# UNIFORM SINGULAR CONTINUOUS SPECTRUM FOR THE PERIOD DOUBLING HAMILTONIAN

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Abstract. We consider the ergodic family of Schrödinger operators generated by the period doubling substitution and we prove that every element of this family has purely singular continuous spectrum.

## 1. Introduction

The discovery of quasicrystals by Shechtman et al. in 1984 [25] has motivated continuing interest by both physicists and mathematicians in adequate models describing these structures. A class of models that has attracted particular attention in this context is provided by one-dimensional Schrödinger operators with potentials generated by so-called primitive substitution sequences, the Fibonacci sequence being the most prominent example. A considerable amount of knowledge about the spectral properties of these operators has since been accumulated, and, from a mathematical point of view, their study is also motivated by the fact that they exhibit rather unusual properties such as purely singular continuous spectral measures which are supported on Cantor sets of Lebesgue measure zero. In particular the apparent tendency of the spectral measures to be always purely singular continuous seems to reflect that substitution sequences give rise to potentials being intermediate between periodic (leading to absolutely continuous spectrum) and random (leading to pure point spectrum). While absence of absolutely continuous spectrum follows in full generality from works of Kotani [21] and Last-Simon [22], the problem of excluding eigenvalues has not yet been solved in similar generality. The spectral theory of substitution Hamiltonians is most conveniently studied within the context of ergodic families of Schrödinger operators since primitive substitution sequences naturally induce strictly ergodic subshifts [24] which serve as a family of potentials associated with a substitution model. One can then employ the powerful general results from this framework; see [8] for the general theory of ergodic Schrödinger operators and [11] for an introduction to one-dimensional quasicrystal models and their spectral theory.

The results on absence of eigenvalues for ergodic families of Schrödinger operators generated by primitive substitutions can be classified, in increasing generality, as *generic* (absence of eigenvalues for a dense $G_\delta$ set of elements of the subshift), *almost sure* (absence of eigenvalues for almost every element with respect to the unique ergodic measure $\mu$ on the subshift), and *uniform results* (absence of eigenvalues for all elements of the subshift). Generic results for certain classes of substitution models were established in [5, 7, 19], almost sure results can be found in [9, 10, 12], and [13, 14] contain uniform results.

Essentially all the known results rely on criteria that deduce absence of $\ell^2$-solutions from the presence of local symmetries, namely, local repetitions [18] or palindromes [19]. This explains why our current understanding of the problem is limited to models that exhibit such symmetries. Explicit models are known (e.g., the Rudin-Shapiro substitution) which give rise to models that do not have the required local symmetries. Moreover, from [18, 19] one can extract three common criteria: the palindrome method of Hof et al. [19] (based on a criterion of Jitomirskaya and Simon [20] which was developed in the context of uniformly almost periodic models) and the two-block [27] and three-block [17] versions of Gordon's method [18]. While the palindrome method is an excellent tool to establish generic results for a large class of primitive substitution models [19], it was shown in [16] that one cannot prove a stronger (i.e., almost sure or uniform) result — its scope is always limited to a set of zero $\mu$-measure; see [1, 3] for other negative results on the (non-)applicability of palindromic criteria. On the other hand, the three-block version of Gordon's method allows for a very simple proof of an almost sure result in the case where the substitution sequence contains a sufficiently high power (slightly more than a third power is enough) [10, 12], whereas the criterion is not sufficient to prove uniform results [12] — there are always elements in the subshift to which one cannot apply the three-block criterion. The only uniform results that are known so far have therefore been established by the two-block version of Gordon. However, the two-block method requires an additional input, namely, sufficient control on a dynamical system (the so-called trace map) that is naturally associated with a substitution model. Sufficient control essentially means that one has to establish boundedness of its orbits (for energies from the spectrum). Unfortunately, such a strong result is known only for a very small subclass of substitution models, namely, those of minimal combinatorial complexity, that is, models which are *Sturmian* (e.g., the Fibonacci case). This indicates that it might be hard to establish uniform results outside this small subclass.

Our goal here is to establish uniform absence of eigenvalues for a prominent model, the family of Schrödinger operators generated by the period doubling substitution (see, e.g., [4, 9]), which indeed lies outside this small subclass and for which the strong trace map result does not hold [6]. Namely, on the alphabet $A = \{a, b\}$, consider the *period doubling substitution* $S(a) = ab$, $S(b) = aa$. Iterating on $a$, we obtain a one-sided sequence $u = abaaabab\ldots$ which is invariant under the substitution process. Define the associated *subshift* $\Omega$ to be the set of all two-sided sequences which have all their finite subblocks occurring in $u$. Choose some non-constant function $f : A \to \mathbb{R}$ and define for $\omega \in \Omega$, a discrete one-dimensional Schrödinger operator $H_\omega$, acting in $\ell^2(\mathbb{Z})$, by

$$(H_\omega \psi)(n) = \psi(n+1) + \psi(n-1) + f(\omega_n)\psi(n).$$

We will prove the following theorem.

**Theorem 1.** *For every $\omega \in \Omega$, the operator $H_\omega$ has empty point spectrum.*

*Remarks.* (a) Note that the result is valid for all replacement functions $f$, that is, it is robust with respect to variation of a potential coupling constant. This is a general phenomenon in the spectral theory of one-dimensional substitution Hamiltonians which stems from the fact that the proofs are mostly combinatorial.

(b) If we combine Theorem 1 with the results of Kotani [21] and Last-Simon [22], we get that for every $\omega \in \Omega$, the operator $H_\omega$ has purely singular continuous spectrum.

(c) Theorem 1 extends [9] where purely singular continuous spectrum was established for almost all $\omega \in \Omega$ with respect to the unique ergodic measure $\mu$ on $\Omega$ (see also [10]).

(d) Our proof of Theorem 1 uses a combination of the two-block and three-block versions of Gordon's criterion along with partitions of the elements in the hull $\Omega$ and results for the trace map obtained by Bellissard et al. [4].

After recalling some more or less known concepts and results in Section 2, we give a proof of Theorem 1 in Section 3. Since the result is somewhat surprising, by virtue of our discussion preceding the statement of the theorem, we also discuss in Section 3 to what extent the approach of the present paper is likely to apply to other substitution models.

## 2. The Trace Map, Partitions, and Gordon's Criterion

In this section we recall some useful results and methods that will be used in our proof of Theorem 1. Among these are the trace map, a dynamical system which is directly induced by the substitution rule, partitions of the elements of the hull into products of canonical words, and criteria for absence of eigenvalues of general one-dimensional Schrödinger operators which are based on Gordon's work [18].

Let us first recall that the sequence $u$ can be regarded as a limit of a sequence of words $s_n$ which obey recursive relations. Namely, with $s_n = S^n(a)$ we have with obvious notation and meaning, $u = \lim_{n\to\infty} s_n$. Moreover, the words $s_n$ obey the recursion $s_n = s_{n-1}s_{n-2}^2$. More transparently, we have with $t_n = S^n(b)$

$$s_n = s_{n-1}t_{n-1},\ t_n = s_{n-1}s_{n-1}. \tag{1}$$

Notice that $s_n$ and $t_n$ both have length $m = 2^n$. Moreover, the words $s_n$ and $t_n$ are almost identical [10]:

**Proposition 2.1.** *For every $n \in \mathbb{N}$, the words $s_n$ and $t_n$ are the same except for their respective rightmost symbol.*

For $s_n = u_1 \dots u_m$ and $t_n = v_1 \dots v_m$ with $u_i, v_i \in A$ and $E \in \mathbb{R}$, we define the matrices $M_n = M_n(E)$ and $N_n = N_n(E)$ by

$$M_n = T(E, u_m) \times \cdots \times T(E, u_1),\ N_n = T(E, v_m) \times \cdots \times T(E, v_1),$$

where for $c \in A$ and $E \in \mathbb{R}$,

$$T(E, c) = \begin{pmatrix} E - f(c) & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $x_n = x_n(E) = \mathrm{tr}(M_n)$ and $y_n = y_n(E) = \mathrm{tr}(N_n)$. Bellissard et al. derived in [4] the recursion

$$x_n = x_{n-1}y_{n-1} - 2,\ y_n = x_{n-1}^2 - 2 \tag{2}$$

which is called the *trace map* (cf. [2]). This dynamical system on $\mathbb{R}^2$ is the central tool in the investigation of the spectral properties of the operators $H_\omega$. Trace maps are induced by all substitutions (see [7] and references therein) and their study in this context is natural and very useful.

It is a standard result that there is a compact set $\Sigma \subseteq \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$. This follows essentially from the minimality of $\Omega$ which results from the fact that $u$ is uniformly recurrent, that is, every finite subblock of $u$ occurs in $u$ infinitely often and with bounded gaps. It follows from the analysis of the trace map performed by Bellissard et al. in [4] that for every $E \in \Sigma$, we have the following: If $|x_n(E)| > 2$ for some $n$, then $|x_{n+1}(E)| \le 2$. We can therefore state the following proposition.

**Proposition 2.2.** *For every $E \in \Sigma$ and every $n \in \mathbb{N}$, we have*

$$\min\{|x_n(E)|, |x_{n+1}(E)|\} \le 2.$$

The next crucial concept we want to recall is the fact that for every $n$, every $\omega \in \Omega$ can be uniquely decomposed into an infinite product of blocks of the form $s_n$ or $t_n$. Let us call this decomposition the *$n$-partition* of $\omega$. We summarize the properties we shall need in the following proposition.

**Proposition 2.3.** *For every $n$, every $\omega \in \Omega$ has a unique $n$-partition. In this product representation, a $t_n$-block is always isolated, and between two consecutive $t_n$-blocks there are either one or three $s_n$-blocks.*

*Proof.* By definition, $u$ can be written as a product of blocks of the form $s_0$ and $t_0$. Moreover, by the self-similarity property $S(u) = u$, we have, for every $n \in \mathbb{N}$, an analogous decomposition into blocks of the form $s_n$ and $t_n$. It is easily checked for $u$ that $t_n$-blocks are isolated and that $s_n$-blocks have multiplicity either one or three. It is then a result of [23] (see [26] for an extension to higher dimensions) that these properties are inherited by the subshift elements $\omega \in \Omega$ and that their canonical decompositions are in fact unique (this follows from aperiodicity of $u$). □

Finally, we discuss Gordon-type criteria which establish a link between combinatorial properties of the sequences $\omega \in \Omega$ and non-decay properties of the solutions to

$$(H_\omega - E)\phi = 0. \tag{3}$$

We do not provide proofs and refer the reader to [11, 17, 27] for proofs, discussions, and applications. Fix some $\omega \in \Omega$ and some $E \in \mathbb{R}$. Let $\phi$ be a two-sided sequence that solves (3) and obeys the normalization condition

$$|\phi(-1)|^2 + |\phi(0)|^2 = 1. \tag{4}$$

Denote $\Phi(n) = (\phi(n), \phi(n-1))^T$. Then we have the following proposition.

**Proposition 2.4.** (a) *If for some $m \in \mathbb{N}$, we have $\omega_{-m+j} = \omega_j = \omega_{m+j}$, $0 \le j \le m-1$, then*

$$\max\left(\|\Phi(-m)\|, \|\Phi(m)\|, \|\Phi(2m)\|\right) \ge \tfrac{1}{2}.$$

(b) *If for some* $m = 2^n \in \mathbb{N}$, *we have that* $\omega_0 \dots \omega_{2m-1}$ *is a cyclic permutation of* $s_n s_n$, *then*

$$\max\left(|x_n(E)| \cdot \|\Phi(m)\|, \|\Phi(2m)\|\right) \geq \tfrac{1}{2}.$$

*Analogous conclusions hold if the assumptions in* (a) *and* (b) *are reflected at the origin.*

We see that we obtain useful estimates for the solutions $\phi$ of (3) if we exhibit appropriate squares and cubes in $\omega$.

## 3. The Proof of Theorem 1

Let us turn to the proof of Theorem 1. Fix $\omega \in \Omega$, $E \in \Sigma$, and a solution $\phi$ to (3) obeying (4). We want to prove that $\phi$ is not square-summable. We shall show that given any $k \in \mathbb{N}$, there exists $m \in \mathbb{Z}$ with $|m| \geq k$ such that $\|\Phi(m)\| \geq \frac{1}{4}$. From this the assertion clearly follows.

So let $k \in \mathbb{N}$ be fixed and pick $n \in \mathbb{N}$ such that $2^n \geq k$. Consider the $n$-partition of $\omega$.

*Case 1: The site* $0 \in \mathbb{Z}$ *is contained in an* $s_n$*-block and this* $s_n$*-block is followed to the right by an* $s_n$*-block.* Because of Proposition 2.3 there are two subcases.

*Case 1.1: The* $n$*-partition looks at the origin locally like* $t_n s_n \hat{s}_n s_n t_n$. Here and in the following, the hat-symbol marks the block that contains the site $0 \in \mathbb{Z}$. Because of Proposition 2.1 we can conclude by applying Proposition 2.4 (a) with $m = 2^n$.

*Case 1.2: We have* $t_n \hat{s}_n s_n s_n t_n$. Then either $|x_n(E)| \leq 2$, and we are done in this case by Proposition 2.4 (b), or $|x_n(E)| > 2$ and then $|x_{n+1}(E)| \leq 2$ by Proposition 2.2. Let us therefore consider the $(n+1)$-partition where we must have $s_{n+1} \hat{t}_{n+1} s_{n+1}$. Note that the origin is not the rightmost site in $\hat{t}_{n+1}$.

*Case 1.2.1: We have* $s_{n+1} s_{n+1} \hat{t}_{n+1} s_{n+1}$. In this case we can conclude immediately by applying Proposition 2.1 and Proposition 2.4 (b) (reflected at the origin) because we have $|x_{n+1}(E)| \leq 2$.

*Case 1.2.2: We have* $t_{n+1} s_{n+1} \hat{t}_{n+1} s_{n+1}$. We pass to the $(n+2)$-partition where we must have $s_{n+2} \hat{s}_{n+2}$.

*Case 1.2.2.1: We have* $s_{n+2} s_{n+2} \hat{s}_{n+2} t_{n+2}$. In this case we can apply Proposition 2.4 (a) (reflected at the origin) with $m = -2^{n+2}$ using Proposition 2.1.

*Case 1.2.2.2: We have* $t_{n+2} s_{n+2} \hat{s}_{n+2} s_{n+2} t_{n+2}$. Again using Proposition 2.1 we can apply Proposition 2.4 (a) with $m = 2^{n+2}$. This closes Case 1.

*Case 2: In the* $n$*-partition we have* $s_n \hat{s}_n$. This case can be treated analogously to Case 1.2.2. This closes Case 2.

*Case 3: In the* $n$*-partition we have* $t_n \hat{s}_n t_n$. We pass to the $(n+1)$-partition where we must have $s_{n+1} \hat{s}_{n+1}$ and we can then proceed analogously to Case 1.2.2. This closes Case 3.

*Case 4: In the* $n$*-partition we have* $\hat{t}_n$. Thus in the $(n+1)$-partition we have $\hat{s}_{n+1}$ and we are in one of the Cases 1–3 with all indices increased by one. In particular, we obtain a sufficient solution estimate. This closes Case 4 and hence concludes the proof. □.

*Remark.* Let us briefly discuss the result of this paper and the obstacles one encounters when one tries to tackle related questions. We have shown that despite the absence of a uniform trace map bound, which was a crucial tool in the corresponding proof in the Fibonacci (and, more generally, Sturmian) case [14], we can nevertheless prove absence of eigenvalues for all elements in the hull. This raises two questions: Can we carry over some of the other uniform results that were proven in the Fibonacci case [13, 15], all of which relied heavily on the uniform trace map bound as well, and is it feasible that we can prove uniform absence of eigenvalues for non-Sturmian substitution models other than the period doubling substitution?

(a) Virtually all further results in [13, 15], in particular $\alpha$-continuity (i.e., continuity with respect to $\alpha$-dimensional Hausdorff measure) of spectral measures, require power-law upper bounds on the solutions to the difference equation (3) for energies $E$ in the spectrum. A proof of this property, uniformly in the energy, seems to be out of reach in the period doubling case since one only has the weak trace map bound

$$|x_n(E)| \leq c \exp(d\gamma^n) \tag{5}$$

for $E \in \Sigma$, where $\gamma$ is any number greater than $\sqrt{2}$ and $c, d$ are constants depending on $\gamma$ [4]. While this bound is sufficient to prove that the Lyapunov exponent vanishes on the spectrum (see [4]), it is clearly insufficient to prove power-law upper bounds on the solutions. One may try to improve this bound within certain energy ranges in order to study, for example, local $\alpha$-continuity, but this requires a more detailed understanding of the trace map dynamics in the period doubling case.

(b) While [10, 12] established purely singular continuous spectrum for almost all elements in the hull, the three-block version of the Gordon argument used there is not capable of proving uniform results [12]. The proofs of uniform results in [14] and the present paper therefore made essential use of the two-block version of Gordon along with suitable trace map bounds. Since boundedness of trace map orbits for energies from the spectrum is only known for Sturmian models, any uniform result outside the class of Sturmian models has to be considered somewhat surprising. What came to our rescue in the period doubling case, beside the weak uniform bound given in Proposition 2.2, is essentially Proposition 2.1 which says that for every $n$, any element in $\Omega$ is almost $2^n$-periodic with the only "defects" being the rightmost symbols in the $t_n$-blocks in the $n$-partition. Such a property is of course a very special feature of the period doubling case (one can certainly construct other examples with this property, such as $a \mapsto a^k b$, $b \mapsto a^k a$ for $k \in \mathbb{N}$, but this is not really the point), so that it is currently not obvious how to extend the result of this paper to other non-Sturmian models.

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