An existence theorem for Vlasov gas dynamics in regions with moving boundaries

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Abstract

We prove a global existence theorem for a class of deterministic infinite-dimensional Hamiltonian systems in which a Vlasov gas is coupled to a finite-dimensional Hamiltonian system. The later is the motion of the rigid and macroscopic boundary. The boundary motion is due to pressure forces of the gas and determined by the law of total momentum conservation of the coupled system. Our existence result shows that if the gas density is initially a smooth function on the phase space, the boundary moves continuously. We consider specific examples of such a gas dynamics and formulate some open mathematical questions for these Hamiltonian systems.

Keywords: Vlasov dynamics, billiards, Hamiltonian systems

1 Introduction

The dynamics of a free gas evolving on a fixed manifold M with boundary is a billiard problem. The motion of each individual particle moves independently of the other particles and is given by a symplectic flow X^t. The particle density in the phase space is given by a measure P. If P is a finite Dirac measure, this describes the position and momentum of finitely many particles. The measure P^t in the phase space at time t is obtained as the push-forward $X^t_* P^0$ of the map X^t. If the particles interact with an inter-particle potential V and no collisions take place, the individual particle paths $X^t(x, y)$ for $(x, y) \in T^*M$ are still described by a symplectic flow X^t. But now, the curve of symplectomorphisms X^t depends on the measure P^0. Noninteracting particles can interact indirectly when the boundary of M is allowed to move rigidly under the pressure forces of the gas. This is interesting when finitely many particles move say in a circular billiard table. The table experiences kicks from the particles and the billiard gas inside the shaken table is no more integrable. Even for $n \geq 2$ particles in a circular table, the dynamics is a nontrivial mixture of stable and unstable behaviour even so the particles interact only through the wall (see Figure 1). (The dynamics of a single particle shaking a billiard table can be shown to be conjugated to the billiard dynamics when the table is fixed).

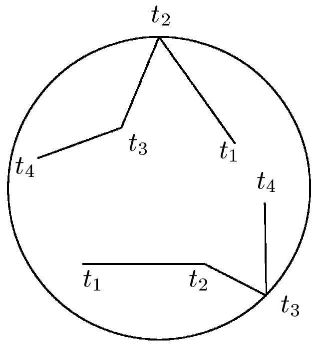

Figure 1. A n-particle billiard where the table is kicked around by the particles. Total momentum of particles and table is conserved. (For a web implementation see http://www.dynamical-systems.org/vlasov). In a coordinate system in which the table is at rest, particles move like in usual billiards but experience nonlocal forces from the other particles whenever one of the particles hits the boundary (at times t_2, t_3 in the illustration). In this article we consider also such billiards when the particle density is smooth.

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In the continuous Vlasov limit, where the gas at time t is described by a continuous density P^t in the phase space, the table moves smoothly. The infinite-dimensional Hamiltonian system is then coupled to a finite-dimensional system. As a first step to study such systems mathematically we establish in this paper an existence result.

When an interacting gas moves on a manifold M without boundary, the particle density P on the cotangent bundle T^*M satisfies the Vlasov equation

$$\dot{P}^t(x,y) + y \cdot \nabla_x P^t(x,y) - E(x,P^t) \cdot \nabla_y P^t(x,y) = 0 \;, \tag{1}$$

where $E(x,P) = \nabla W(x) + \int_{T^*M} \nabla V(x,x') \; dP(x',y'))$ is the sum of the external force and the inter-particle forces. We get the solutions P^t by pushing forward the initial density P^0 with a symplectic transformation $X^t = (f^t, g^t)$ on T^*M which satisfies the characteristic differential equation

$$\dot{f}^t(x,y) = g^t(x,y), \dot{g}^t(x,y) = E(f^t(x,y), P^0(x,y)) \;. \tag{2}$$

The Vlasov description of the gas dynamics is not yet any approximation of the real physical problem as long the class of measures P and potentials V is not narrowed: in the case when P is a Dirac measure describing the motion of finitely many particles which are pair interacting with a potential V, then this is the real finite-dimensional system in classical mechanics. Choosing V suitably allows to incorporate all inter-particle and inter-molecular interactions. We stress that V does not need to be the natural Newton potential determined by the Laplacian on the manifold in which case one deals with the Vlasov-Poisson system. An interesting example on $M = \mathbf{R}$, where V is different from the Poisson potential $V(x) = |x|$ on M is the potential $V(x) = x^{-2} + x^2$ which leads for smooth initial measures P^0 still to an integrable nonlinear Vlasov gas dynamics. It is the Calogero-Moser-Vlasov system which extends the classical Calogero-Moser system, in which case P is a finite Dirac measure.
The characteristic system (2) still describes the system in the case when M has a boundary and also if the boundary is allowed to move. We prove here an existence theorem in the case, when P is initially smooth and M has a boundary which can move with finitely many degrees of freedom. We assume furthermore that the particles interact with a fixed smooth potential V and that they are possibly exposed to an external potential W.
To get the existence theorem one has to deal with the facts that the phase space of the particles is time-dependent and that the momentum is not smooth at the boundary. This can produce difficulties when trying to set up an existence result in a function space containing discontinuous functions. However, discontinuous functions can be avoided.

An existence and uniqueness theorem for the Vlasov equation in a manifold without boundary for smooth interactions and general Borel measures P of compact support has been established by Brown-Hepp-Neunzert-Dobrushin [1, 2]. Our result should be seen as an extension to this existence theorem, however only for the relatively narrow class of smooth measures P. An other, different problem of a Vlasov gas in a manifold with fixed boundary is the Vlasov-Poisson system, where the potential V is determined by the boundary and where little is known besides the existence of equilibrium solutions [10]. We don't address any Vlasov-Poisson situation in this paper.

In the concrete examples we do consider here, the inter-particle potential of the gas vanishes. The gas is then an ideal gas interacting with a macroscopic boundary. In all cases, we expect the boundary motion to converge to an equilibrium with a speed of convergence depending on the dynamical properties of the billiard obtained when the boundary is at rest.

Our setup and the existence theorem is general in principle but we have in mind concrete examples:

Example 1. The Wojtkowski gas. The first example is a system of noninteracting non-colliding particles which fall onto a floor. The particles are accelerated by a constant gravitational force and the floor experiences a constant opposite force matching the pressure force so that the total momentum of the coupled particle-floor system is kept zero at all times. In the case of finitely many particles, the system is completely hyperbolic in the sense that all possible Lyapunov exponents are positive [14]. We consider the system in the case when a particle gas has a continuous density P^t and where the gas dynamics is coupled with the now smoothly moving floor.

Example 2. The Birkhoff gas. As a second example, we consider a convex Birkhoff billiard in which a gas with a smooth density evolves according to the billiard law. The gas interacts with the rigid, macroscopic boundary. The later moves with translational degrees of freedom in such a way that the total momentum and energy of gas and boundary are conserved. Two prototype examples are the case of the square, where the billiard dynamics is integrable and the case of the Bunimovich billiard, where the billiard dynamics is ergodic. Unrelated to our examples are billiards with moving boundaries, where the motion of the boundary is decoupled from the motion of the gas and where the boundary moves in the normal direction [6]. The square billiard is especially appealing when the initial density factorizes because the problem reduces then to a one-dimensional Vlasov billiard which is the simplest possible system of the class we consider here.

Example 3. A Vlasov-thermometer. In the third example, we have a container, in which a macroscopic freely sliding wall separates two regions. The container is fixed and the particles in the gas are reflected at the ends of the container. While this is a decoupled set of billiards systems if the separating wall is held at rest, the gas pressure difference at the wall makes the wall move and different gas particles interact through the macroscopic boundary. A limiting case is the 'ideal gas thermometer', where the gas on one side is assumed to produce a constant pressure force onto the wall. The position of the piston measures then the temperature of the gas on the other side.

Example 4: The Sinai-Vlasov gas. In the fourth example, a disc moves as an eddie in a two-dimensional torus under the pressure forces of the gas. Again the gas has a continuous density and each gas particle is reflected elastically from the obstacle as in the Sinai billiard. In the special case, when the phase space density of the gas is invariant with respect to rotations at the center of the disc, then the disc does not move and each particle of the gas moves according to the Sinai billiard.

These four examples provide prototypes and motivation to look at more general systems, where global existence of the dynamics is expected to hold. Despite the apparent simplicity and the assumption to have no inter-particle forces in those examples, the long time behavior of the four chosen systems is largely open: almost nothing mathematical seems to be known. Most promising for an mathematical analysis is the gas in a one-dimensional box which is a special case of Example 2. We expect in all examples that the motion of the boundary converges to a constant motion, that the energy of the gas and the wall do converge when averaged over a fixed time interval and that the Lyapunov exponent of the gas is the Lyapunov exponent of the billiard obtained in the case when the boundaries are fixed.

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2 The setup

Let M be an open region in Euclidean space $\mathbf{R}^d$ or the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$. We assume that M has a smooth orientable boundary $\delta M \subset M$. (We could more generally assume that M is a smooth Riemannian manifold with orientable boundary in which case the free motion is the geodesic flow in the interior. Everything below would extend under suitable restrictions of the metric to such a generalization. For simplicity, we stick to the flat case.)

A gas of particles moves in M, each particle is reflected at the moving boundary δM^t. The gas is given by a continuous measure P on the phase space $T^*M = M \times \mathbf{R}^d$. The gas can interact with a smooth inter-particle force V. The total momentum change of the gas at the boundary produces a force on the wall. Each particle of the gas which is initially at (x, y) has at the time t the position $f^t(x, y) \in \overline{M^t}$ and the momentum $g^t(x,y) \in \mathbf{R}^d$. Writing $X^t = (f^t, g^t)$, the symplectic deformation $X^t : M^0 \times \mathbf{R}^d \to M^t \times \mathbf{R}^d$ describes completely the motion of the gas. The density P^t on $M^t \times \mathbf{R}$ is the push-forward of P^0 under this map. In order to find a set of differential equations for X^t and the motion of the boundary, it is often convenient to go into a coordinate system, where the boundary is at rest. Assume $M = \phi^{-t}(M^t)$, where $\phi^t : M \to M^t$ is a diffeomorphism for fixed t. The transformation ϕ^t in the position space can be extended to a symplectic transformation $\Phi : T^*M^t \to T^*M$ on the cotangent bundle. This change of variables produces space-dependent forces for the particles. These forces appear additionally to the internal forces from the interaction potential V and the external potential W.

We restrict us here for simplicity to the case when the motion of the movable part of the boundary δM^t is restricted to translations in M. The position of the moving part of δM^t is denoted by $q(t) \in \mathbf{R}^d$, its momentum is called $p(t) \in \mathbf{R}^d$. The mass of the moving part is assumed to be 1. With one moving boundary, only the mass ratio gas-mass/boundary is relevant and the gas-mass is the total mass of the measure P. In the coordinate system, where the region M is fixed, the particles bounce off from the boundary according to the law of reflection in billiards.

Additionally, we assume that the particles can then be exposed to a smooth external potential W and that the boundary experiences a smooth external force ∇U. In order to obtain total momentum conservation, we need $U + W = 0$. A prototype for such a system is the Wojtkowski gas, where W is the gravitational potential and where ∇U is the constant force holding the floor. If the forces of U and W are translational invariant, it is possible to go into a coordinate system at rest, introducing additional forces from this nonlinear change of coordinates.

Let S be the manifold obtained by taking the manifold T^*M with boundary and identifying the points $(x, v) = (x, -v)$ at boundary points. Any smooth probability measure P on S defines an initial gas density. Together with external and interaction potentials we have so defined a Vlasov dynamical system in S.

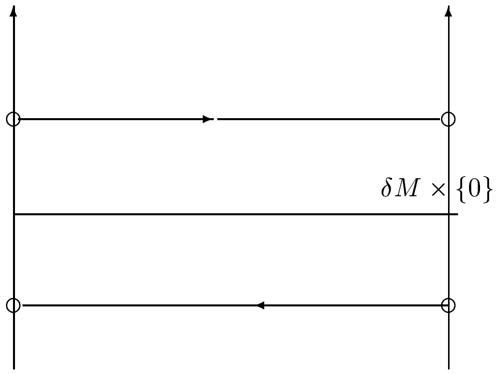

Figure 2.The manifold S in the case, when particles move in the one-dimensional manifold $M = [0,1]$ with boundary. The y-axes represent the left and right walls of $M = [0,1]$. If the boundary δM is kept at rest, then each particle of speed $v \neq 0$ moves on a circle in S.

The flow invariant symplectic form on an open dense subset of S implies that there is a natural Liouville measure which is invariant under the flow X^t.

3 Existence of the dynamics

We consider an initially smooth probability measure P^0 on S. It is the probability distribution for the position and momentum of a particle in the particle gas. Particles are labeled by their initial conditions $(x, y) \in S$. Their motion is determined by a one-parameter family of continuous volume-preserving maps $X^t = (f^t, g^t) : S \to S$. We write $f^t(x,y) \in M$ for the position of the particle and $g^t(x,y) \in \mathbf{R}^d$ for the momentum. The identification at the boundary assures that $t \mapsto X^t(x,y) = (x^t, y^t) = (f^t(x,y), g^t(x,y))$ is continuous for all $(x,y) \in S$.

If X^t exists, the particle density at time t is given by the push-forward $P^t = X^t_* P^0$. This means that $P^t[B] = P^0[X^{-t}B]$ for any Borel set B in S. As usual in existence problems for Vlasov dynamics, we do not try directly to show existence for the partial differential equation for P but instead focus on the existence of the symplectic dynamics of X^t which induces the evolution of the measure P^t and so a solution of the Vlasov equation. If X^t is unique, then P^t is unique.

In the case, when the boundary is exposed to a constant force ∇W and the region M^t moves rigidly as one unit, one can get rid of the macroscopic motion of the boundary and go into a coordinate system, where $q = p = 0$. The particles feel then time-dependent pressure forces from this coordinate change.

The equations of motion for the coupled system can be written entirely in the phase space variables of the gas, eliminating the motion of the boundary. In a fixed chart around $(x, y) \in S$, we have

$$\begin{aligned}
\dot{f}(x,y) &= g(x,y) \\
\dot{g}(x,y) &= [-\nabla W(f(x,y)) - \nabla U - \int_S \nabla V(f(x,y), f(x',y'))\, dP^0(x',y') \\
&- \int_{f(x,y)\in\delta M} n(f(x,y))\, (n(f(x,y)) \cdot g(x,y))\, dP^0(x,y)] \, ,
\end{aligned}$$

where ∇W is the forces acting on the particle described in a chart around the point $(f(x,y), g(x,y))$ and where $n(x)$ is the normal vector of a point $x \in \delta M$. In a coordinate system, where the boundary is fixed, the measure $P^t(x,y) = X^t_*(x,y)P^0(x,y)$ satisfies in each chart of S the Vlasov differential equation

$$\dot{P} + y\nabla_x P - E(x, P^t)\nabla_y P \ = 0$$

with $E(x, P^t) = -\nabla W(x) - \nabla U - \int_S \nabla V(x,x')\, dP^t(x',y') - \int_{x\in\delta M} n(x)\, (n(x)\cdot y)\, dP^t(x,y)$.

Theorem 3.1 (Existence and uniqueness) *Let P^0 be a smooth probability density on S which has compact support. Let $W : S \to \mathbf{R}$ be a smooth external potential for the particles, let $U : \mathbf{R}^d \to \mathbf{R}$ be a smooth external potential for the boundary and let V be a smooth inter-particle potential so that $\nabla V, \nabla W, \nabla U$ are all globally bounded and globally Lipshitz continuous. The differential equation $\dot{X} = F(X)$ for $X^t = (f^t, g^t)$ has global unique solutions in the group of homeomorphisms of S. The particle density $(X^t)^* P^0 = P^t$ stays continuous at all times and the motion of the macroscopic boundary $t \mapsto (q(t), p(t))$ is smooth. P^t is a unique solution of the Vlasov equation.*

Proof. We aim to show that the characteristic system $\dot{X} = F(X)$ has global solutions in the complete metric space $\mathcal{X} = C(S,S)$ of all continuous maps from S to S. This is done with Piccard's existence theorem. Given a curve X^t in $\mathcal{X}$, the evolution of a test particle in that time-dependent situation defines a new curve $\phi(X)^t$. A fixed point of this map $X^t \mapsto \phi(X)^t$ is a solution of $\dot{X} = F(X)$. We will go through the proof, show that the Piccard iteration is defined and verify the contraction property of the Piccard iteration.

(i) Definition of the Piccard map.
The Piccard map ϕ is defined on the space $C([0,\tau], C(S,S))$ of all continuous paths which start at $X^0 = Id$. The map is explicitly given by

$$\phi(X)(\omega)(t) = X^0(\omega) + \int_0^t F(X^s(\omega))\, ds \, .$$

(ii) There exists a constant C such that $||F(Y)|| \leq C$ for all Y in a δ-neighborhood of X^0 because all forces are assumed to be bounded. The force from the boundary acceleration stays bounded because we have assumed that P^0 has compact support (which implies that $X^s_* P^0$ has compact support).

(ii) There exists a constant C such that for all Y_1, Y_2 in a δ-neighborhood of $X^s = X^0 = \mathrm{Id}$ of $C([0,\tau], C(S,S))$, for all $s \in [0,\tau]$ and all $\omega \in S$ on has

$$||F(Y_1^s(\omega)) - F(Y_2^s(\omega))|| \leq C||Y_1 - Y_2|| \, .$$

Proof. It is enough to show separately that
a) the inter-particle force, b) the boundary pressure force, c) the external force for particles as well as d) the external force for the boundary satisfy this Lipshitz property.

a) **Inter-particle force.**
If $||Y_1^s - Y_2^s|| \leq \delta$ and we write $Y_i^s = (f_i^s, g_i^s)$, then the positions of a particle located initially at (x, y) which move along the different paths Y_1^s, Y_2^s satisfy $|f_1(x, y) - f_2(x, y)| \leq \delta$.

If a particle is initially located at (x, y), consider a second particle, initially located at (x', y') in the phase space. The force between the two particles at time s is $\nabla V(f_1^s(x, y) - f_1^s(x', y'))$ along the first path Y_1 and $\nabla V(f_2^s(x, y) - f_2^s(x', y'))$ along the second path Y_2. The difference satisfies

$$|\nabla V(f_1^s(x, y) - f_1^s(x', y')) - \nabla V(f_2^s(x, y) - f_2^s(x', y'))| \leq 2C\delta ,$$

where $C = ||\nabla V||_\infty$ is the Lipshitz constant of V. We used that $|(f_1^s(x, y) - f_1^s(x', y')) - (f_2^s(x, y) - f_2^s(x', y'))| \leq |f_1^s(x, y) - f_2^s(x, y)| + |f_1^s(x', y') - f_2^s(x', y')| \leq 2\delta$. Integration over P^0 gives

$$\int_S |\nabla V(f_1^s(x, y) - f_1^s(x', y')) - \nabla V(f_2^s(x, y) - f_2^s(x', y'))| \, dP^0 \leq 2C\delta .$$

b) **Force from the coordinate changes due to boundary acceleration.**
Extend the normal vector $n(x, y)$ from the boundary set $\delta M \times \mathbf{R}^d$ to a neighborhood in S. The momentum transfer densities $(x, y) \mapsto H_i(x, y) = n(f_i(x, y)) \, (n(f_i(x, y)) \cdot g_i(x, y))$ are continuous in a neighborhood of the sets $\Sigma_i^s = \{(x, y) \in S \mid f_i^s(x, y) \in \delta M \}$ which consists of phase points (x, y) at which a particle hits at the time s the boundary δM. The sets Σ_i^s are continuous deformations of $\delta M \times \mathbf{R}^d$, are piecewise smooth and carry the natural, from P^0 by X^t pushed forwarded measure. Claim: there exists C such that

$$|\int_{\Sigma_1^s} H_1(x, y) \, dP^0(x, y) - \int_{\Sigma_2^s} H_2(x, y) \, dP^0(x, y)| \leq C\delta .$$

Proof. The set Σ_1^s is in a δ-neighborhood of Σ_2^s and vice versa (in other words, their Hausdorff distance is $\leq \delta$). Now, $|H_1(x, y) - H_2(x, y)| \leq C\delta$ in that neighborhood. Let $\psi^s(\Sigma_1^s) = \Sigma_2^s$, so that

$$\begin{aligned} |\int_{\Sigma_1^s} H_1(x, y) \, dP^0(x, y) - \int_{\Sigma_2^s} H_2(x, y) \, dP^0(x, y)| &= |\int_{S_1^s} H_1(x, y) \, dP^0(x, y) - H_2(\psi^s(x, y)) \, dP^0(x, y)| \\ &\leq ||H_1(x, y) - H_2(\psi^s(x, y))||_\infty \leq C\delta . \end{aligned}$$

c) **External force ∇W for particles**.

$$|\nabla W(f_1^s(x, y)) - \nabla W(f_2^s(x, y))| \leq ||\nabla W||_\infty |f_1^s(x, y) - f_2^s(x, y)| \leq C\delta .$$

d) **External force ∇U for the boundary**.
This force is Y-independent.

(iv) From (ii) and (iii), we have a constant C such that

$$\begin{aligned} \sup_{\omega \in S} |\int_0^\tau F(Y_1^s(\omega)) - F(Y_2^s(\omega)) ds| &\leq \int_0^\tau \sup_{\omega \in S} |F(Y_1^s) - F(Y_2^s)| ds . \\ &\leq C||Y_1 - Y_2||\tau . \end{aligned}$$

(v) From (iv) follows that ϕ is a contraction for small τ. Therefore, ϕ has a fixed point in a neighborhood of the constant map $X^0 \in C([0, \tau], C(S, S))$. A fixed point of ϕ provides a locally unique solution of the differential equations.

(vi) Global existence is established with a Gronwall estimate:

Because all forces are globally Lipshitz with some Lipshitz constant C and because

$$g \mapsto \int_{\delta M \times \mathbf{R}^+} n(f(x, y))(n(f(x, y)) \cdot g(x, y)) \, dP^0(x, y)$$

is differentiable with a global Lipshitz constant 1, it follows that the norm of X can not grow faster than exponentially. $\square$

Remark. Actually, we expect a more general result along the same lines, where one has a general smooth d-dimensional manifold M with smooth boundary, where each connected component of the boundary of M can move by translations. A situation where different boundaries can interact is a billiard in which different heavy rigid bodies float inside. A difficulty with such an extended setup is that the topology of M can change during the evolution. Hard-core collisions could be replaced by strong repelling forces however so that the topology of M stays the same. An other extension would be to allow the boundary to rotate, conserving so angular momentum of the gas-container system. Going into a fixed coordinate system introduces Coriolis type forces. Even more ambitious would be to allow the boundary to become deformed continuously. The gas boundary system would then be a coupled system of two infinite-dimensional systems. We expect that there is a global existence theorem in the case of Coulomb interaction, where the Vlasov equation has global existing solutions [8, 9, 11, 5, 13, 4]. The global existence problem for the Vlasov-Poisson dynamics in a bounded domains with fixed boundaries is an open problem.

4 Lyapunov exponent for gas and boundary

The time derivative of the total momentum

$$p + \int_S g(x,y)\, dP^0(x,y) = p + \int_S y\, dP^t(x,y)$$

is the integral of the external forces $\int \nabla W(x)\, dP^t(x,y) + \int_{\mathbf{R}^d} \nabla U(q) dq$ on the gas and the boundary. If $\nabla W + \nabla U = 0$, the total momentum of the system is preserved. The energy

$$\frac{p^2}{2} + \int_S \frac{y^2}{2}\, dP^t(x,y) + \int_{S\times S} V(x-x')\, dP^t(x,y)\, dP^t(x,y) + \int_S W(x)\, dP^t(x,y) + \int_{\mathbf{R}^{2d}} U(x)\, dP^t(x,y)$$

is preserved by the evolution. If $V = 0, W = -U$, the energy consists of the sum of the kinetic energies of the boundary and the gas alone.

At each time, we have a decomposition of the total energy into the energy of the particle and the energy of the boundary. We say that the system has **equipartition of energy**, if the energy of the macroscopic and the microscopic parts averaged over a time interval $[t, t+1]$ are the same in the limit $t \to \infty$. Note that equipartition of energy is not incompatible with convergence to equilibrium: while the wall can come to rest, it can do so by vibrating with higher and higher frequency and smaller and smaller amplitudes.

Denote by $\lambda(x,y)$ the **Lyapunov exponent of the gas**. It is defined as

$$\lambda(x,y) = \limsup_{t\to\infty} \frac{1}{t} \log ||DX^t(x,y)|| \in [0,\infty]$$

(see [7]). It is the usual Lyapunov exponent of a test particle moving in the field generated by the massive particles. With no interaction among the particles (free gas), it is the Lyapunov exponent of the billiard flow in the moving billiard table. If we are additionally in the equilibrium situation, it is the Lyapunov exponent of the usual billiard flow.

We can also consider $\lambda(q,p)$, the **Lyapunov exponent** of the boundary

$$\lambda(q,p) = \limsup_{t\to\infty} \frac{1}{t} \log ||DY^t(q,p)|| \in [0,\infty] \ ,$$

where Y^t is the curve of symplectic maps on R^{2d} mapping $(q^0, p^0) \mapsto (q^t, p^t)$. Both Lyapunov exponents are determined from the initial condition P^0 of the gas and the initial condition (q^0, p^0) of the boundary.

5 Example: The Wojtkowski gas:

In this example of a falling gas, where $M = \mathbf{R}^+$ has one dimensions, we consider noninteracting gas attracted to a floor. The floor is hold up by a constant force determined by total momentum conservation. The gas located above the floor does not interact with itself but it interacts through the floor: the momentum change of the particles depends on the momentum of the floor. The manifold S is obtained by gluing $\delta M \times \mathbf{R}^+$ with $\delta M \times (-\mathbf{R}^+)$ and identifying points (x, y) with $(x, -y)$. In S, the characteristic system is

$$\begin{aligned} \dot{f}(x,y) &= g(x,y) \\ \dot{g}(x,y) &= [-1 + \int_{\{f(x,y)=0\}} |g(x,y)| \, dP^0(x,y)] \, , \end{aligned}$$

where P^0 is an initially smooth density on $M \times \mathbf{R}^+$.

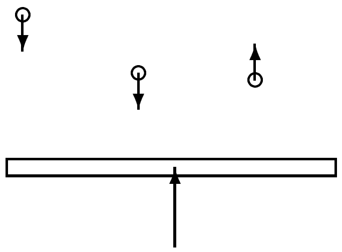

Figure 3. A gas of particles falling onto a floor hold with constant force matching the total weight of the gas. This gas is non-uniformly hyperbolic if P^t is a discrete point measure [14].

We know from Theorem (3.1) that the Vlasov gas of falling balls with smooth initial gas density with bounded momentum has a globally defined dynamics. The floor moves continuously at all times. The Wojtkowski gas is interesting because if P is a finite Dirac measure $\sum_j \delta(x_j, y_j)$, then all Lyapunov exponents are known to be positive [14]. What happens in the case, when P^0 is a smooth measure?

6 Birkhoff-Vlasov billiards

In the second example, the gas is confined to a two-dimensional smooth convex region $M \subset \mathbf{R}^2$. If this region M is fixed, it defines a Birkhoff billiard table. Each particle moves then according to the billiard law and the symplectic flow X^t depends only on the table and is independent of P^0. The dynamics with a moving boundary is already in the case, when the table is a circle or a square, cases in which the dynamics with a fixed boundary are integrable.

If the boundary can move, the total system conserves momentum. We assume no external force nor any inter-particle forces $U = V = W = 0$. The manifold S is four-dimensional. The differential equation is in each local chart of the form

$$\begin{aligned} \dot{f}(x,y) &= g(x,y) \\ \dot{g}(x,y) &= [\int_{f(x,y) \in \delta M} n(f(x,y))(n(f(x,y)) \cdot g(x,y)) \, dP^0(x,y)] \, . \end{aligned}$$

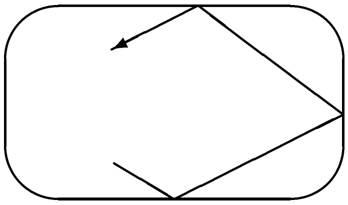

Figure 4. Vlasov billiard. A convex table in which a free gas moves according to the billiard law. The reflections of the particles at the boundary shake the table and the whole system satisfies momentum conservation. If P^0 is a smooth initial density on the phase space S of the gas, the dynamics exists for all times. The boundary moves continuously.

This Birkhoff-Vlasov billiard is a natural generalization of the Birkhoff billiard. The later is the limit, when the boundary has infinite mass. We expect fast convergence $P^t \to P$ to equilibrium in the case of an ergodic billiard like in the case of a Bunimovich billiard, i.e. in a square with circular rounded corners as in Figure 4).

Remarks.
a) If the table and P is symmetric with respect to spatial reflections at the x or y axes, the boundary is not accelerated and each particle moves according to the billiard law. For example, if M is the Bunimovich billiard table and P^0 has these two discrete symmetries, then P^t converges weakly to an equilibrium measure which has constant density $\rho(x) = \int_S P(x,y)\,dy$. The Lyapunov exponent of the Vlasov dynamics is the Lyapunov exponent of the billiard. If the gas is not in equilibrium and the boundary does not move, then from the mixing property of the flow, then P^t still converges weakly to an equilibrium measure. The flow of P^t is then just the push-forward of the measure P^0 under the billiard flow. We expect in general that this happens also because the set $\Sigma^t = \{(x,y) \in S \mid f^t(x,y) \in \delta M\}$ is expected to become more and more dense in the phase space for $t \to \infty$.

b) For some billiards, there are initial conditions P^0, such that the motion is equivalent to a one-dimensional situation. For example, assume that in the Bunimovich billiard the density is located on a hyper-surface in the phase space such that almost all particles are moving perpendicular to two parallel walls. In this case, the motion is equivalent to a one-dimensional Vlasov-billiard in an interval. The dynamics is nontrivial already in this case.

c) There are special initial conditions in which a gas can shake a circular billiard table in such a way that the Lyapunov exponent $\lambda(x,y) = \lim_{t\to\infty} t^{-1} \log ||DX^t||$ becomes zero at some points and positive at others. The example works more generally for a Bunimovich stadium of length $2+a$ and width 1. We allow also $a \in (-2,0)$ in which case the table is lens shaped. A discrete measure P represents two massive particles with the same speed 1 rsp. -1 and mass m traveling along the longest diameter of the table, which has mass M. Their initial condition can be set up so that the table moves with speed v if $t \in [0,1]+4\mathbf{Z}$, with speed 0 if $t \in [1,2] \cup [3,4]+4\mathbf{Z}$ and $-v$ if $t \in [2,3]+4\mathbf{Z}$. If v is not too large, then there are periodic orbits of period 2 whose stability is determined from the corresponding 2-periodic orbit in a Bunimovich stadium of length $2+a-2v/4$ rsp. $2+a+2v/4$. For an open set of a and v's we have $2+a-2v/4 < 2$ and $2+a+2v/4 > 2$. The first orbit is elliptic with zero Lyapunov exponent, the second orbit is hyperbolic with positive Lyapunov exponent.

7 A one-dimensional Vlasov gas

We consider now the example of a convex Vlasov billiard, where convergence of the boundary to equilibrium is plausible. The example is a box, where the initial phase space density P factorizes in each coordinate direction $P(x) = \prod_j P_j(x_j)$. The independence property is invariant under the dynamics and we have a decoupled system of one-dimensional billiards. This one-dimensional billiard is the simplest possible system which fits into our framework.

The initial condition of the gas is given by a probability measure P on $S_1 = [0,1] \times \mathbf{R}$. Assuming that P has compact support means a bound on the initial velocities. By attaching a second copy $S_2 = [-1,0] \times \mathbf{R}$

on which we take the density $P(-x,-y)$ the manifold S is obtained. It is a cylinder $[-1,1] \times \mathbf{R}$ and on which the dynamics of the particles is continuous. The characteristic system is in S given by

$$\dot{f}^t = g^t, \dot{g}^t = F(f^t) \; ,$$

where

$$F(f^t) = 2\left[\int_{\Sigma_+^t(f)} y \; dP^0(x,y) - \int_{\Sigma_-^t(f)} y \; dP^0(x,y)\right] \; ,$$

where $\Sigma_+^t = \{(x,y) \mid f^t(x,y) = 1\}$ and where $\Sigma_-^t = \{(x,y) \mid f^t(x,y) = 0\}$. The set Σ_+^t is the set of points in the phase space S, which are mapped at time t to the right wall. The set Σ_-^t is the set of initial coordinates for particles which are at time t at the left wall. The first term of the force $F(f^t)$ is the change of momentum at the right wall, while the second term is the change of momentum at the left wall. These changes of momenta produce time-dependent forces in a time-independent region. All particles at time t experience the same force $F(f^t)$ in S_1 and $-F(f^t)$ in S_2.

This is the simplest example of a deterministic infinite-dimensional gas dynamics coupled to a macroscopic finite dimensional macroscopic wall system. It is possible a priori that particles can be "Fermi accelerated" to arbitrary large velocities, despite the fact that the total energy and the total momentum stays constant.

For all smooth initial measures P^0 on S with compact support, we expect the position q^t of the wall to go to 0 for $|t| \to \infty$ and that the measure P^t converges weakly to the motion of the billiard gas where the table is at rest.

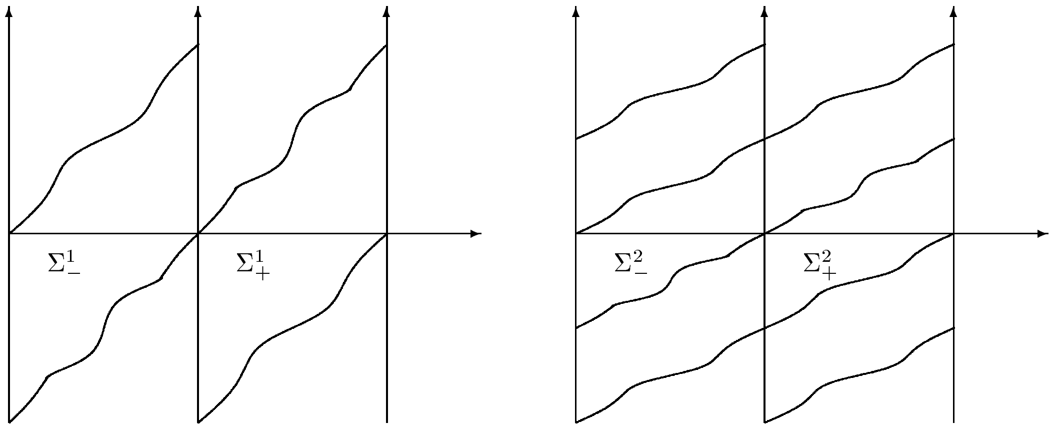

Figure 5. The argument is that the one-dimensional sets Σ_+^t get wrapped around the manifold S (shown here unfolded) tighter and tighter and get more and more dense in every fixed bounded open set. Call P_+^t the push-forward $P_\pm^t$ of the measure $P^0|\Sigma_\pm^0$. It is located on $\Sigma_\pm^t$. If the boundary is hold fixed (i.e. if it has infinite mass), we know that $dP_+^t - dP_-^t$ converges weakly to zero. This is expected to persist when the boundary is released. A fixed point argument or an implicit function argument should be able to show this because the tighter wrapping leads to smaller forces from the boundary acceleration. A proof has still to be done.

The motion of the gas is completely determined by the evolution of the two surfaces $\Sigma_\pm^t$ which satisfy the integro-differential equations

$$\frac{d}{dt}\Sigma_\pm = 2\left[\int_{\Sigma_+^t} y \; dP^0 - \int_{\Sigma_-^t} y \; dP^0\right] \; .$$

This evolution of the geometric object $\Sigma_\pm^t$ is simpler to simulate than the evolution of the measure P^t. For numerical simulations (over short time at least) one would approximate $\Sigma_\pm$ by polygons with finitely many vertices $(x_k^\pm, y_k^\pm)$. Initially, they form the surfaces $\Sigma_\pm^0$. The evolution of the vertices is given by a nonlocal interaction

$$\ddot{x}_j = 2[\sum_k P^0(x_k^+, \dot{x}_k^+) - P^0(x_k^-, \dot{x}_k^-)] \; .$$

The points of the left boundary and of the right boundary behave like particles of opposite charge and their contribution to the force on each of them depends on place and position.

8 A container with a moving wall

In the third example, where $d = 2$, the region M models a two-dimensional container, where a moving wall of some fixed mass separates two regions of a container. Each region is a billiard table if the wall does not move. The gas moves freely and is reflected at the boundaries of the container. Again we assume absence of interaction and external forces: $U = V = W = 0$. This system does not conserve momentum because of the reflections at the two fixed ends of the container and does not quite fit into the setup done for the existence theorem. New in this situation is that the pressure forces are not homogeneous: if the volume is squeezed then the velocities in that region become larger because the map $X^t : [0, q(0)] \times \mathbf{R} \mapsto [0, q(t)] \times \mathbf{R}$ is canonical. The gas becomes therefore "hotter" on the squeezed side and produces larger pressure forces.

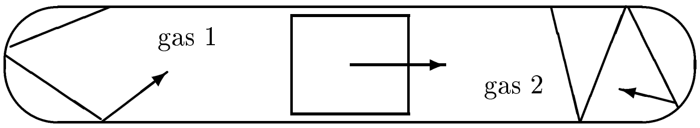

Figure 6. A piston which separates two gas regions. In this situation the particle dynamics on each side is non-uniformly hyperbolic and ergodic when the piston is at rest.

The piston experiences from the left boundary the force $2\int_{f^t=a} g^t(x,y)\ dP^0(x,y) = 2\int_{x=a} y\ dP^t(x,y)$ and similarly from the right boundary the force $2\int_{x=b} y\ dP^t(x,y)$. These forces can become large if the momenta of the particles are large.

Assume the piston is at $[q(t) - a, q(t) + a]$ at the time t. The gas is a billiard flow in a time-dependent chamber on each side. The piston moves according to

$$\ddot{q} = 2[\int_{f_+^t = q(t)-a} g^t(x,y)\ dP^0(x,y) - \int_{f_-^t = q(t)+a} g^t(x,y)\ dP^0(x,y)]\ .$$

Because of energy conservation, the piston position $q(t)$ is confined to a bounded interval and the existence theorem extends to this situation.

9 Sinai-Vlasov billiards

As a final example, we consider a Sinai billiard [12, 3] in which the scatterer K moves on the torus under the pressure forces of a Vlasov gas with continuous density in the phase space $M^t = \mathbf{T} \setminus K^t$.

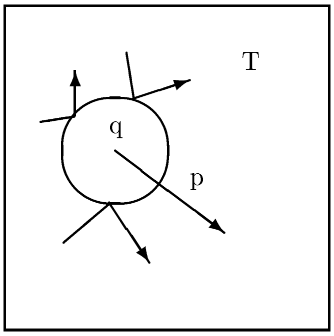

Figure 7. Sinai-Vlasov billiards. A convex obstacle floats in a gas moving in a region with periodic boundary conditions.

In equilibrium, when the obstacle is at rest, the motion of the particles is decoupled and is given by the

Sinai billiard which is known to be a partially hyperbolic ergodic flow. Unlike in the other examples, the boundary can now move with two degrees of freedom.

The manifold S is diffeomorphic to the cotangent bundle of a surface of genus 2, a four dimensional manifold.

A simplified modification is when the obstacle is a thickened homotopically nontrivial straight wall winding around the torus. This example is equivalent to a one-dimensional example of an eddie in a circle which is itself equivalent to the gas in a one dimensional box from the previous section.

10 Discussion

Having established an existence result, questions pop up. Some of these questions might sound too obvious from the physical point of view. They are what one could expect from thermodynamics, heuristic argumentation or numerical experiments. Proving them however could turn out to be rather difficult.

A gas coupled to a macroscopic boundary is **in equilibrium**, if the boundary and the density P are time-independent. P is then an **equilibrium configuration**. In all examples, it is possible to find equilibrium configurations if there is no interaction between the particles.

Question 1 (Convergence to equilibrium):
We expect in all four examples that the macroscopic motion comes to rest $q^t \to 0$, that some partial derivatives of P^t become indefinitely large. We expect that P^t converges weakly to an equilibrium measure if the billiard flow in the chamber is hyperbolic (meaning that all Lyapunov exponents are nonzero almost everywhere). We expect the momentum function $t \in [0,1] \mapsto p^t_{t\in[T,T+1]}$ of the boundary to converge weak-* to a constant function in $L^\infty([0,1])$ for $T \to \infty$. The speed of convergence should depend on the Lyapunov exponent of the billiard in the stationary situation.

Remark. The dynamics is reversible and running the dynamics backwards in time would show the same convergence. For discrete measures, we have a finite dimensional Hamiltonian system and so Poincaré recurrence.

Question 2 (Separation of energy):
We expect in all examples that both $\int g^{T+t}(x,y)^2 \; dP^0(x,y)$ and $(p^{T+t})^2_{t\in[0,1]}$ converges for $T \to \infty$ weakly in $L^\infty([0,1])$ to a constant. The limiting energy of the wall could be positive. This is not in contradiction to weak convergence to equilibrium. The floor can have a fixed nonzero energy (temperature) while having an arbitrary small vibration amplitude. Marek Rychlik has written a program to simulate the Wojtkowski gas numerically with finitely many particles [1].

Remark. Energy conservation says that $\dot{p}^2 + \int g^t(x,y)^2 \; dP^0(x,y) = E$ is constant in the case of zero potential. We know therefore that $|p^t|$ and $||g_t||_{L^2(S)}$ stays bounded above by $2\sqrt{E}$, where E is the total energy of the system. Nothing prevents to grow $||g^t||_{L^\infty}$ indefinitely however.

Question 3 (Growth of fluctuations):
We expect that the force $\dot{p}^t$ on the floor is oscillating faster and faster: $\limsup_{t\to\infty} |\ddot{p}^t| = \infty$. Does the amplitude of the oscillations (the maximal force difference $\limsup_{t\to\infty} \dot{p}^t - \liminf_{t\to\infty} p^t$) stay bounded for $|t| \to \infty$?

Remark. The reason for this question is some particles will experience Fermi acceleration. This is already true if the wall moves in a forced deterministic way, not influenced by the gas and where it is already a nontrivial question, whether particles can be accelerated arbitrarily fast.

Question 4 (Lyapunov exponents as in stationary case):
We expect that the Lyapunov exponent of the gas is the Lyapunov exponent of the billiard in the sta-

[1] Software Balls-1.0, by Marek Rychlik (http://alamos.math.arizona.edu/)

tionary situation, when the boundary is fixed.

Remark. Energy conservation does not exclude the possibility that some of the particles gain more and more speed for $t \to \infty$. It is not impossible a priory that the Lyapunov exponent as we defined it, could become infinite due to super exponential growth of $||DX^t(\omega)||$ at some points $\omega \in S$. An affirmative answer to Question 4) would imply that the Lyapunov exponent is a finite number.

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