\le\frac{3k-1}{5}+\left(1-\frac{6}{5}\right)\le\frac{3k-1}{5}, \tag{A2.16}$$

so that (A2.6) follows once more. If $m'=0$ the discussion becomes trivial: one has $V\ge 2$, hence $k\ge 2$, while $N_\alpha(\theta)=1$, so that (A2.6) follows.

Finally if $m=0$ then again one has $V\ge 2$, hence $k\ge 2$, while $N_\alpha(\theta)=1$, so that (A2.6) trivially holds. ∎

A2.4. *Improvement of the bound on the radius of convergence.* With respect to [BG] we can modify the multi-scale decomposition by defining the scales starting from n such that $C_0 2^n<\alpha/8\le C_0 2^{n+1}$.

Then one can reason as in [GGM] in order to show that, by using the lemma A2.3, instead of a factor C_0^{-k} one has a factor $(8\alpha^{-1})^k C_0^{-3k/5}$ (here the exponent k for $8\alpha^{-1}$ is a bound on the number of lines which do not contribute to $N_\alpha(\theta)$).

This implies that the radius of convergence ε_0 of the series defining the functions $\mathbf{h},\mathbf{H},\eta$ can be bounded by $\varepsilon_0=E'C_0^{3/5}$, for some constant E' depending on α but not on C_0.

A2.5. *Persistence of KAM tori.* Under the condition on (A2.1) on $\boldsymbol{\omega}$ the analysis of [BG] can be repeated for the Hamiltonian (2.5) with f as in (2.7) and (2.10). By the discussion in §A2.4 one finds $\varepsilon_0\equiv E'C_0^{3/5}$, so that one can choose $\varepsilon_*\in[0,\varepsilon_0]$ and repeat the same argument in sect. 5 of [BG]: the main difference is that one can fix the interval I of size $|I|=aC_0^{3/5}$ and, as $C_0^{3/5}\gg C_0$ for C_0 small enough (if C_0 is not small one can choose the constant b in the theorem 1.4 of [BG] so that the same conclusions hold), one can easily prove that there are infinitely many $\mu_0\in I$ such that the corresponding $\boldsymbol{\omega}_0$ verify the Diophantine condition (1.11) of [BG].

A2.6. *Remark.* Note that in proving the lemma A2.3 for the case (2.10) what is really needed is that

$$\mu\nu\pm 1\neq 0 \qquad \forall|\nu|\le 3, \tag{A2.17}$$

which is weaker than (A2.1). If one wants to deal with the case (2.8), the analysis can be performed with no essential change, simply imposing the non-resonance condition

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|\neq 0 \qquad \forall\boldsymbol{\nu}\in\{\boldsymbol{\nu}=(\nu_1,\nu_2)\in\mathbb{Z}^2\ :\ |\nu_1|\le 3\}, \tag{A2.18}$$

as it is straightforward to check.

A2.7. *Case of the upside-down pendulum.* In the case of the model introduced in §3.1, so that the perturbation is of the form (3.16), if one requires

$$|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \neq 0 \qquad \forall\boldsymbol{\nu} \text{ such that } |\nu_1| \leq 2 \text{ and } \quad \nu_2 \in \mathbb{Z}, \tag{A2.19}$$

then one can reason as in §A2.3, to prove that if $|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}| > \alpha/2$ for all $\boldsymbol{\nu}$ such that $|\nu_1| \leq 2$, then the number of lines ℓ such that $|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}| < \alpha/8$ is bounded as in (A2.5). Therefore we can reason as in the previous sections and conclude the existence of invariant tori for any value of $\boldsymbol{\omega}_0$ near to $\boldsymbol{\omega}$ and verifying a suitable Diophantine condition.

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