Splitting of separatrices near resonant periodic orbits

Vassili Gelfreich
E-mail: gelf@math.fu-berlin.de

October 12, 2000

Abstract

We consider an analytic family of area-preserving maps F_ε with an elliptic fixed point. We assume that for $\varepsilon = 0$ the fixed point is resonant of an order $n = 1, 2$ or 3. In each of these cases the fixed point can be unstable at the exact resonance, and close to the exact resonance there is a hyperbolic periodic orbit. The resonant normal form is integrable and its separatrices form a small loop. Separatrices of the map F_ε are close to the separatrices of the normal form but can intersect transversally.

Asymptotic formulae for the splitting of separatrices are provided. The splitting is exponentially small compared to ε and can not be detected by Melnikov method.

This problem is equivalent to studying a generic family of close-to-resonant elliptic periodic orbits in an analytic Hamiltonian system with two degrees of freedom.

1 Introduction

In a Hamiltonian system closed orbits are not isolated but form one-parametric families. As a rule, nearby members of the family belong to different energy levels. Here we consider an analytic Hamiltonian system with two degrees of freedom only. In such a system small oscillations about the closed orbits can be described by a time-periodic system with one degree of freedom which depends on a parameter. Let the periodic orbits be of elliptic type. It has a complex multiplicator μ with $|\mu| = 1$. We say that a periodic orbit is resonant if $\mu^n = 1$ for some integer n. We take the smallest positive n to be the order of the resonance. The motion near the resonant periodic orbit can be described using a resonant normal form. The resonant normal form can be reduced to an autonomous system with one degree of freedom and its phase portrait can be drawn. If coefficients of the lowest-order terms in the normal form are in general position, then there are only finitely many different types of phase portraits [AKN93].

The resonances of the order $n = 1, 2, 3$, and 4 have a special property: the linearly stable periodic orbit may become unstable at exact resonance. At resonance of the order $n = 1$ or 3 the elliptic periodic orbit is generically unstable. At resonance of the order $n = 2$ or 4 both stable and unstable cases are possible in general position depending on coefficients of the resonant normal form.

It is more convenient to consider an equivalent problem formulated in terms of families of area-preserving maps, which are obtained by considering Poincaré section near the closed orbits.

Let F_ε denote an one-parametric analytic family of area-preserving maps. Assume that there is a family of elliptic fixed points $\mathbf{p}_\varepsilon$ with a (complex) multiplicator μ_ε, $|\mu_\varepsilon| = 1$. Let $\mu_0^n = 1$, we say that $\mathbf{p}_0$ is resonant. In this paper we study the resonances of the order $n = 1, 2$, and 3. The case $n = 4$ has some special properties and it is to be studied in a separate paper.

The n^{th} iterate F_ε^n can be considered as a time-one map of an autonomous Hamiltonian system with one degree of freedom with an error, which is smaller than any power of ε [AKN93]. It is possible to simplify the form of the corresponding Hamiltonian by a canonical transformation. In this way the system is transformed to a *resonant normal form*. The lower order normal form Hamiltonian H_n depends on 2 parameters: δ describes deviation from the exact resonance (and it is proportional to a power of ε), and A depends on the map F_0 only. It is not too difficult to provide an explicit formula for A and δ as a function of several first coefficients of the Taylor series of F_ε in two space variables and ε. These formulae are not of big importance for the purposes of the present paper since we will formulate our results in terms of quantities, which allow coordinate independent definitions.

The phase portraits and the main oder terms of the normal forms are given on Fig. 1 and 2. This is a complete list of non-degenerate cases for resonances of the order $n = 1, 2$, and 3 [AKN93].

The phase portrait of the map F_ε looks quite similar to its normal form, but still some explanations are necessary.

In the cases $n = 1$ and 2 the elliptic fixed point $\mathbf{p}_\varepsilon$ exists only on one side of the resonance. Exactly at resonance the multiplicator of $\mathbf{p}_0$ is real (± 1, respectively), and the fixed point is not elliptic but parabolic.

In the case $n = 1$ shown on Fig. 1 a the elliptic fixed point collides with a hyperbolic fixed point and disappear. This corresponds to a Hamiltonian saddle-center (Bogdanov-Tackens) bifurcation.

In the case $n = 2$ we see a pitch-fork period doubling bifurcation. There are two possibilities: a) (Fig. 1 b) the elliptic fixed point collides with a hyperbolic period-two trajectory, after that it becomes hyperbolic and the period-two trajectory disappears; b) (Fig. 2) the elliptic fixed point becomes hyperbolic and a period two elliptic trajectory arises.

In the case $n = 3$ (see Fig. 1 c)) the elliptic fixed point exists on both sides of the resonance. At the resonance it collides with a period 3 hyperbolic periodic orbit.

In each of these cases separatrices of the normal form make a small loop

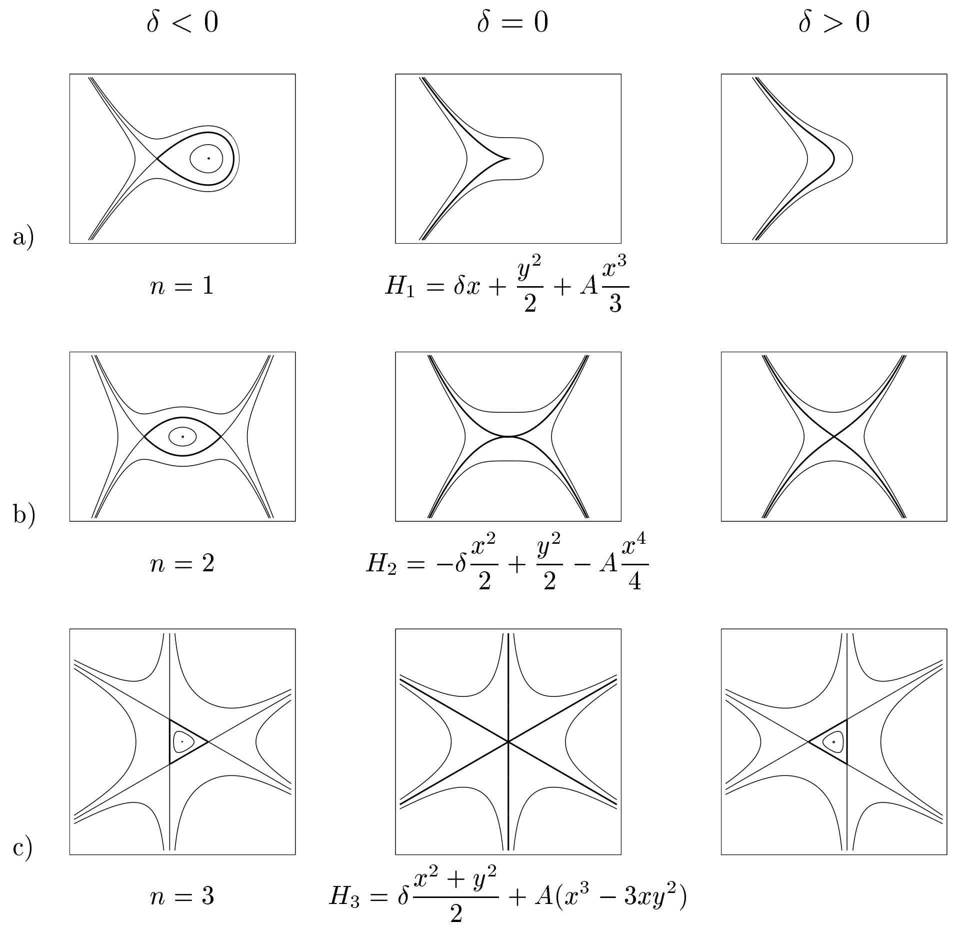

Figure 1: Bifurcations of equilibria of the resonant normal forms ($A > 0$)

around the elliptic equilibrium. At any order the resonant normal form is integrable, and the corresponding stable and unstable separatrices coincide. This is not generically true for the map F_ε—as it was already pointed in [AKN93]—the separatrices of the map may intersect transversally. Detecting this is a very difficult analytical problem since in the analytical case the splitting of separatrices is exponentially small in ε. Before providing the asymptotic formulae, which describe the splitting of the separatrices, we need in some preliminary definitions.

There are many different ways in which one may quantitatively describe the separatrices splitting. We prefer Lazutkin homoclinic invariant, which is defined in the following way. Let the separatrices of the hyperbolic n-periodic orbit be parameterized by solutions of the following finite-difference equation

$$\psi^{\pm}(t+h) = F_{\varepsilon}^{n}(\psi^{\pm}(t)), \qquad \psi^{\pm}(t) \to \mathbf{x}_{\varepsilon} \quad \text{as } t \to \pm\infty,$$

where $\mathbf{x}_{\varepsilon}$ is one of the n points of the hyperbolic periodic trajectory. The "+" corresponds to the stable separatrix and the "-" to the unstable one. We let $h = \log \lambda_{\varepsilon}$ and assume that the multiplicator of the hyperbolic periodic orbit $\lambda_{\varepsilon} > 1$. The image $\psi^{\pm}(\mathbf{R})$ is one whisker of the separatrix. Under an additional condition ($2\pi\mathrm{i}$ periodicity in t) these solutions are unique up to a substitution $t \mapsto t + \text{const}$. The derivative $\vec{e}_{\pm}(t) = \frac{d\psi^{\pm}}{dt}(t)$ defines a vector field tangent to the stable and unstable separatrices, respectively. These vector fields are F_{ε}^{n}-invariant. It is easy to see that the vector fields are independent from the choice of the normalization constant in the definition of $\psi^{\pm}$. Let $\mathbf{x}_h$ denote a homoclinic point. Then Lazutkin homoclinic invariant, $\omega := \vec{e}_{-} \wedge \vec{e}_{+}$, equals to the area of a parallelogram defined by the tangent vectors $\vec{e}_{-}$ and $\vec{e}_{+}$ at $\mathbf{x}_h$. Lazutkin homoclinic invariant has remarkable properties: 1) it takes the same value for all points of the homoclinic trajectory of $\mathbf{x}_h$; 2) it is invariant with respect to canonical substitutions; 3) for a fixed $\mathbf{x}_h$ it is proportional to the sine of the splitting angle.

We have chosen the homoclinic invariant in order to describe the splitting by one invariantly defined number, which is easy to compute with high precision. This permits a careful comparison of the theory with results of numerical experiments. Of course, inside the proofs there are expressions, which describe the behavior of the separatrices as curves, i.e. much detailed description of the separatrices is also possible by the same method.

The main result of the present paper is the following asymptotic formula for the Lazutkin homoclinic invariant. Let the map F_{ε} afford one of the nondegenerate resonant normal forms shown on Fig. 1. Assume ε be sufficiently small, then F_{ε} has exactly one n-periodic hyperbolic trajectory in a neighborhood of $\mathbf{p}_{\varepsilon}$ (only on one side of the resonance if $n = 1$ or 2). Separatrices of the hyperbolic periodic trajectory form a small loop around $\mathbf{p}_{\varepsilon}$. There are at least two different primary homoclinic orbits. The homoclinic invariant of one primary homoclinic orbit is given asymptotically by

$$\omega = \frac{4\pi}{h^2} \, \mathrm{e}^{-\frac{2\pi^2}{h}} \big(|\Theta| + O(h) \big), \tag{1}$$

where $h = \log \lambda_{\varepsilon}$, λ_{ε} is the multiplicator of the hyperbolic periodic orbit. It vanishes together with ε, so the splitting is exponentially small compared to ε. The preexponential factor $|\Theta|$ is a constant, which depends on the map at the moment of the resonance only. If Θ does not vanish there are exactly two primary homoclinic orbits with different orientation of intersection of separatrices, and both of them are transversal.

A similar statement is valid if the map F_{ε} has the resonant normal form shown on Fig. 2, but the asymptotic formula for the homoclinic invariant has

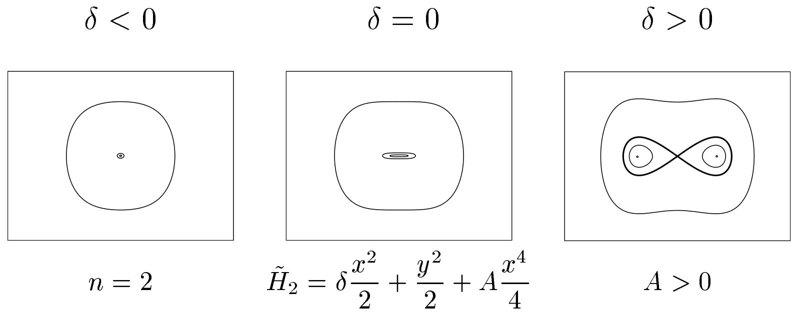

Figure 2: Resonant normal form.

to be replaced by:

$$\omega = \frac{4\pi}{h^2} \, \mathrm{e}^{-\frac{\pi^2}{h}} \big(|\Theta| + O(h) \big), \tag{2}$$

where $h = \log \lambda_\varepsilon^2$, $\lambda_\varepsilon < -1$ is the multiplicator of the hyperbolic fixed point, which appears after the bifurcation. The only difference compared to the previous case is in the exponent.

An important ingredient of the asymptotic formulae is the constant Θ. The asymptotic formulae above imply transversality of the separatrices for small ε only provided Θ does not vanish. This constant can be defined as a symplectic invariant associated with the corresponding map F_0. The constant Θ can vanish. For example, if the map F_0 is time one map of the corresponding normal form, or if F_0 is integrable. Numerical experiments show that Θ may vanish in a nonintegrable case too. We conjecture, that for any map F_0, any $\Theta_0 \in \mathbf{C}$ and any natural N there is another map $\tilde{F}_0$, which have the same Taylor expansion as F_0 up to the order N and $\Theta(\tilde{F}_0) = \Theta_0$. In other words, the constant Θ is not defined by any final jet of the function F_0. On the other hand there are efficient numerical method for its evaluation [LST89, GS99]. In some cases it is possible to prove analytically that Θ does not vanish [GS99], making in this way the proof of transversality of separatrices purely analytical even for concrete families and not only in generic sense.

If the map F_ε has additional symmetries its separatrix splitting can be much smaller compared to the above-described general case. For example, consider $n = 2$ and let the map F_ε be odd. Then the constant $\Theta(F_0) = 0$ due to the symmetry, and the splitting of separatrices is essentially smaller compare to the general non-odd case. Nevertheless an asymptotic formula for the splitting of separatrices can be derived by the same method for this case too. Instead of F_ε we have to consider an auxiliary map $G_\varepsilon = -F_\varepsilon$. Note that $G_\varepsilon^2 = F_\varepsilon^2$, so the separatrices of these two maps coincide. The corresponding statements and asymptotic asymptotic formula (2) remain valid for G_ε if we use $h = \log(-\lambda_\varepsilon)$. Note that the multiplicator of the hyperbolic fixed point of G_ε equals to $-\lambda_\varepsilon$. Thus the new h is a half of the corresponding value for F_ε, and the order of the splitting is much smaller (approximately square of the general case); moreover if $\Theta(G_0) \neq 0$ then the map F_ε has exactly 4 different primary homoclinic orbits.

2 Separatrices of the resonant normal form

In all the cases of $n = 1, 2, 3$ the separatrix W_0 of the main order normal form can be found explicitly. This separatrix plays an important role in the study. It provides the first order approximation for the separatrices of F_ε. The behavior of the separatrices of the map near singularities of W_0 gives a key to detecting the exponentially small transversality of the separatrices. The following table contains a list of the corresponding separatrix solutions.

$n=1$	$X_0 = -\dfrac{3\delta}{2A}\dfrac{1}{\cosh^2\frac{\sqrt{-\delta}\,t}{2}}$	$Y_0 = \dfrac{3\sqrt{-\delta^3}}{2A}\dfrac{\sinh\frac{\sqrt{-\delta}\,t}{2}}{\cosh^3\frac{\sqrt{-\delta}\,t}{2}}$	$\delta < 0$
$n=2$	$X_0 = \sqrt{\dfrac{-\delta}{A}}\tanh\sqrt{\frac{-\delta}{2}}\,t$	$Y_0 = \dfrac{\sqrt{\delta^2/(2A)}}{\sinh^2\sqrt{\frac{-\delta}{2}}\,t}$	$\delta < 0$ (for H_2)
	$X_0 = \dfrac{\sqrt{2\delta/A}}{\cosh\sqrt{\delta}\,t}$	$Y_0 = -\dfrac{\delta\sqrt{2/A}\sinh\sqrt{\delta}\,t}{\cosh^2\sqrt{\delta}\,t}$	$\delta > 0$ (for $\tilde{H}_2$)
$n=3$	$X_0 = \dfrac{\delta}{6A}$	$Y_0 = \dfrac{-\delta}{2\sqrt{3A}}\tanh\dfrac{\sqrt{3\delta}\,t}{2}$	

3 Definition of Θ

We give a short description of a problem, which leads to the definition of the splitting constant Θ. We consider the finite-difference equation

$$\begin{pmatrix} u \\ v \end{pmatrix}(\tau+1) = F_0^n\left(\begin{pmatrix} u \\ v \end{pmatrix}(\tau)\right). \tag{3}$$

This system has a formal solution in class of formal series in powers of τ^{-1}. It is defined up to a translation $\tau \mapsto \tau + \tau_0$ with $\tau_0 \in \mathbf{R}$. The equation has two analytic solution $(u^\pm(\tau), v^\pm(\tau))$, which are asymptotic to the single formal solution as $\tau \to \pm\infty$, respectively. We consider their analytical continuation on complex τ. They have the same asymptotic behavior as $\Im\tau \to -\infty$. Moreover, their difference decreases *exponentially* as $\Im\tau \to -\infty$. The constant Θ is defined as the following limit

$$\Theta = \lim_{\Im\tau\to-\infty} \mathrm{e}^{2\pi i\tau} \det\begin{pmatrix} \frac{du^-}{d\tau} & u^+ - u^- \\ \frac{dv^-}{d\tau} & v^+ - v^- \end{pmatrix}(\tau). \tag{4}$$

It is not difficult to check that $|\Theta|$ is a symplectic invariant of the map F_0—its definition is invariant with respect to area-preserving changes of coordinates.

4 Final remarks

An example of a degenerate Hamiltonian bifurcation was considered in the pioneer paper [HMS88]. This example can be treated using a modification of the Melnikov method as described in [Gel97]. In particular in this degenerate case the constant Θ can be computed analytically by the Melnikov method. In general case, which is described in the present note, this is impossible. As far as the author knows, all attempts to solve the problems described in the present paper and some similar ones using modifications of the Melnikov method have failed. Although recent numerical studies [DR99] showed that in some other simple degenerate cases Melnikov method also provides an accurate estimate for the splitting.

The case $n = 1$ was considered previously in the paper [Gel00].

In order to prove asymptotic formulae (1)–(2) the author used the method proposed by Lazutkin [Laz84] and developed later by Lazutkin and his collaborators for the study of the separatrix splitting of the standard map (see [Gel99], where a detailed description of the method is given, and [Gel00] for a shorter exposition in context of an application to a bifurcation problem).

The method is based on an accurate study of the analytic continuation of the map F_ε and its separatrices in the complex phase space. In a complex domain the splitting of separatrices is not small and can be detected by a specially developed complex matching method. The splitting of complex separatrices is already present for F_0 at the moment of the resonance, although it is not visible on the real pictures but leaves in a complex domain only. The value of this separatrix splitting is described by the complex constant Θ, and it is inherited near the resonance. A "discrete flow fox theorem" ("theorem on analytic integral") is used to come back from complex dynamics to the real one.

Acknowledgments

This work is partially supported by the INTAS grant 97-10771. The author thanks B. Fiedler for stimulating discussions.

References

[AKN93] Arnold, V.I., Kozlov, V.V., Neishtadt, A.I., *Mathematical aspects of classical and celestial mechanics.* 2nd ed., Springer, 291 p. (1993)

[DR99] Delshams, A., Ramírez-Ros, R., Singular separatrix splitting and Melnikov method: An experimental study *Experiment. Math.*, 8(1):29-48, 1999.

[Gel97] V.G. Gelfreich. Melnikov method and exponentially small splitting of separatrices // *Physica D*, 101:227–248, 1997.

[Gel99] V. G. Gelfreich. A proof of the exponentially small transversality of the separatrices for the Standard Map *Comm. Math. Phys.* 201, 155-216, 1999

[Gel00] V. G. Gelfreich. Splitting of a small separatrix loop near the saddle-center bifurcation in area-preserving maps. *Physica D* 136, 266–279, 2000

[GS99] V.Gelfreich, D.Sauzin, Borel summation and the splitting of separatrices for the Henon map. *Notes Scientifuqes et techniques du Bureau des Longitudes*, S067, Mai 1999. 48 p.

[HMS88] P. Holmes, J. Marsden, and J. Scheurle. Exponentially small splitting of separatrices with applications to KAM theory and degenerate bifurcations, *Contemporary Mathematics*, 81:213–244, 1988.

[Laz84] V. F. Lazutkin. Splitting of separatrices for the Chirikov's standard map. VINITI no. 6372/84, 1984, (Russian).

[LST89] V.F. Lazutkin, I.G. Schachmanski, and M.B. Tabanov. Splitting of separatrices for standard and semistandard mappings, *Physica D*, 40:235–348, 1989.