Exponential Decay of Eigenfunctions of the Bethe-Salpeter Operator

Volker Bach and Oliver Matte
Fachbereich Mathematik
Johannes Gutenberg-Universitt
55099 Mainz
email: matte@informatik.uni-mainz.de

September 22, 2000

Abstract

Exponential decay of the bound states of the Bethe-Salpeter operator is proven. Explicit estimate on the decay rates are derived, and several norm estimates of operators involving the free Dirac operator and exponential weight functions are given.

Keywords: Dirac Operator, Bethe and Salpeter, Brown and Ravenhall, Exponential Decay.

I Introduction

In 1928, Dirac postulated [7, 8] the operator

$$D_0 \; := \; \vec{\alpha} \cdot (-i\vec{\nabla}) \, + \, \beta \; := \; \sum_{j=1}^{3} -i \, \alpha_j \, \frac{\partial}{\partial x_j} \, + \, \beta \tag{I.1}$$

to be the generator of the dynamics of a free, relativistic electron. This operator, thus called the *free Dirac operator*, is a self-adjoint, matrix-valued operator with domain

$$\mathrm{dom}(D_0) \; = \; \mathrm{H}^1(\mathbb{R}^3; \mathbb{C}^4) \; \subseteq \; \mathcal{H} \; := \; L^2(\mathbb{R}^3; \mathbb{C}^4) \, , \tag{I.2}$$

the Sobolev space of 1^{st} order with values in $\mathbb{C}^4$. Representing $\mathbb{C}^4$ as $\mathbb{C}^2 \otimes \mathbb{C}^2$, the 4×4 Dirac matrices α_j and β are defined as $\alpha_j := \sigma_1 \otimes \sigma_j$ and $\beta = \sigma_3 \otimes \mathbf{1}$, respectively, where σ_j are the usual Pauli matrices,

$$\sigma_1 \; = \; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 \; = \; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 \; = \; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{I.3}$$

The unit of energy in (I.1) is by the rest energy mc^2 of the electron, where m is the electron mass, and c is the speed of light. Lengths are measured in multiples of the Compton wavelength, $\hbar/mc$, so that in these units, the square of the elementary charge e^2 equals the fine structure constant $e^2 = \alpha \approx 1/137$.

By (componentwise) Fourier transformation, $\mathcal{F}$, the free Dirac operator D_0 becomes a 4×4 matrix-valued multiplication operator,

$$\big(\mathcal{F} \, D_0 \, \mathcal{F}^{-1} \big)[\vec{\xi}] \; = \; \vec{\alpha} \cdot \vec{\xi} \, + \, \beta \, , \tag{I.4}$$

and from (I.4) it is easy to see that the spectrum of D_0 is absolutely continuous and is the union of two half-lines,

$$\sigma[D_0] \; = \; \sigma_{\mathrm{ac}}[D_0] \; = \; (-\infty, -1] \cup [1, \infty) \, . \tag{I.5}$$

The lack of semiboundedness of D_0 has been a source of controversy in physics, as it seems to be inconsistent with the existence of an energy minimizing ground state. Indeed, in his original proposal Dirac himself declared all negative energy levels to be filled, and this lead to important concepts such as normal-ordering and anti-particles. The lack of semiboundedness is the price to pay for having a square root of the Klein-Gordon operator, $-\Delta + 1$ which is a "local", i.e., differential operator.

One way to restore the semiboundedness of the electron Hamiltonian is to study a simplified operator obtained from restricting D_0 to its positive spectral subspace $\mathcal{H}_+ := \Lambda_+ \mathcal{H}$, where

$$\Lambda_+ \ := \ \chi_{[0,\infty)}[D_0] \ = \ \chi_{[1,\infty)}[D_0] \ , \tag{I.6}$$

and, in Fourier representation,

$$\big(\mathcal{F}\, \Lambda_+\, \mathcal{F}^{-1}\big)[\vec{\xi}] \ = \ \frac{1}{2} \ + \ \frac{\vec{\alpha}\cdot\vec{\xi} + \beta}{\xi^2 + 1} \, . \tag{I.7}$$

As a semibounded relativistic electron model that accommodates for bound states, Bethe and Salpeter proposed [3] to add a potential V to D_0 and restrict it again to the positive subspace $\mathcal{H}_+$ of the free Dirac operator. The two-electron analogon of this procedure was introduced earlier by Brown and Ravenhall [4].

We assume V to satisfy the following quadratic form bound,

$$\langle\, \psi, V^2\, \psi \,\rangle \ \le \ \beta^2\, \langle\, \psi, (-\Delta + 1)\, \psi \,\rangle \ , \tag{I.8}$$

for all $\psi \in H^2(\mathbb{R}^3; \mathbb{C}^4)$ and some $\beta < 1$, and furthermore

$$\|V(x)\|_{\mathcal{L}(\mathbb{C}^4)} \ \longrightarrow \ 0 \, , \quad \text{as} \quad |x| \ \longrightarrow \ \infty \ . \tag{I.9}$$

Then the operator

$$B_V \ = \ \Lambda_+\, (D_0 - V)\, \Lambda_+ \ , \tag{I.10}$$

resulting from this procedure, is defined on the same domain as B_0, i.e., $\mathrm{dom}[B_V] \ = \ \mathrm{dom}[B_0] \ = \ \Lambda_+ \mathrm{dom}[D_0]$, since V is B_0-bounded with relative bound β. Moreover, B_V is bounded from below by $1-\beta > 0$ and in particular invertible. To locate its essential spectrum we use a standard argument (see, e.g., [15]). Condition (I.9) implies that $B_0^{-1} - B_V^{-1}$ is the norm limit of the operators $B_0^{-1} - B_{\chi_R V}^{-1}$ as R tends to infinity, where χ_R denotes a smoothed characteristic function of the set $\{|x| \le R\}$. But for each $R > 0$ we get

$$B_0^{-1} - B_{\chi_R V}^{-1} \ = \ -B_{\chi_R V}^{-1}\, \Lambda_+ \chi_R V\, \Lambda_+\, B_0^{-1} \ = \ -B_{\chi_R V}^{-1}\, \Lambda_+ V\, \chi_R\, |D_0|^{-1}\, \Lambda_+ \ . \tag{I.11}$$

Now $B_{\chi_R V}^{-1}\, \Lambda_+ V$ is bounded because of (I.8) and $\chi_R\, |D_0|^{-1}$ is compact because of the local compactness property of the free Dirac operator. Thus, $B_0^{-1} - B_{\chi_R V}^{-1}$ is compact, for each $R > 0$, and so is its norm limit $B_0^{-1} - B_V^{-1}$. By Weyl's essential spectrum theorem, we therefore obtain

$$\sigma_{\mathrm{ess}}[B_V] \ = \ \sigma_{\mathrm{ess}}[B_0] \ = \ [1, \infty) \ . \tag{I.12}$$

Our concern in this paper is to prove the following exponential decay estimate for the eigenfunctions of B_V including an estimate on their decay rate.

Theorem I.1 *Suppose V satisfies (I.8) and (I.9). Furthermore, let $0 < b < 1 - a < 1$. Then*

$$\left\| e^{a|x|} \chi_{(b\,,\,1-a)}[B_V] \right\| \;<\; \infty\,, \tag{I.13}$$

where $\chi_{(b\,,\,1-a)}[B_V]$ denotes the spectral projection of B_V onto the interval $(b\,,\,1-a)$.

Although its spectrum resembles the standard spectrum of Schrödinger operators, B_V is not a differential operator, and the exponential decay of eigenfunctions does not *directly* follow from a standard Combes-Thomas argument. We succeed, however, in adapting a Combes-Thomas argument to prove Theorem I.1, above.

For the Klein-Gordon operator with external potential of the form $K := \sqrt{-\Delta + m^2} - V(x)$, exponential decay of eigenfunctions was proved in [5, 6], using path integral techniques. Later, in a study [10] of the tunnel effect in a two-well potential in the semiclassical limit, a Combes-Thomas argument, or Agmon estimate, was applied to prove exponential decay of eigenfunctions of K. The operator B_V, which we study in the present paper, can be written as $\Lambda_+ \, K \, \Lambda_+$, and, in fact, the main problem addressed here is to prove that the insertion of Λ_+ does not effect the exponential decay in an essential way.

The case of the relativistic hydrogen-like atom, for which $V(x) = \alpha Z/|x|$ is a Coulomb potential, has received special attention. In [9] the quadratic form associated to B_V is shown to be bounded from below on smooth spinors, provided $\alpha Z \leq \alpha Z_c := 2/(\pi/2 + 2/\pi)$ and unbounded from below otherwise. The corresponding Friedrichs extension in case $Z \leq Z_c$ is denoted by the same symbol, and it is shown in [9] and later refined in [16] that $\sigma_{\mathrm{ess}} = [1, \infty)$ and $\sigma_{\mathrm{sc}} = \emptyset$. Due to a result in [17], $B_V \geq 1 - \alpha Z$ is even strictly positive, for $Z \leq Z_c$. Furthermore, for $Z < Z_c$, there is an infinite number of eigenvalues of B_V below 1, accumulating only at 1 [17]. These results cover all known elements because $Z_c \approx 124$.

In order to apply Theorem I.1 to Coulomb potentials, $V(x) = \alpha Z/|x|$, we have to require that $Z \leq 68$, because otherwise $\alpha Z/|x|$ is relatively D_0-bounded with relative bound ≥ 1, due to the fact that Kato's inequality is sharp [12].

Our interest in the exponential decay of eigenfunctions of B_V originates from the question of existence of a ground state of a pseudorelativistic electron modelled by B_V and coupled to the quantized electromagnetic field, similar to a result for nonrelativistic quantum electrodynamics in [1, 2]. We have thus arranged the material in this paper in such a way that Theorem I.1 is a consequence of Corollary III.2 and Lemma III.1. In a forthcoming publication on pseudorelativistic quantum electrodynamics [13], Corollary III.2

and Lemma III.1 will serve as basic input for the proof of the existence of a ground state.

II Exponential decay

Proof of Theorem I.1: Our proof proceeds along the lines of Theorem II.1 in [1]. We show that the assertion holds on $\mathcal{H}$ and hence also on $\mathcal{H}_+ = \Lambda_+\mathcal{H}$. Considering B_V as an operator on $\mathcal{H}$, zero is an additional eigenvalue of infinite multiplicity. Our requirement $b > 0$ insures that this spectral point is avoided and that $\dim \operatorname{Ran} \chi_\Delta[B_V] < \infty$, where $\Delta := (b\,,\, 1-a)$. We may assume that $\sigma[B_V] \cap \Delta \neq \emptyset$. Let $b'' := \min\{\sigma[B_V] \cap \Delta\}$, $1-a'' := \max\{\sigma[B_V] \cap \Delta\}$, and set $\Delta'' := [b''\,,\, 1-a''] \subset \Delta$. We pick $b < b' < b''$ and $a < a' < a''$, and we set $\Delta' := (b'\,,\, 1-a')$, so we have

$$\Delta'' \;\subset\; \Delta' \;\subset\; \overline{\Delta'} \;\subset\; \Delta\,. \tag{II.1}$$

We may replace χ_Δ by $\widetilde{\chi}_{\Delta'} \in C_0^\infty(\Delta'; [0,1])$, which is chosen such that $\widetilde{\chi}_{\Delta'}[B_V] \equiv 1$ on Δ'', implying that $\widetilde{\chi}_{\Delta'}[B_V] = \chi_\Delta[B_V]$.

Define $B_{\chi_R V} := \Lambda_+(D_0 - \chi_R V)\Lambda_+$, where χ_R is a smooth characteristic function of the set $\{|x| \geq R\}$, e.g., $\chi_R(x) := \chi_1(x/R)$, $1 - \chi_1 \in C_0^\infty(B_2(0); [0,1])$, and $\operatorname{supp} \chi_1 \subseteq \{|x| \geq 1\}$. Then, for any $\varepsilon > 0$, there is an $R < \infty$ such that $\|\chi_R V\|_\infty < \varepsilon$, and we have

$$\langle \psi | B_{\chi_R V} \psi \rangle \;=\; \langle \psi|\, |D_0|\psi\rangle \;-\; \langle \psi | \chi_R V \psi \rangle \;\geq\; 1 - \|\chi_R V\|_\infty \;\geq\; 1 - \varepsilon\,, \tag{II.2}$$

for any normalized $\psi = \Lambda_+\psi$, and certainly $\langle \psi | B_{\chi_R V} \psi\rangle = 0$, for $\psi = \Lambda_-\psi$. Choosing $0 < \varepsilon < a$, we thus have that $\sigma[B_{\chi_R V}] \cap \Delta = \emptyset$ which implies that $\widetilde{\chi}_{\Delta'}[B_{\chi_R V}] = 0$, or

$$\chi_\Delta[B_V] \;=\; \widetilde{\chi}_{\Delta'}[B_V] \;-\; \widetilde{\chi}_{\Delta'}[B_{\chi_R V}]\,. \tag{II.3}$$

Since $\widetilde{\chi}_{\Delta'} \in C_0^\infty(\Delta'; [0,1])$, it has an almost analytic extension $f \in C_0^\infty(\mathbb{C})$. That is, given an arbitrary $\delta > 0$ and $N \in \mathbb{N}$, one can construct (see, e.g., [11]) a smooth function f on $\mathbb{C}$ such that

$$\mathcal{U}(\Delta') \;:=\; \Delta' + \{|\zeta| \leq \delta\} \;\supset\; \operatorname{supp} f\,, \tag{II.4}$$

$$f|_\Delta \;=\; \widetilde{\chi}_{\Delta'}\,, \tag{II.5}$$

$$\big|\partial_{\overline{z}} f(z)\big| \;=\; \mathcal{O}\big(|\operatorname{Im} z|^N\big)\,. \tag{II.6}$$

For our application below, we choose $N = 1$ and $\delta := \min\{(a'-a)\,,\,(b'-b)\}/2$. An almost analytic extension can be used to define a functional calculus by means of the following indentity,

$$\widetilde{\chi}_{\Delta'}(x) \;=\; \frac{i}{2\pi} \int_{\mathcal{U}(\Delta)} (z-x)^{-1} \, \frac{\partial f}{\partial \overline{z}} \, dz \wedge d\overline{z} \, . \tag{II.7}$$

Using the second resolvent equation and the abreviations $df(z) := \frac{i}{2\pi}\frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z}$ and $\overline{\chi_R} := 1 - \chi_R$, we can write

$$\begin{aligned} \widetilde{\chi}_{\Delta}(B) \;&=\; \int df(z) \, \Big[(B_V - z)^{-1} \;-\; (B_{\chi_R V} - z)^{-1} \Big] \\ &=\; \int df(z) \, (B_{\chi_R V} - z)^{-1} \, \Lambda_+ \, \overline{\chi_R} \, V \, \Lambda_+ \, (B_V - z)^{-1} \, . \end{aligned} \tag{II.8}$$

Now, we pick a smooth, convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $\phi \equiv 0$ on $(0,2]$ and $\phi(r) = r - 3$, for $r \geq 4$, so $|\phi'| \leq 1$. Defining $F : \mathbb{R}^3 \to \mathbb{R}_+$, $F(x) = aR \, \phi(|x|/R)$, we infer that $F \equiv 0$ on $\{|x| \leq 2R\}$ and $F(x) = a|x| - 3aR$ for $|x| \geq 4R$. Furthermore, $|\nabla F(x)| \leq a$, for all $x \in \mathbb{R}^3$.

Note that, since $a|x| - \mathrm{const} \leq F(x) \leq a|x|$, Eq. (I.13) is equivalent to

$$\big\| e^F \, \widetilde{\chi}_{\Delta'}[B_V] \big\| \;<\; \infty \, . \tag{II.9}$$

We define

$$B_{R,F} \;:=\; e^F \, B_{\chi_R V} \, e^{-F} \quad \text{and} \quad \Lambda_F \;:=\; e^F \, \Lambda_+ \, e^{-F} \, , \tag{II.10}$$

so that $e^F (B_{\chi_R V} - z)^{-1} e^{-F} = (B_{R,F} - z)^{-1}$. Furthermore, we observe that $e^F \overline{\chi_R} = \overline{\chi_R}$, and hence we have

$$\begin{aligned} \big\| e^F \, \chi_\Delta[B_V] \big\| \;&=\; \big\| e^F \, \widetilde{\chi}_{\Delta'}[B_V] \chi_\Delta[B_V] \big\| \\ &=\; \Big\| e^F \int df(z) \, (B_{\chi_R V} - z)^{-1} \, \Lambda_+ \, \overline{\chi_R} \, V \, \Lambda_+ \, (B_V - z)^{-1} \chi_\Delta[B_V] \Big\| \\ &\leq\; \int |df(z)| \, \Big\| (B_{R,F} - z)^{-1} \, \Lambda_F \, \overline{\chi_R} \, V \, \Lambda_+ \, \chi_\Delta[B_V] \, (B - z)^{-1} \Big\| \\ &\leq\; \int |df(z)| \, \Big\| (B_{R,F} - z)^{-1} \, \Lambda_F \Big\| \, \Big\| V \, (D_0 + i)^{-1} \Big\| \\ &\qquad \Big\| (D_0 + i) \, \Lambda_+ \, \chi_\Delta[B_V] \Big\| \, \Big\| (B - z)^{-1} \Big\| \, . \end{aligned} \tag{II.11}$$

In order to estimate the norm of $(B_{R,F} - z)^{-1} \Lambda_F$ in the integral on the right side of (II.11), we first remark that $\Lambda_F = \Lambda_F^2$ is a bounded projection, due to Corollary III.2. Hence its image, $\Lambda_F \mathcal{H}$, is closed and left invariant under the action of the resolvent $(B_{R,F} - z)^{-1}$, since $B_{R,F}$ commutes with Λ_F. For $\phi = \Lambda_F \phi \in \Lambda_F \mathcal{H}$, we have

$$\begin{aligned}
&\big\| \Lambda_F \, (B_{R,F} - z) \, \Lambda_F \, \phi \big\| \\
&\quad = \; \big\| \Lambda_F \, e^F \, \Lambda_+ \, (D_0 - \chi_R V) \, \Lambda_+ \, e^{-F} \, \Lambda_F \, \phi \; - \; z \, \phi \big\| \\
&\quad = \; \big\| e^F \, D_0 \, \Lambda_+ \, e^{-F} \, \phi \; - \; \Lambda_F \, e^F \, \Lambda_+ \, e^{-F} \, \chi_R \, V \, e^F \, \Lambda_+ \, e^{-F} \, \Lambda_F \, \phi \; - \; z \, \phi \big\| \\
&\quad = \; \big\| (D_0 + i\vec{\alpha} \cdot \vec{\nabla} F) \, \Lambda_F \, \phi \; - \; \Lambda_F \, \chi_R \, V \, \phi \; - \; z \, \phi \big\| \\
&\quad = \; \big\| D_0 \, \phi \; + \; i\vec{\alpha} \cdot \vec{\nabla} F \, \phi \; - \; \Lambda_F \, \chi_R \, V \, \phi \; - \; z \, \phi \big\| \\
&\quad \geq \; \Big(1 - a - \| \chi_R V \| \, \| \Lambda_F \| - \sup_{\mathcal{U}(\Delta')} |z| \Big) \, \| \phi \| \, .
\end{aligned} \tag{II.12}$$

We observe that upon choosing $\delta := (a' - a)/3$, we have $\sup_{\mathcal{U}(\Delta')} |z| \leq 1 - a' + \delta = 1 - 2a'/3 - a/3$. By Corollary III.2, $\|\Lambda_F\| = 1 + O(a)$ and thus $\|\chi_R V\| \, \|\Lambda_F\| \leq (a' - a)/3$, provided $R < \infty$ is chosen sufficiently large. Hence

$$\big\| \Lambda_F \, (B_{R,F} - z) \, \Lambda_F \, \phi \big\| \; \geq \; \frac{a' - a}{3} \, \| \phi \| \, . \tag{II.13}$$

Using again $\|\Lambda_F\| = 1 + O(a)$, we thus obtain that

$$\| (B_{R,F} - z)^{-1} \, \Lambda_F \| \; \leq \; \frac{3 + \mathcal{O}(a)}{a' - a} \; < \; \infty \, . \tag{II.14}$$

Next, V is relatively D_0-bounded, by (I.8), hence

$$\big\| V \, (D_0 + i)^{-1} \big\| \; < \; \infty \, , \tag{II.15}$$

and furthermore $D_0 \, \Lambda_+$ is relatively B_V bounded. Together with the boundedness of $\Delta \subset [0, 1]$, this implies

$$\begin{aligned}
&\Big\| (D_0 + i) \, \Lambda_+ \, \chi_\Delta[B_V] \Big\| \\
&\quad \leq \; \Big\| (D_0 + i) \, \Lambda_+ \, (B_V + iC)^{-1} \Big\| \cdot \Big\| (B_V + iC) \, \chi_\Delta[B_V] \Big\| \; < \; \infty \, ,
\end{aligned} \tag{II.16}$$

for $0 < C < \infty$ sufficiently large.

Finally, our choice $N = 1$ for the almost analytic contiunuation insures that $|df(z)| = \mathcal{O}(\mathrm{Im}\, z)$ compensates for the singularity arising from $\|(B - z)^{-1}\| \leq |\mathrm{Im}\, z|^{-1}$. Hence

$$|df(z)| \, \big\| (B - z)^{-1} \big\| \; \leq \; \mathcal{O}(1) \, dz \wedge d\overline{z} \, . \tag{II.17}$$

The desired estimate (II.9) now follows from inserting (II.14)–(II.17) into (II.11). □

III Estimates on Commutators

The purpose of this section is to estimate the norms of various operators that involve the resolvent of $|D_0|$ and an exponential weight function e^F (acting as $(e^F\psi)(x) := e^{F(x)}\psi(x)$),

$$K_1 \;:=\; e^F\,|D_0|^{-1}\,e^{-F}\,, \tag{III.1}$$
$$K_2 \;:=\; e^F\left[\,|D_0|^{-1}\,,\, e^{-F}\right] \;=\; \left[e^F\,,\,|D_0|^{-1}\right]e^{-F}\,, \tag{III.2}$$
$$K_3 \;:=\; \left[\,|D_0|^{-1}\,,\, e^{-F}\right]e^F \;=\; e^{-F}\left[e^F\,,\,|D_0|^{-1}\right], \tag{III.3}$$

$$D_0\,K_2\,, \qquad D_0\,K_3\,. \tag{III.4}$$

Our norm estimates below on (III.1)–(III.4) serve as a main input for proving Theorem I.1, especially the boundedness of Λ_F.

Lemma III.1 *Assume that $F \in C^1(\mathbb{R}^3;\mathbb{R}_+)$ is such that $F(0) = 0$ and $a := \sup_x |\vec\nabla F(x)| < 1$. Then the operators K_1, K_2, K_3, D_0K_2, and D_0K_3 are defined on $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$, the space of spinor-valued Schwartz test functions, and extend to bounded operators on $\mathcal{H}$, denoted by the same symbols, with norms bounded by*

$$\|K_1\| \;=\; \mathcal{O}\big((1-a)^{-2}\big)\,, \tag{III.5}$$
$$\|K_2\|\,,\ \|K_3\| \;=\; \mathcal{O}\big(a\,(1-a)^{-2}\big)\,, \tag{III.6}$$
$$\|D_0K_2\|\,,\ \|D_0K_3\| \;=\; \mathcal{O}\big(a\,(1-a)^{-2}\big)\,. \tag{III.7}$$

Proof: We first remark that $|D_0|^{-1}\psi = G*\psi$, where $G*$ denotes convolution with

$$G(x) \;:=\; \frac{1}{4\pi^{3/2}\Gamma(1/2)}\int_0^\infty \exp\Big[-r/2 - x^2/2r\Big]\,\frac{dr}{r^2}\,, \tag{III.8}$$

the inverse Fourier transform of $(\xi^2+1)^{-1}$ [14]. Hence, $K_1 : \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4) \to \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$ can be written as an integral operator with Schwartz kernel

$$K_1(x,y) \;=\; \gamma\int_0^\infty \exp\Big[-r/2 - (x-y)^2/2r + F(x) - F(y)\Big]\,\frac{dr}{r^2}\,, \tag{III.9}$$

where $\gamma^{-1} := 4\pi^{3/2}\Gamma(1/2)$. Note that $0 \le K_1(x,y) \le \widetilde{K}_1(x-y)$, where

$$\widetilde{K}_1(x) \;:=\; \gamma\int_0^\infty \exp\Big[-r/2 - x^2/2r + a|x|\Big]\,\frac{dr}{r^2}\,, \tag{III.10}$$

since $|\vec{\nabla}F| \leq a$. Furthermore, observe that

$$\int \widetilde{K}_1(x)\, d^3x \;\leq\; \gamma \int_0^\infty \frac{dr}{r^2} \int d^3x \, \exp\Big[- \frac{(1-a)r}{2}\Big(1 + \frac{x^2}{r^2}\Big)\Big] \;\leq\; \mathcal{O}\big((1-a)^{-2}\big) \, , \tag{III.11}$$

which implies that K_1 extends to an operator $\in \mathcal{B}(\mathcal{H})$, with norm bounded by

$$\|K_1\| \;\leq\; \Big(\sup_x \int K_1(x,y)\, d^3y \Big)^{1/2} \Big(\sup_y \int K_1(x,y)\, d^3x \Big)^{1/2} \;\leq\; \int \widetilde{K}_1(x)\, d^3x \;=\; \mathcal{O}(1) \, . \tag{III.12}$$

Likewise, we define K_2 and K_3 first as integral operators $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^4) \to \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$, with Schwartz kernel

$$K_2(x,y) \;=\; -K_3(y,x) \;:=\; \gamma \int_0^\infty \exp\Big[- r/2 - (x-y)^2/2r \Big] \Big(e^{F(x)-F(y)} \;-\; 1 \Big) \frac{dr}{r^2} \, . \tag{III.13}$$

Then, using that

$$\big| e^{F(x)-F(y)} - 1 \big| \;\leq\; e^{|F(x)-F(y)|} - 1 \;\leq\; |F(x) - F(y)| \, e^{|F(x)-F(y)|} \;\leq\; a|x-y| \, e^{a|x-y|} \, , \tag{III.14}$$

we find that $|K_2(x,y)|, |K_3(y,x)| \leq \widetilde{K}_2(x-y) := a|x-y| \, \widetilde{K}_1(x-y)$, and similarly to (III.11)–(III.12), this yields

$$\|K_2\| \, , \;\; \|K_3\| \;\leq\; \int \widetilde{K}_2(x)\, d^3x \;=\; \mathcal{O}(a) \, . \tag{III.15}$$

Finally, the Schwartz kernel of $D_0 K_2 : \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4) \to \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$ is given by

$$\big[D_0 K_2\big](x,y) \;=\; -i\vec{\alpha} \cdot \vec{\nabla}_x \, K_2(x,y) \;+\; \beta \, K_2(x,y) \, . \tag{III.16}$$

Furthermore, Leibnitz' rule yields

$$\begin{aligned} &\big|\vec{\nabla}_x \, K_2(x,y)\big| \\ &= \gamma \bigg| \int_0^\infty \frac{dr}{r^2} \exp\Big[- r/2 - (x-y)^2/2r \Big] \\ &\qquad \Big\{ - \frac{x-y}{r}\Big(e^{F(x)-F(y)} \;-\; 1 \Big) \;+\; \vec{\nabla}F(x) \, e^{F(x)-F(y)} \Big\} \bigg| \\ &\leq a\,\gamma \int_0^\infty \frac{dr}{r^2} \exp\Big[- r/2 - (x-y)^2/2r + a|x-y| \Big] \Big(\frac{|x-y|^2}{r} + |x-y| \Big) \, , \end{aligned} \tag{III.17}$$

from which we conclude $\|D_0 K_2\| = \mathcal{O}(a)$ and, analogously, $\|D_0 K_3\| = \mathcal{O}(a)$. $\square$

Corollary III.2 *Λ_F, $1-\Lambda_F$ are bounded (non-orthogonal) projections on $\mathcal{H}$ with $\|\Lambda_F\|, \|\mathbf{1}-\Lambda_F\| = 1+\mathcal{O}(a)$, and $\|\Lambda_+ - \Lambda_F\| = \mathcal{O}(a)$.*

Proof: The idempotency of Λ_F and $\mathbf{1}-\Lambda_F$ is trivial, and the norm estimates on both Λ_F and $\mathbf{1}-\Lambda_F$ follow from $\|\Lambda_+ - \Lambda_F\| = \mathcal{O}(a)$ which hence remains to be proven.

To this end, we observe that on $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$, the following identity holds,

$$\begin{aligned}
\Lambda_+ - \Lambda_F &= \Lambda_+ \;-\; e^F\,\Lambda_+\,e^{-F} \;=\; D_0\,|D_0|^{-1} \;-\; e^F\,D_0\,|D_0|^{-1}\,e^{-F} \\
&= D_0\,e^F\,e^{-F}\,|D_0|^{-1} \;-\; e^F\,D_0\,|D_0|^{-1}\,e^{-F} \\
&= D_0\,e^F\left[e^{-F}, |D_0|^{-1}\right] \;-\; \left[e^F, D_0\right]|D_0|^{-1}\,e^{-F} \\
&= -D_0\,K_2 \;-\; i\,\vec{\alpha}\cdot\vec{\nabla}F\,e^F\,|D_0|^{-1}\,e^{-F} \\
&= -D_0\,K_2 \;-\; i\,\vec{\alpha}\cdot\vec{\nabla}F\,K_1\,.
\end{aligned} \tag{III.18}$$

Hence, due to Lemma III.1,

$$\|\Lambda_+ - \Lambda_F\| \;\le\; \|D_0\,K_2\| \;+\; a\,\|K_1\| \;=\; \mathcal{O}(a)\,.\square \tag{III.19}$$

References

[1] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. *Adv. in Math.* , 137:299–395, 1998.

[2] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Commun. Math. Phys.*, 207(2):249–290, 1999.

[3] H. Bethe and E. Salpeter. Quantum mechanics of one- and two-electron atoms. In S. Flügge, editor, *Handbuch der Physik, XXXV*, pages 88–436. Springer, Berlin, 1957.

[4] G. E. Brown and D. G. Ravenhall. On the interaction of two electrons. *Proc. Roy. Soc. London A*, 208:552–559, 1951.

[5] R. Carmona. Path integral for relativistic Schrödinger operators. In H. Holden and A. Jensen, editors, *Schrödinger Operators*, volume 345 of *Lecture Notes in Physics*, pages 65–92, Heidelberg, 1988. Springer-Verlag.

[6] R. Carmona, W. Masters, and B. Simon. Relativistic Schrödinger operators: Asymptotic behaviour of the eigenfunctions. *J. Funct. Anal.*, 1992.

[7] P.A.M. Dirac. The quantum theory of the electron. *Proc. Royal Soc. London*, 117(A):610–624, 1928.

[8] P.A.M. Dirac. The quantum theory of the electron II. *Proc. Royal Soc. London*, 118(A):351–361, 1928.

[9] D. Evans, P. Perry, and H. Siedentop. The spectrum of relativistic one-electron atoms according to Bethe and Salpeter. *Commun. Math. Phys.*, 178(3):733–746, July 1996.

[10] B. Helffer and B .Parissee. Comparaison entre la décroissance de fonctions propres pour les opérateurs de Dirac et de Klein-Gordon. Application à l'étude de l'effet tunnel. *Ann. Inst. H. Poincaré*, 60(2):147–187, 1994.

[11] B. Helffer and J. Sjöstrand. Semi-classical analysis for Harper's equation iii: Cantor structure of the spectrum. *Suppl. Bull. Soc. Math. France*, 39(117), 1989.

[12] T. Kato. Smooth operators and commutators. *Stud. Math. Appl.* , 31:535–546, 1968.

[13] O. Matte. One-electron atoms in the Brown-Ravenhall model coupled to the quantized radiation field. Master's thesis, Universität Mainz (Diploma Thesis), In Preparation.

[14] G. Schleinkofer. Introduction to pseudodifferential operators. Posgrado en Matematicas Universidad Nacional de Colombia, 1996.

[15] B. Thaller. *The Dirac Equation.* Springer-Verlag, Heidelberg, New York, 1992.

[16] C. Tix. Self-adjointness and spectral properties of a pseudo-relativistic Hamiltonian due to Brown and Ravenhall. *Preprint mp-arc 97-441*, 1997.

[17] C. Tix. Strict positivity of a relativistic Hamiltonian due to Brown and Ravenhall. *Bull. London Math. Soc.*, 30(3):283–290, 1998.