The logarithmic Sobolev constant of Kawasaki dynamics under a mixing condition revisited

N. Cancrini [1], F. Martinelli [2] C. Roberto [3]

[1] Dipartimento di Energetica, Università dell'Aquila, Italy and INFM Unità di Roma "La Sapienza"
e-mail: nicoletta.cancrini@roma1.infn.it

[2] Dipartimento di Matematica, Università di Roma Tre, Italy
e-mail: martin@mat.uniroma3.it

[3] Département de Mathématiques, Laboratoire de Statistique et Probabilités, Université Paul–Sabatier, Toulouse, France
e-mail: roberto@cict.fr

Abstract. We consider a conservative stochastic spin exchange dynamics reversible with respect to the canonical Gibbs measure of a lattice gas model. We assume that the corresponding grand canonical measure satisfies a suitable strong mixing condition. Following previous work by two of us for the spectral gap, we provide an alternative and quite natural, from the physical point of view, proof of the well known result of Yau stating that the logarithmic Sobolev constant in a box of side L grows like L^2.

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1. Introduction

In this paper we provide a new, independent proof of the famous result of Yau [Y] which states that the logarithmic Sobolev constant of a spin exchange dynamics in a box of side L of $\mathbb{Z}^d$, reversible w.r.t. the canonical Gibbs measure of a finite range lattice gas, grows like L^2, provided that the corresponding grand canonical Gibbs measure satisfies a suitable "high temperature" condition. We thus complete the program that was begun in [CM2] were a similar scaling law was proved for the inverse spectral gap.

The problem of computing the relaxation time of stochastic Monte Carlo algorithms for models of classical spin systems in $\mathbb{Z}^d$ has attracted in the last years considerable attention and many new rigorous techniques have been developed giving rise to nice progresses in probability theory and statistical mechanics. If, for simplicity, we confine ourselves to ± 1 (or 0–1 in the lattice gas picture) spins, the two most studied random dynamics have been non– conservative Glauber type algorithms, in which a spin at a time flips its value with a rate satisfying the detailed balance condition w.r.t. the grand canonical Gibbs measure, and conservative Kawasaki dynamics in which nearest neighbors spins exchange their values with a rate satisfying the detailed balance condition w.r.t. the canonical Gibbs measure.

For Glauber dynamics the general picture is relatively clear for a wide class of models both in the one phase and in the phase coexistence region with the notable exception of the critical point (see e.g. [M] and references therein).

For Kawasaki dynamics, instead, the presence of a conservation law makes the analysis of the relaxation properties much more difficult than in the non–conservative case, with interesting analogies with the problem of the Goldstone mode in quantum mechanics [A], and many interesting questions are still open both in the one phase region as well as in the presence of phase coexistence (see [CCM]).

One important class of results for Kawasaki dynamics concerns the relaxation behaviour under a suitable "high temperature condition". Such a condition (see section 2.4 below for a precise description) requires in particular the exponential decay of the grand canonical covariances *uniformly* in the chemical potential, *i.e.* in the particle density, and therefore it cannot be true at low temperature where, in general, phase coexistence takes place for certain values of the density.

Under the validity of such a condition, the most important result is the so called diffusive scaling L^2 for the relaxation time (in what follows identified with the inverse of the spectral gap of the generator) of the Kawasaki dynamics in a box of side L, first proved in [LY] and then, much later and by different methods, in [CM2] (see also [CM3] for an extension to a spin system with random interactions). Such a result is a key stone in the study of the hydrodynamical limit of the Ising model [VY] and it plays an essential role in the proof of the power law L^2–decay to equilibrium of local observables (see [CM2]). Quite interesting its proof poses new, non trivial problems in the theory of canonical Gibbs measures, their large deviations properties and their detailed equivalence to grand canonical ones (see [BZ1], [BZ2], [CM1], [BCO] and [CZ]). Some time after the basic paper [LY], the above result was considerably strengthen in [Y] in that the same diffusive scaling was proved also for the logarithmic Sobolev constant (see also [LeY], [Y1] for related work).

The basic strategy of [Y] is the martingale approach developed in [LY] together with entropy

bounds, large deviations estimate for canonical Gibbs measure and partial averaging on mesoscopic scales. Unfortunately the paper is particularily intricate to read, partly because the problem of equivalence of ensembles is entangled with the analysis of the dynamics, and, as a consequence, the main physical mechanism behind the diffusive scaling does not appear clearly.
With this motivation in mind and with the desire to understand the result by our own methods, we decided to reprove it following the "bisection" technique envisaged in [CM2].

Our method works as follows. Let $c(L)$ be the largest (over the boundary conditions and number of particles) among the logarithmic Sobolev constants in a cube of side L with given boundary conditions and fixed number of particles. The real hard part is to prove an upper bound for $c(L)$ of the right order; the lower bound is readily obtained by plugging into the logarithmic Sobolev inequality a suitable test function (a slowly varying function of the local density). In order to prove the correct upper bound we look for a recursive inequality of the form

$$c(2L) \leq \frac{3}{2}c(L) + kL^2 \tag{1.1}$$

which, upon iteration, proves the bound $c(L) \leq k'L^2$.
For this purpose, let Λ be the cube of side $2L$ and let us divide it into two (almost) halves Λ_1, Λ_2 in such a way that the overlap between Λ_1 and Λ_2 is a thin layer of width δL, $\delta \ll 1$. Let us denote by ν the canonical Gibbs measure on Λ with some given number of particles and let $\mathrm{Ent}_\nu(f^2) = \nu\left(f^2 \log\left(\frac{f^2}{\nu(f^2)}\right)\right)$ be the entropy of f^2 w.r.t ν. If the two σ–algebras $\mathcal{F}_1 := \mathcal{F}_{\Lambda_1^c}$ and $\mathcal{F}_2 := \mathcal{F}_{\Lambda_2^c}$, namely the σ–algebras generated by the lattice gas variables outside Λ_1 and Λ_2 respectively, were *weakly dependent* in the sense that for some $\epsilon(L) \ll 1$

$$\|\nu(g \,|\, \mathcal{F}_2) - \nu(g)\|_\infty \leq \epsilon(L)\nu(g) \tag{1.2}$$

for all non–negative functions g measurable w.r.t. $\mathcal{F}_1$ (weak dependence on the boundary conditions), then it would follow (see [Ce] and section 3.3 below for more details) that (*almost factorization of the entropy*)

$$\mathrm{Ent}_\nu(f^2) \leq (1 + \epsilon(L))\nu\big(\mathrm{Ent}_\nu(f^2 \,|\, \mathcal{F}_1) + \mathrm{Ent}_\nu(f^2 \,|\, \mathcal{F}_2)\big)$$

where $\mathrm{Ent}_\nu(f^2 \,|\, \mathcal{F}_1)$ is a shorthand notation for the entropy of f^2 w.r.t. $\nu(\cdot \,|\, \mathcal{F}_1)$. Notice that $\nu(\cdot \,|\, \mathcal{F}_1)$ is nothing but the canonical measure on the smaller set Λ_1.
If the canonical measure were replaced by the grand canonical one then, under our mixing condition, (1.2) would follow immediately with $\epsilon(L) = O(e^{-m\delta L})$ for some positive m and few lines more would suffice to prove uniformly boundedness of the logarithmic Sobolev constant for a "high temperature" Glauber dynamics [Ce] (see also [M] for a similar discussion for the spectral gap).
For the canonical measure instead, (1.2) cannot hold precisely because of the conservation of the number of particles. Even in the absence of any interaction, the Kawasaki dynamics on two nearby disjoint sets does not factorize into two independent dynamics because the particles may migrate from one set to the other one. In particular, the relaxation time in Λ (intuitively proportional to $c(L)$) is related to the relaxation time of the modified Kawasaki dynamics in which the number of

particles in the three sets Λ_1, Λ_2 and $\Lambda_1 \cap \Lambda_2$ is conserved *and* to the relaxation time of the process of exchange of particles between Λ_1, Λ_2 and $\Lambda_1 \cap \Lambda_2$. This suggests to try to separate the two effects which are, apriori, strongly interlaced and to analyze them separately. In some sense this idea is the heart of our approach and technically it can be achieved by elementary conditioning as follows. Let n_0 and n_1 be the random variables counting the number of particles in $\Lambda_1 \cap \Lambda_2$ and in $\Lambda \setminus \Lambda_2$ respectively and let $\mathrm{Ent}_\nu(f^2 \mid n_0, n_1)$ be the entropy of f^2 w.r.t canonical measure ν conditioned on n_0, n_1. Then we can write

$$\mathrm{Ent}_\nu(f^2) = \nu\big(\mathrm{Ent}_\nu(f^2 \mid n_0, n_1)\big) + \mathrm{Ent}_\nu\big(\nu(f^2 \mid n_0, n_1)\big) \tag{1.3}$$

The second term in (1.3) can in turn be expanded as

$$\mathrm{Ent}_\nu\big(\nu(f^2 \mid n_0, n_1)\big) = \nu\Big(\mathrm{Ent}_\nu\big(\nu(f^2 \mid n_0, n_1) \mid n_0\big)\Big) + \mathrm{Ent}_\nu\big(\nu(f^2 \mid n_0)\big) \tag{1.4}$$

Notice that in the first term in the r.h.s of (1.3) we need to bound the entropy with respect a *multi canonical* measure in which the number of particles in each atom of the partition $\{\Lambda \setminus \Lambda_1, \Lambda_1 \cap \Lambda_2, \Lambda \setminus \Lambda_2\}$ is frozen. As shown in [CM1] (see also section 3.2 below) such a new measure has better chances to satisfy the "weak dependence" condition (1.2) than the original measure ν precisely because of the extra conservation laws. Thus, by the previous reasoning, we may hope to bound the first term in the r.h.s of (1.3) by the largest among the logarithmic Sobolev constant of each of the three sets times the Dirichlet form of the Kawasaki dynamics. Notice that for each of the three sets the linear dimension in one direction has been (at least) almost halved. Thus the first term in the r.h.s. of (1.3) should be the responsible for the first term in the r.h.s of (1.1).

Let us now examine the pieces that come from the second term in the r.h.s. of (1.3). As one can observe in (1.4), in each of them one has to bound an entropy with respect to the distribution of a one dimensional discrete random variable e.g. the number of particles n_0 in the second one. Although such a distribution is difficult to compute exactly, one has a sufficiently good control to be able to establish, via Hardy inequality (see [Mi] and [An]), a sharp logarithmic Sobolev inequality with respect to the Dirichlet form of a reversible Metropolis birth and death process. Physically such a process corresponds to the creation of an extra particle in e.g. $\Lambda_1 \cap \Lambda_2$ and the contemporary annihilation of a particle in e.g. $\Lambda \setminus \Lambda_1$ that is to the *exchange* of particles among the three sets. Since each particle moves, essentially, by a sort of perturbed random walk, and on average it has to travel a distance $O(L)$, it is not surprising that the second term in the r.h.s. of (1.3) is the responsible for the L^2 term in the r.h.s. of (1.2).
Of course and unfortunately, the story is quite involved and we have deliberately hidden here several technical extra conditions that would have obscured the whole discussion without adding any relevant information.

We conclude with some technical comments and a short road map of the paper.
In section 4 and 5, the real technical core of the paper, we need to bound from above canonical (and multi canonical) Laplace transform of the form $\nu_\Lambda\left(\exp(t\sum_{x\in\Lambda} g_x)\right)$, where g_x is the translated by x of a local function around the origin. In principle that could be a difficult task because, due to

the conservation law, canonical measures do not have good mixing properties. Instead of entering into the analysis of the large deviations properties of ν_Λ, we use the simple, but quite useful bound (see [CM2]),

$$\nu_\Lambda\left(e^{t\sum_{x\in\Lambda} g_x}\right) \le A\mu_\Lambda\left(e^{t\sum_{x\in\Lambda} g_x}\right)$$

where A is some universal constant and μ_Λ the grand canonical Gibbs measure. Such a bound, valid under our mixing condition, simplifies enormously the problem (compare for instance with [Y]) because grand canonical Laplace transforms can be easily bounded using concentration inequalities and logarithmic Sobolev inequalities w.r.t. Glauber dynamics.

Road map:

- In section 2 we define the setting, the mixing condition we need and we state the main result.
- In section 3 we collect several technical results, based on the bounds obtained in section 5 and in the appendix, that will be needed later on. A detailed description of these results is given at the beginning of the section.
- In section 4 we study in some detail grand canonical and canonical Laplace transforms via the so called Herbst's approach. This section is the key technical input for the next one.
- In section 5 we bound the square of covariances of the form $\nu_\Lambda(f^2, \sum_{x\in\Lambda} g_x)$, mainly via the entropy inequality and the bounds obtained in section 4.
- In section 6 we finally prove recursively the main theorem on the diffusive scaling of the logarithmic Sobolev constant.
- In the appendix we discuss the logarithmic Sobolev inequality for one dimensional random walks via Hardy's inequalities with some application to our problem.

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2. Notation and results

In this section we first define the setting in which we will work (spin model, Gibbs measure, dynamics), then we define the basic mixing condition on the Gibbs measure and subsequently state the main theorem on this work.

2.1 The lattice and the configuration space

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = (x_1, \dots, x_d)$ and norms

$$|x|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p} \quad p \geq 1 \quad \text{and} \qquad |x| = |x|_\infty = \max_{i \in \{1,\dots,d\}} |x_i| \,.$$

The associated distance functions are denoted by $d_p(\cdot,\cdot)$ and $d(\cdot,\cdot)$. By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball (w.r.t $d(\cdot,\cdot)$) of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \dots, -L))$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. $[x, y]$ is the *closed segment* with endpoints x and y. The *edges* of $\mathbb{Z}^d$ are those $e = [x, y]$ with x, y nearest neighbors in $\mathbb{Z}^d$. We denote by $\mathcal{E}_\Lambda$ the set of all edges such that both endpoints are in Λ.

Given $\Lambda \subset \mathbb{Z}^d$ we define its interior and exterior boundaries as respectively, $\partial^- \Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq 1\}$ and $\partial^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq 1\}$, and more generally we define the boundaries of width n as $\partial_n^- \Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq n\}$, $\partial_n^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq n\}$.

Regular sets. A finite subset Λ of $\mathbb{Z}^d$ is said to be *l–regular*, $l \in \mathbb{Z}_+$, if Λ is the union of a finite number of cubes $Q_l(x^i)$ where $x^i \in l\mathbb{Z}^d$. We denote the class of all such sets by $\mathbb{F}_l$. Notice that any set is 1–regular *i.e.* $\mathbb{F}_{l=1} = \mathbb{F}$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{0, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \tau_V$ is the configuration on $U \cup V$ which is equal to σ on U and τ on V. Given $V \in \mathbb{F}$ we define the *number of particles* $N_V : \Omega \mapsto \mathbb{N}$ as

$$N_V(\sigma) = \sum_{x \in V} \sigma(x) \tag{2.1}$$

while the *density* is given by $\rho_V = N_V / |V|$.

If f is a function on Ω, Δ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Δ_f}. f is called *local* if Δ_f is finite. The *l–support* of a function $\Delta_f^{(l)}$, $l \in \mathbb{Z}_+$, is the smallest l–regular set V such that $\Delta_f \subset V$. $\mathcal{F}_\Lambda$ stands for the σ–algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{0, 1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x. Finally $\mathrm{Osc}(f) = \sup_{\sigma,\eta} |f(\sigma) - f(\eta)|$.

2.2 The interaction and the Gibbs Measures.

Definition 2.1. *A finite range, translation–invariant potential $\{\Phi_\Lambda\}_{\Lambda\in\mathbb{F}}$ is a collection of real, local functions on Ω with the following properties*

1. *$\Phi_\Lambda(\sigma) = \Phi_{\Lambda+x}(\theta_x\sigma$ for all σ, all $\Lambda \in \mathbb{F}$ and all $x \in \mathbb{Z}^d$, where θ_x denotes the shift operator by $x \in \mathbb{Z}^d$.*
2. *For each Λ the support of Φ_Λ coincides with Λ*
3. *There exists $r > 0$ such that $\Phi_\Lambda = 0$ if $\mathrm{diam}\,\Lambda > r$. r is called the range of the interaction.*
4. *$\|\Phi\| := \sum_{\Lambda\ni 0} \|\Phi_\Lambda\|_\infty < \infty$*

Given a collection of real numbers $\underline{\lambda} = \{\lambda_x\}_{x\in\mathbb{Z}^d}$ and a *potential* Φ, we define $\Phi^{\underline{\lambda}}$ as

$$\Phi^{\underline{\lambda}}_\Lambda(\sigma) = \begin{cases} (h+\lambda_x)\sigma(x) & \text{if } \Lambda = \{x\} \\ \Phi_\Lambda(\sigma) & \text{otherwise} \end{cases}$$

where h is the chemical potential (one body part of Φ).
Given a *potential* Φ ($\Phi^{\underline{\lambda}}$) and $V \in \mathbb{F}$, we define the Hamiltonian $H_V^\Phi : \Omega \mapsto \mathbb{R}$ by

$$H_V^\Phi(\sigma) = -\sum_{\Lambda:\,\Lambda\cap V\neq\emptyset} \Phi_\Lambda(\sigma)$$

For $\sigma, \tau \in \Omega$ we also let $H_V^{\Phi,\tau}(\sigma) = H_V^\Phi(\sigma_V\tau_{V^c})$ and τ is called the *boundary condition*. For each $V \in \mathbb{F}$, $\tau \in \Omega$ the (finite volume) conditional Gibbs measure on $(\Omega, \mathcal{F})$, are given by

$$d\mu_V^{\Phi,\tau}(\sigma) = \begin{cases} \left(Z_V^{\Phi,\tau}\right)^{-1} \exp[\,-H_V^{\Phi,\tau}(\sigma)\,] & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

where $Z_V^{\Phi,\tau}$ is the proper normalization factor called partition function. Notice that in (2.2) we have absorbed in the interaction Φ the usual inverse temperature factor β in front of the Hamiltonian. In most notation we will drop the superscript Φ if that does not generate confusion. Moreover, whenever we consider $\Phi^{\underline{\lambda}}$ instead of Φ, we will write $H_V^{\tau,\underline{\lambda}}$ for the finite volume Hamiltonian and $\mu_V^{\tau,\underline{\lambda}}$ for the corresponding finite volume Gibbs measure.

Given a measurable bounded function f on Ω, $\mu_V(f)$ denotes the *function* $\sigma \mapsto \mu_V^\sigma(f)$ where $\mu_V^\sigma(f)$ is just the average of f w.r.t. μ_V^σ. Analogously, for any event X, $\mu_V^\tau(X) := \mu_V^\tau(\mathbb{I}_X)$, where

$\mathbb{1}_X$ is the characteristic function of X. $\mu_V^\tau(f,g)$ stands for the covariance or *truncated correlation* (with respect to μ_V^τ) of f and g. The set of measures (2.2) satisfies the DLR compatibility conditions

$$\mu_\Lambda^\tau(\mu_V(X)) = \mu_\Lambda^\tau(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \subset \Lambda \subset\subset \mathbb{Z}^d \tag{2.3}$$

Definition 2.2. *A probability measure μ on $(\Omega, \mathcal{F})$ is called a Gibbs measure for Φ if*

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F} \tag{2.4}$$

see e.g. [G].

We introduce the *canonical Gibbs measures* on $(\Omega, \mathcal{F})$ defined as

$$\nu_{\Lambda,N}^\tau = \mu_\Lambda^\tau(\cdot \mid N_\Lambda = N) \qquad N \in \{0, 1, \ldots, |\Lambda|\} \tag{2.5}$$

2.3 The dynamics

We consider the so–called Kawasaki dynamics in which particles (spins with $\sigma(x) = +1$) can jump to nearest neighbor empty ($\sigma(x) = 0$) locations, keeping the total number of particles constant. For $\sigma \in \Omega$, let σ^{xy} be the configuration obtained from σ by exchanging the spins $\sigma(x)$ and $\sigma(y)$. Let $t_{xy}\sigma = \sigma^{xy}$ and define $(T_{xy}f)(\sigma) = f(t_{xy}\sigma)$. The stochastic dynamics we want to study is determined by the Markov generators L_V, $V \subset\subset \mathbb{Z}^d$, defined by

$$(L_V f)(\sigma) = \sum_{[x,y] \in \mathcal{E}_V} c_{xy}(\sigma)\,(\nabla_{xy} f)(\sigma) \qquad \sigma \in \Omega\,, \quad f : \Omega \mapsto \mathbb{R} \tag{2.6}$$

where $\nabla_{xy} = T_{xy} - \mathbb{1}$. The nonnegative real quantities $c_{xy}(\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Finite range.* $c_{xy}(\sigma)$ depends only on the spins $\sigma(z)$ with $d(\{x,y\}, z) \le r$

(2) *Detailed balance.* For all $\sigma \in \Omega$ and $[x,y] \in \mathcal{E}_{\mathbb{Z}^d}$

$$\exp\bigl[-H_{\{x,y\}}(\sigma)\bigr]\, c_{xy}(\sigma) = \exp\bigl[-H_{\{x,y\}}(\sigma^{xy})\bigr]\, c_{xy}(\sigma^{xy}) \tag{2.7}$$

(3) *Positivity and boundedness.* There exist positive real numbers $c_m(\beta)$ $c_M(\beta)$ such that

$$c_m \le c_{xy}(\sigma) \le c_M \qquad \forall x, y \in \mathbb{Z}^d, \sigma \in \Omega\,. \tag{2.8}$$

We denote by $L_{V,N}^\tau$ the operator L_V acting on $L^2(\Omega, \nu_{V,N}^\tau)$ (this amounts to choosing τ as the boundary condition and N as the number of particles). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L_{V,N}^\tau$, and whose semigroup we denote by $(T_t^{V,N,\tau})_{t\ge 0}$. $L_{V,N}^\tau$ is a bounded operator on $L^2(\Omega, \nu_{V,N}^\tau)$ and $\nu_{V,N}^\tau$ is its unique invariant measure. Moreover $\nu_{V,N}^\tau$ is *reversible* with respect to the process, *i.e.* $L_{V,N}^\tau$ is self–adjoint on $L^2(\Omega, \nu_{V,N}^\tau)$.

A first fundamental quantity associated with the dynamics of a reversible system is the spectral gap of the generator, *i.e.*

$$\operatorname{gap}(L^\tau_{V,N}) = \inf \operatorname{spec}\left(-L^\tau_{V,N} \restriction \mathbb{1}^\perp\right)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega, \nu^\tau_{V,N})$ orthogonal to the constant functions. We let $\mathcal{E}$ to be the Dirichlet form associated with the generator $L^\tau_{V,N}$,

$$\mathcal{E}^\tau_{V,N}(f,f) = \langle f, -L^\tau_{V,N} f\rangle_{L^2(\Omega,\nu^\tau_{V,N})} = \frac{1}{2} \sum_{[x,y]\in\mathcal{E}_V} \nu^\tau_{V,N}\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right] \tag{2.9}$$

and $\operatorname{Var}^\tau_{V,N}$ the variance relative to the probability measure $\nu^\tau_{V,N}$. Then the gap can also be characterized as

$$\operatorname{gap}(L^\tau_{V,N}) = \inf_{\substack{f\in L^2(\Omega,\nu^\tau_{V,N}),\\ \operatorname{Var}^\tau_{V,N}(f)\neq 0}} \frac{\mathcal{E}^\tau_{V,N}(f,f)}{\operatorname{Var}^\tau_{V,N}(f)}\,. \tag{2.10}$$

A second relevant quantity is the logarithmic Sobolev constant $c^\tau_{V,N}$ defined as the smallest constant c such that

$$\operatorname{Ent}^\tau_{V,N}(f^2) \le \frac{c}{2}\,\mathcal{E}^\tau_{V,N}(f,f) \tag{2.11}$$

for all non negative functions f with $\nu^\tau_{V,N}(f^2) = 1$, where $\operatorname{Ent}^\tau_{V,N}(f^2) = \nu^\tau_{V,N}(f^2 \ln f^2)$. For the connection between spectral gap, logarithmic Sobolev constant and speed of relaxation to equilibrium we refer the reader to [DiSa].

2.4 Definition of the mixing condition and main results.

In order to formulate our basic mixing condition on the two (or more) body part of the interaction Φ we fix positive numbers C, m, l with $l \in \mathbf{N}$. We then say that a collection of real numbers $\underline{\lambda} := \{\lambda_x\}_{x\in\mathbb{Z}^d}$ is l–regular if, for all $i \in \mathbb{Z}^d$ and all $x \in Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$, $\lambda_x = \lambda_{x^i}$.

Definition of property $USMT(C,m,l)$**.** *For any l–regular set Λ, any l–regular $\underline{\lambda}$, any boundary condition τ and any pair of bounded local functions f and g*

$$|\mu^{\tau,\underline{\lambda}}_\Lambda(f,g)| \le C \sup_\tau \mu^{\tau,\underline{\lambda}}_\Lambda(|f|)\, \sup_\tau \mu^{\tau,\underline{\lambda}}_{\Lambda\setminus\Delta^{(l)}_f}(|g|) \sum_{x\in\partial_r^-\Delta^{(l)}_f}\ \sum_{y\in\partial_r^-\Delta^{(l)}_g} e^{-m|x-y|}$$

provided that $d(\Delta^{(l)}_f, \Delta^{(l)}_g) \ge l$.

Remark. The expert reader may have noticed that our condition is different, and in principle stronger, than the one used in [LY] and [Y] because we require the exponential decay of covariances uniformly in the chemical potential even when the latter *varies* over the atoms of a partition of Λ while in the above mentioned papers the chemical potential is assumed to be constant over Λ. In two dimension, one can prove either by cluster expansion methods (see [BCO]) or followig the ideas of [MOS], that the two conditions are equivalent. In higher dimension one can construct examples (see the appendix A2 of [BCO]) in which a kind of phase transition occurs along the interface between two subsets with different chemical potential, even if for all l–regular sets Λ the covariances decay exponentially fast uniformly w.r.t. to constant chemical potentials.

We are finally in a position to formulate the main result of this paper on the logarithmic Sobolev constant of Kawasaki dynamics in a finite volume.

Theorem 2.3. *Assume that there exist positive numbers C, m, l, with $l \in \mathbf{N}$, such that property $USMT(C, m, l)$ holds. Then there exist positive constants c_1, c_2 such that*

$$c_1 L^2 \leq c^{\tau}_{Q_L, N} \leq c_2 L^2 \tag{2.12}$$

for all boundary condition τ and particle number N.

3. Technical results.

Here we collect several preliminary technical results that will be used in the future. Although the reader may skip this section during a first reading and come back when these results are needed, we still think it is useful to give a short roadmap of the section.

- In paragraph §3.1, §3.2 we recall some useful results on the comparison between finite volume multicanonical Gibbs measures, namely grand canonical Gibbs measures conditioned to have a specified number of particles in the atoms of a given partition of a finite set Λ, and the corresponding unconditioned measures.
- In paragraph §3.3, we prove a "baby" version of a nice, new inequality for the entropy (see [Ce]), that, very roughly speaking, allows us to bound the multicanonical entropy in terms of the multicanonical average of the sum of local entropies and of the multicanonical variance. For the expert reader we just mention that the importance of such an inequality resides in the fact that one is spared from the cumbersome computation of quantities like $\left[\nabla_{xy}\, \nu^{\tau}_{\Lambda}(f^2)^{\frac{1}{2}}\,\right]^2$.
- In paragraphs §3.4, §3.5, §3.6, we first show how to compute and then how to estimate quantities like $\nu\Big(\big[\frac{d}{dn}\, \nu\left(f^2 \mid N_V = n\right)^{\frac{1}{2}}\,\big]^2\Big)$, where ν is a multi canonical measure over the atoms $\{\Lambda_i\}_{i=1}^k$ of a partition of a finite set Λ, V is a subset of a given atom Λ_j and $\frac{d}{dn}$ is the discrete derivative. As it will appear clearly in section 6, terms like the one above naturally appear in the recursive bound of the spectral gap and, roughly speaking, they measure the influence on the logarithmic Sobolev constant of the exchange of particles between different atoms of the partition.
- In paragraph §3.7 we recall some results proved in [CM2] on the distribution of the number of particles in an atom of a given partition of a finite set Λ under a multicanonical measure. These results will then be used to prove sharp bound on the logarithmic Sobolev inequality for such a measure via Hardy inequality (see the appendix).

THE GENERAL SETTING.

Throughout all this section our setting and notation will be as follows.

Fix $\delta_0 \in (0,1)$ and two integers $j_{\max}, l$ such that $\delta_0 j_{\max} < 1$. Let $L_1, \dots, L_{j_{\max}}$ be large multiples of the basic length scale l, let $L = \sum_j L_j$ and assume that $L_j \geq \delta_0 L$ for any j. We then choose one coordinate direction, e.g. the d direction, and we take $\Lambda = Q_L$, Λ_1 equal to the first slice of Λ orthogonal to d–direction of width L_1, *i.e.* $\Lambda_1 = \{x \in \Lambda \ : \ 0 \leq x_d < L_1\}$, Λ_2 equal to the slice of Λ on top of Λ_1 of width L_2 and so on. Let also $\mathbf{N} = \{N_j\}_{j=1}^{j_{\max}}$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_j}\}_{j=1}^{j_{\max}}$, $\rho_j = \frac{N_j}{|\Lambda_j|}$ the corresponding densities and let us assume, for a given boundary condition τ, that $\underline{\lambda} = \underline{\lambda}(\tau, \mathbf{N})$ is constant on each set Λ_j and such that $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_j}) = N_j$, $j = 1, \dots, j_{\max}$ (see the appendix of [CM1] for a proof of the existence of $\underline{\lambda}$). Notice that, as far as the tilting fields $\underline{\lambda}$ is concerned, the variables N_j could also be *any* set of real numbers in the range $[0, |\Lambda_j|]$. In particular we will be allowed to compute the derivative of $\underline{\lambda}$ w.r.t. any of them. We simply write μ for $\mu_\Lambda^{\tau,\underline{\lambda}}$ and ν for the multi canonical Gibbs measure $\mu\big(\cdot \,|\, N_{\Lambda_j} = N_j,\, j = 1, \dots, j_{\max}\big)$.

Finally, in all this section k will denote a generic constant, depending on the interaction, on the dimensions of the lattice, on δ_0 and on $j_{\max}$, whose value may vary from line to line.

3.1 ON THE TILTING FIELDS.

We begin by recalling the following quite general result on the relationship between particle numbers, boundary condition and the chemical potential (see the appendix in [CM1]). We assume here $j_{\max} = 1$ so that we can set, for notation convenience, $N_1 = n$, $\rho = \frac{n}{L^d}$. In order to be more clear, in the following lemma we will write explicitly the dependence on the boundary conditions and the chemical potential of the grand canonical Gibbs state.

Lemma 3.1. *Assume property $USMT(C,m,l)$ and let $\alpha \in (0,1)$. Then there exists a constant k independent on L such that for any L large enough and any f with $\|f\|_\infty = 1$*

(1) If $d(\Delta_f, \Lambda^c) \geq L^\alpha$

$$i) \qquad \sup_{y \in \partial_r^+ \Lambda} \|\nabla_y \mu^{\tau,\lambda}(f)\|_\infty \leq k\, \rho \frac{|\Delta_f|}{|\Lambda|}$$

$$ii) \qquad \sup_{y \in \partial_r^+ \Lambda} \|\nabla_y \frac{d}{dn} \mu^{\tau,\lambda}(f)\|_\infty \leq k \,\frac{|\Delta_f|}{|\Lambda|^2}$$

(2) If $\Delta_f \subset \Lambda$

$$i) \qquad \| \frac{d}{dn} \mu^{\tau,\lambda}(f) \|_\infty \leq k\, \frac{|\Delta_f|}{|\Lambda|}$$

$$ii) \qquad \| \frac{d^2}{dn^2} \mu^{\tau,\lambda}(f) \|_\infty \leq k\, \frac{1}{n} \frac{|\Delta_f|}{|\Lambda|}$$

Proof. We omit the proof since is practical the same of an almost identical result (Proposition 3.1) of [CM2]. $\square$

3.2 Equivalence of ensembles

Here we recall some fine results on the finite volume equivalence of ensembles that will be crucial in most of our future arguments. We refer the reader to sections 5, 7.2 and 7.3 of [CM1].

Given $M > 0$ and $\Delta \subset \Lambda$, we say that Δ is good if there exists $j \in [1, j_{\max}]$ such that either

$$\{x \in \Lambda :\ d(x,\Delta) \le M \ln L\} \subset \Lambda_j \quad \text{and} \quad \rho_j \ge |\Lambda|^{-\epsilon}$$

$$\text{or}$$

$$\{x \in \Lambda :\ d(x,\Delta) \le M\} \subset \Lambda_j \quad \text{and} \quad \rho_j \le |\Lambda|^{-\epsilon}$$

A set is bad if it is not good. For good sets $\Delta \subset \Lambda_j$, $j = 1, \dots, j_{\max}$, we define

$$\bar\Delta = \begin{cases} \{x \in \Lambda :\ d(x,\Delta) \le M \ln L\,\} & \text{if } \rho_j \ge |\Lambda|^{-\epsilon} \\ \{x \in \Lambda :\ d(x,\Delta) \le M\,\} & \text{if } \rho_j \le |\Lambda|^{-\epsilon} \end{cases}$$

while for bad sets

$$\bar\Delta = \{x \in \Lambda :\ d(x,\Delta) \le M \ln L\}$$

With these notation the results on the finite volume equivalence of ensembles that will be essential for the rest of this paper read as follows.

Proposition 3.2. *Assume condition $USMT(C,m,l)$. Then, for any l, M large enough and ϵ small enough independent of $\{\rho_j\}_{i=1}^{j_{\max}}$, there exists $L_0 = L_0(C, m, \|\Phi\|, l, M, \delta_0, \epsilon)$ such that, for any $L \ge L_0$ the following holds. Fix two l–regular sets Δ_1, Δ_2 in Λ such that $|\Delta_i| \le |\Lambda|^{1-\epsilon}$, $i = 1, 2$, and two functions f, g with $\|f\|_\infty = \|g\|_\infty = 1$, $\Delta_f^{(l)} = \Delta_1$, $\Delta_g^{(l)} = \Delta_2$.*

(1) Assume $j_{\max} = 1$. Then

$$|\,\nu(f,g)\,| \le C(f,g)\rho \left[\frac{1}{|\Lambda|} + e^{-md(\Delta_1,\Delta_2)}\right]$$

where $C(f,g) = k\,|\Delta_1|^2\,|\Delta_2|^2$. Moreover

$$|\,\nu(f) - \mu(f)\,| \le k \frac{|\Delta_1|}{|\Lambda|}$$

(2) Assume $j_{\max} \ge 2$. Then

$$|\,\nu(f,g)\,| \le C(f,g) A(\Delta_1,\Delta_2) \begin{cases} \nu(|f|) \wedge \nu(|g|) & \textit{if } \Delta_1 \textit{ and } \Delta_2 \textit{ are both good or bad} \\ \nu(|f|) & \textit{if } \Delta_1 \textit{ is bad} \\ \nu(|g|) & \textit{if } \Delta_2 \textit{ is bad} \end{cases}$$

$$\textit{where} \quad A(\Delta_1,\Delta_2) = \begin{cases} \frac{1}{|\Lambda|} + e^{-md(\Delta_1,\Delta_2)} & \textit{if } \Delta_1 \textit{ or } \Delta_2 \textit{ is good} \\ \frac{1}{|\Lambda|}\left(\frac{|\bar\Delta_2|}{|\Delta_2|}\right)^2 \left(\frac{|\bar\Delta_1|}{|\Delta_1|}\right)^2 + e^{-md(\Delta_1,\Delta_2)} & \textit{otherwise} \end{cases}$$

(3) Assume $j_{\max} \ge 2$ and $\Delta_1 \subset \Lambda_j$, $j \le j_{\max}$. Then

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \le k \begin{cases} |\Delta_1| \left[\frac{1}{|\Lambda|} + \frac{1}{L^{k-n+1}} \right] & \text{if } \Delta_1 \text{ is good} \\ \frac{|\Delta_1|}{|\Lambda|} \left(\frac{|\bar\Delta_1|}{|\Delta_1|} \right)^2 + \max_{j=n,n\pm 1} \left[\frac{|\bar\Delta_1 \cap \Lambda_j|^{\frac{1}{2}}}{L^{j_{\max}-j+1}} \right] & \text{if } \Delta_1 \text{ is bad} \end{cases}$$

where Ω_τ is the set of configurations τ' that coincide with τ in the half space $\{x \in \mathbb{Z}^d \ : \ x_d < L\}$.

Remark. Actually the first part of the proposition holds in a much more general geometric context (see §7.3 of [CM1]).

For future purposes the next result is stated in a slightly more general form. The proof is given in [CM2].

Proposition 3.3. *Assume condition $USMT(C,m,l)$ and let f be such that $|\Lambda_j \setminus \Delta_f| \ge \delta_0 |\Lambda_j|$ for any $j = 1, \dots j_{\max}$. Then, for any l large enough, there exists a constant A depending only on $C, m, \|\Phi\|, l, j_{\max}, \delta_0$ such that*

$$\nu(|f|) \le A\, \mu(|f|)$$

In particular

$$\mathrm{Var}_\nu(f,f) \le A \ \mathrm{Var}_\mu(f,f)$$

We conclude this paragraph with a final result that plays a crucial role in our approach (see §3.3). For simplicity we consider only the two dimensional case and at the end we explain how to generalize it to higher dimensions.

Assume that the number of layers $j_{\max}$ is greater than 4, fix $3 \le j_0 \le j_{\max}$ and let $A = \cup_{j=1}^{j_0} \Lambda_j$, $B = \cup_{j_0-1}^{j_{\max}} \Lambda_j$ and $S = \Lambda_{j_0-1} \cup \Lambda_{j_0}$. Let also $\nu_X(\cdot) := \nu(\cdot \,|\, \mathcal{F}_{X^c})$ for $X = A, B, S$. Notice that ν–almost surely N_A, N_B and N_S are constant.

Lemma 3.4. *Assume condition $USMT(C,m,l)$ and let g be a positive function measurable w.r.t. $\mathcal{F}_{A^c}$. Then for any $\epsilon > 0$ there exists $L_0 = L_0(\epsilon, C, m, \|\Phi\|_\infty, l, \delta)$ such that, if $L \ge L_0$,*

$$\|\nu_B(g) - \nu(g)\|_\infty \le \epsilon\, \nu(g)$$

In other words the Radon–Nycodim derivative of the marginal on $\mathcal{F}_{A^c}$ of ν_B w.r.t. the same marginal of ν is close to one in the L^∞–norm.

Proof. Fix $\epsilon \in (0,1)$ and η and let $h_x(\eta) := e^{-\nabla_x H(\eta)}/\nu_B^\eta(e^{-\nabla_x H})$. Using the definition of h_x and the DLR equations (valid because the numbers of particles in A, B, S are constant) we have

$$\begin{aligned} |\nu_B^{\eta^x}(g) - \nu_B^\eta(g)| &= |\nu_B^\eta\,(g, h_x)\,| \\ &= |\nu_B^\eta\big(g, \nu_S(h_x)\big)| \\ &\le \nu_B^\eta(g) \sup_{\tau' \in \Omega_\tau} |\nu_S^\tau(h_x) - \nu_S^{\tau'}(h_x)| \end{aligned} \tag{3.1}$$

where Ω_τ is the set of configurations τ' which differs from τ only on $\partial^+ S \cap B$. By point (3) of proposition 3.2 there exists a positive constant k such that

$$\sup_{\tau' \in \Omega_\tau} |\nu_S^\tau(h_x) - \nu_S^{\tau'}(h_x)| \le k \, \frac{(\ln L)^{\frac{1}{2}}}{L^2} \tag{3.2}$$

Thus

$$\nu_B^{\eta^x}(g) \le \left[1 + k \, \frac{(\ln L)^{\frac{1}{2}}}{L^2}\right] \nu_B^\eta(g) \tag{3.3}$$

As any two boundary configurations η and η' differ at most in kL sites, by iterating (3.3), we have

$$\nu_B^\eta(g) \le \left[1 + C' \, \frac{(\ln L)^{\frac{1}{2}}}{L^2}\right]^{kL} \nu_B^{\eta'} g \le (1+\epsilon) \, \nu_B^{\eta'}(g) \tag{3.4}$$

for L large enough. It suffices now to integrate (3.4) w.r.t. $d\nu(\eta')$ and use the arbitrariness of η. $\square$

Remark. The restriction of $d = 2$ comes from point (3) of proposition 3.2. In fact, in e.g. three dimensions, the bound (3.2) becomes useless. The way out is to have the “safety belt” S divided into more layers (just three in $d = 3$). It is interesting at this point to observe that a similar problem occurs also in the recursive study of the spectral gap (see [CM2]). In that case however, the safety belt S in $d = 2$ consisted of just one atom and not of two as in our case. The reason is that, in the spectral gap analysis, a weaker form of lemma 3.4 was necessary in which the r.h.s. of the basic inequality is $\epsilon \|g\|_\infty$ and not $\epsilon \nu(g)$.

3.3 A TWO–BLOCK INEQUALITY ON THE ENTROPY.

Here we give a result that is a key step in our recursive bound of the spectral gap of Kawasaki dynamics. For simplicity we discuss the next estimates in two dimensions and at the end we explain how to generalize it to higher dimensions.

Assume that the number of layers $j_{\max}$ is greater than 4, fix $3 \le j_0 \le j_{\max}$ and let $A = \cup_{j=1}^{j_0} \Lambda_j$, $B = \cup_{j_0-1}^{j_{\max}} \Lambda_j$ and $S = \Lambda_{j_0-1} \cup \Lambda_{j_0}$. Let also $\nu_X(\cdot) := \nu(\cdot \,|\, \mathcal{F}_{X^c})$ for $X = A, B, S$. Notice that ν–almost surely N_A, N_B and N_S are constant.

Proposition 3.5. *Assume $USMT(C, m, l)$. Then, for any $\epsilon > 0$ there exists L_0 depending only on $\epsilon, C, m, \|\Phi\|_\infty, l, \delta_0$ such that, if $L \ge L_0$,*

$$\mathrm{Ent}_\nu(f^2) \le \nu\left(\mathrm{Ent}_{\nu_A}(f^2) + \mathrm{Ent}_{\nu_B}(f^2)\right) + \epsilon \, \mathrm{Var}_\nu(f) \tag{3.5}$$

for any f with $\nu(f) = 0$.

Remark. The origin of this inequality is the following simple bound valid for an arbitrary product measure $\nu = \nu_1 \otimes \nu_2$:

$$\mathrm{Ent}_\nu(f^2) \le \nu\left(\mathrm{Ent}_{\nu_1}(f^2) + \mathrm{Ent}_{\nu_2}(f^2)\right) \qquad \forall\, f \ge 0$$

and, in some sense, the term $\epsilon\,\mathrm{Var}_\nu(f)$ in (3.5) takes into account the correction due to the presence of the overlapping strip S. As we will see the proof of the proposition is rather simple. Quite a non trivial task is to remove the restriction $\nu(f) = 0$. If one tries to use the Rothaus inequality (see e.g. [An])

$$\mathrm{Ent}_\nu(f^2) \le 2\,\mathrm{Var}_\nu(f) + \mathrm{Ent}_\nu\big(\big[f - \nu(f)\big]^2\big)$$

together with (3.5), one gets immediately

$$\mathrm{Ent}_\nu(f^2) \le \nu\left(\mathrm{Ent}_{\nu_A}(f^2) + \mathrm{Ent}_{\nu_B}(f^2)\right) + (2+\epsilon)\ \mathrm{Var}_\nu(f) \tag{3.6}$$

which is much worse than (3.5) because of the factor $2+\epsilon$. Quite remarkably, in [Ce] the restriction $\nu(f) = 0$ has been removed in great generality and moreover the error term $\epsilon\,\mathrm{Var}_\nu(f)$ has been replaced by $\epsilon\,\mathrm{Ent}_\nu(f^2)$. The resulting inequality has been then applied to provide a very simple and natural proof of the logarithmic Sobolev inequality for grand canonical Gibbs measures under a mixing condition. For a "variance" counterpart of (3.5) (that is $\mathrm{Ent}_\nu(f^2) \mapsto \mathrm{Var}_\nu(f)$) we refer the reader to proposition 3.4 in [CM2] (see also section 3.7 in [M] and [BCC] for the Glauber case).

Proof. The proof is based on lemma 3.4 above. We write

$$\mathrm{Ent}_\nu(f^2) = \nu\Big(f^2 \log\frac{f^2}{\nu_A(f^2)}\Big) + \nu\Big(f^2\log\frac{\nu_A(f^2)}{\nu_B\big(\nu_A(f^2)\big)}\Big) + \nu\Big(f^2\log\frac{\nu_B\big(\nu_A(f^2)\big)}{\nu(f^2)}\Big) \tag{3.7}$$

The first term in (3.7) is equal to $\nu\big(\mathrm{Ent}_{\nu_A}(f^2)\big)$. By using the variational definition of the entropy $\mathrm{Ent}_{\nu_B}(f^2)$

$$\mathrm{Ent}_{\nu_B}(f^2) = \sup_{g\,:\,\nu_B(e^g)=1} \nu_B(f^2 g)$$

together with

$$\nu_B\Big(\frac{\nu_A(f^2)}{\nu_B\big(\nu_A(f^2)\big)}\Big) = 1$$

the second term in (3.7) can be bounded from above by $\nu\big(\mathrm{Ent}_{\nu_B}(f^2)\big)$. Finally, the argument of the logarithm in the third term is smaller than

$$\nu_B\Big(\frac{\nu_A(f^2)}{\nu(f^2)}\Big) = \frac{\nu_B\big(\nu_A(f^2)\big)}{\nu\big(\nu_A(f^2)\big)} \le 1+\epsilon$$

for any L large enough independent of f because of lemma 3.4. Thus the third term can be bounded from above by $\epsilon\,\nu(f^2) = \epsilon\,\mathrm{Var}_\nu(f)$. $\square$

3.4. Computing the gradient with respect to the particle number.

Let V and W be such that $V \cap W = \emptyset$, $\Lambda_j = V \cup W$ for some $j \in \{1, \ldots, j_{\max}\}$ and $0 < \delta_0 \le \frac{|V|}{|W|} \le \delta_0^{-1}$. Here δ_0 is the same as the one entering in the general multicanonical setting described at the beginning of the section.

Our goal is to compute, for an arbitrary f, the gradient w.r.t. to n of $\big(\nu(f^2 \,|\, N_V = n)\big)^{\frac{1}{2}}$. For simplicity we set $\nu(\cdot \,|\, n) := \nu(\cdot \,|\, N_V = n)$ and $\gamma(n) := \nu(N_V = n)$.

For $x, z \in \mathbb{Z}^d$, we define the events

$$E_{xz} = \{\sigma \in \Omega : \sigma(x) = 1,\ \sigma(z) = 0\}\,. \tag{3.8}$$

In Proposition 3.10 of [CM2] the following result was proved.

Proposition 3.6. *Let V and W be such that $V \cap W = \emptyset$ and $\Lambda_j = V \cup W$ for some $j \in \{1, \dots, k\}$. Let also $c_n = n(|W| - N_j + n)$, that is $\big($ number of particles in $V\big) \times \big($ number of holes in $W\big)$, and let $c'_n = n(|V| - N_j + n)$. Then, for all functions f on Ω we have*

(1)

$$\begin{aligned}\nu(f^2 \,|\, n) - \nu(f^2 \,|\, n-1) =& \\ &\frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu\left((\nabla_{zx} f^2) \mathbb{1}_{E_{zx}} e^{-\nabla_{xz} H_\Lambda} \,|\, n-1\right) \\ &+ \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu\left(\left(e^{-\nabla_{xz} H_\Lambda} - 1\right) \mathbb{1}_{E_{zx}}, f^2 \,|\, n-1\right)\ ,\end{aligned}$$

(2)

$$\begin{aligned}\nu(f^2 \,|\, n) - \nu(f^2 \,|\, n-1) =& \\ &- \frac{1}{c'_{N-n+1}} \frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x \in V \\ z \in W}} \nu\left((\nabla_{xz} f^2) \mathbb{1}_{E_{xz}} e^{-\nabla_{xz} H_\Lambda} \,|\, n\right) \\ &- \frac{1}{c'_{N-n+1}} \frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x \in V \\ z \in W}} \nu\left(\left(e^{-\nabla_{xz} H_\Lambda} - 1\right) \mathbb{1}_{E_{xz}}, f^2 \,|\, n\right)\ .\end{aligned}$$

Let now $n_{\max}$, $n_{\min}$ be the maximum and minimum value of the particle number in V under the constraint that $N_{\Lambda_j} = N_j$. Let $u = \lfloor \rho_j |V| \rfloor$, where $\rho_j = \frac{N_j}{|\Lambda_j|}$, and let, for $n \in (n_{\min}, n_{\max})$,

$$\begin{aligned} A(n) &= \begin{cases} \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu\left((\nabla_{zx} f^2) \mathbb{1}_{E_{zx}} e^{-\nabla_{xz} H_\Lambda} \,|\, n-1\right) & \text{if } n \le u \\ \frac{1}{c'_{N-n+1}} \frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x \in V \\ z \in W}} \nu\left((\nabla_{xz} f^2) \mathbb{1}_{E_{xz}} e^{-\nabla_{xz} H_\Lambda} \,|\, n\right) & \text{otherwise,} \end{cases} \\ B(n) &= \begin{cases} \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu\left(\left(e^{-\nabla_{xz} H_\Lambda} - 1\right) \mathbb{1}_{E_{zx}}, f^2 \,|\, n-1\right) & \text{if } n \le u \\ \frac{1}{c'_{N-n+1}} \frac{\gamma(n)}{\gamma(n-1)} \sum_{\substack{x \in V \\ z \in W}} \nu\left(\left(e^{-\nabla_{xz} H_\Lambda} - 1\right) \mathbb{1}_{E_{xz}}, f^2 \,|\, n\right) & \text{otherwise.} \end{cases} \end{aligned} \tag{3.9}$$

With this definition, and using $(\sqrt{x} - \sqrt{y})^2 = \left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)^2 \le \frac{(x-y)^2}{x \vee y}$, we have immediately the following corollary

Corollary 3.7. *In the same setting of proposition 3.6*

$$\sum_n \gamma(n) \wedge \gamma(n-1) \left(\sqrt{\nu(f^2 \,|\, n)} - \sqrt{\nu(f^2 \,|\, n-1)} \right)^2 \le 2 \sum_n \gamma_f(n) \left(A(n)^2 + B(n)^2 \right)$$

where $\gamma_f(n) := \frac{\gamma(n)\wedge\gamma(n-1)}{\nu(f^2 \,|\, n) \vee \nu(f^2 \,|\, n-1)}$

3.5 Bound on $\sum_n \gamma_f(n) A(n)^2$.

Here we show how to bound from above the term $\sum_n \gamma_f(n) A(n)^2$ of corollary 3.7.

Proposition 3.8.

$$\sum_n \gamma_f(n) \, A(n)^2 \le k \, \frac{L^2}{\rho_j |\Lambda_j|} \mathcal{E}_\nu(f, f)$$

Proof. Let $u = \lfloor \rho_j |V| \rfloor$ and assume, without loss of generality, $\rho_j \le \frac{1}{2}$ and $n \le u$. Then we observe that, because of the conservation law,

$$\frac{1}{c_n} \, \frac{\gamma(n-1)}{\gamma(n)} \sum_{\substack{x \in V \\ z \in W}} \nu\big(e^{-\nabla_{xz} H_\Lambda} \mathbb{1}_{E_{zx}} \,|\, n-1 \big) = 1 \; ,$$

so that

$$e^{-2\|\Phi\|} \, \frac{1}{c'_{N_j - n+1}} \le \frac{1}{c_n} \, \frac{\gamma(n-1)}{\gamma(n)} \le e^{2\|\Phi\|} \, \frac{1}{c'_{N_j - n+1}} \; .$$

Now, using twice the Cauchy-Schwarz inequality together with the identity $a^2 - b^2 = (a-b)(a+b)$ on $\nabla_{xz} f^2$ gives

$$\begin{aligned}
A(n)^2 = {} & \Big[\frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{(x,z) \in V \times W} \nu\big(\, e^{-\nabla_{xz} H} \nabla_{xz} f^2 \,|\, n-1, E_{zx} \, \big) \nu\big(\, E_{zx} \,|\, n-1 \, \big) \Big]^2 \\
\le {} & k \, \frac{1}{c'_{N_j - n+1}} \sum_{(x,z) \in V \times W} \nu\big(\, (\nabla_{xz} f)^2 \,|\, n-1, E_{zx} \, \big) \nu\big(\, E_{zx} \,|\, n-1 \, \big) \\
\times {} & \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{(x,z) \in V \times W} \nu\big(\, f^2 + (T_{xz} f)^2 \,|\, n-1, E_{zx} \, \big) \nu\big(\, E_{zx} \,|\, n-1 \, \big)
\end{aligned} \qquad (3.10)$$

We recall that $\nu \, (\cdot \,|\, n-1)$–almost surely, $\sum_{\substack{x \in V \\ z \in W}} \mathbb{1}_{E_{zx}} = c'_{N-n+1}$. So, one can see that

$$\begin{aligned}
& \frac{1}{c_n} \frac{\gamma(n-1)}{\gamma(n)} \sum_{(x,z) \in V \times W} \nu\big(\, f^2 \,|\, n-1, E_{zx} \, \big) \nu\big(\, E_{zx} \,|\, n-1 \, \big) \\
& \qquad \le k \frac{1}{c'_{N_j - n+1}} \sum_{(x,z) \in V \times W} \nu\big(\, f^2 \mathbb{1}_{E_{zx}} \,|\, n-1 \, \big) \\
& \qquad = k \nu\big(\, f^2 \,|\, n-1 \, \big) \; .
\end{aligned} \qquad (3.11)$$

On the other hand, using the change of variable $\sigma \mapsto \sigma^{xz}$ and the equality $\nu(fg) = \nu(f,g)+\nu(f)\nu(g)$, we get

$$\begin{aligned}
&\frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \sum_{(x,z)\in V\times W} \nu\big((T_{xz}f)^2 \,|\, n-1, E_{zx} \big)\nu\big(E_{zx} \,|\, n-1 \big) \\
&= \frac{1}{c_n}\frac{\gamma(n-1)}{\gamma(n)} \sum_{(x,z)\in V\times W} \nu\big((T_{xz}f)^2 \mathbb{1}_{E_{zx}} \,|\, n-1 \big) \\
&\le \frac{1}{c_n} \sum_{(x,z)\in V\times W} \nu\big(f^2 \mathbb{1}_{E_{xz}} e^{-\nabla_{zx}H} \,|\, n \big) \\
&\le k\nu\big(f^2 \,|\, n \big)
\end{aligned} \tag{3.12}$$

where for the last inequality, we exploited the fact that $\nu\,(\cdot\,|\,n)$–almost surely $\sum_{\substack{x\in V\\ z\in W}} \mathbb{1}_{E_{xz}} = c_n$. Now, since c'_n is increasing in n, we obtain from (3.10), (3.11) and (3.12) that

$$A(n)^2 \le k\frac{1}{c'_{N_j-u+1}}\{\nu(f^2\,|\,n) \vee \nu(f^2\,|\,n-1)\} \sum_{(x,z)\in V\times W} \nu\big((\nabla_{xz}f)^2 \,|\, n-1 \big)\ , \tag{3.13}$$

and similarily for $n > u$. Notice that $c'_{N_j-u+1} \ge c_u = k\rho_j|\Lambda_j|^2$. Thus, it follows from (3.13) that

$$\begin{aligned}
\sum_n \gamma_f(n)\, A(n)^2 &\le k\frac{1}{\rho_j\,|\Lambda_j|^2} \sum_{(x,z)\in V\times W} \nu\big((\nabla_{xz}f)^2 \big) \\
&\le k\frac{L^2}{\rho_j|\Lambda_j|}\mathcal{E}_\nu(f,f)
\end{aligned} \tag{3.14}$$

because

$$\sum_{(x,z)\in V\times W} \nu\big((\nabla_{xz}f)^2 \big) \le kL^{d+2}\mathcal{E}_\nu(f,f) \tag{3.15}$$

In order to prove (3.15) we need the following definition.

Definition 3.9. Given a finite connected subset Λ of $\mathbb{Z}^d$ a *path choice in* Λ is a collection $\lambda = \{\lambda_{xz} : (x,z) \in \Lambda\times\Lambda\}$ such that λ_{xz} is a self–avoiding path from x to z inside Λ.

Given a path choice λ in Λ_j and V, W as above, we let

$$\begin{aligned}
\mathcal{G}_V(\lambda) &= \max_{e\in\mathcal{E}_\Lambda} \#\{(x,z)\in V\times W : \lambda_{xz} \sqsupset e\} \\
\mathcal{D}_V(\lambda) &= \max_{(x,z)\in V\times W} |\lambda_{xz}|
\end{aligned}$$

Let now λ be any path choice. Thanks to Lemma 4.3 in [Y] we get that

$$\nu\big((\nabla_{xz}f)^2 \big) \le k\,|\lambda_{xz}| \sum_{e\sqsubset\lambda_{xz}} \nu\big((\nabla_e f)^2\big)$$

which, together with the definition of $\mathcal{G}_V(\lambda)$ and $\mathcal{D}_V(\lambda)$, implies

$$\sum_{(x,z)\in V\times W} \nu\big((\nabla_{xz}f)^2\big) \le k\,\mathcal{G}_V(\lambda)\mathcal{D}_V(\lambda)\mathcal{E}_\nu(f,f)$$

A sensible path choice is the following. Given $x \in V$ and $z \in W$, start increasing (or decreasing) the first coordinate of x until it is equal to the first coordinate of z. Repeat for the remaining coordinates until you get to z. With this particular path choice it is easy to see that $\mathcal{G}_V(\lambda)\mathcal{D}_V(\lambda) \le kL^{d+2}$ (see section 5.2 in [CM2]) and the proof of (3.15) follows at once. $\square$

3.6 Bound on $\sum_n \gamma_f(n)B(n)^2$.

Here we show how to bound from above the term $\sum_n \gamma_f(n)B(n)^2$ of corollary 3.7.

Proposition 3.10. *Assume property $USMT(C,m,l)$. Then for any $\epsilon > 0$ and $\rho_0 \le \frac{1}{2}$ there exists C_ϵ such that, for any f with $\nu(f^2) = 1$,*

(i) if $\rho_j \le \rho_0$

$$\sum_n \gamma_f(n)\,B(n)^2 \le \frac{1}{\rho_j|\Lambda_j|}\left[C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\right]$$

(ii) if $\rho_0 \le \rho_j \le \frac{1}{2}$

$$\sum_n \gamma_f(n)\,B(n)^2 \le \frac{1}{\rho_j^2|\Lambda_j|}\left[C_\epsilon + C_\epsilon\,L^2\mathcal{E}_\nu(f,f) + \epsilon\,\mathrm{Ent}_\nu(f^2)\right]$$

Proof.
(i) Let $u = \lfloor \rho_j|V| \rfloor$ and assume, without loss of generality, $n \le u$. Then, by proceeding as in the proof of proposition 3.8,

$$B(n)^2 \le k\,\nu\Big(f^2, \frac{1}{c'_{N_j-u+1}} \sum_{\substack{x\in V\\ z\in W}} \big(e^{-\nabla_{xz}H_\Lambda} - 1\big)\mathbb{1}_{E_{zx}}\,|\,n-1\Big)^2 \tag{3.16}$$

where $c'_{N_j-u+1} = (N_j - u + 1)(|V| - u + 1) \ge k\rho_j|V||W|$.
Let now $h_x(\sigma) := (e^{-\nabla_x H_\Lambda(\sigma)} - 1)(1-\sigma(x))$ and $g_x(\sigma) := (e^{-\nabla_x H_\Lambda(\sigma)} - 1)\sigma(x)$. Then

$$\begin{aligned}
B(n)^2 &\le k\,\nu\Big(f^2, \frac{1}{\rho_j|V||W|} \sum_{\substack{x\in V\\ z\in W}} \big(e^{-\nabla_{xz}H_\Lambda} - 1\big)\mathbb{1}_{E_{zx}}\,|\,n-1\Big)^2 \\
&\le k\,\nu\Big(f^2, \frac{1}{\rho_j|V||W|} \sum_{\substack{x\in V\\ z\in W}} \big[(e^{-\nabla_{xz}H_\Lambda} - 1)\mathbb{1}_{E_{zx}} - h_x g_z\big]\,|\,n-1\Big)^2 \\
&\quad + k\,\nu\Big(f^2, \frac{1}{\rho_j|V||W|} \sum_{\substack{x\in V\\ z\in W}} h_x g_z\,|\,n-1\Big)^2
\end{aligned} \tag{3.17}$$

Notice that

$$(e^{-\nabla_{xz}H_\Lambda}-1)\mathbb{1}_{E_{zx}}-g_z\,h_x=g_z(1-\sigma_x)+h_x\sigma_z\qquad\text{if}\quad d(x,z)>r$$

Thus the first term in the r.h.s. of (3.17) can be bounded from above by

$$\begin{aligned}&k\,\nu\Big(f^2,\frac{1}{\rho_j|V||W|}\sum_{\substack{(x,z)\in V\times W\\ d(x,z)\le r}}\big[(e^{-\nabla_{xz}H_\Lambda}-1)\mathbb{1}_{E_{zx}}-h_xg_z\big]\,|\,n-1\Big)^2+\\ &k\,\nu\Big(f^2,\frac{|V|-n+1}{\rho_j|W||V|}\sum_{z\in W}g_z\,\,|\,n-1\Big)^2+k\,\nu\Big(f^2,\frac{N_j-n+1}{\rho_j|W||V|}\sum_{x\in V}h_x\,\,|\,n-1\Big)^2\end{aligned}\tag{3.18}$$

In order to bound the second and third term in (3.18) we can apply proposition 5.2 to the functions $\sum_{z\in W}g_z$ and $\sum_{x\in V}\rho_jh_x$. The result is that the sum of second and third term in the r.h.s. of (3.18) is smaller than

$$k\frac{1}{\rho_j|\Lambda_j|}\nu(f^2\,|\,n-1)\left[C_\epsilon\nu(f^2\,|\,n-1)+\epsilon\,\mathrm{Ent}_{\nu(\cdot\,|\,n-1)}(f^2)\right]\tag{3.19}$$

Notice that, if we average (3.19) over $\gamma_f(n)$ we get, after a Jensen inequality, the sought bound. Finally we consider the first term in the r.h.s. of (3.18). A simple L^∞ bound gives that this term is smaller than

$$k\frac{1}{L^2}\,\frac{1}{\rho_j^2|\Lambda_j|^2}\,\nu(f^2\,|\,n-1)^2$$

and thus, by recalling the definition of $\gamma_f(n)$, the result is obtained.

(ii). We can proceed as in the previous case using proposition 5.3 instead of proposition 5.2 and point (ii) of proposition 5.5 instead of point (i). $\square$

3.7 On the distribution of the number of particles inside one block.

The aim of this paragraph is first to recall some general result on the distribution of the number of particles inside one block Λ_j and secondly to derive an (optimal) upper bound on its logarithmic Sobolev constant (see proposition 3.17 below).

The setting is that of §3.4 above. We assume without loss of generality $\rho_j\le\frac{1}{2}$ and we denote by $\bar n=\mu(N_V)$. We also set $n_{\min}=\max\{0,N_j-|W|\}$ and $n_{\max}=\min\{|V|,N_j\}$ to be the smallest and the largest value of $N_V(\sigma)$ under the constraint that $N_{\Lambda_j}(\sigma)=N_j$. It is easy to check that $\frac{1}{k}\,\bar n\le n_{\max}-\bar n\le k\,\bar n$ and similarly for $\bar n-n_{\min}$.

In what follows we will consider the distribution of the number of particles in V under the measure ν. More precisely we define $\gamma=\{\gamma(n)\}$ to be the probability measure on $I=\{n\in[n_{\min},n_{\max}]:$ n is an integer $\}$, given by

$$\gamma(n):=\nu\,(N_V=n)$$

In order to obtain sharp bounds on $\gamma(n)$, we modify the chemical potential $\underline{\lambda}$ appearing in $\mu := \mu_\Lambda^{\tau,\underline{\lambda}}$ in an n–dependent way in such a way that the value n becomes the new average value of N_V under $\mu_\Lambda^{\tau,\underline{\lambda}}$. More precisely, given $n \in [n_{\min}, n_{\max}]$, let $\underline{\lambda}(n) = \{\lambda_1, \ldots, \lambda_{j-1}, \lambda_V, \lambda_W, \lambda_{j+1}, \ldots, \lambda_{j_{\max}}\}$ be a new chemical potential constant on the atoms of the new partition and such that

$$\begin{aligned} \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_{\Lambda_i}) &= N_i, \quad i = 1, \cdots, j_{\max} \\ \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_V) &= n \\ \mu_\Lambda^{\tau,\underline{\lambda}(n)}(N_W) &= N_j - n. \end{aligned} \tag{3.20}$$

It is then easy to check that γ can be written in the Gibbsian form, $\gamma(n) = e^{-H(n)}\varphi(n)$, where

$$\begin{aligned} H(n) &:= \sum_{i \neq j} (\lambda_i(n) - \lambda_i)\, N_i + \lambda_V\, n + \lambda_W (N_j - n) - \lambda_j N_j - \log\left(\frac{Z_\Lambda^{\tau,\underline{\lambda}(n)}}{Z_\Lambda^{\tau,\underline{\lambda}}}\right), \\ \varphi(n) &:= \frac{\mu_\Lambda^{\tau,\underline{\lambda}(n)}\,(\mathbf{N}_\Lambda = \mathbf{N}\,;\, N_V = n)}{\mu_\Lambda^{\tau,\underline{\lambda}}\,(\mathbf{N}_\Lambda = \mathbf{N})}. \end{aligned} \tag{3.21}$$

Finally, given $\epsilon \in (0,1)$, we consider for technical reasons the "ϵ–regularization" (ϵ here must be thougth to be close to one) of γ defined by

$$\gamma_\epsilon(n) := \begin{cases} \frac{1}{Z} e^{-H(n)} & \text{if } n \in I_\epsilon \\ \\ \gamma(n) & \text{otherwise} \end{cases} \tag{3.22}$$

where

$$\begin{aligned} Z &:= \frac{\sum_{n \in I_\epsilon} e^{-H(n)}}{\sum_{n \in I_\epsilon} \gamma(n)} \\ n_{\max}^\epsilon &= \bar{n} + \epsilon(n_{\max} - \bar{n}) \\ n_{\min}^\epsilon &= \bar{n} + \epsilon(n_{\min} - \bar{n}) \\ I_\epsilon &:= [n_{\min}^\epsilon, n_{\max}^\epsilon] \cap I \end{aligned}$$

We now recall, without any proof, some results on γ_ϵ established in section 3.4 and 4 of [CM2] under the mixing hypothesis $USMT(C,m,l)$ that will turn out to be useful. Then, we complete these properties by some others in order to prove that γ_ϵ satisfies a *convex condition*. Once convexity is established, we can appeal to proposition A.5 in the appendix to give a sharp bound on its logarithmic Sobolev constant.

The first result concerns the behaviour of the "potential" $H(n)$. As usual k will denote an arbitrary constant depending only on $C, m, l, \delta, \epsilon, \|\Phi\|_\infty$ and whose value varies from time to time.

Lemma 3.11. *For all $n \in I_\epsilon$,*

$$\begin{aligned} \frac{1}{k}\,\frac{n - \bar{n}}{\bar{n}} &\leq H(n+1) - H(n) \leq k\,\frac{n - \bar{n}}{\bar{n}} \qquad \text{for all } n \geq \bar{n} \\ \frac{1}{k}\,\frac{\bar{n} - n}{\bar{n}} &\leq H(n-1) - H(n) \leq k\,\frac{\bar{n} - n}{\bar{n}} \qquad \text{for all } n \leq \bar{n} \end{aligned}$$

The next two results show that the relative density between γ and its regularization is bounded uniformly in the size of I and that the tails of $\gamma(n)$ are at least exponential. More precisely

Lemma 3.12.

$$\frac{1}{k} \leq \inf_{n\in I} \frac{\gamma(n)}{\gamma_\epsilon(n)} \leq \sup_{n\in I} \frac{\gamma(n)}{\gamma_\epsilon(n)} \leq k$$

Lemma 3.13. *There exists a positive constant $\epsilon_0 = \epsilon_0(\delta, \|\Phi\|_\infty)$ such that $\forall \epsilon \in (\epsilon_0, 1)$*

$$\frac{\gamma(n+1)}{\gamma(n)} \leq \frac{1}{2} \quad \text{if} \quad n \in [n^\epsilon_{\text{max}}, n_{\text{max}}]$$
$$\frac{\gamma(n-1)}{\gamma(n)} \leq \frac{1}{2} \quad \text{if} \quad n \in [n_{\text{min}}, n^\epsilon_{\text{min}}]$$

Corollary 3.14.

$$\frac{\gamma_\epsilon(n+1)}{\gamma_\epsilon(n)} \leq k e^{-k\,\frac{n-\bar{n}}{\bar{n}}} \qquad \forall n \geq \bar{n} \tag{3.23}$$

and similarily for $\frac{\gamma_\epsilon(n-1)}{\gamma_\epsilon(n)}$, $n \leq \bar{n}$.

The next result shows that the normalization factor Z appearing in the definition of γ_ϵ is close to $\bar{n}^{\frac{1}{2}}$.

Lemma 3.15.

$$\frac{1}{k}\sqrt{\bar{n}} \leq Z \leq k\sqrt{\bar{n}}$$

Proof. We must consider two terms, $\sum_{n\in I_\epsilon} e^{-H(n)}$ and $\sum_{n\in I_\epsilon} \gamma(n)$.
By using lemma 3.11 one can easily get for all $n \in I_\epsilon$

$$\frac{1}{k} e^{-\frac{k}{\bar{n}}(\bar{n}-n)^2} \leq e^{-H(n)} \leq k\, e^{-\frac{1}{k\bar{n}}(\bar{n}-n)^2} \tag{3.24}$$

Thus, using a comparison with integrals and a change of variable, it easily follows that

$$\frac{1}{k}\sqrt{\bar{n}} \leq \sum_{n\in I_\epsilon} e^{-H(n)} \leq k\sqrt{\bar{n}} \tag{3.25}$$

On the other hand, it follows from lemma 3.12 that

$$\begin{aligned}
\sum_{n=n_{\text{min}}}^{n^\epsilon_{\text{min}}} \gamma(n) &= \gamma(n^\epsilon_{\text{min}}) \sum_{n=n_{\text{min}}}^{n^\epsilon_{\text{min}}} \frac{\gamma(n)}{\gamma(n^\epsilon_{\text{min}})} \\
&= \gamma(n^\epsilon_{\text{min}}) \sum_{n=n_{\text{min}}}^{n^\epsilon_{\text{min}}} \prod_{j=n+1}^{n^\epsilon_{\text{min}}} \frac{\gamma(j-1)}{\gamma(j)} \\
&\leq \gamma(n^\epsilon_{\text{min}}) \sum_{n=n_{\text{min}}}^{n^\epsilon_{\text{min}}} \left(\frac{1}{2}\right)^{n^\epsilon_{\text{min}}-n} \\
&\leq 2\,\gamma(n^\epsilon_{\text{min}})
\end{aligned}$$

A similar statement for $\sum_{n=n^\epsilon_{\max}}^{n_{\max}} \gamma(n)$ yields

$$\sum_{n\in I_\epsilon} \gamma(n) \geq 1 - 2\left(\gamma(n^\epsilon_{\min}) + \gamma(n^\epsilon_{\max})\right)$$

We are left with the problem of bounding from above $\gamma(n^\epsilon_{\min})$ and $\gamma(n^\epsilon_{\max})$. Because of lemma 3.13 it's enough to control $\gamma_\epsilon(n^\epsilon_{\min})$ and $\gamma_\epsilon(n^\epsilon_{\max})$. By using lemma 3.11 together with corollary 3.14, we get

$$\begin{aligned}
\gamma_\epsilon(n^\epsilon_{\min}) &= \gamma_\epsilon(\bar n) \prod_{j=n^\epsilon_{\min}+1}^{\bar n} \frac{\gamma_\epsilon(j-1)}{\gamma_\epsilon(j)} \\
&\leq k\, e^{-\frac{k}{2\bar n}(\bar n - n^\epsilon_{\min})^2} \\
&\leq k\, e^{-\frac{k}{\bar n}(n_{\min} - \bar n)^2} \\
&\leq k\, e^{-k(\bar n - n_{\min})}
\end{aligned}$$

In conclusion, we get that $\gamma(n^\epsilon_{\min}) \leq \frac{1}{8}$ provided that $\bar n$ is large enough. A similar statement for $\gamma(n^\epsilon_{\max})$ gives finally that

$$1 \geq \sum_{n\in I_\epsilon} \gamma(n) \geq \frac{1}{2}$$

This result together with (3.25) completes the proof. $\square$

From lemma 3.11 and 3.15, one can easily see that γ_ϵ is bell shaped around $\bar n$ at least if we restrict it to the interval I_ϵ. Our last result shows that, in some sense, this property holds also outside I_ϵ.

Lemma 3.16.

$$\frac{1}{k\sqrt{\bar n}} e^{-\frac{k}{\bar n}(\bar n - n)^2} \leq \gamma_\epsilon(n) \leq \frac{k}{\sqrt{\bar n}} e^{-\frac{1}{k\bar n}(\bar n - n)^2} \tag{3.26}$$

Proof. Thanks to lemma 3.11, 3.15 and the fact that $n_{\max} - \bar n$ is proportional to $\bar n$ one can see that (3.26) holds for all $n \in I_\epsilon$. Moreover, by using corollary 3.14 the upper bound of (3.26) holds for all $n \in I$. It is therefore enough to prove the lower bound of (3.26) for $n \in I \setminus I_\epsilon$.
For this purpose, we let $n \in [n^\epsilon_{\max}, n_{\max}]$ (the case $n \in [n_{\min}, n^\epsilon_{\min}]$ being similary) and we recall that, thanks to Proposition 3.6 of [CM2]),

$$k \frac{(|V| - n)(N_j - n)}{(n+1)(|W| - N_j + n + 1)} \leq \frac{\gamma(n+1)}{\gamma(n)}$$

Therefore

$$\begin{aligned}
\gamma(n) &\geq \gamma(n^\epsilon_{\max})\, e^{-k(n - n^\epsilon_{\max})} \prod_{n^\epsilon_{\max}}^{n-1} \frac{(|V| - i)(N_j - i)}{(i+1)(|W| - N_j + i + 1)} \\
&\geq \frac{k}{\sqrt{\bar n}} e^{-\frac{1}{k\bar n}(\bar n - n)^2}
\end{aligned}$$

where we have used (3.26) to get the exponential bound for $\gamma(n^\epsilon_{\max})$, Stirling formula and a some straightforward calculation to bound from below the product. The proof is complete. $\square$

We are finally in a position to state the main result of this paragraph.

Proposition 3.17. *Assume property $USMT(C,m,l)$. Then, for all $f : \Omega_\Lambda \mapsto \mathbb{R}$ that depend only on $N_V(\sigma)$, the following logarithmic Sobolev inequality holds*

$$\mathrm{Ent}_\gamma(f) \le k\,\bar{n} \sum_{n\in I} (\gamma(n) \wedge \gamma(n-1))\,[f(n) - f(n-1)]^2$$

Proof. Because of lemma 3.12 it is sufficient to prove the logarithmic Sobolev inequality for the regularized measure γ_ϵ.
Pick ϵ sufficiently close to one in such a way that lemma 3.13 holds. Then, thanks to corollary 3.14 and lemma 3.16, the regularized measure γ_ϵ satisfies the *convex hypothesis* $CONV(c,\bar{n})$ described in the appendix, for some c independent of $\bar{n}$ and I. Therefore we can apply Proposition A.5 in the appendix and get the result. □

4. On the grand canonical Laplace transform.

In this section we seek Gaussian bounds on quantities of the form $\mu\left(e^{tf}\right)$ where μ is the grand canonical Gibbs measure on some finite set and f is a mean zero function, namely bounds of the type

$$\mu\left(e^{tf}\right) \le e^{t^2 K_f}$$

Once bounds like the one above are proved, then we can transfer them to the *canonical* Laplace transform by means of proposition 3.3.
We first explain in some general terms our approach based on a combination of the so–called Herbst's approach and the DLR equation and then we consider some concrete cases that will play a key role in the recursive analysis of the logarithmic Sobolev constant for the canonical measure.

Our setting is as follows. Let $\Lambda \in \mathcal{F}_l$ be a finite set and, for a given boundary configuration τ and (possible vector) chemical potential $\underline{\lambda}$, let $\mu := \mu_\Lambda^{\tau,\underline{\lambda}}$. Let $\{V_\alpha\}_{\alpha\in I}$ be a collection of subsets of Λ such that $\mathrm{dist}(V_\alpha, V_\beta) \ge r+1$ for $\alpha \neq \beta$, r being the range of the interaction, and let $V = \cup_\alpha V_\alpha$. Let also $f : \Omega_\Lambda \mapsto \mathbb{R}$ be such that $\mu(f) = 0$ and define $\mathrm{Osc}_\alpha(f) := \|f - \mu_{V_\alpha}(f)\|_\infty$ and $\mathrm{Var}_\alpha(f) := \|\mu^\tau_{V_\alpha}(f,f)\|_\infty$.

Proposition 4.1. *Fix $t_0 > 0$. Then, for all $t \in [0, t_0]$*

(i)

$$\mu(e^{tf}) \le e^{t^2 K_f}$$

where $K_f = e^{2\|\Phi\|} \sum_{x\in\Lambda} \|\nabla_x f\|_\infty^2\, c_\mu$ and c_μ is the logarithmic Sobolev constant of μ w.r.t. to the Heat Bath rates.

(ii) Let $f_V(\tau) := \mu_V^\tau(f)$. Then

$$\mu(e^{tf}) \le e^{t^2 K'_f}$$

where

$$K'_f(t_0) = e^{2\|\Phi\|} \sum_{x\in\Lambda\setminus V} \|\nabla_x f_V\|_\infty^2\, c_\mu + \sum_\alpha \mathrm{Var}_\alpha(f)\, e^{t_0\, \mathrm{Osc}_\alpha(f)}$$

Remark. Notice that if we make the trivial choice $V_\alpha = \Lambda$, then (ii) becomes $\mu(e^{tf}) \le e^{t^2 K'_f}$ with $K'_f = \mu(f,f)e^{t_0 \operatorname{Osc}(f)}$. Such a choice makes sense when $\operatorname{Osc}(f)$ is independent of $|\Lambda|$. However, in most of the subequent application, $\operatorname{Osc}(f) = O(|\Lambda|)$ and a more refined choice of the sets V_α will be necessary.

Proof. In order to prove part (i) we appeal to the following lemma known as the Herbst's argument (see for istance section 6 in[An]).

Lemma 4.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite probability space and f a function on Ω. Assume that there exists $K > 0$ such that for all $t \in [0, t_0]$,*

$$\operatorname{Ent}_\mu(e^{tf}) \le K t^2 \mu(e^{tf})$$

Then, for all $t \in [0, t_0]$,

$$\mu(e^{tf}) \le e^{t\mu(f) + Kt^2}$$

Proof of the Lemma. Let $H(t) = \mu(e^{tf})$ be the Laplace transform of f. A simple computation gives

$$tH'(t) - H(t)\ln H(t) = \operatorname{Ent}(e^{tf}) \le K t^2 H(t) \ .$$

Dividing by $t^2 H(t)$ and writing $K(t) = \frac{\ln H(t)}{t}$, we get

$$K'(t) \le K \ .$$

Notice now that $K(t) \to \mu(f)$ as $t \to 0$. Consequently, a simple integration achieves the proof. $\square$

Back to the proof of the proposition, if we apply the definition of the logarithmic Sobolev constant c_μ we get

$$\operatorname{Ent}_\mu(e^{tf}) \le c_\mu \sum_{x\in\Lambda} \mu\big(c_x\,(\nabla_x e^{\frac{t}{2}f})^2\big) \le \big(e^{2\|\Phi\|} c_\mu \sum_{x\in\Lambda} \|\nabla_x f\|_\infty^2\big)\, t^2 \mu(e^{tf}) \tag{4.1}$$

where $c_x(\sigma) = \mu^\sigma_{\{x\}}(\sigma^x)$ are the flip rates for the Heat Bath dynamics. Thus, thanks to lemma 4.2, part (i) follows at once.

In order to prove part (ii) we proceed as follows. Denote by μ_V^τ the Gibbs measure on V with particle configuration τ in $\Lambda \setminus V$. By construction μ_V^τ is a product measure over the sets $\{V_\alpha\}_{\alpha\in I}$ that, for simplicity and omitting the superscript τ, we write as $\prod_\alpha \mu_\alpha$. Then we have

$$\mu(e^{tf}) = \mu\Big((\prod_\alpha \mu_\alpha)(e^{tf}) \Big)$$

Let us examine a generic term $\mu_\alpha(e^{tf})$. A simple Taylor expansion up to the second order gives

$$\mu_\alpha(e^{tf}) \le \exp\Big(t\mu_\alpha(f) + t^2 \operatorname{Var}_\alpha(f) e^{t_0 \operatorname{Osc}_\alpha(f)} \Big) \tag{4.2}$$

In order to iterate (over the index α) the above bound, we simply observe that $\mathrm{Var}_\beta(\mu_\alpha(f)) \leq \mathrm{Var}_\beta(f)$ and $\mathrm{Osc}_\beta(\mu_\alpha(f)) \leq \mathrm{Osc}_\beta(f)$. Thus

$$\mu\Big(\big(\prod_\alpha \mu_\alpha\big)(e^{tf})\Big) \leq e^{t^2 \sum_\alpha \mathrm{Var}_\alpha(f) e^{t_0 \,\mathrm{Osc}_\alpha(f)}} \mu\big(e^{t\mu_V^\tau(f)}\big) \tag{4.3}$$

It suffices now to apply part (i) to the new term $\mu(e^{t\mu_V^\tau(f)})$. $\square$

Remark. It is important to understand the difference between the two results given in proposition 4.1. Assume that condition $UMST(C,m,l)$ holds so that the logarithmic Sobolev constant c_μ is bounded from above uniformly in Λ (see e.g. [M] and references therein) and let $f = \sum_{x\in\Lambda} f_x$, where f_x is a mean zero local function with small support around x. For small values of the parameter t one expects, on the basis of a second order Taylor expansion, a bound of the form

$$\mu\left(e^{tf}\right) \leq e^{t^2 K_f}$$

with $K_f \approx \mu(f,f) \approx \sum_x \mu(f_x, f_x)$ because of the mixing assumption. Part (i) of the proposition proves the above Gaussian bound for *all* t, but with a new constant $K_f \approx c_\mu \sum_x \|\nabla_x f_x\|_\infty^2$. In "normal" situations that is quite satisfactory because K_f becomes of the same order of the heuristic guess. The problem arises when, anomalous, very large values (maybe depending on the size of Λ) of the chemical potential $\underline{\lambda}$ are considered. In that case the Gibbs measure μ becomes very close to the product of highly unfair Bernoulli measures and its logarithmic Sobolev constant c_μ gets very large (see e.g. [DiSa]) so that the Herbst's bound deteriorates. Part (ii) of the proposition partially takes care of these extreme situation, at least for not too large values of t, as we now explain (see also proposition 4.3 below for more technical details). Assume that each set V_α is a cube of side l_0 multiple of the basic length scale l and suppose that $f = \sum_\alpha f_\alpha$ where $f_\alpha = \sum_{\substack{x\in V_\alpha \\ d(x,V_\alpha^c)\geq \frac{1}{2}l_0}} f_x$. Then the constant K_f' appearing in part (ii) of the proposition becomes

$$K_f' = e^{2\|\Phi\|} c_\mu \sum_\alpha \sum_{x\notin V_\alpha} \|\nabla_x \mu_{V_\alpha}(f_\alpha)\|_\infty^2 + \sum_\alpha \mathrm{Var}_\alpha(f_\alpha)\, e^{t_0 \,\mathrm{Osc}_\alpha(f)}$$

The second term in the above expression is, apart from the constant $e^{t_0 \,\mathrm{Osc}_\alpha(f)}$, exactly the contribution given by a naive Taylor expansion. The first term is instead similar to the constant K_f discussed above but with the following important difference. The logarithmic Sobolev constant c_μ is now multiplied a term, $\sum_x \|\nabla_x \mu_{V_\alpha}(f_\alpha)\|_\infty^2$, which, apart from trivial constants, is different from zero only if $x \notin V_\alpha$ and in that case is proportional to $|V|\, e^{-ml_0}$ because of the mixing assumptions. Moreover, for extreme values of the chemical potential, the "mass" m can be taken proportional to the chemical potential itself so that the "effective" logarithmic Sobolev constant $e^{-ml_0} c_\mu$ becomes very small and the Gaussian bound becomes more precise.

4.1 An application at low density.

Here we discuss an application of our bound to a concrete case that will be important in the next section.

The setting is the standard multicanonical setting described at the beginning of section 3.
Given now a local function g with support Λ_g containing the origin and of diameter smaller than $2r$, r being the range of interaction, and an integer $j \le j_{\max}$, we define $G = \sum_{x\in\Lambda_j}[g_x - \mu(g_x)]$, where g_x is the translate of g by x. Let also $\rho_j := \frac{N_j}{|\Lambda_j|}$ and assume without loss of generality that $\rho_j < 1/2$.

Proposition 4.3. *Assume $USMT(C,m,l)$ and that there exists a constant $k = k(\|\Phi\|_\infty, C, m, l) > 0$ such that*

$$\mu(|g_x|) \le k\,\rho_j^2\,\|g\|_\infty \qquad \forall\, x \in \Lambda_j \text{ such that } \operatorname{dist}(x,\Lambda_j^c) \ge 2r$$
$$\mu(|g_x|) \le k\,\rho_j\,\|g\|_\infty \qquad \forall\, x \in \Lambda_j \text{ such that } \operatorname{dist}(x,\Lambda_j^c) \le 2r$$

Then, for any $t_0 > 0$ there exist positive constants $A = A(\|\Phi\|_\infty, C, m, l, t_0, r, \|g\|_\infty, \delta_0)$ and $B = B(\|\Phi\|_\infty, C, m, l, r, \delta_0)$ such that

$$\mu(e^{t\,G}) \le e^{t^2 AL^d \rho_j[\rho_j + \frac{\ln L}{L}]} \qquad \forall\, t \in [0,t_0]$$
$$\nu(e^{t\,G}) \le Be^{t^2 AL^d \rho_j[\rho_j + \frac{\ln L}{L}]} \qquad \forall\, t \in [0,t_0]$$

Remark. Although the proposition is stated for any value of the density $\rho_j \le \frac{1}{2}$, the most interesting application is when ρ_j is very small (possibly depending on L).

Proof. In what follows k will always denote a generic numerical constant depending only on $\|\Phi\|, C, m, l, t_0, r, d, \delta_0, \|g\|_\infty$ and whose value may vary in different estimates.
Let $\{C_\alpha\}_{\alpha\in I}$ be a chessboard–like partition of Λ_j into cubes of side $l_0 = 6l$, l being the basic length scale. A moment of thought suffices to convince oneself that we can partition the index set I into $2d$ subsets $\{I_i\}_{i=1}^{2d}$ in such a way that $\min_{\{\alpha\neq\alpha'\in I_i\}} \operatorname{dist}(C_\alpha, C_{\alpha'}) \ge l_0$ and $\min_{\alpha\in I_i} \operatorname{dist}(C_\alpha, \Lambda_j^c) \ge l_0$.
Accordingly we write $G = \sum_{i=1}^{2d} G^{(i)} + G^{\text{ext}}$, where $G^{\text{ext}} = \sum_{\substack{x\in\Lambda_j \\ \operatorname{dist}(x,\Lambda_j^c)\le l_0}} [g_x - \mu(g_x)]$ and $G^{(i)} = \sum_{\alpha\in I_i} g_\alpha$ with $g_\alpha = \sum_{x\in C_\alpha}[g_x - \mu(g_x)]$.
Thus, using Hölder inequality, we arrive at

$$\mu(e^{tG}) \le \mu(e^{(2d+1)\,t\,G^{\text{ext}}})^{\frac{1}{2d+1}} \prod_{i=1}^{2d} \mu(e^{(2d+1)\,t\,G^{(i)}})^{\frac{1}{2d+1}}$$

It is thus enough to bound from above a generic term $\mu(e^{(2d+1)\,t\,G^{(i)}})$ and the boundary term $\mu(e^{t\,G^{\text{ext}}})$.
Let us thus set $f := G^{(i)}$ and, for any $\alpha \in I_i$, let V_α be a cube of side $2l_0$ and having the same center of C_α. Thanks to our assumptions on the function g and the strog mixing, the constants appearing in the second part of proposition 4.1 satisfy the bounds

$$\begin{aligned} \operatorname{Var}_\alpha(f) &\le k\, l_0^d\, \rho_j^2 \\ \operatorname{Osc}_\alpha(f) &\le k\, l_0^d \\ \sup_{x\notin V_\alpha} \|\nabla_x \mu_\alpha(f)\|_\infty^2 &\le k\, \rho_j^{3l} \end{aligned} \tag{4.4}$$

where the latter inequality follows from any standard low activity expansion if ρ_j is small and from the mixing assumption if ρ_j is "moderate".
Let now $\mu_j := \mu^\tau_{\Lambda_j}$ and write $\mu(e^{tf}) = \mu\left(\mu_j(e^{tf})\right)$. It is not difficult to check that the logarithmic Sobolev constant of the measure μ_j, c_{μ_j}, is not larger than $k\ln(\frac{1}{\rho_j})$ uniformly in the boundary condition τ, because of the mixing assumption and the fact that $-\ln(p)$ is the order of magnitude of the logarithmic Sobolev constant of a Bernoulli measure of parameter p as $p \to 0$ (see e.g. [DiSa]). If we now apply the bound (ii) of proposition 4.1 to $\mu_j(e^{tf})$ we get

$$\mu\left(\mu_j(e^{tf})\right) \le \mu\left(e^{t\mu_j(f)}\right) e^{t^2\, k\rho_j^2|\Lambda_j|} \qquad \forall t \in [0, t_0]$$

It remains to bound the correction term $\mu\left(e^{t\mu_j(f)}\right)$. It suffices to apply part (i) of proposition 4.1 together with the observation that $c_\mu \le k \ln L$ because of the previous remarks and that

$$\sum_{x\in\Lambda_j^c} \|\nabla_x \mu_j(f)\|^2_\infty \le kL^{d-1}\rho_j^{6l}$$

because of the mixing assumption and the hypothesis $\min_{\alpha\in I_i} \mathrm{dist}(C_\alpha, \Lambda_j^c) \ge l_0$ (see also (4.4)). In conclusion

$$\mu\left(e^{t\,\mu_j(f)}\right) \le e^{t^2 k\rho_j^{6l} L^{d-1}\ln L} \le e^{t^2 k\rho_j^2|\Lambda_j|}$$

so that

$$\mu(e^{\,(2d+1)\,t\,G^{(i)}}) \le e^{\,t^2 k\rho_j^2|\Lambda_j|} \tag{4.5}$$

Let us finally analyze the boundary term $\mu(e^{\,t\,G^{\rm ext}})$. We could repeat word by word the argument that was used to bound the term $\mu\left(e^{t\mu_j(f)}\right)$ to get

$$\mu(e^{\,t\,G^{\rm ext}}) \le e^{\,t^2\, kL^{d-1}\ln L}$$

which is certainly smaller than the r.h.s. of (4.5) for $\rho_j^2 \ge \frac{\ln(L)}{L}$. If instead $\rho_j^2 \le \frac{\ln(L)}{L}$ then we proceed as follows. We decompose $G^{\rm ext}$ into the sum of a finite (only depending on r) number of terms $G^{{\rm ext},i}$, each of the form $G^{{\rm ext},i} = \sum_{x\in E_i}[g_x - \mu(g_x)]$ where E_i is a subset of the "layer" $\{x \in \Lambda_j$ such that $\mathrm{dist}(x, \Lambda_j^c) \le l_0\}$ and it has the property that $\min_{x,y\in E_i} \mathrm{dist}(x,y) \ge 2r$. The choice of the sets E_i is somewhat arbitrary but it causes no problem (just write $\mathbb{Z}^d = \cup_{x\in 2r\mathbb{Z}^d} Q_{2r}(x)$). Given $x \in E_i$, we let $V_x = \Lambda_g + x$ and we bound from above the quantity $\mu(e^{\,t\,G^{{\rm ext},i}})$ using part (ii) of proposition 4.1 with the sets $\{V_x\}_{x\in E_i}$ playing the role of the sets $\{V_\alpha\}_{\alpha\in I}$. With the obvious notation and thanks to our hypotheses on the function g we get

$$\begin{aligned} \mathrm{Var}_x(g_x) &\le k\|g\|^2_\infty\, \rho_j \\ \mathrm{Osc}_x(g_x) &\le 2\|g\|_\infty \\ \sup_{y\notin V_x} \|\nabla_y\, \mu_{V_x}(g_x)\|^2_\infty &\le k\|g\|^2_\infty\, \rho_j \end{aligned}$$

Thus

$$\mu(e^{\,t\,G^{{\rm ext},i}}) \le e^{t^2 c_\mu\, \rho_j\, L^{d-1}} \le e^{t^2 k\, \rho_j\, L^{d-1}\ln L} \qquad \forall\, t \in [0, t_0] \tag{4.6}$$

In the last inequality we have used the bound $c_\mu \le k \ln L$. In conclusion, by putting together (4.5) and (4.6) we get

$$\mu(e^{tG}) \le e^{t^2 AL^d \rho_j [\rho_j + \frac{\ln L}{L}]} \qquad \forall\, t \in [0, t_0]$$

for a suitable positive constant $A = A(\|\Phi\|_\infty, C, m, l, t_0, r, d, \|g\|_\infty, \delta_0)$.

In order to prove the analogous bound for the multicanonical measure ν, we observe that the function G can be written as $G = G_1 + G_2$, where G_1 and G_2 have the same expression of G but with the sum over x restricted to two halves of the set Λ_j. Then

$$\nu(e^{t\,G}) \le \nu(e^{2t\,G_1})^{\frac{1}{2}} \nu(e^{2t\,G_2})^{\frac{1}{2}}$$

and we can apply to each of the factors the bound (see proposition 3.3)

$$\nu(e^{2t\,G_i}) \le B\mu(e^{2t\,G_i}), \qquad i = 1, 2$$

The final result follows at once from the bound on the grand canonical expectation. □

4.2 An application to "normal" densities

Here we discuss another situation, similar to the one just analyzed, that will play an important role in the forthcoming sections. Contrary to proposition 4.3, the interest of the results stated below is when the density ρ_j is not too small. That should explains why, in most of the estimates given below, we allow certain numerical constant to depend in some unspecified way on ρ_j, without worrying about the case when $\rho_j \to 0$ as $L \to \infty$.

The setting is that already described in the previous paragraph but the function G is different. In order to define it, let $\{C_\alpha\}_{\alpha \in I}$ be a collection of cubes in Λ_j of sides $l_0 \gg r$ multiple of l, such that for any $\alpha \ne \beta$, $d(C_\alpha, C_\beta) \ge l_0^{\frac{1}{2}}$ and $d(C_\alpha, \partial\Lambda) \ge l_0^{\frac{1}{2}}$. Let $C_\alpha^{\rm int} := \{x \in C_\alpha : d(x, C_\alpha^c) \ge l_0^{1/4}\}$, let $C^{\rm int} = \cup_\alpha C_\alpha^{\rm int}$ and denote by $n_c = |I|$ be the number of such cubes.

Given $\eta \in \Omega_\Lambda$ and $s \in [0, l_0^d]$, write $\mu_\alpha^{\eta,s}(\cdot) = \mu_{C_\alpha}^{\eta,\lambda(\eta,s)}(\cdot)$ to denote the grand canonical Gibbs measure on C_α with boundary condition η and constant chemical potential $\lambda(\eta, s)$ such that $\mu_{C_\alpha}^{\eta,\lambda(\eta,s)}(N_{C_\alpha}) = s$. Whenever s is also an integer, say $s = n \in [0, 1, \dots, \lfloor l_0^d \rfloor]$, we will use the standard notation $\nu_{C_\alpha,n}^\eta$ for the corresponding canonical Gibbs measure.

Consider now a local function g with support Λ_g containing the origin and of diameter smaller than $2r$ such that $g(\underline{0}) = 0$ where $\underline{0}$ is the configuration identically equal to zero. Then we define $g^\delta(\sigma) = g_x(\sigma) - \delta\sigma(x)$ and

$$\begin{aligned}
\xi_\alpha^\delta(\eta, s) &= \sum_{x \in C_\alpha^{\rm int}} \mu_\alpha^{\eta,s}(g_x^\delta)\,, \quad \xi_\alpha^\delta(\eta) := \xi_\alpha^\delta(\eta, N_{C_\alpha}(\eta))\,, \quad \xi_\alpha^{\delta,\underline{0}}(\eta) := \xi_\alpha^\delta(\underline{0}, N_{C_\alpha}(\eta)) \\
g_\alpha^\delta(\eta, n) &= \sum_{x \in C_\alpha^{\rm int}} \nu_{C_\alpha,n}^\eta(g_x^\delta) \quad ; \quad g_\alpha^\delta(\eta) := g_\alpha^\delta(\eta, N_{C_\alpha}(\eta))
\end{aligned} \tag{4.7}$$

where δ is such that for one (and therefore all) cube C_α $\frac{d}{ds}\xi_\alpha^{\delta,\underline{0}}(s)|_{s=\bar n} = 0$, $\bar n = \rho_j l_0^d$. By definition

$$\delta = \frac{\mu_{C_\alpha}^{\underline{0},\lambda(\underline{0},\bar n)}(\sum_{x \in C_\alpha^{\rm int}} g_x, N_{C_\alpha})}{\mu_{C_\alpha}^{\underline{0},\lambda(\underline{0},\bar n)}(N_{C_\alpha^{\rm int}}, N_{C_\alpha})}$$

so that, if condition $USMT(C,m,l)$ holds, $\delta \leq \delta(C,m,l,\|g\|_\infty)$ uniformly in L.
Finally we define

$$
\begin{aligned}
G_\nu(\eta) &= \sum_\alpha [\, g_\alpha^\delta(\eta) - \mu(g_\alpha^\delta)\,] \\
G_\mu(\eta) &= \sum_\alpha [\, \xi_\alpha^\delta(\eta) - \mu(\xi_\alpha^\delta)\,] \\
G_{\mu}^{\underline{0}}(\eta) &= \sum_\alpha [\, \xi_\alpha^{\delta,\underline{0}}(\eta) - \mu(\xi_\alpha^{\delta,\underline{0}})\,] \\
G^{\rm ext}(\eta) &= \sum_{x\in\Lambda_j\setminus C^{\rm int}} [\, g_x^\delta - \mu(g_x^\delta)\,]
\end{aligned}
\tag{4.8}
$$

Proposition 4.4. *Assume $USMT(C,m,l)$ and fix $t_0>0$. Then there exists a constant $A=A(C,m,l,t_0,\|g\|_\infty,\rho_j)$ such that*

$$
\begin{array}{lll}
(i) & \mu(e^{t\,G^{\rm ext}}) \leq e^{t^2 L^d A l_0^{-\frac12}} & \forall\, t \\
(ii) & \mu\big(e^{t\,(G_\nu - G_\mu)}\big) \leq e^{t^2 L^d A l_0^{-d}} & \forall\, t\in[0,t_0] \\
(iii) & \mu\big(e^{t\,(G_\mu^{\underline{0}} - G_\mu)}\big) \leq e^{t^2 L^d A l_0^{-1}} & \forall\, t \\
(iv) & \mu\big(e^{t\,G_\mu^{\underline{0}}}\big) \leq e^{t^2 L^d A l_0^{-d}} & \forall\, t\in[0,t_0\, l_0^{-d}]
\end{array}
$$

The same bounds hold, but with an extra factor $B=B(\|\Phi\|_\infty,C,m,l,r,\delta)$ in front of the exponential, if μ is replaced by the multicanonical measure ν.

Proof. In what follows k will always denote a generic numerical constant depending only on $\|\Phi\|,C,m,l,t_0,r,d,\|g\|_\infty$ and whose value may vary in different estimates.

(i). Let $\mu_j := \mu^\tau_{\Lambda_j}$ and write $\mu(e^{t\,G^{\rm ext}}) = \mu\left(\mu_j(e^{t\,G^{\rm ext}})\right)$. We simply apply part (i) of proposition 4.1 to the function $\mu_j(e^{t\,G^{\rm ext}})$ together with the observation that, thanks to the mixing condition, $\sup_\tau c_{\mu_j^\tau} \leq k(C,m,l,\rho_j,\delta)$. The reason why we refrain to apply directly such an argument to $\mu(e^{t\,G^{\rm ext}})$ is that it is possible that the global logarithmic Sobolev constant c_μ is very large, up to order $k\ln L$, because in some atom Λ_j, $j\neq i$, the density is either very small or very close to one. As we will see, conditioning on the exterior of Λ_j is a simple way to take care of such a nasty possibility.
Since

$$
\sum_{x\in\Lambda_j} \|\nabla_x G^{\rm ext}\|_\infty^2 \leq kL^d l_0^{-\frac12}
$$

we get

$$
\mu\big(\mu_j(e^{t\,G^{\rm ext}})\big) \leq \mu(e^{t\mu_j(G^{\rm ext})})e^{t^2 kL^d l_0^{-\frac12}}
\tag{4.9}
$$

The term $\mu(e^{t\mu_j(G^{\rm ext})})$ can also be estimated by the Herbst argument if we recall the observation that, under the mixing assumption, $c_\mu \leq k\ln L$ (see the proof of proposition 4.3). We get

$$
c_\mu \sum_{x\in\Lambda} \|\nabla_x \mu_j^\tau(G^{\rm ext})\|_\infty^2 \leq kL^{d-1}\ln L
$$

so that

$$\mu\big(e^{t\mu_j(G^{\text{ext}})}\big) \le e^{t^2 k L^{d-1}\ln L} \tag{4.10}$$

Clearly (4.9) and (4.10) complete the proof.

(ii). First we observe that, because of the equivalence of ensembles (see 3.2 point (1)), $\sup_\alpha \|\big(g^\delta_\alpha - \xi^\delta_\alpha\big)\|_\infty \le k$. Next, given α, let V_α be an $(\frac{1}{2}l_0^{\frac{1}{2}} - 2r)$–neighborhood of the cube C_α. For l_0 large enough the sets $\{V_\alpha\}$ satisfy the condition of proposition 4.1. Moreover

$$\|\nabla_x \mu_{V_\alpha}\big(g^\delta_\alpha - \xi^\delta_\alpha\big)\|_\infty \le \begin{cases} k\, e^{-\frac{1}{2}m l_0^{\frac{1}{2}}} & \text{if } x \in \partial_r^+ V_\alpha \\ 0 & \text{otherwise} \end{cases}$$

for l_0 large enough.
Thus, part (ii) of proposition 4.1 together the usual conditioning $\mu(\cdot) = \mu\big(\mu_j(\cdot)\big)$ give (see also the argument leading to formula (4.10))

$$\begin{aligned} \mu\big(e^{t(G_\nu - G_\mu)}\big) &\le e^{t^2 L^d k\, l_0^{-d}} \mu\big(e^{t\mu_j(G_\nu - G_\mu)}\big) \\ &\le e^{t^2 L^d k\,(l_0^{-d} + L^{-1}\ln L)} \\ &\le e^{t^2 L^d A l_0^{-d}} \qquad \forall\, t \in [0, t_0] \end{aligned} \tag{4.11}$$

for L large enough.

(iii). Notice that

$$\|\nabla_x\big(\xi^\delta_\alpha(\eta) - \xi^{\delta,\underline{0}}_\alpha(\eta)\big)\|_\infty \le \begin{cases} k & \text{if } x \in \partial_r^+ C_\alpha \\ k\, l_0^{-1} & \text{if } x \in C_\alpha \end{cases}$$

because of lemma 3.1 point (i) of (1) and (2). Therefore we can apply the same reasoning leading to (4.11) to get

$$\mu\big(e^{t\,(G^{\underline{0}}_\mu - G_\mu)}\big) \le e^{t^2 L^d A\,(l_0^{-1} + L^{-1}\ln L)} \qquad \forall\, t \tag{4.12}$$

and the sought bound follows for L large enough.

(iv). We want to use part (ii) of proposition 4.1.

As in the proof of point (ii), let V_α be an $(\frac{1}{2}l_0^{\frac{1}{2}} - 2r)$–neighborhood of the cube C_α. For l_0 large enough the sets $\{V_\alpha\}$ satisfy the condition of proposition 4.1 and because of the mixing assumption

$$\|\nabla_x \mu_{V_\alpha}\big(\xi^{\delta,\underline{0}}_\alpha\big)\|_\infty \le \begin{cases} k\, e^{-\frac{1}{2}m l_0^{\frac{1}{2}}} & \text{if } x \in \partial_r^+ V_\alpha \\ 0 & \text{otherwise} \end{cases}$$

for l_0 large enough.
Let us now compute $\mathrm{Var}_\alpha(\xi^{\delta,\underline{0}}_\alpha)$ and $\mathrm{Osc}_\alpha(\xi^{\delta,\underline{0}}_\alpha)$. For this purpose we first observe, that, by the very definition of δ and the constance of the chemical potential inside Λ_j, we have (see point (ii) of part (2) of lemma 3.1)

$$\xi^\delta_\alpha(\underline{0}, n) - \xi^\delta_\alpha(\underline{0}, \bar{n}) = \int_{\bar{n}}^{N_\alpha} ds \int_{\bar{n}}^{s} dt\, \frac{d^2}{dt^2}\xi^\delta_\alpha(\underline{0}, t) \le k\,\frac{(n - \bar{n})^2}{\bar{n}} \tag{4.13}$$

Therefore $\mathrm{Osc}_\alpha(\xi_\alpha^{\delta,\underline{0}}) \le k l_0^d$ and $(n_\alpha := N_{C_\alpha}(\eta))$

$$\begin{aligned}
\mathrm{Var}_\alpha(\xi_\alpha^{\delta,\underline{0}}) &\le \frac{k}{l_0^{2d}} \,\|\mu_{V_\alpha}\big((n_\alpha - \bar n)^4\big)\|_\infty \\
&\le \frac{k}{l_0^{2d}} \,\|\mu_{V_\alpha}\Big(\big(n_\alpha - \mu_{V_\alpha}(n_\alpha)\big)^4\Big)\|_\infty + \frac{k}{l_0^{2d}} \,\|\mu_{V_\alpha}(n_\alpha) - \bar n\|_\infty^4 \\
&\le k
\end{aligned} \tag{4.14}$$

again because of the mixing condition. The statement now follows at once from point (ii) of proposition 4.1 applied to $f = G_{\mu}^{\underline{0}}$.

Finally the statements for the multicanonical measure follow exactly as in the proof of proposition 4.3. $\square$

5. On the covariance of f^2 with sums of local functions.

In this section we discuss some important bounds on covariances of the form $\nu(f^2, G)$, where ν is a multicanonical measure on $\Lambda = \cup_{j=1}^{j_{\max}} \Lambda_j$ as in the standard multicanonical setting of section 3, f an arbitrary function with $\nu(f^2) = 1$ and $G = \sum_{x\in\Lambda_j} g_x$ for some $1 \le j \le j_{\max}$, where g_x is the translate by x of a local function g with small support around the origin.

Roughly speaking our main goal is to bound $\nu(f^2, G)^2$ in terms of the only quantities that enter in the logarithmic Sobolev inequality, namely the entropy $\mathrm{Ent}_\nu(f^2)$ and the Dirichelt form $\mathcal{E}_\nu(f,f)$, but we want to keep track of the right dependence on the volume or, more precisely, on the number of particles N_j in Λ_j. Moreover, for reasons that will become more clear later in the paper, we have another constraint in that $\mathrm{Ent}_\nu(f^2)$ must appear multiplied by a very small constant times the volume.

In order to appreciate the difficulty of the problem, we notice that, since f enters as f^2, one of the natural tool to bound covariances, namely Schwarz inequality, becomes useless since no L^p–norm of f, $p > 2$, enters into the logarithmic Sobolev inequality. This is precisely one of the main technical difference and new challenge between the Poincaré inequality (where f appears linearly) and the logarithmic Sobolev inequality for conservative stochastic dynamics.

A natural counterpart to Schwarz inequality in this context is the so called *entropy inequality* that can be stated as follows.

Lemma 5.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite probability space. Then, for any $t > 0$ and any real valued functions f, G on Ω with $\mu(f^2) = 1$,*

$$\mu(f^2 G) \le \frac{1}{t} \ln\big(\mu(e^{tG})\big) + \frac{1}{t}\,\mathrm{Ent}_\mu(f^2)\ . \tag{5.1}$$

Proof. It is an immediate consequence of the following Young's inequality, valid for all $u > 0$ and all v : $uv \le u \ln u + e^v - u$. $\square$

In our case we can assume G of zero mean w.r.t. μ (because we are taking the covariance of G with f^2) so that it is natural to expect (see section 4) a Gaussian bound of the form $\nu(e^{tG}) \le e^{t^2 K|\Lambda_j|}$

for all $t > 0$ and some constant K that may depend on the particle density. If that is the case, lemma 5.1 gives

$$\nu(f^2 G) \le tK|\Lambda_j| + \frac{1}{t}\,\mathrm{Ent}_\nu(f^2) \qquad \forall\, t>0 \tag{5.2}$$

If we finally optimize over the free parameter t, namely we take $t^2 = \frac{1}{K|\Lambda_j|}\,\mathrm{Ent}_\nu(f^2)$, and we assume, without loss of generality, that $\nu(f^2G) \ge 0$, we get

$$\nu(f^2G)^2 \le 4K|\Lambda_j|\,\mathrm{Ent}_\nu(f^2) \tag{5.3}$$

Let us pause for a moment to clarify (for the alert reader only) more explicitely how we plan to use the results of section 4. In that section in fact, Gaussian bounds on the canonical Laplace transform of G were established under the hypothesis of vanishing grand canonical (not of the canonical) average of G. Moreover the resulting estimates are distorted Gaussian bounds because of the presence of an extra constant B in front of the exponential. Thus, in reality, (5.2) is slightly more complicated and looks like

$$\nu(f^2, G) \le \mu(G) + \frac{1}{t}\ln B + tK|\Lambda_j| + \frac{1}{t}\,\mathrm{Ent}_\mu(f^2) \qquad \forall\, t>0$$

The two extra terms, $\mu(G)$ and $\frac{1}{t}\ln B$ are just a little nuisance; the first one can in fact be handled quite easily using the equivalence of ensembles while the second one becomes harmless by choosing the "optimal" t always greater than $|\Lambda_j|^{-\frac{1}{2}}$.

Let us continue our informal discussion. Let $t_*^2 = \frac{1}{|\Lambda_j|}\,\mathrm{Ent}_\nu(f^2)$ and let t_0 be some large number independent of the volume and particle density. If $t_* \ge t_0$, *i.e.* $|\Lambda_j| \le \frac{1}{t_0^2}\,\mathrm{Ent}_\nu(f^2)$, then trivially

$$\nu(f^2G)^2 \;\le\; \|G\|_\infty^2 \;\le\; \|g\|_\infty^2|\Lambda_j| \;\le\; \|g\|_\infty^2\frac{1}{t_0^2}\,|\Lambda_j|\;\mathrm{Ent}_\nu(f^2) \tag{5.4}$$

which is like (5.3) but with a smaller constant in front of $|\Lambda_j|\,\mathrm{Ent}_\nu(f^2)$ if t_0 is large enough. In other words, if $t_* \ge t_0$ a simple L_∞ estimate gives a better result than the entropy bound.

In order to understand this point we remark that, on the basis of the central limit theorem and for "normal" values of the particle density in Λ_j, one expects the distribution $dP(G)$ of the random variable $\sum_{x\in\Lambda_j} g_x$ to be close to a centered gaussian with variance proportional to $|\Lambda_j|$. If this is the case, then, for t large enough, the distorted distribution $dP_t(G) \propto e^{tG}dP(G)$ becomes concentrated on the largest value of G and the gaussian bound $\nu(e^{tG}) \le e^{t^2K|\Lambda_j|}$ becomes unnatural and certainly worst than the trivial one $\nu(e^{tG}) \le e^{t\|G\|_\infty}$. On the contrary, for "moderate" values of t, the distortion only moves the center of the gaussian and in this case the entropy inequality will perform better.

Thus, in what follows, our strategy will always be, roughly speaking, the following. Depending on the ratio between the entropy and the volume, we will either apply the entropy bound with the optimal t and appeal several times to the results of section 4 or we will apply the trivial L_∞ bound. It remains to explain how we get in both cases a small constant in front of the entropy. For small values of the density or large values of the entropy it will follow quite easily from the results of section 4 (see proposition 4.3). In all the other cases we will have to appeal to a partial average argument, almost identical to the one used in [LY] and [CM2] under the name of "two–blocks estimates", in order to reduce the fluctuations of the function G.

We now explain more precisely our results.

5.1 Low density case.

Here we discuss our first result in the low density regime.

Proposition 5.2. *Assume $USMT(C,m,l)$ and that there exists a constant $k = k(\|\Phi\|_\infty, C, m, l) > 0$ such that*

$$\begin{aligned} &\mu(|g_x|) \le k\,\rho_j^2\,\|g\|_\infty \qquad \forall\, x \in \Lambda_j \text{ such that } \operatorname{dist}(x, \Lambda_j^c) \ge 2r \\ &\mu(|g_x|) \le k\,\rho_j\,\|g\|_\infty \qquad \forall\, x \in \Lambda_j \text{ such that } \operatorname{dist}(x, \Lambda_j^c) \le 2r \\ &\|G\|_\infty \le kN_j \end{aligned}$$

Then, for any $\epsilon > 0$, there exists C_ϵ and ρ_0 such that for all $\rho_j < \rho_0$, for all function f with $\nu(f^2) = 1$,

$$\nu(f^2, G)^2 \le N_j\left(C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\right)$$

for L large enough.

Proof. As usual, in what follows, k will always denote a generic positive numerical constant depending only on $\|\Phi\|, C, m, l, r, d, \delta_0, \|g\|_\infty$ and whose value may vary in different estimates.
Fix $\epsilon > 0$ and f with $\nu(f^2) = 1$. Let $M > 0$ be an arbitrary, large constant that will be fixed further on and let $t_*^2 = \frac{1}{N_j}\left(1 \vee M\,\mathrm{Ent}_\nu(f^2)\right)$. We distinguish between two cases.

In the first one, we suppose $t_* \ge M$. According to the general discussion we can safely apply an L_∞ bound to get

$$\nu\left(f^2, G\right)^2 \le 2\|G\|_\infty^2 \le kN_j^2 \le \frac{k}{M} N_j\,\mathrm{Ent}_\nu(f^2) \tag{5.5}$$

where we have used the hypotheses on $\|G\|_\infty$ and the definition of t_*.

In the second case, we suppose $t_* \le M$ and we first write

$$\nu(f^2, G)^2 \le 2\nu\left(f^2 G\right)^2 + 2\nu(G)^2 \tag{5.6}$$

Using the equivalence of ensembles, (see point 1 of proposition 3.3) and the fact that $\mu(G) = 0$, $|\nu(G)| \le k$ for some constant k. Thus, we can focus our attention on the first term $\nu\left(f^2 G\right)^2$.
By applying proposition 4.3 with $t_0 = M$, we get that there exist constants A, B depending on $\|\Phi\|_\infty, C, m, l, t_0, r, \|g\|_\infty, \delta_0$ such that

$$\nu(e^{t_* G}) \le Be^{t_*^2 AL^d \rho_j\left[\rho_j + \frac{\ln L}{L}\right]} = Be^{t_*^2 AN_j\left[\rho_j + \frac{\ln L}{L}\right]}$$

Thus, by applying lemma 5.1 with $t = t_*$, we get

$$\begin{aligned}\nu\left(f^2G\right)^2 &\le \left(\frac{1}{t_*}\ln B + t_* A N_j\,[\rho_j + \frac{\ln L}{L}] + \frac{1}{t_*}\operatorname{Ent}_\nu(f^2)\right)^2 \\ &\le kN_j\left(1 + \left(MA^2[\rho_j + \frac{\ln L}{L}]^2 + \frac{1}{M}\right)\operatorname{Ent}_\nu(f^2)\right)\end{aligned} \tag{5.7}$$

Now, from (5.6), (5.5) and (5.7), one can choose first M large enough and then ρ_j small enough and L large enough in order to conclude the proof. □

5.2 Normal density.

Here we treat instead the case of "normal" density ρ_j, namely we assume that $\rho_j \ge \rho_0$ for some constant ρ_0 independent of L.

Proposition 5.3. *Assume $USMT(C, m, l)$. Then, for any $\epsilon > 0$ and any $0 < \rho_0 < \frac{1}{2}$, there exists C_ϵ such that for all $\rho_0 \le \rho_j \le \frac{1}{2}$,*

$$\nu(f^2, G)^2 \le |\Lambda_j|\left[C_\epsilon + \epsilon\operatorname{Ent}_\nu(f^2) + L^2\mathcal{E}_\nu(f, f)\right]$$

Proof. Fix f with $\nu(f^2) = 1$, together with $\epsilon > 0$, $\rho_0 > 0$ and M large enough (how large will be specified later on).

As in proposition 5.2 we define $t_*^2 = \frac{1}{|\Lambda_j|}\left(1 \vee M\operatorname{Ent}_\nu(f^2)\right)$ where the number of particles N_j has been replaced by the volume $|\Lambda_j|$ because we are assuming that $\rho_j \ge \rho_0$.

If $t_* \ge M$, then we can proceed as in (5.5) to get

$$\nu\left(f^2, G\right)^2 \le \frac{k}{M}|\Lambda_j|\operatorname{Ent}_\nu(f^2) \tag{5.8}$$

and the proof is finished provided that M was taken large enough.

Let us now examine the much more complicate case of $t_* \le M$.

We define $\{C_\alpha\}_{\alpha\in I}$ to be a collection of cubes of side $l_0 \gg r$ multiple of l inside Λ_j, such that

$$\begin{aligned} &i) && \operatorname{dist}(C_\alpha, C_\beta) \ge l_0^{\frac{1}{2}} && \forall\,\alpha \ne \beta \\ &ii) && \operatorname{dist}(C_\alpha, \partial\Lambda_j) \ge l_0^{\frac{1}{2}} && \forall\,\alpha \\ &iii) && |\Lambda_j \setminus \cup_\alpha C_\alpha| \le l_0^{-\frac{1}{2}}|\Lambda_j| && \end{aligned}$$

Clearly such a construction is always possible. Define also $C_\alpha^{\rm int} = \{x \in C_\alpha \,:\, d(x, C_\alpha^c) \ge l_0^{1/4}\}$ and $C^{\rm int} = \cup_\alpha C_\alpha^{\rm int}$.

Next we observe that, without loss of generality, we can replace G by $G - \delta\, N_{\Lambda_j}$, δ being an arbitrary constant independent of x, because $N_{\Lambda_j} = N_j$ almost surely. Our choice of δ will be made later (see

(5.16)) but we anticipate that it will be almost independent of l_0 and that under $USMT(C,m,l)$, there exists a constant $\delta(C,m,l,\|g\|_\infty)$ such that $\delta \le \delta(C,m,l,\|g\|_\infty)$ uniformly in L. Finally we set $g_x^\delta(\sigma) := g_x(\sigma) - \delta\sigma(x)$ and

$$\begin{aligned} G^{\rm ext} &= \sum_{x\in\Lambda_j\setminus C^{\rm int}} \left[\, g_x^\delta - \mu(g_x^\delta)\,\right] \\ G^{\rm int} &= \sum_{x\in C^{\rm int}} \left[\, g_x^\delta - \mu(g_x^\delta)\,\right] \end{aligned}$$

Then we write

$$\begin{aligned} \nu\left(f^2, G^{\rm ext}\right)^2 &\le 2\,\nu\left(f^2 G^{\rm ext}\right)^2 + 2\,\nu\left(G^{\rm ext}\right)^2 \\ &\le k|\Lambda_j|\left[1 + \big(l_0^{-1}K + \frac{1}{M}\big)\,{\rm Ent}_\nu(f^2)\right] + k l_0^{-1} \\ &\le |\Lambda_j|\,\big(k + \epsilon\,{\rm Ent}_\nu(f^2)\big) \end{aligned}$$

where K is some constant independent of l_0 but possibly dependent on M and the last inequality holds if M and l_0 are chosen appropriately.
Above we have applied lemma 5.1 with $t = t_*$ together with proposition 4.4 point (i) to bound the first term and the equivalence of ensembles (see proposition 3.2 point 1) together with iii) above to bound the second one.

We now turn to the relevant term $\nu(f^2, G^{\rm int})^2$.
Let $\mathcal{F}_0$ be the σ–algebra generated by the the random variables $\{\sigma(x)\}_{x\in\Lambda\setminus\cup_\alpha C_\alpha}$, $\{N_\alpha\}_{\alpha\in I}$, where $N_\alpha(\sigma) := \sum_{x\in C_\alpha}\sigma(x)$. Then, by the formula for the conditonal covariance, we get

$$\nu\left(f^2, G^{\rm int}_\delta\right)^2 \le 2\,\nu\left(\nu\left(f^2, G^{\rm int}\,|\,\mathcal{F}_0\right)\right)^2 + 2\,\nu\left(f^2, \nu\left(G^{\rm int}\,|\,\mathcal{F}_0\right)\right)^2 \;. \tag{5.9}$$

For simplicity let $\nu_0(\cdot) := \nu(\cdot\,|\,\mathcal{F}_0)$, $f_0^2 := f^2/\nu_0(f^2)$ and $s_*^2 := \frac{M}{|\Lambda_j|}\,{\rm Ent}_{\nu_0}(f_0^2)$. Then the entropy inequality (Lemma 5.1) gives

$$\nu_0(f_0^2, G^{\rm int}) \le \frac{1}{s_*}\ln\left(\nu_0\left(e^{s_*(G^{\rm int} - \nu_0(G^{\rm int}))}\right)\right) + \frac{1}{s_*}\,{\rm Ent}_{\nu_0}(f_0^2)$$

Notice that ν_0 is the product of standard canonical measures on each cube C_α with a certain number of particles and boundary conditions. Thus a simple Taylor expansion gives

$$\nu_0\left(e^{s_*(G^{\rm int} - \nu_0(G^{\rm int}))}\right) \le e^{K s_*^2 |\Lambda_j|}$$

where K now may depend on l_0. Thus

$$\nu_0(f_0^2, G^{\rm int}) \le K|\Lambda_j| s_* + \frac{1}{s_*}\,{\rm Ent}_{\nu_0} f_0^2$$

and similarly for $G^{\rm int}$ replaced by $-G^{\rm int}$. The definition of s_* yields

$$\nu_0(f_0^2, G^{\rm int})^2 \le 2K^2\big(M + \frac{1}{M}\big)|\Lambda_j|\,{\rm Ent}_{\nu_0}(f_0^2)$$

Finally, since ν_0 is a product measure it certainly satisfies the logarithmic Sobolev inequality $\mathrm{Ent}_{\nu_0}(f_0^2) \le C(l_0)\mathcal{E}_{\nu_0}(f_0, f_0)$, and we can conclude that

$$\nu\left(\nu\left(f^2, G^{\mathrm{int}} \mid \mathcal{F}_0\right)\right)^2 \le K'\mathcal{E}_\nu(f,f) \tag{5.10}$$

where $K' = K'(\|\Phi\|_\infty, \|g\|_\infty, r, l_0, M)$.

The second term in the r.h.s. of (5.9) needs some more reductions. We recall first some definitions introduced in section 4.
Given $\eta \in \Omega_{\Lambda_j}$ and $n \in [0, \dots, l_0^d]$, write $\mu_\alpha^{\eta,n}(\cdot) := \mu_{C_\alpha}^{\eta,\lambda(\eta,n)}(\cdot)$ to denote the grand canonical Gibbs measure on C_α with boundary condition η and constant chemical potential $\lambda(\eta,n)$ such that $\mu_{C_\alpha}^{\eta,\lambda(\eta,n)}(N_{C_\alpha}) = n$. We will use the standard notation $\nu_{C_\alpha,n}^\eta$ for the corresponding canonical Gibbs measure.
With this notation we define (see (4.7)),

$$\xi_\alpha^\delta(\eta,n) = \sum_{x \in C_\alpha^{\mathrm{int}}} \mu_\alpha^{\eta,n}(g_x - \delta\sigma(x)) \quad ; \quad \xi_\alpha^\delta(\eta) := \xi_\alpha^\delta(\eta, N_{C_\alpha}(\eta))$$

$$g_\alpha^\delta(\eta,n) = \sum_{x \in C_\alpha^{\mathrm{int}}} \nu_{C_\alpha,n}^\eta(g_x - \delta\sigma(x)) \quad ; \quad g_\alpha^\delta(\eta) := g_\alpha^\delta(\eta, N_{C_\alpha}(\eta))$$

and

$$G_\nu(\eta) := \nu\left(G^{\mathrm{int}} \mid \mathcal{F}_0\right) = \sum_\alpha [\, g_\alpha^\delta(\eta) - \mu(g_\alpha^\delta)\,]$$

$$G_\mu(\eta) := \sum_\alpha [\, \xi_\alpha^\delta(\eta) - \mu(\xi_\alpha^\delta)\,]$$

Then we write

$$\begin{aligned}\nu(f^2, \nu(G^{\mathrm{int}} \mid \mathcal{F}_0))^2 &= \nu\left(f^2, G_\nu\right)^2 \\ &\le 4\nu\left(f^2(G_\nu - G_\mu)\right)^2 + 4\left(\nu(G_\nu - G_\mu)\right)^2 + 2\nu\left(f^2, G_\mu\right)^2\end{aligned} \tag{5.11}$$

Let us examine the three terms separately.
Using lemma 5.1 with $t = t_*$ combined with (ii) of proposition 4.4, we can bound the first term by

$$4\nu\left(f^2(G_\nu - G_\mu)\right)^2 \le k|\Lambda_j| \left[1 + \left(\frac{MA^2}{l_0^{2d}} + \frac{1}{M}\right) \mathrm{Ent}_\nu(f^2)\right] \tag{5.12}$$

where $A = A(C, m, l, M, \|g\|_\infty, \rho_0)$ is the constant appearing in proposition 4.4.
Because of the equivalence of ensembles (see point (1) of proposition 3.2) and the fact that $\mu(G_\nu - G_\mu) = 0$

$$4\left(\nu(G_\nu - G_\mu)\right)^2 \le k l_0^{2d} \tag{5.13}$$

In conclusion, by a suitable choice of M and l_0 we get

$$4\nu\left(f^2(G_\nu - G_\mu)\right)^2 + 4\left(\nu(G_\nu - G_\mu)\right)^2 \le |\Lambda_j| \left(C_\epsilon + \epsilon\, \mathrm{Ent}_\nu f^2\right) \tag{5.14}$$

for a suitable constant C_ϵ.

So, it remains to bound the third term in the r.h.s. of (5.11) $\nu\left(f^2, G_\mu\right)^2$. Here we are (unfortunately) forced to distinguish between two subcases.

a) $t_* \le \frac{M}{l_0^d}$.

We can appeal to lemma 5.1 with $t = t_*$ combined with (iii), (iv) of proposition 4.4 to get

$$\begin{aligned}\nu\left(f^2, G_\mu\right)^2 &\le 2\nu\left(f^2\, G_\mu\right)^2 + 2|\nu(G_\mu) - \mu(G_\mu)|^2 \\ &\le k|\Lambda_j|\left[1 + \left(\frac{MA^2}{l_0} + \frac{1}{M}\right)\operatorname{Ent}_\nu(f^2)\right] + k\, l_0^{2d} \\ &\le |\Lambda_j|\left(C_\epsilon + \epsilon \operatorname{Ent}_\nu f^2\right)\end{aligned} \tag{5.15}$$

for a suitable choice of M, l_0 and for all L large enough.

b) $\frac{M}{l_0^d} \le t_* \le M$.

In this case we can assume, without loss of generality, that $|\Lambda_j|$ is so large that $t_*^2 = \frac{1}{|\Lambda_j|} M \operatorname{Ent}_\nu(f^2)$. We first need to simplify a bit the random variable G_μ by reducing it to a sum of variables each of which depends only on the number of particles in C_α^{int} and not also on the boundary condition outside C_α.

Let $\xi_\alpha^{\delta,\underline{0}}(n) := \xi_\alpha^\delta(0,n)$ and let $G_\mu^{\underline{0}}(\eta) = \sum_\alpha \left[\, \xi_\alpha^{\delta,\underline{0}}(N_{C_\alpha}(\eta)) - \mu(\xi_\alpha^{\delta,\underline{0}})\,\right]$. Notice that, by translation invariance inside Λ_j, $\xi_\alpha^{\delta,\underline{0}}(n)$ is independent of α.

Using once more proposition 3.2 we have $|\nu(G_\mu - G_\mu^{\underline{0}})| \le k l_0^d$. Therefore

$$\nu\left(f^2, G_\mu\right)^2 \le k\,\left[\nu\left(f^2(G_\mu - G_\mu^{\underline{0}})\right)^2 + k l_0^{2d} + \nu\left(f^2, G_\mu^{\underline{0}}\right)^2\right]$$

Now, using Lemma 5.1 with $t = t_*$ together with Proposition 4.4 point (iii), we get

$$\nu\left(f^2(G_\mu - G_\mu^{\underline{0}})\right)^2 \le kN_j\left(1 + \frac{MA^2}{l_0^2} + \frac{1}{M}\operatorname{Ent}_\nu f^2\right)$$

In conclusion

$$\nu\left(f^2, G_\mu\right)^2 \le |\Lambda_j|\left(C_\epsilon + \epsilon \operatorname{Ent}_\nu f^2\right) + k\,\nu\left(f^2, G_\mu^{\underline{0}}\right)^2$$

for suitably chosen M and l_0.

We are left with the estimate of the term $\nu\left(f^2, G_\mu^{\underline{0}}\right)^2$. It is at this point that the substraction with the free parameter δ made at the beginning becomes important.

Let δ be such that for one (and therefore all) cube C_α, $\frac{d}{ds}\xi_\alpha^{\delta,\underline{0}}(s)|_{s=\bar{n}} = 0$ where $\bar{n} = \rho_j|C_\alpha|$. By definition, δ is independent of α and it is given by the following expression

$$\delta = \frac{\mu_{C_\alpha}^{\underline{0},\lambda(\underline{0},\bar{n})}\left(\sum_{x\in C_\alpha^{\text{int}}} g_x, N_{C_\alpha}\right)}{\mu_{C_\alpha}^{\underline{0},\lambda(\underline{0},\bar{n})}\left(N_{C_\alpha^{\text{int}}}, N_{C_\alpha}\right)} \tag{5.16}$$

so that, thanks to the mixing hypothesis, $|\delta| \leq k$ uniformly in L.
Notice that with this choice,

$$| \, \xi_\alpha^{\delta,\underline{0}}(n) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, | = | \int_{\bar{n}}^{n} ds \int_{\bar{n}}^{s} dt \, \frac{d^2}{dt^2} \xi_\alpha^{\delta}(0,t) \, | \leq k \, \frac{(n-\bar{n})^2}{\bar{n}} \tag{5.17}$$

because of lemma 3.1 (point (ii) of (2)). In turn (5.17) implies, in particular, that

$$\nu \left(|\xi_\alpha^{\delta,\underline{0}} - \xi_\alpha^{\delta,\underline{0}}(\bar{n})| \right) \leq k\mu \left(|\xi_\alpha^{\delta,\underline{0}} - \xi_\alpha^{\delta,\underline{0}}(\bar{n})| \right) \leq k \tag{5.18}$$

again because of the mixing condition.

Lemma 5.4. *Assume $USMT(C,m,l)$. Then, for any $\epsilon > 0$, there exists a constant C_ϵ such that for all function f,*

$$\nu \left(f^2, G_{\mu}^{\underline{0}} \right)^2 \leq |\Lambda_j| \left[C_\epsilon + C_\epsilon \, L^2 \mathcal{E}_\nu(f,f) + \epsilon \, \mathrm{Ent}_\nu(f^2) \right]$$

Proof. First write ($n_\alpha \equiv N_{C_\alpha}(\eta)$)

$$\begin{aligned}
\nu \left(f^2, G_{\mu}^{\underline{0}} \right)^2 &\leq 2\nu \Big(f^2 \sum_{\alpha \in I} \left[\, \xi_\alpha^{\delta,\underline{0}}(n_\alpha) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, \right] \Big)^2 + 2\nu \Big(\sum_{\alpha \in I} \left[\, \xi_\alpha^{\delta,\underline{0}}(n_\alpha) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, \right] \Big)^2 \\
&\leq 2\nu \Big(f^2 \sum_{\alpha \in I} \left[\, \xi_\alpha^{\delta,\underline{0}}(n_\alpha) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, \right] \Big)^2 + k|I|^2 \\
&\leq 2\nu \Big(f^2 \sum_{\alpha \in I} \left[\, \xi_\alpha^{\delta,\underline{0}}(n_\alpha) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, \right] \Big)^2 + k \, \frac{|\Lambda_j|}{M} \, \mathrm{Ent}_\nu(f^2)
\end{aligned}$$

where we have used once more the equivalence of ensembles, proposition 3.2 point (1), together with (5.17) and the fact that $t_* \geq \frac{M}{l_0^d}$.

We can thus focus our attention on the relevant term $\nu \left(f^2 \sum_{\alpha \in I} \left[\, \xi_\alpha^{\delta,\underline{0}}(n_\alpha) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, \right] \right)^2$. Thanks to (5.17) we can write

$$\nu \Big(f^2 \sum_{\alpha \in I} \left[\, \xi_\alpha^{\delta,\underline{0}}(n_\alpha) - \xi_\alpha^{\delta,\underline{0}}(\bar{n}) \, \right] \Big)^2 \leq k \, \nu \Big(f^2 \sum_{\alpha \in I} \frac{(n_\alpha - \bar{n})^2}{\bar{n}} \Big)^2 \tag{5.19}$$

We now analyze the r.h.s. of (5.19). Let $\langle \cdot \rangle$ denote the average (normalized sum) over the cubes $\{C_\beta\}_{\beta \in I}$. Thanks to the conservation law we expect $\langle n_\beta \rangle$ to be quite close to $\bar{n}$. In fact, it is easy to check that $\langle n_\beta \rangle - \bar{n} = \frac{1}{|I|} \sum_{x \in \Lambda_j \setminus C} \left[\rho_j - \eta(x) \right]$. Notice that

$$\sum_y \| \nabla_y \big[\sum_{x \in \Lambda_j \setminus C} (\rho_j - \eta(x)) \big]^2 \|_\infty^2 \leq k \, |\Lambda_j \setminus C|^3$$

Moreover, using the mixing condition together with the Poincaré inequality for μ, we also have

$$|\mu\big(\sum_{x\in\Lambda_j\setminus C}[\rho_j-\eta(x)]\big)|\le k\frac{\ln L}{L}|\Lambda_j|$$
$$\mathrm{Var}_\mu\big(\sum_{x\in\Lambda_j\setminus C}[\rho_j-\eta(x)]\big)\le k|\Lambda_j\setminus C|^3$$

Thus

$$\begin{aligned}\nu\Big(f^2\sum_\alpha\frac{\big(\langle n_\beta\rangle-\bar n\big)^2}{\bar n}\Big)^2&\le k\frac{1}{|\Lambda_j|^2}\,\nu\Big(f^2\big[\sum_{x\in\Lambda_j\setminus C}\big(\rho_j-\eta(x)\big)\big]^2\Big)^2\\&\le k\Big(\frac{\ln L}{L}\Big)^4|\Lambda_j|^2+k\frac{1}{|\Lambda_j|^2}|\Lambda_j\setminus C|^3\,\mathrm{Ent}_\nu(f^2)\\&\le\epsilon\,|\Lambda_j|\,\mathrm{Ent}_\nu(f^2)\end{aligned}\tag{5.20}$$

provided that l_0 and L are large enough.
Above we have used lemma 5.1 together with part i) of proposition 4.1 applied to the function $\hat F=F-\mu(F)$, $F=\big[\sum_{x\in\Lambda_j\setminus C}\big(\rho_j-\eta(x)\big)\big]^2$, the bound $\nu(e^{t\hat F})\le A\mu(e^{t\hat F})$ and the hypothesis $t_*\ge\frac{M}{l_0^d}$.
Therefore we can safely add and subtract $\langle n_\beta\rangle$ inside the r.h.s. of (5.19) and confine ourselves to the estimate of $\nu\Big(f^2\sum_{\alpha\in I}\frac{(n_\alpha-\langle n_\beta\rangle)^2}{\bar n}\Big)^2$. We get

$$\nu\Big(f^2\sum_{\alpha\in I}\frac{(n_\alpha-\langle n_\beta\rangle)^2}{\bar n}\Big)^2\le\frac{2}{|I|^2}\Big[\sum_{\alpha,\beta}\nu\big(f^2,\frac{(n_\alpha-n_\beta)^2}{\bar n}\big)\Big]^2+\frac{2}{|I|^2}\Big[\sum_{\alpha,\beta}\nu\big(\frac{(n_\alpha-n_\beta)^2}{\bar n}\big)\Big]^2\tag{5.21}$$

Since, by the strong mixing assumption and proposition 3.3, $\nu\big(\frac{(n_\alpha-n_\beta)^2}{\bar n}\big)\le k\mu\big(\frac{(n_\alpha-n_\beta)^2}{\bar n}\big)\le k$, the second term in the r.h.s. of (5.21) is bounded by $k|I|^2\le k\frac{|\Lambda_j|}{M}\,\mathrm{Ent}_\nu(f^2)$ because of the hypothesis $t_*\ge\frac{M}{l_0^d}$.

It now remains to bound the first term of (5.21). At this stage we cannot appeal to the same old argument based on the entropy inequality and we must proceed differently.

Following [Yau] we introduce $\mathcal{F}_{\alpha,\beta}$, the σ-algebra generated by the random variables $\{\sigma(x)\}_{x\in\Lambda\setminus(C_\alpha\cup C_\beta)}$, and we write

$$\begin{aligned}|\nu\Big(f^2,\frac{(n_\alpha-n_\beta)^2}{\bar n}\Big)|&\le|\nu\Big(\nu\big(f^2,\frac{(n_\alpha-n_\beta)^2}{\bar n}\,|\,\mathcal{F}_{\alpha,\beta}\big)\Big)|+|\nu\Big(f^2,\nu\big(\frac{(n_\alpha-n_\beta)^2}{\bar n}\,|\,\mathcal{F}_{\alpha,\beta}\big)\Big)|\\&\le|\nu\Big(\nu\big(f^2,\frac{(n_\alpha-n_\beta)^2}{\bar n}\,|\,\mathcal{F}_{\alpha,\beta}\big)\Big)|+k\end{aligned}$$

where we have used once more the equivalence of ensembles (Proposition 3.2 point (1), the mixing assumption and the fact that n_α (n_β) being the number of particles in C_α (C_β) is weakly sensitive

to the boundary conditions outside C_α (C_β), to bound the second term.
Define now

$$\mathcal{E}_{\alpha,\beta}(f,f) = \sum_{x,y\in C_\alpha\cup C_\beta} \nu\Big(\big(\nabla_{xy} f\big)^2 \,|\, \mathcal{F}_{\alpha,\beta}\Big)$$

Obviously there exists a "spectral gap" constant $C(l_0)$ such that

$$\nu\big(f,f \,|\, \mathcal{F}_{\alpha,\beta}\big) \le C(l_0)\mathcal{E}_{\alpha,\beta}(f,f) \qquad \forall\, f$$

Thus

$$\begin{aligned}|\nu\big(f^2, \frac{(n_\alpha - n_\beta)^2}{\bar n} \,|\, \mathcal{F}_{\alpha,\beta}\big)| &\le 2\|\frac{(n_\alpha - n_\beta)^2}{\bar n}\|_\infty \Big(\nu\big(f,f \,|\, \mathcal{F}_{\alpha,\beta}\big)\Big)^{\frac12}\Big(\nu\big(f^2 \,|\, \mathcal{F}_{\alpha,\beta}\big)\Big)^{\frac12} \\ &\le l_0^d C(l_0)^{\frac12}\Big(\mathcal{E}_{\alpha,\beta}(f,f)\Big)^{\frac12}\Big(\nu\big(f^2 \,|\, \mathcal{F}_{\alpha,\beta}\big)\Big)^{\frac12}\end{aligned}$$

Schwarz inequality yields

$$\nu\Big(|(\nu\big(f^2, \frac{(n_\alpha - n_\beta)^2}{\bar n} \,|\, \mathcal{F}_{\alpha,\beta}\big)|\Big)^2 \le C'(l_0) \sum_{x,y\in C_\alpha\cup C_\beta} \nu\big(\nabla_{xy} f\big)^2 \tag{5.22}$$

Finally, from (5.22), we get that the first term in the r.h.s. of (5.21) is bounded from above by

$$\begin{aligned}\frac{2}{|I|^2}\Big[\sum_{\alpha,\beta}\nu\big(f^2, \frac{(n_\alpha - n_\beta)^2}{\bar n}\big)\Big]^2 &\le C''(l_0) \sum_{x,y\in\Lambda_j} \nu\left(\nabla_{xy} f\right)^2 + k\Big(\frac{|\Lambda_j|}{l_0^d}\Big)^2 \\ &\le C''(l_0) \sum_{x,y\in\Lambda_j} \nu\left(\nabla_{xy} f\right)^2 + \frac{k}{M}|\Lambda_j| \operatorname{Ent}_\nu(f^2) \\ &\le C'''(l_0)|\Lambda_j| L^2 \mathcal{E}_\nu(f,f) + \frac{k}{M}|\Lambda_j| \operatorname{Ent}_\nu(f^2)\end{aligned}$$

since $t_* \ge \frac{M}{l_0^d}$.

Above we have used once more the "path" bound $\sum_{x,y\in\Lambda_j}\nu\left(\nabla_{xy} f\right)^2 \le kL^{d+2}\mathcal{E}_\nu(f,f)$ given in (3.15). By choosing M large enough we get the sought result also in this case. $\square$

5.3 Applications.

Here we discuss an application of our results which is directly relevant for the proof of the diffusive scaling of the logarithmic Sobolev constant for the Kawasaki dynamics.

Fix $i,j \in \{1,\dots,j_{\max}\}$ with $i \ne j$ and let $\rho = \frac{N_j + N_j}{|\Lambda_j| + |\Lambda_j|}$. Clearly $\frac{\delta_0}{2}(\rho_i + \rho_j) \le \rho \le \rho_i + \rho_j$ where ρ_i and ρ_j are the densities in Λ_i and Λ_j. Without loss of generality we assume that $\rho_i \le \rho$ which implies $\rho_j \ge \rho_i$. Let also

$$g_x(\sigma) = \big[e^{-\nabla_x H^\tau(\sigma)} - 1\big]\sigma(x) \qquad , \qquad G := \sum_{x\in\Lambda_i} g_x$$

$$h_x(\sigma) = \big[e^{-\nabla_x H^\tau(\sigma)} - 1\big]\big(1 - \sigma(x)\big) \qquad , \qquad \mathcal{H} := \sum_{z\in\Lambda_j} h_z$$

Notice that g_x satisfies the hypotheses of proposition 4.3 simply because $g_x = 0$ if there are less than two particles (spins equal to 1) inside its support. Similarly $h_z = 0$ if there is less than one particle inside its support. In particular $\|\mathcal{H}\|_\infty \le kN_j$.

Proposition 5.5. *Assume condition $USMT(C,m,l)$. Then for any $\epsilon > 0$ there exist C_ϵ and ρ_0 such that for any f with $\nu(f^2) = 1$*

(i) if $\rho \le \rho_0$

$$\nu\big(f^2, \sum_{\substack{x\in\Lambda_i\\ z\in\Lambda_j}} g_x h_z\big)^2 \le \rho\,|\Lambda|^3\,\left[C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\right]$$

(ii) if $\rho > \rho_0$

$$\nu\big(f^2, \sum_{\substack{x\in\Lambda_i\\ z\in\Lambda_j}} g_x h_z\big)^2 \le |\Lambda|^3\,\left[C_\epsilon + C_\epsilon\, L^2 \mathcal{E}_\nu(f,f) + \epsilon\,\mathrm{Ent}_\nu(f^2)\right]$$

provided that $|\Lambda|$ is large enough.

Proof. (i). Fix $\epsilon > 0$ and observe first that, thanks to proposition 5.2, we can safely replace $\mathcal{H}$ and G by by $\mathcal{H} - \mu(\mathcal{H})$ and $G - \mu(G)$ respectively. The extra terms we get are in fact bounded by

$$\begin{aligned}
|\mu(\mathcal{H})|^2\,\big(\nu(f^2,G)\big)^2 &\le |\Lambda_j|^2 N_i\,\big(C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\big) \le \rho|\Lambda|^3\,\big(C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\big)\\
|\mu(G)|^2\,\big(\nu(f^2,\mathcal{H})\big)^2 &\le \Big(\frac{\rho_i^2|\Lambda_i| + \rho_i|\partial^+\Lambda_i|}{\rho_j}\Big)^2 N_j\,\big(C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\big) \le \rho|\Lambda|^3\,\big(C_\epsilon + \epsilon\,\mathrm{Ent}_\nu(f^2)\big)
\end{aligned}$$

provided that ρ is small enough. Above we have applied proposition 5.2 to G and $\rho_j\mathcal{H}$ respectively. Note that in the case $\rho_i \ge \rho$ which implies $\rho_j \le \rho_i$ every thing is the same except that we apply proposition 5.2 to $\rho\mathcal{H}$. From now on we will write G for $G - \mu(G)$ and similarly for $\mathcal{H}$.

Next we write $\mathcal{H} = \mathcal{H}^{\mathrm{in}} + \mathcal{H}^{\mathrm{ext}}$ where $\mathcal{H}^{\mathrm{in}}$ is the sum over those z's in Λ_j such that $\Lambda_{h_z} \subset \Lambda_j$ and $\mathcal{H}^{\mathrm{ext}}$ the rest. Similarily for G. Thus we have to bound four terms

$$\nu\big(f^2, G^{\mathrm{int}}\mathcal{H}^{\mathrm{int}}\big)^2,\quad \nu\big(f^2, G^{\mathrm{int}}\mathcal{H}^{\mathrm{ext}}\big)^2,\quad \nu\big(f^2, G^{\mathrm{ext}}\mathcal{H}^{\mathrm{ext}}\big)^2,\quad \nu\big(f^2, G^{\mathrm{ext}}\mathcal{H}^{\mathrm{int}}\big)^2$$

All the mix terms can be treated similarily and therefore we will only analyze one of them e.g. $\nu\big(f^2, G^{\mathrm{int}}\mathcal{H}^{\mathrm{ext}}\big)^2$.

If we combine part *(i)* of proposition 4.1 together with lemma 5.1 we get

$$\begin{aligned}
\nu\big(f^2, G^{\mathrm{int}}\mathcal{H}^{\mathrm{ext}}\big)^2 &\le k \inf_{t\ge 0}\Big(\frac{k}{t} + t\,k\,c_\mu|\Lambda|^2|\partial^+\Lambda_j| + \frac{1}{t}\,\mathrm{Ent}_\nu(f^2)\Big)^2\\
&\le k|\Lambda|^2\,|\partial^+\Lambda_j|\,c_\mu\,\mathrm{Ent}_\nu(f^2)\\
&\le k\rho\,|\Lambda|^3\,\mathrm{Ent}_\nu(f^2)
\end{aligned}\tag{5.23}$$

for $|\Lambda|$ large enough, where c_μ denotes the logarithmic Sobolev constant of μ w.r.t. to the Heat Bath rates and we have used the fact (see the proof of proposition 4.3) that $c_\mu \le k \ln L$ uniformly in the (vector) chemical potential $\underline{\lambda}$.

We are left with the estimate of the "diagonal" term $\nu\bigl(f^2, G^{\rm int}\mathcal{H}^{\rm int}\bigr)^2$. Let $\mathcal{F}_i = \mathcal{F}_{\Lambda_i^c}$. Then

$$\begin{aligned}\nu\bigl(f^2, G^{\rm int}\mathcal{H}^{\rm int}\bigr)^2 &\le k\,N_j^2\nu\left(\nu\bigl(f^2, G\,|\,\mathcal{F}_i\bigr)^2\right) + 2\nu\Bigl(f^2, \mathcal{H}^{\rm int}\nu\left(G^{\rm int}\,|\,\mathcal{F}_i\right)\Bigr)^2 \\ &\le k\,N_j^2 N_i\left(C_\epsilon + \epsilon\,{\rm Ent}_\nu(f^2)\right) + 2\nu\Bigl(f^2, \mathcal{H}^{\rm int}\nu(G^{\rm int}\,|\,\mathcal{F}_i)\Bigr)^2\end{aligned} \tag{5.24}$$

provided that ρ is small enough. In order to estimate the last term in the r.h.s. of (5.24), we observe that, thanks to the strong mixing assumption, the fact that $\mu(G) = 0$, and points (1) and (2) of proposition 3.2 one has

$$\|\nabla_x \mathcal{H}^{\rm int}\nu(G^{\rm int}\,|\,\mathcal{F}_i)\|_\infty \le \begin{cases} k\,\rho_i|\partial^+\Lambda_i| & \text{if } {\rm dist}(x,\Lambda_i) \ge r \\ k\,N_j & \text{if } {\rm dist}(x,\partial^+\Lambda_i) \le r \end{cases}$$

In particular

$$\sum_{x\in\Lambda_j} \|\nabla_x \mathcal{H}^{\rm int}\nu(G^{\rm int}\,|\,\mathcal{F}_i)\|_\infty^2 \le k\rho|\Lambda||\partial^+\Lambda_i|^2$$

Thus, we can proceed as in (5.23) and get that also $\nu\Bigl(f^2, \mathcal{H}^{\rm int}\nu(G^{\rm int}\,|\,\mathcal{F}_i)\Bigr)^2$ is smaller than $k\rho|\Lambda|^3\left(C_\epsilon + \epsilon\,{\rm Ent}_\nu(f^2)\right)$.

To prove point (ii) the argument is unchanged with the only difference that now we must use proposition 5.3 instead of proposition 5.2. $\square$

6. Recursive estimate of the logarithmic Sobolev constant.

In this section we prove the main result of the paper, theorem 2.3, via a recursive analysis on the behaviour of the logarithmic Sobolev constant when the linear size of the volume under consideration is doubled. For simplicity we carry out our analysis in two dimensions but the extension to higher dimension is straightforward (see remark at the end of §3.3). We confine ourselves with the proof of the upper bound since the lower bound is easily proved by plugging a suitable test function (a slowly varying function of the local density) inside the logarithmic Sobolev inequality (2.11).

Definition. *$\mathcal{R}_L$ will denote the class of rectangles in $\mathbb{Z}^2$, which, modulo translations and permutations of the coordinates, can be written as $R(l_1,l_2) = [0,l_1-1]\times[0,l_2-1]$ with $0.1\,l_2 \le l_1 \le l_2 \le L$. We also set*

$$c_s(L) = \max_{R\in\mathcal{R}_L}\,\max_{N,\tau}\,c^\tau_{R,N}$$

where $c^\tau_{R,N}$, the logarithmic Sobolev constant in R with boundary condition τ and N particles, has been defined in (2.11).

With the above notation we will prove the following recursive bound.

Theorem 6.1. *Assume $USMT(C,m,l)$. Then there exists a positive constant $k = k(d,r,\|\Phi\|)$ such that*

$$c_s(2L) \le \frac{3}{2}\, c_s(L) + kL^2$$

for all L large enough. In particular $\sup_{L\ge l} L^{-2} c_s(L) < +\infty$.

Proof. Let us consider a rectangle $\Lambda := R(l_1,l_2) \in \mathcal{R}_{2L} \setminus \mathcal{R}_L$ with $l_1 \le l_2$ and let us fix a small number $\delta_0 \in (0, 10^{-2})$ and we set $d = \lfloor \delta_0 L \rfloor$. Given an integer $i \in [1, \lfloor \frac{1}{10\delta_0} \rfloor - 1]$, we partition Λ into four disjoint sub–rectangles $\{\Lambda_j\}_{j=1}^4$ as follows

$$\begin{aligned}
\Lambda_1 &= \{x \in \Lambda;\ 0 \le x_2 \le l_2/2 + (i-1)d\} \\
\Lambda_2 &= \{x \in \Lambda;\ (i-1)d < x_2 \le id\} \\
\Lambda_3 &= \{x \in \Lambda;\ id < x_2 \le (i+1)d\} \\
\Lambda_4 &= \{x \in \Lambda;\ (i+1)d < x_2\}
\end{aligned} \tag{6.1}$$

and we set $A = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, $B = \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$, $S = \Lambda_2 \cup \Lambda_3$.
Fix now a boundary condition τ outside Λ, a number of particles $N \in [0, \dots, |\Lambda|]$ and let $\nu := \nu^\tau_{\Lambda,N}$. We will then use three times the formula relating the entropy of f^2 w.r.t. the measure ν, $\mathrm{Ent}_\nu(f^2)$, to the entropy of f^2 w.r.t. the measure $\nu(\cdot \,|\, \mathcal{F}_0)$ conditioned to a sub σ–algebra $\mathcal{F}_0$, $\mathrm{Ent}_\nu(f^2 \,|\, \mathcal{F}_0)$:

$$\mathrm{Ent}_\nu(f^2) = \nu\left(\mathrm{Ent}_\nu(f^2 \,|\, \mathcal{F}_0)\right) + \mathrm{Ent}_\nu\left(\nu[\, f^2 \,|\, \mathcal{F}_0\,]\right) \tag{6.2}$$

to write

$$\begin{aligned}
\mathrm{Ent}_\nu(f^2) &= \nu\left(\mathrm{Ent}_\nu(f^2 \,|\, N_A)\right) + \mathrm{Ent}_\nu\left(\nu[\, f^2 \,|\, N_A\,]\right) \\
&= \nu\left(\mathrm{Ent}_\nu(f^2 \,|\, N_A, N_S)\right) + \nu\left(\mathrm{Ent}_\nu(\nu[\, f^2 \,|\, N_A, N_S\,] \,|\, N_A)\right) + \mathrm{Ent}_\nu(\nu[\, f^2 \,|\, N_A\,]) \\
&= \nu\left(\mathrm{Ent}_\nu(\, f^2 \,|\, N_{\Lambda_1}, N_{\Lambda_2}, N_{\Lambda_3}\,)\right) + \nu\left(\mathrm{Ent}_\nu(\nu[\, f^2 \,|\, N_{\Lambda_1}, N_{\Lambda_2}, N_{\Lambda_3}\,] \,|\, N_A, N_S)\right) \\
&\quad + \nu\left(\mathrm{Ent}_\nu\left(\nu[\, f^2 \,|\, N_A, N_S\,] \,|\, N_A\right)\right) + \mathrm{Ent}_\nu\left(\nu[\, f^2 \,|\, N_A\,]\right)
\end{aligned} \tag{6.3}$$

where, we recall, N_V denotes the number of particles in the region V. Formula (6.3) will represent our basic starting point. We will now examine separately each term in the r.h.s. of (6.3).
As usual, in what follows, k will denote a generic constant depending on the interaction, on the dimensions of the lattice and on δ_0, whose value may vary from line to line.

6.1 Analysis of the first term in the r.h.s. of (6.3).

For any small ϵ and large enough L, we can use proposition 3.5 to bound from above the first term in the r.h.s. of (6.3) by

$$\begin{aligned}
\nu\left(\mathrm{Ent}_\nu(\, f^2 \,|\, N_{\Lambda_1}, N_{\Lambda_2}, N_{\Lambda_3}\,)\right) &\le \nu\left(\mathrm{Ent}_{\nu_A}(f^2) + \mathrm{Ent}_{\nu_B}(f^2)\right) + (2+\epsilon)\,\mathrm{Var}_\nu(f) \\
&\le \nu\left(\mathrm{Ent}_{\nu_A}(f^2) + \mathrm{Ent}_{\nu_B}(f^2)\right) + kL^2 \mathcal{E}_\nu(f,f)
\end{aligned} \tag{6.4}$$

where the average is over the number of particles in A (B) and over the boundary conditions outside $A,(B)$. Above we have used the trivial bound $\nu\,(\mathrm{Var}_\nu(f\,|\,N_A, N_{\Lambda_2}, N_{\Lambda_3}\,)) \le \mathrm{Var}_\nu(f)$ together with (3.6) and the spectral gap estimate (see [LY] and [CM2])

$$\mathrm{Var}_\nu(f) \le kL^2\mathcal{E}_\nu(f,f)$$

Let us now examine the geometry of the bottom rectangle A, the reasoning being similar for the top one.
There are two cases to analyze:

a) $l_1 \le \frac{3}{2}L$. In this case one easily verifies that $A \in \mathcal{R}_{\frac{3}{2}L}$.

b) $l_1 > \frac{3}{2}L$. In this case $A \in \mathcal{R}_{2L}$ but now the *longest* side is l_1 and the *shortest* one is smaller than $l_2/2 + (i+1)d$ which in turn is smaller than $1.2L$ by construction.

Therefore $\max\limits_{\tau,N} c^\tau_{A,N} \le \max\{c_s(\frac{3}{2}L),\, \hat{c}_s(2L)\}$ where

$$\hat{c}_s(2L) = \max_{\substack{R\,\in\,\mathcal{R}_{2L}\\ l_1\,\le\,1.2L,\\ l_2\ge\frac{3}{2}L}} \max_{\tau,N} c^\tau_{R,N}$$

In other words

$$\nu\left(\mathrm{Ent}_{\nu_A}(f^2)\right) \le \max\{c_s(\frac{3}{2}L),\, \hat{c}_s(2L)\}\mathcal{E}_\nu(f,f)$$

and similarily for B.
In conclusion, we obtain that the r.h.s. of (6.4) is smaller than

$$\max\{\,c_s(\frac{3}{2}L),\,\hat{c}_s(2L)\,\}\left[\mathcal{E}_\nu(f,f) + \frac{1}{2}\sum_{[x,y]\in\mathcal{E}_S}\nu[\,c_{xy}\,(\nabla_{xy}f)^2\,]\right] + kL^2\mathcal{E}_\nu(f,f) \tag{6.5}$$

uniformly in $i \in [1, \frac{1}{10\delta}]$. Notice that the "spurious" term $\frac{1}{2}\sum_{[x,y]\in\mathcal{E}_S}\nu\left[\,c_{xy}\,(\nabla_{xy}f)^2\,\right]$ comes from the fact that $A\cap B = S$.

6.2 Analysis of the remaining terms in the r.h.s. of (6.3).

Here we bound from above the other three terms in (6.3). The necessary steps are almost identical for all of them and therefore, for shortness, we treat only the second one (the last one enjoys some minor simplifications not shared by the other two). Later on we will state without further comments the analogous result for the first and third one.

For a given value N_A of the number of particles in A, let $\rho_A := \frac{N_A}{|A|}$ and assume, without loss of generality, that $\rho_A \le \frac{1}{2}$. Let $\hat{\nu}(\cdot) := \nu(\cdot\,|\,N_A)$ be the associated multicanonical measure and let $\hat{\mu}$ be the corresponding (multi)–grand canonical measure. Let also $N^*_S = \hat{\mu}\,(N_S)$, and let

$\gamma(n) := \hat{\nu}\,(N_S = n)$. Notice that $N_S^* \le k\rho_A L^d$ (see proposition 3.1 of [CM1]). Then, using corollary 3.7, we can write

$$\begin{aligned}
\mathrm{Ent}_\nu\big(\nu[\, f^2 \,|\, N_A, N_S \,] \,|\, N_A\big) = \\
&= \mathrm{Ent}_{\hat{\nu}}\,\big(\hat{\nu}[\, f^2 \,|\, N_S \,]\big) \\
&= \mathrm{Ent}_{\gamma}\,\big(\hat{\nu}[\, f^2 \,|\, N_S \,]\big) \\
&\le k\rho_A L^d \sum_n \gamma(n) \wedge \gamma(n-1) \left(\sqrt{\hat{\nu}(f^2 \,|\, N_S = n)} - \sqrt{\hat{\nu}(f^2 \,|\, N_S = n-1)}\right)^2 \\
&\le k\rho_A L^d \sum_n \gamma_f(n) \left(A(n)^2 + B(n)^2\right)
\end{aligned} \tag{6.6}$$

where $A(n)$ and $B(n)$ have been defined in (3.9).
Thanks to proposition 3.8 and proposition 3.10, for any $\epsilon > 0$ there exists a constant C_ϵ independent of ρ_A such that the r.h.s. of (6.6) is smaller than

$$C_\epsilon\, \hat{\nu}(f^2) + C_\epsilon\, L^2 \mathcal{E}_{\hat{\nu}}(f,f) + \epsilon\; \mathrm{Ent}_{\hat{\nu}}(f^2) \tag{6.7}$$

Thus, if we average w.r.t. the canonical measure ν the l.h.s. of (6.6) and use the simple inequality $\nu\left(\mathrm{Ent}_\nu(f^2 \,|\, \mathcal{F}_0)\right) \le \mathrm{Ent}_\nu(f^2)$ for any $\mathcal{F}_0$, we get

$$\begin{aligned}
\nu\left(\mathrm{Ent}_\nu(\nu[\, f^2 \,|\, N_A, N_S \,] \,|\, N_A)\right) &= \nu\left(\mathrm{Ent}_{\hat{\nu}}(\nu[\, f^2 \,|\, N_A, N_S \,])\right) \\
&\le C_\epsilon\, \nu(f^2) + C_\epsilon\, L^2 \mathcal{E}_\nu(f,f) + \epsilon\; \mathrm{Ent}_\nu(f^2)
\end{aligned} \tag{6.8}$$

Similar bounds hold also for the first and third term in the r.h.s. of (6.3).

6.3 The recursion completed.

We are finally in a position to complete the proof of theorem 6.1. If we put together (6.8) and (6.5) we get that, for any ϵ small enough

$$\begin{aligned}
\text{r.h.s. of (6.3)} \;\le \max\{\, c_s(\tfrac{3}{2}L),\, \hat{c}_s(2L)\,\} &\Big[\mathcal{E}_\nu(f,f) + \frac{1}{2} \sum_{[x,y]\in\mathcal{E}_S} \nu\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right]\Big] \\
&+ C_\epsilon \nu(f^2) + C_\epsilon L^2\, \mathcal{E}_\nu(f,f) + \epsilon\; \mathrm{Ent}_\nu(f^2)
\end{aligned} \tag{6.9}$$

that is

$$\begin{aligned}
\mathrm{Ent}_\nu(f^2) \le \left(\frac{1}{1-\epsilon}\right) \max\Big\{\, c_s(\tfrac{3}{2}L)^{-1},\, \hat{c}_s(2L)^{-1}\,\Big\} &\Big[\mathcal{E}_\nu(f,f) + \frac{1}{2} \sum_{[x,y]\in\mathcal{E}_S} \nu\left[\, c_{xy}\,(\nabla_{xy} f)^2\,\right]\Big] \\
+ C_\epsilon \nu(f^2) + C_\epsilon L^2\, \mathcal{E}_\nu(f,f)
\end{aligned} \tag{6.10}$$

for a suitable constant C_ϵ.
Finally, following [M], we average the above inequality w.r.t. to the integer i (see (6.1)) and use the observation that, as i varies in $[1, \frac{1}{10\delta_0}]$, the strips $S \equiv S_i$ are disjoint. In particular

$$\frac{1}{2} \sum_{j \in [1, \frac{1}{10\delta_0}]} \sum_{[x,y] \in \mathcal{E}_{S_i}} \nu\left[c_{xy} \, (\nabla_{xy} f)^2 \right] \le \mathcal{E}_\nu(f,f)$$

so that

$$\begin{aligned} &\mathrm{Ent}_\nu(f^2) \le \\ &\left(\frac{1}{1-2\epsilon}\right)(1+\lfloor 10\delta_0 \rfloor) \max\Big\{ c_s(\frac{3}{2}L),\, \hat{c}_s(2L) \Big\} \mathcal{E}_\nu(f,f) + C_\epsilon \nu(f^2) + C_\epsilon L^2 \, \mathcal{E}_\nu(f,f) \end{aligned} \tag{6.11}$$

Notice that if we write $f = [f - \nu(f)] + \nu(f)$ and we use the Poincaré bound $\mathrm{Var}_\nu(f) \le kL^2 \mathcal{E}_\nu(f,f)$ we get

$$\begin{aligned} \mathrm{Ent}_\nu(f^2) &\le \mathrm{Ent}_\nu([f-\nu(f)]^2) + 2\,\mathrm{Var}_\nu(f) \\ &\le \left(\frac{1}{1-2\epsilon}\right)(1+\lfloor 10\delta_0 \rfloor) \max\Big\{ c_s(\frac{3}{2}L),\, \hat{c}_s(2L) \Big\} \mathcal{E}_\nu(f,f) + k\, C_\epsilon \, L^2 \mathcal{E}_\nu(f,f) \end{aligned} \tag{6.12}$$

where in the first line we have used once more the Rothaus inequality (see [An]).
In other words

$$c^\tau_{\Lambda,N} \le \left(\frac{1}{1-2\epsilon}\right)(1+\lfloor 10\delta_0 \rfloor) \max\Big\{ c_s(\frac{3}{2}L),\, \hat{c}_s(2L) \Big\} + k\, C_\epsilon L^2 \tag{6.13}$$

Notice that if the original rectangle Λ was such that $l_1 \le 1.2L$ while $l_2 \ge \frac{3}{2}L$, *i.e.* Λ was chosen in the sub–class of $\mathcal{R}_{2L}$ entering in the definition of $\hat{c}_s(2L)$, then we would have obtained the inequality (6.13) with the factor $\max\{ c_s(\frac{3}{2}L), \hat{c}_s(2L) \}$ replaced by $c_s(\frac{3}{2}L)^{-1}$ simply because case *b)* right after (6.4) would have been impossible. Thus

$$\hat{c}_s(2L) \le \left(\frac{1}{1-2\epsilon}\right)(1+\lfloor 10\delta_0 \rfloor)\, c_s(\frac{3}{2}L)^{-1} + k\, C_\epsilon L^2 \tag{6.14}$$

If we combine (6.13) with (6.14) we finally get

$$c^\tau_{\Lambda,N} \le \left(\frac{1}{1-2\epsilon}\right)^2 (1+\lfloor 10\delta_0 \rfloor)^2 \; c_s(\frac{3}{2}L) + C'_\epsilon L^2 \tag{6.15}$$

for another constant C'_ϵ. Thus

$$c_s(2L) \le \left(\frac{1}{1-2\epsilon}\right)^2 (1+\lfloor 10\delta_0 \rfloor)^2 \; c_s(\frac{3}{2}L) + C'_\epsilon L^2$$

and two more iterations prove the recursive inequality of the theorem provided that the two parameters ϵ, δ_0 were chosen small enough.
Finally the fact that $\max_L \left(c_s(L)\, L^{-2}\right) > 0$ is a trivial consequence of the recursive bound. $\square$

Appendix. Logarithmic Sobolev inequality for one dimensional discrete random walks

Let $\Omega = \{n_{\min}, n_{\min}+1, \dots, n_{\max}\}$ be a finite subset of the integers and let γ be a positive probability measure on Ω. We present a result, based on Hardy type inequalities, to bound in a precise way the logarithmic Sobolev constant of γ with respect to the Dirichlet form of discrete random walks reversible w.r.t. γ. After that we discuss a concrete example related to the distribution of the number of particles under a multicanonical measure (see §7).

Proposition A.1. *For all functions f on Ω we have*

$$\mathrm{Ent}_\gamma(f^2) \le 20B \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n) \wedge \gamma(n-1)) \left[f(n) - f(n-1)\right]^2$$

where $B := \inf_i \, [B_+(i) \vee B_-(i)]$, with

$$B_+(i) := \sup_{n \ge i+1} \left(\sum_{k=n}^{n_{\max}} \gamma(k)\right) \ln\left(\frac{1}{\sum_{k=n}^{n_{\max}} \gamma(k)}\right) \left(\sum_{k=i+1}^{n} \frac{1}{\gamma(k) \wedge \gamma(k-1)}\right)$$

$$B_-(i) := \sup_{n \le i-1} \left(\sum_{k=n_{\min}}^{n} \gamma(k)\right) \ln\left(\frac{1}{\sum_{k=n_{\min}}^{n} \gamma(k)}\right) \left(\sum_{k=n}^{i-1} \frac{1}{\gamma(k) \wedge \gamma(k+1)}\right)$$

Proof. The proof is a straight forward application of Hardy type inequalities (see [Mi] section 3). Indeed, it's enough to consider the birth and death continuous time Markov chain (Metropolis chain) with transition rates

$$c(n,j) = \begin{cases} \frac{\gamma(j)}{\gamma(n)} \wedge \frac{1}{2} & \text{if } j = n \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

□

We now define a convexity assumption for the probability measure under consideration that will allow us to bound rather sharply the constant B described above.

Definition of condition $CONV(c, \bar n)$. *We say that γ satisfies the convexity hypothesis with parameters $c > 0$ and $\bar n \in \Omega$, in what follows denoted by $CONV(c, \bar n)$, if $c^{-1}\bar n \le n_{\max} - \bar n \le c\bar n$ and similarily for $\bar n - n_{\min}$,*

$$\frac{\gamma(n+1)}{\gamma(n)} \le c\, e^{-\frac{(n-\bar n)}{c\bar n}} \qquad \text{for all } n \ge \bar n\ ,$$

$$\frac{\gamma(n-1)}{\gamma(n)} \le c\, e^{-\frac{(\bar n-n)}{c\bar n}} \qquad \text{for all } n \le \bar n$$

$$\frac{1}{c\sqrt{\bar n}} e^{-\frac{c(\bar n-n)^2}{\bar n}} \le \gamma(n) \le \frac{c}{\sqrt{\bar n}} e^{-\frac{(\bar n-n)^2}{c\bar n}} \qquad \text{for all } n \in \Omega \tag{A.2}$$

Remark. Since γ is assumed to be a positive probability measure, we can always write it in the Gibbsian form $\gamma(n) = Z^{-1}\exp(-H(n))$. Then, if H is strictly convex with "second derivative" bounded from below and from above by c^{-1} and c respectively, and the minimum in $\bar{n}$, *i.e.*

$$\begin{aligned} \frac{1}{c\bar{n}} \le H(n+2) - 2H(n+1) + H(n) \le \frac{c}{\bar{n}} \qquad \forall n \in \Omega \\ H(\bar{n}+1) = H(\bar{n}) \end{aligned} \tag{A.3}$$

then, if $(c, \bar{n}, n_{\max}, n_{\min})$ are in the right proportion described in $CONV(c,\bar{n})$, γ satisfies condition $CONV(c,\bar{n})$.

Our aim now is to prove that under $CONV(c,\bar{n})$ the measure γ satisfies a logarithmic Sobolev inequality with optimal constant proportional (depending on c) to $\bar{n}$ (see Proposition A.5 below). For this purpose, let us first discuss some simple lemma that deal with the different terms appearing in the constants $B_+(i)$ and $B_-(i)$ of Proposition A.1. Since obviously $B \le B_+(\bar{n}) \vee B_-(\bar{n}))$ we can restrict us to the case $i = \bar{n}$.

Lemma A.2. *Let γ satisfies $CONV(c,\bar{n})$. Then*

$$\begin{aligned} \sum_{k=n}^{n_{\max}} \gamma(k) &\le C \frac{\bar{n}}{n-\bar{n}} \gamma(n) \qquad \text{for all} \quad n \ge \bar{n}+1, \\ \sum_{k=n_{\min}}^{n} \gamma(k) &\le C \frac{\bar{n}}{\bar{n}-n} \gamma(n) \qquad \text{for all} \quad n \le \bar{n}-1\ , \end{aligned}$$

for a suitable numerical constant C depending only on c and not on $\bar{n}$.

Proof. Fix $n \ge \bar{n}+1$. By using the definition of $CONV(c,\bar{n})$ together with a simple telescopic argument we get

$$\begin{aligned} \sum_{k=n}^{n_{\max}} \frac{\gamma(k)}{\gamma(n)} &= \sum_{k=n}^{n_{\max}} \prod_{j=n}^{k-1} \frac{\gamma(j+1)}{\gamma(j)} \\ &\le c \sum_{k=n}^{n_{\max}} \prod_{j=n}^{k-1} e^{-\frac{(j-\bar{n})}{c\bar{n}}} \\ &\le c \sum_{k \ge n} e^{-\frac{(k-n)(n-\bar{n})}{2c\bar{n}}} \\ &= c \frac{1}{1 - e^{-\frac{(n-\bar{n})}{2c\bar{n}}}} \end{aligned}$$

But, $(n-\bar{n})/\bar{n} \le (n_{\max}-\bar{n})/\bar{n} \le c$ and therefore $\left(1 - e^{-\frac{(n-\bar{n})}{2c\bar{n}}}\right)^{-1} \le C\frac{\bar{n}}{n-\bar{n}}$ for some constant C depending only on c. A similar computation gives the corresponding result for $n \le \bar{n}-1$. This achieves the proof. $\square$

Lemma A.3. *Let γ satisfies $CONV(c,\bar n)$. Then*

$$\sum_{k=\bar n+1}^{n} \frac{1}{\gamma(k)} \le C\frac{\bar n}{n-\bar n}\frac{1}{\gamma(n)} \qquad \text{for all} \quad n\ge \bar n+1$$

$$\sum_{k=n}^{\bar n-1} \frac{1}{\gamma(k)} \le C\frac{\bar n}{\bar n-n}\frac{1}{\gamma(n)} \qquad \text{forall} \quad n\le \bar n-1$$

for a suitable numerical constant C depending only on c.

Proof. The proof is practically the same as that of lemma A.2 and it is omitted. □

Lemma A.4. *Let γ satisfies $CONV(c,\bar n)$. Then*

$$\ln\left(\frac{1}{\sum_{k=n}^{n_{\max}}\gamma(k)}\right) \le C\left(1+\frac{(\bar n-n)^2}{\bar n}\right) \qquad \text{for all} \quad n\ge \bar n+\sqrt{\bar n}$$

$$\ln\left(\frac{1}{\sum_{k=n_{\min}}^{n}\gamma(k)}\right) \le C\left(1+\frac{(\bar n-n)^2}{\bar n}\right) \qquad \text{for all} \quad n\le \bar n-\sqrt{\bar n}$$

where C is a suitable numerical constant depending only on c.

Proof. If $\bar n$ is smaller than some large constant depending on c there is nothing to be proved. Therefore we can assume that $\bar n$ is large enough and we consider two cases.
In the first one, we suppose $\bar n+\sqrt{\bar n}\le n\le n_{\max}-\sqrt{\bar n}$. By using the definition of $CONV(c,\bar n)$ we get

$$\begin{aligned}\sum_{k=n}^{n_{\max}}\gamma(k) &\ge \frac{1}{c}\sum_{k=n}^{n_{\max}}\frac{1}{\sqrt{\bar n}}e^{-\frac{c}{\bar n}(\bar n-k)^2}\\ &\ge \frac{1}{c}\sum_{k=n}^{n+\sqrt{\bar n}}\frac{1}{\sqrt{\bar n}}e^{-\frac{c}{\bar n}(\bar n-k)^2}\\ &\ge \frac{1}{c}\sqrt{\bar n}\frac{1}{\sqrt{\bar n}}e^{-\frac{c}{\bar n}(n+\sqrt{\bar n}-\bar n)^2}\\ &\ge \frac{1}{c}e^{-\frac{4c}{\bar n}(n-\bar n)^2},\end{aligned} \tag{A.4}$$

where, in the last inequality, we have used the fact that $(n+\sqrt{\bar n}-\bar n)^2\le 4(n-\bar n)^2$ for all $n\ge\bar n+\sqrt{\bar n}$.
In the second case, we suppose $n\ge n_{\max}-\sqrt{\bar n}$. An easy computation gives

$$\sum_{k=n}^{n_{\max}}\gamma(k)\ge\frac{1}{c}\sum_{k=n}^{n_{\max}}\frac{1}{\sqrt{\bar n}}e^{-\frac{c}{\bar n}(\bar n-k)^2}\ge\frac{1}{c}\frac{1}{\sqrt{\bar n}}e^{-\frac{c}{\bar n}(\bar n-n)^2}\ge C^{-1}e^{-\frac{C}{\bar n}(\bar n-n)^2} \tag{A.5}$$

for some constant C depending only on c. Finally the first statement of the lemma follows at once from (A.4) and (A.5) by taking the logarithm. Similarly one proceeds for the proof of the second part of the lemma. □

We can now state the main result of the appendix.

Proposition A.5. *Let γ satisfies $CONV(c,\bar n)$. Then there exists a constant C depending only on c such that for all real functions f on Ω*

$$\mathrm{Ent}_\gamma(f) \le C\,\bar n \sum_{n=n_{\min}+1}^{n_{\max}} (\gamma(n)\wedge\gamma(n-1))\,\left[f(n)-f(n-1)\right]^2 \tag{A.6}$$

Proof. Thanks to Proposition A.1 it is enough to bound from above the constant B or, what is enough $B_+(\bar n)\vee B_-(\bar n)$. First notice that by symmetry we can only consider one of them e.g.

$$B_+(\bar n) = \sup_{n\ge \bar n+1} \left(\sum_{k=n}^{n_{\max}}\gamma(k)\right)\ln\left(\frac{1}{\sum_{k=n}^{n_{\max}}\gamma(k)}\right)\left(\sum_{k=\bar n+1}^{n}\frac{1}{\gamma(k)\wedge\gamma(k+1)}\right)$$

Let us consider two cases.

In the first one, we suppose $\bar n+1\le n\le \bar n+\sqrt{\bar n}$. Then, because $\sum_{k=n}^{n_{\max}}\gamma(k)\le 1$ and $x\ln(1/x)\le 1$ for all $x\in[0,1]$,

$$\left(\sum_{k=n}^{n_{\max}}\gamma(k)\right)\ \ln\left(\frac{1}{\sum_{k=n}^{n_{\max}}\gamma(k)}\right)\le 1$$

Moreover, by (A.2)

$$\sum_{k=\bar n+1}^{n}\frac{1}{\gamma(k)}\le c\sum_{k=\bar n+1}^{n}\frac{1}{\sqrt{\bar n}}\,e^{\frac{c}{\bar n}(\bar n-k)^2}\le c\sqrt{\bar n}\frac{1}{\sqrt{\bar n}}\,e^{\frac{C}{\bar n}(\bar n-n+1)^2}\le C$$

for some constant C depending only on c.

In the second case, $n\ge \bar n+\sqrt{\bar n}$, we apply lemma A.2, A.3 and A.4 to get

$$\begin{aligned}\sum_{k=n}^{n_{\max}}\gamma(k)\ \ln\left(\frac{1}{\sum_{k=n}^{n_{\max}}\gamma(k)}\right)\sum_{k=\bar n+1}^{n}\frac{1}{\gamma(k)} &\le C\frac{\bar n}{n-\bar n}\left(1+\frac{(\bar n-n)^2}{\bar n}\right)\frac{\bar n}{n-\bar n}\\ &\le C\bar n\end{aligned}$$

This achieves the proof. $\square$

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