Spectral density estimation from random sampling for multiplicative stationary processes.

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Abstract

In this paper the spectral density estimation of a nonstationary class of processes is investigated. Although these processes are not stationary with respect to an additive binary operation i.e. in the classical weak sense, they are stationary with respect to a multiplicative binary operation. These processes exist naturally as continous-time processes. To anwer many questions in practical situations using these processes, we develop a random sampling method for estimating their spectral density by using a discrete-time process. Some simulations are given.

A.M.S. classification: Primary 62M15; secondary 62G20.

Key-words: Stationary, nonstationary processes, Spectral representation, Cumulants, Consistency, Spectral analysis.

Estimation spectrale pour les processus M-stationnaires par échantillonnage aléatoire.

Résumé : Dans ce travail, l'estimation de la densité spectrale pour une classe de processus non stationnaires est étudiée. Bien que ces processus ne soient stationnaires relativement à la loi additive c'est à dire faiblement stationnaire au sens classique, ils sont stationnaires par rapport à la loi multiplicative. Ces processus existent naturellement comme des enregistrements à temps continu. Pour tenter de répondre à un certain nombre de questions posées dans des situations pratiques, nous adoptons l'échantillonnage aléatoire afin d'estimer la densité spectrale en utilisant un processus à temps discret. Des simulations sont données.

Mots-clefs : Processus non stationnaires, Representation spectrale, Cumulants, Convergence en moyenne quadratique, Analyse spectrale.

1 Introduction

In this paper, the spectral density estimation of a class of nonstationary processes is investigated. The considered nonstationary processes are refered to as multiplicative stationary processes. Although these processes are not stationary with respect to the usual additive operation, they are stationary with respect to the multiplicative operation. This property ensures essentially the same structure as is available for stationary processes, especially the spectral representation.

The concept of stationarity under a general group composition operation has been considered extensively by Hannan (1965). It may even be generalized to a semi-group operation, see Girardin & Senoussi (2000). The M-stationary processes are defined and exist naturally as continuous-time processes. They are studied thoroughly in Gray and Zhang (1988). However, no particular advantage seems to have been taken yet of the generality afforded by this generalized concept of stationarity. A particular aspect of their study is considered here: for practical applications, a sampling of the time becomes suitable for estimating the spectral density. We develop a random sampling technique, showing that the scheme developped by Masry (1978) and Messaci (1986) may be used here. Consistency in mean-square of the estimate is established. A numeric example is given, for an M-stationary Gauss-Markov process.

This paper is organized as follows. In Section 2, we recall some properties and examples of M-stationary processes. The sampling technique and the linked spectral density estimate construction are exposed in Section 3. We give the results of convergence of the estimate in Section 4. A short simulation study is given in section 5. The proofs of the main results are developpped in Section 7. And a wider concept of stationarity is introduced in Section 6, extending the M-stationarity.

2 Properties and examples of M-stationary processes

Definition 1 *Let $X = \{X(t),\, t > 0\}$ be a stochastic process such that, for all positive real numbers t and τ.*

(i) $\mathbb{E}\{X(t)\} = \mu$ *is independent of* t.

(ii) *var*$\{X(t)\} < +\infty$.

(iii) $\mathbb{E}\{(X(t)-\mu)(X(t.\tau)-\mu)\} = R_X(\tau)$ *exists and is independent of* t.

Then X is said to be a weakly multiplicative-stationary (or M-stationary) process.

In the classical sense, M-stationary processes are nonstationary, but, to each M-stationary process corresponds a stationary process as stated in the following theorem due to Gray and Zhang (1988).

Theorem 1 *Let $Y = \{Y(u),\, u \in \mathbb{R}\}$ be a stochastic process such that $Y(u) = X(t)$ where $t = \exp(u)$. Then X is weakly M-stationary if and only if Y is weakly stationary.*

Y is referred to as the dual stationary process of X. Let us denote by C_Y the autocovariance function of Y; then for all positive real number τ, we have

$$R_X(\tau) = R_X(\tau^{-1}) = R_Y(\ln(\tau)).$$

As will be seen in the following, this dual process will the basis of the developement of the theory of spectral density estimation of an M-stationary process.

Let us give some examples of M-stationary processes to set up ideas.

Example 1 Consider the process X, defined for $t > 0$ by

$$X(t) = a\,\cos(\beta\,\ln(t) + \omega),$$

where a is a constant and ω is a uniformly distributed on $[0, 2\pi]$ random variable; then X is M-stationary, with zero mean, and covariance function $R_X(\tau) = \dfrac{A^2}{2}\cos(\beta\,\ln(\tau))$ for all $\tau > 0$.

Example 2 Consider the process X, defined for $t > 0$ by

$$X(t) = \sum_{i=1}^{N}\left[A_i\,\cos(a_i\,\ln t) + B_i\,\sin(a_i\,\ln t)\right],$$

where, for all i and j, A_i and B_i are zero-mean uncorrelated random variables such that, for all $i \in \{1,\ldots,n\}$, $\mathbb{E}\{A_i^2\} = \mathbb{E}\{B_i^2\}$. Then X is a zero-mean, M-stationary process with

$$R_X(\tau) = \sum_{i=1}^{N}\mathbb{E}\{A_i^2\}\cos(a_i\,\ln(\tau)).$$

Example 3 A process ε is said to be an M-white noise if its dual process is a white noise. An M-white noise is thus defined by

$$\mathbb{E}\{\varepsilon(t)\} = 0 \text{ and } \mathbb{E}\{\varepsilon(t)\varepsilon(t\tau)\} = c\,\delta(\ln(\tau))$$

where δ is the Kronecker symbol.

Example 4 Consider the process X, defined for $t>0$ by $X(t)=\int_0^t H(\frac{1}{u})\varepsilon(u)d\ln(u)$, where

- $\int_0^{+\infty} |H(t)|^2\, d\ln t > \infty$ and $H(t)=0$ for $0<t<1$.
- ε is an M-white noise.

Then X is an M-stationary process called M-linear process.

Example 5 The Euler process is a particular case of M-linear processes, sometimes called long memory process, with

$$H(t)=\begin{cases} \sum_{i=1}^M \sum_{j=1}^{m_i} c_{ij}(\ln t)^j t^{-a_i} & \text{for } t\geq 1 \\ 0 & \text{for } 0<t<1, \end{cases}$$

where c_{ij} and a_i are complex numbers such that $\mathcal{R}e(a_i)\geq 0$, where $\mathcal{R}e(z)$ denote the real part of the the complex number z. We have then

$$R_X(\tau)=\sum_{i=1}^{M}\sum_{j=1}^{m_i} b_{ij}(\ln\tau)^j\tau^{-a_i},$$

where the constants b_{ij} are uniquely determined from the constants a_i and c_{ij}.
The integer $k=\sum_{i=1}^M(m_i+1)$ is called the order of the process. Note that the Euler processes play the same part in the M-stationary processes theory as the auto-regressive stationary models in the usual case.

Let X be an M-stationary process. The M-spectral density G_X of X is defined as the Mellin transform of the autocovariance R_X if it does exists, namely

$$G_X(x)=\int_0^{+\infty}\tau^{-2i\pi x}R_X(\tau)d\ln(\tau). \tag{1}$$

It may also be written as

$$G_X(x)=\int_{-\infty}^{+\infty} e^{-2i\pi x u}R_Y(u)du,$$

where R_Y is the autocovariance function of the dual process Y of X.
We have tacitly assumed that the Mellin transform in (1) exists. We make this assumption too for the inverse Mellin transform. For details on the Mellin transform, see Morettin (1974) and (1981). From (1), it follows at once that $G_X(x)=G_Y(x)$, where G_Y is the spectral density of Y.

In the sequel, we shall use the term spectral density for M-spectral density, and assume that the process X is zero-mean, M-stationary up to order four i.e., that, for all positive real numbers t_1, t_2, t_3 and t $\mathbb{E}\{X(t)X(t.t_1)X(t.t_2)X(t.t_3)\}$ is independent of t. In this case we call M-cumulant of order four of X, the function

$$\begin{aligned} Q(t_1,t_2,t_3) &= \mathbb{E}\{X(t)X(t.t_1)X(t.t_2)X(t.t_3)\}-R_X(t_1)R_X(\frac{t_2}{t_3}) \\ &\quad -R_X(t_2)R_X(\frac{t_3}{t_1})-R_X(t_3)R_X(\frac{t_2}{t_1}). \end{aligned}$$

Remark 1 Let X be an M-stationary process and let Y be its dual process. By the relationship between X and Y, we get

$$R_X(\tau)=R_X(\tau^{-1}) \text{ and } R_X(\tau)=R_Y(\ln(\tau)).$$

If we assume that R_X is continuous and that $\tau^{-\alpha}R_X(\tau)\in L^{1/\alpha}(\mathbb{R}+)$ for $\alpha=\frac{1}{2}$ or 1, then R_Y is continuous, absolutely integrable and square integrable.

3 Time sampling and spectral density estimation

For sampling the time, we consider the scheme $\{t_n\}_{n\in\mathbb{N}}$ proposed by Messaci (1986), namely

$$\begin{aligned} & t_0 = 1 \\ \text{and} \qquad & t_n = t_{n-1}\beta_n \end{aligned} \tag{2}$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ is a sequence of Pareto distributed random variables, with density

$$f_{\beta_n}(x) = \beta x^{-\beta-1} 1_{[1,+\infty[}(x)\,;\ \beta > 0.$$

Consider then the scheme $\tau_n = \ln(t_n)$ for all $n \in \mathbb{N}$. The sequence $\{\tau_n\}_{n\in\mathbb{N}}$ verifies $\tau_0 = 0$ and $\tau_n = \tau_{n-1} + \alpha_n$ for $n \in \mathbb{N}^*$ where $\alpha_n = \ln(\beta_n)$ is a sequence of exponentially distributed random variables with parameter $\beta > 0$, with density

$$f_{\alpha_n}(x) = \beta \exp(-\beta x) 1_{[0,+\infty[}(x).$$

Note that since $t_n \geq 1$ a.s., we have also $\beta_n \geq 1$ a.s..

Consider the discrete process $\{X(t_k)\}_{k\in\mathbb{N}}$, which is zero-mean too. Since the sequence $\{t_n\}$ and the continuous process X are independent, the autocovariance function of the sampled process is

$$\rho_X(n) = \mathbb{E}\{X(t_{k+n})X(t_k)\} = \int_0^{+\infty} R_X(t) f_n(t)dt.$$

Let $X = \{X(t),\, t \geq 0\}$ be an M-stationary process and let $Y = \{Y(u),\, u \in \mathbb{R}\}$ be its dual process, Y is then zero mean, stationary in the classical sens, of order four, and with cumulant function C given by

$$C(u_1, u_2, u_3) = Q(\exp(u_1), \exp(u_2), \exp(u_3)).$$

Since X and Y have the same spectral density, then the problem of spectral density estimation can be transfered from an M-stationary process to a stationary one.

Remark 2 For some $k \geq 1$, the difference $u_{n+k} - u_n = \sum_{i=1}^{k} \alpha_{n+i}$, admits as probability density

$$f_n(t) = \beta \frac{(\beta t)^{n-1}}{(n-1)!} \exp(-\beta t) 1_{\mathbb{R}_+}(t), \text{ (Gamma of order } n \text{ and parameter } \beta).$$

Consider the discrete time process $\{X(t_n)\}_{n\in\mathbb{N}}$. This process is of zero mean and of autocovariance function ρ_X such that

$$\begin{aligned} \rho_X(n) &= \mathbb{E}\{X(t_{k+n})X(t_k)\} = \mathbb{E}_{\{t_n\}} \circ \mathbb{E}_X\{X(t_{k+n})X(t_k)\}, \\ &= \mathbb{E}_{\{t_n\}}\{R_X(t_{k+n}/t_k)\}, \\ &= \int_0^{+\infty} R_X(t) f_{\alpha_n}(t)dt. \end{aligned}$$

since the process X and the scheme $\{t_n\}$ are independent. More, the system $\{f_{\alpha_n}(t)\}_{n\in\mathbb{N}^*}$ is complete in $L^2(0,+\infty)$, and, if orthonormalized, it becomes

$$g_n(t) = \sqrt{2\beta} L_{n-1}^{(0)}(2\beta t) \exp(-\beta t) 1_{\mathbb{R}_+}(t),$$

where $L_n^{(0)}$ is the n-th Laguerre polynomial. For details on Laguerre polynomials, see [26] for notions on these polynomials.

Conversely, the expansion of $g_n(t)$ in the basis $\{f_k(t)\}_{k=1}^{n}$, is

$$g_n(t) = \sum_{k=1}^{n} \theta_{n,k} f_k(t) \text{ where } \theta_{n,k} = \sqrt{2/\beta}(-2)^{k-1} \mathcal{C}_{n-1}^{k-1}.$$

Hence we obtain $R_X(t) = \sum_{n=1}^{+\infty} a_n g_n(|t|)$ in $L^2(\mathbb{R})$, with

$$a_n = \int_{\mathbb{R}_+} R_X(t) g_n(t) dt = \sum_{k=1}^{n} \theta_{n,k} \int_{\mathbb{R}_+} R_X(t) f_k(t) dt = \sum_{k=1}^{n} \theta_{n,k} \rho_X(k).$$

And the spectral density can be written as

$$\begin{aligned} G_X(\lambda) &= \int_{\mathbb{R}} \exp(-2i\pi\lambda u) R_Y(u) du \\ &= \sum_{n=1}^{+\infty} a_n G_n(\lambda), \end{aligned}$$

with $G_n(\lambda) = \int_{\mathbb{R}_+} \exp(-2i\pi\lambda u) g_n(|u|) du$. But $t_{k+1} - t_k \sim f_{\alpha_n}$ and $t_{k+n} - t_k = \sum_{i=1}^{n} (t_{k+i} - t_k)$ where for each $i \in \{1, \ldots, n\}$, the random variables $(t_{k+i} - t_k)$ are i.i.d with distribution $Exp(\beta)$. The characteristic function of an exponential distribution of parameter β, is $\phi(\lambda) = \beta/(\beta - i\lambda)$, thus the characteristic function of $t_{k+n} - t_k$ is

$$\phi_{t_{k+n} - t_k}(\lambda) = \prod_{i=1}^{n} \phi_{t_{k+i} - t_k}(\lambda) = \left(\frac{\beta}{\beta - i\lambda} \right)^n.$$

Hence

$$\begin{aligned} G_n(\lambda) &= \int_{\mathbb{R}} \exp(-2i\pi\lambda u) \sum_{k=1}^{n} \theta_{n,k} f_k(|u|) du \\ &= 2\mathcal{R}e \left(\sum_{k=1}^{n} \theta_{n,k} \phi_{t_{k+n} - t_k}(2\pi\lambda) \right) \\ &= 2\mathcal{R}e \left(\sum_{k=1}^{n} \theta_{n,k} \left(\frac{\beta}{\beta - 2i\pi\lambda} \right)^n \right) \end{aligned}$$

or

$$G_n(\lambda) = 2(-1)^{n-1} \sqrt{\frac{b\beta}{\beta^2 + 4\pi^2\lambda^2}} \cos\left((2n-1) \arctan(2\pi\lambda/\beta) \right).$$

Remark 3 Some simple algebra shows that $\int_{\mathbb{R}} G_n(\lambda) G_p(\lambda) d\lambda = 2\delta_{n,p}$, where $\delta_{n,p}$ is the Kronecker symbol.

The expansion of the spectral density in this basis becomes

$$G_X(\lambda) = \sum_{n=1}^{+\infty} a_n G_n(\lambda) \text{ in } L^2(\mathbb{R}),$$

with $a_n = \sum_{k=1}^{n} \theta_{n,k} \rho_X(k)$ and $\theta_{n,k} = \sqrt{2/\beta}(-2)^{k-1} \mathcal{C}_{n-1}^{k-1}$.

For estimating G_X, let us be given the observations $X(t_k)$, $k = 1, \ldots, N$ for $N > 0$. First, $\rho_X(n)$ may be estimated by

$$\widehat{\rho}_{X,n}(N) = \begin{cases} \frac{1}{N} \sum_{k=1}^{N-n} X(t_{k+n}) X(t_k) & \text{if} \quad 1 \le n < N \\ 0 & \text{otherwise.} \end{cases}$$

Then, we estimate a_n by

$$\widehat{a}_n(N) = \sum_{k=1}^{n} \theta_{n,k} \widehat{\rho}_{X,n}(N).$$

Thus, the spectral density estimate is finally given by

$$\widehat{G}_{X,n}(\lambda) = \sum_{n=1}^{M_N} y_n(N) \widehat{a}_n(N) G_n(\lambda), \tag{3}$$

with the bandwidth $M_N = [b \ln(N)/(2a)] + 1$ and with $y_n(N) = h\left(\exp(\alpha n)/N^b\right)$, where $\alpha > \ln(3)$, $0 < b\alpha/(2\ln(3))$, and the real function h is a Lipschitz function of order 1 such that $h(u) \leq h(0) = 1$, $\forall u \in \mathbb{R}$.

4 Mean square consistency

For studying the asymptotic behaviour of the spectral density estimate (3), we will assume that the following conditions are fulfilled.

CONDITION 1. Let $AC^{r-1}[1, +\infty[$ be the set of r-times differentiable functions with absolutely continuous successive derivatives. We assume that $R_X \in AC^{r-1}[1, +\infty[$ and that $[\ln^r(t) t^{2k-1}]^{1/2} R_X^{(k)}(t) \in L^2(1, +\infty)$ for $k \in \{0, \ldots, r\}$, with $r \geq 3$.

CONDITION 2.

- $(\ln(t)/t)^{1/2} R_X(t) \in L^2(1, +\infty)$
- $|Q(t_1, t_2, t_3)| \leq h_1(t_1, t_2, t_3)$, for all positive numbers t_1, t_2 and t_3, where h_1 is a non-increasing function on $[1, +\infty[$ for each argument, and verifies

$$\begin{aligned} h_1(t_1, t_2, t_3) = h_1(\frac{1}{t_1}, t_2, t_3) &= h_1(t_1, \frac{1}{t_2}, t_3) = h_1(t_1, t_2, \frac{1}{t_3}) \\ \int_1^{+\infty} h_1(1, \tau, 1) d\ln(\tau) &< +\infty. \end{aligned}$$

Note that conditions 1 and 2 are equivalent to those assumed by Messaci (1986).

In order to study the behavior of the estimate $\widehat{G}_{X,n}$, we have to study first the behavior of the estimates $\widehat{a}_n$ and $\widehat{\rho}_{X,n}$. All this is given in the four following theorems which are proven in Section 7. The first result concerns $\widehat{\rho}_{X,n}$.

Theorem 2 *Under Condition, $\widehat{\rho}_{X,n}$ is consistent in mean-square. That is*

1. $\mathbb{E}\{\widehat{\rho}_{X,n}(N)\} = (1 - n/N)\rho_X$.
2. *There exits a uniform on n and N positive constant A_2 such that $var\{\widehat{\rho}_{X,n}(N)\} \leq A_2/N$.*

The estimate $\widehat{a}_n$ of a_n behaves as follows.

Theorem 3 *Under Condition 2, we have*

1. $\mathbb{E}\{\widehat{a}_n(N)\} = a_n - \frac{1}{N}(n a_n - (n-1)a_{n-1})$.
2. $var\{\widehat{a}_n(N)\} \leq A_3 \frac{3^{2n}}{N}$ *with* $A_3 = \frac{2A_2}{9\beta}$.

The mean-square error of the estimate $\widehat{G}_{X,n}$ is given in the following theorem.

Theorem 4 *If Condition 1 and Condition 2 are satisfied, we have*

$$\mathbb{E}\{\widehat{G}_{X,n}(\lambda) - G_X(\lambda)\}^2 = O\left(\frac{1}{\ln(N)}\right)^{r-2}.$$

And finally the mean square error is given below.

Theorem 5 *Under the same conditions, the integrated mean square error of the estimate $\widehat{G}_{X,n}$ satisfies*

$$\mathbb{E}\Big[\int_{-\infty}^{+\infty} |\widehat{G}_{X,n}(\lambda) - \phi(\lambda)|^2 d\lambda\Big] = O\left(\frac{1}{\ln(N)}\right)^{r-1}.$$

5 Short simulation study

The calculus of the defined estimates is given in this section on some simulated samples. Let X be a second-order M-stationary Gaussian process with zero mean, continuous covariance function $R_X \in L^1$, and spectral density function G_X. We study the finite-sample size performance of the spectral estimate (3), by Monte-Carlo simulation, for one representative spectral density functions

$$R_X(t) = \exp(-\ln(t)) \text{ and } G_X(\lambda) = \frac{1}{\pi(1+\lambda^2)}, \tag{4}$$

Eq. (4) is the lowpass spectral density of the Gauss-Markov process. We will use two numbers of samples $N = 500$ and $N = 1000$, and consider Parzen's spectral window for which

$$h(t) = \begin{cases} 1 - 6t^2 + 6|t|^3, & \text{if } \quad |t| \le \frac{1}{2}, \\ 2(1-|t|)^3, & \text{if } \quad \frac{1}{2} < |t| \le 1, \\ 0, & \text{if } \quad 1 < |t|. \end{cases}$$

The estimate was evaluated for several values of the sample rate β in order to test its inference on the performance of the estimate.

For each sampling rate, the estimate (3) was initially evaluated for several values of M_N and for each estimate the optimal M_N was chosen by a Cross-Validation procedure as in [24]. The optimal M_N minimizes a Cross-Validated creterion depending only on the observations, which is proved to minimize asymptotically the integrated mean square error between G_X and $\widehat{G}_{X,n}$, over the range of graphes. The smallest sampling rate β one could use is $\beta = 1/\pi,\ 2/\pi$ and 1

The sampling instants $\{t_k\}_{k=1}^N$ and the observations $\{X(t_k)\}_{k=1}^N$ were generated recursively as follows. The $\{t_k\}_{k=1}^N$ are already given by (2) in a recursive form. Sample values of the $\{\alpha_k\}_k$ were obtained by

$$\alpha_k = -\frac{1}{\beta}\ln(\theta_k),\ k = 1, \ldots, N,$$

where the $\{\theta_k\}_k$ are uniform on $[0,1]$ random numbers. The observations $\{X(t_k)\}_{k=1}^N$ are then obtained via the dynamical representation for process X (see, [**?**]). We have

$$X(t) = \exp(-(t-\tau))X(\tau) + \sqrt{2}\int_\tau^t \exp(-(t-s))dW(s),\ t > \tau \ge 0,$$

where $\{W(t),\ t \ge 0\}$ is the real valued Wiener process with zero mean, and covariance $R_X(t,s) = \min(t,s)$.

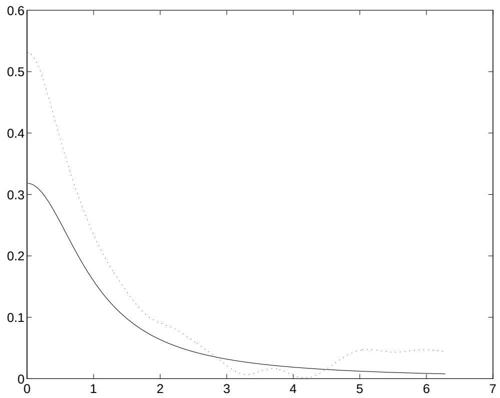

Fig 1. The estimate and the spectrum : $\beta = 1/\pi$, $N = 500$

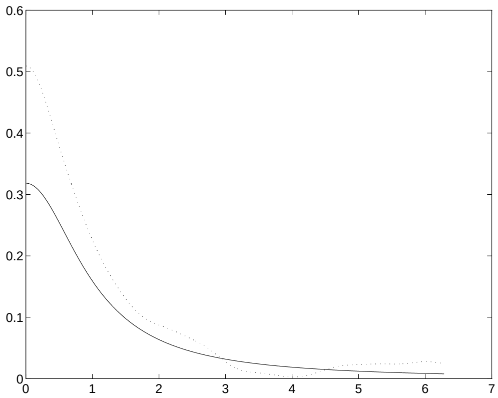

Fig 2. Gauss-Markov spectrum and its estimate : $\beta = 1/\pi, N = 1000$.

Then, it is easily seen that

$$X(t_0) = 0,$$
$$X(t_{k+1}) = \exp(-(t_{k+1} - t_k))X(t_k) + [1 - \exp(-2(t_{k+1} - t_k))]^{1/2}\, \gamma_{k+1},\ k = 0, 1, \ldots.$$

where the $\{\gamma_k\}_k$ are i.i.d. zero mean unit variance, Gaussian random variables.

Sample values of the $\{\gamma_k\}_k$ were obtained from a Gaussian random number generator (under Matlab). The estimate (3) was then evaluated and the results are shown in Figs. 1-2. It is seen that the estimate performs well. The simulation results indicate that the sample size performance of the estimate (3) is excellent, that is insensitive to the sampling rate β.

6 Conclusion

This technique is not uniquely linked to the definition of the M-stationarity. It will be extended in Girardin & Rachdi (2000), to other generalized forms of stationarity, where a suitable time transformation replaces the logarithm used here. The spectral density is then given by the corresponding theory of group representation coming in place of the Mellin transform. Examples of such processes may be derived from those given above, especially the linear ones, extending so the notion of stationary auto-regressive models. More generally, their extensive study may be developped from the study of M-stationary processes done by Gray& Zhang (1988).

The bandwidth selection is an important problem. In fact, a bad selection of this parameter allows to a bad rate of convergence of the estimates and do not minimizes optimally the mean square error. Some techniques as Cross-Validation developped by Rachdi (1998), for continuous time stationary process can solve this problem as used in the simulation study.

7 Proofs

Proof of Theorem 2.

(1) First, $\mathbb{E}\{\widehat{\rho}_{X,n}(N)\} = \dfrac{1}{N}\displaystyle\sum_{k=1}^{N-n} \mathbb{E}\{X(t_{k+n}) - X(t_k)\} = (1 - n/N)\rho_n$. Thus the order of magnitude of the bias of $\widehat{\rho}_{X,n}$ is $O(1/N)$.

(2)

$$var\{\widehat{\rho}_{X,n}(N)\} = \frac{1}{N^2}\sum_{k,l=1}^{N-n} cum\{X(t_{k+n})X(t_k), X(t_{l+n})X(t_l)\},$$

where cum means the cumulant function. By some simple manipulation of the cumulant functions, the quantity (3) becomes

$$\frac{1}{N^2}\sum_{k,l=1}^{N-n} \mathbb{E}_{\{t_n\}}\{R_Y(t_{k+n}-t_k)R_Y(t_{l+n}-t_l)+R_Y(t_l-t_k)R_Y(t_{l+n}-t_{k+n})$$
$$+R_Y(t_{k+n}-t_k)R_Y(t_{k+n}-t_l)+Q(t_{k+n},t_k,t_{l+n},t_l)\},$$

where $\mathbb{E}_{\{t_n\}}$ denotes the mean conditonal to the distribution of the scheme $\{t_n\}$. Together with the stationarity up to order four of the process

$$Q(t_{k+n},t_k,t_{l+n},t_l)\}=Q(t_{k+n}/t_k,t_{l+n}/t_k,t_l/t_k)\}=cum(t_{k+n}-t_k,t_{l+n}-t_k,t_l-t_k)\},$$

the technique used in Messaci (1986), pages 18-25, ensures that $var\{\widehat{\rho}_{X,n}(N)\}=O\left(\frac{1}{N}\right)$.

Proof of Theorem 3.
Recall that $\widehat{a}_n=\sum_{k=1}^{N}\theta_{n,k}\widehat{\rho}_{X,k}(N)$. From Theorem 2, we get both following points.

(1)

$$\begin{aligned}\mathbb{E}\{\widehat{a}_n(N)\} &= \sum_{k=1}^{N}\theta_{n,k}\mathbb{E}\{\widehat{\rho}_{X,k}(N)\}\\ &= \widehat{a}_n-\frac{1}{N}\sum_{k=1}^{N}\theta_{n,k}\rho_k\frac{k}{N}\end{aligned}$$

Denote by $Bias(\widehat{a}_n(N))$ the bias of $\widehat{a}_n(N)$. We get then from (5)

$$\begin{aligned}Bias\{\widehat{a}_n(N)\} &= -\frac{1}{N}\sum_{k=1}^{N}\theta_{n,k}k\int_0^{+\infty}R_Y(t)f_k(t)dt\\ &= -\frac{1}{N}\int_0^{+\infty}\left(\sum_{k=1}^{N}\theta_{n,k}kf_k(t)\right)R_Y(t)f_k(t)dt=-\frac{1}{N}\int_0^{+\infty}R_Y(t)q_n(t)dt\end{aligned}$$

where $\sum_{k=1}^{n}k\theta_{n,k}f_k(t)=nq_n(t)-(n-1)q_{n-1}(t)$, with $q_n(t)=\sum_{k=1}^{n}\theta_{n,k}f_k(t)$. Thus

$$\begin{aligned}Bias\{\widehat{a}_n(N)\} &= -\frac{n}{N}\int_0^{+\infty}R_Y(t)g_n(t)+\frac{n-1}{N}\int_0^{+\infty}R_Y(t)g_{n-1}(t)dt\\ &= -\frac{n}{N}\widehat{a}_n+\frac{n-1}{N}a_{n-1}\\ &= \widehat{a}_n-\frac{1}{N}\left(n\widehat{a}_n-(n-1)a_{n-1}\right).\end{aligned}$$

(2) The autocovariance function of the estimate $\widehat{a}_n(N)$ behaves as follows

$$var\{\widehat{a}_n(N)\}\le\left(\sum_{k=1}^{n}|\theta_{n,k}|\sqrt{var\{\widehat{\rho}_{X,n}(N)\}}\right)^2,$$

and from theorem 2, this becomes, by some simple calculation

$$\begin{aligned}var\{\widehat{a}_n(N)\} &\le \left(\sum_{k=1}^{n}\sqrt{\frac{2A_2}{\beta N}}2^{k-1}C_{n-1}^{k-1}\right)^2\\ &\le 3^{2n}\frac{A_3}{N}\text{ with }A_3=\frac{2A_2}{9},\end{aligned}$$

which completes the proof of the theorem.

Proof of Theorem 4.
Recall that the mean square error of $\widehat{G}_{X,n}$ is expanded as

$$\mathbb{E}\{\widehat{G}_{X,n}(\lambda) - G_X(\lambda)\}^2 = var\{\widehat{G}_{X,n}(\lambda)\} + Bias^2(\widehat{G}_{X,n}(\lambda)).$$

The proof of the theorem will take two main steps.

First step: Majoration of the terms $var\{\widehat{G}_{X,n}(\lambda)\}$.
From the expression of the estimate and theorem 3, we get

$$\begin{aligned}\left(var\{\widehat{G}_{X,n}(\lambda)\}\right)^{1/2} &\leq \sum_{n=1}^{M_N} |y_n(N)||G_n(\lambda)| \left(\frac{A_3 3^{2n}}{N}\right)^{1/2} \\ &\leq 2\sum_{n=1}^{M_N} h\left(\frac{\exp(na)}{N^b}\right)\sqrt{\frac{2A_3\beta 3^{2n}}{N(\beta^2+4\pi^2\lambda^2)}}. \qquad (5)\end{aligned}$$

From the expression of the bandwidth M_N, the quantity (5) is less than $9\sqrt{\dfrac{2A_3}{\beta}}N^{-1/2(1-\frac{b}{2\alpha}\ln(3))}$, thus $var\{\widehat{G}_{X,n}(\lambda)\} = O(N^{-p})$ with $p = 1 - \dfrac{b}{2\alpha}\ln(3)$.
Second step: Bound for the bias.
We have

$$\begin{aligned}Bias\{\widehat{G}_{X,n}(\lambda)\} &= \sum_{n=1}^{M_N} y_n(\lambda)G_n(\lambda)a_n - \sum_{n=1}^{M_N} a_nG_n(\lambda) - \sum_{n=M_N+1}^{+\infty} a_nG_n(\lambda) \\ &\quad -\frac{1}{N}\sum_{n=1}^{M_N} G_n(\lambda)y_n(N)(na_n-(n-1)a_{n-1})\end{aligned}$$

which implies

$$\begin{aligned}|Bias\{\widehat{G}_{X,n}(\lambda)\}| &\leq \sum_{n=1}^{M_N}|G_n(\lambda)||a_n||1-y_n(\lambda)| + \frac{1}{N}\sum_{n=1}^{M_N}|G_n(\lambda)||y_n(\lambda)|(n|a_n|+(n-1)|a_{n-1}|) \\ &\quad +\sum_{n=1}^{M_N}|G_n(\lambda)||a_n|. \qquad (6)\end{aligned}$$

We are going to proceed in three sub-steps:

(i) we have that

$$\sum_{n=1}^{M_N}|G_n(\lambda)\,a_n\,(1-y_n(N))| \leq \sqrt{\frac{2}{\pi^2\beta}}\sum_{n=1}^{M_N}|a_n|\,|1-y_n(N)|.$$

By application of Lemma 1, in Messaci (1989), and the imposed conditions on h, we obtain

$$\sum_{n=1}^{M_N}|G_n(\lambda)\,a_n\,(1-y_n(N))| \leq \sqrt{\frac{2}{\pi^2\beta}}k_2A_1(r)\sum_{n=1}^{M_N} n^{-r/2}\frac{\exp(n\alpha)}{N^b}.$$

Since $\sum_{n=1}^{M_N} n^{-r/2} \leq 1 + \int_1^{+\infty} x^{-r/2}dx = \dfrac{r}{r-2}$ and $M_N \leq \dfrac{b}{2\alpha}\ln(N)+1$, we deduce that

$$\sum_{n=1}^{M_N}|G_n(\lambda)\,a_n\,(1-y_n(N))| \leq \sqrt{\frac{2}{\pi^2\beta}}k_2A_1(r)\frac{r}{r-2}\,e\,N^{-b/2} = O\left(\frac{}{N^{b/2}}\right),$$

which is of order of magnitude $O\left(\dfrac{1}{\ln(N)}\right)^{(r-2)/2}$.
(ii) By simple calculations, we have that

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{M_N}|y_n(N)G_n(\lambda)|\,|na_n+(n-1)a_{n-1}| &\leq \frac{2}{N}\sqrt{\frac{2}{\pi^2\beta}}\sum_{n=1}^{+\infty}M_{N}A_1(r)n^{-r/2} \\ &\leq 2A_1(r)\sqrt{\frac{2}{\pi^2\beta}}\sum_{n=1}^{+\infty}n^{-r/2}\left(\frac{b}{2\alpha}\ln(N)+1\right)\frac{1}{N^{-p+1}\,N^p},\end{aligned}$$

where $p = 1-\dfrac{b}{\alpha}\ln(3)$. The imposed hypothesis on b and α ensure that $0<p<1$, which implies that in the first hand $(\dfrac{b}{2\alpha}\ln(N)+1)\dfrac{1}{N^{-p+1}} = o(1)$, and in the other hand

$$\frac{1}{N}\sum_{n=1}^{M_N}|y_n(N)G_n(\lambda)|\,|na_n+(n-1)a_{n-1}| = O(1/N)^p = O(1/\ln(N))^{\frac{r-2}{2}}.$$

(iii) From lemma 2.1 of Masry (1980), under Condition 1 and since $|a_n|\leq A_1(r)n^{-r/2}$ with $A_1(r) = \sqrt{\dfrac{1}{2\beta}}||t^{r/2}\dfrac{d^r}{d\,t^r}\left(R_{Y}(t/2\beta)\exp(-t/2)\right)||_{L^2(\mathbb{R}_+)}$, the term (6) is less than

$$\begin{aligned}\sqrt{\frac{2}{\pi^2\beta}}A_1(r)\sum_{n=M_N+1}^{+\infty}n^{-r/2} &\leq \sqrt{\frac{2}{\pi^2\beta}}A_1(r)\int_{M_N}^{+\infty}x^{-r/2}dx \\ &\leq \sqrt{\frac{2}{\pi^2\beta}}A_1(r)\frac{2}{r-2}M_{N}^{\frac{-r+2}{2}} \\ &\leq \sqrt{\frac{2}{\pi^2\beta}}A_1(r)\left(\frac{2\alpha}{b\ln(N)}\right)^{\frac{r-2}{2}}\end{aligned}$$

thus $\displaystyle\sum_{n=M_N+1}^{+\infty}|a_nG_n(\lambda)| = O\left(\frac{1}{\ln(N)}\right)^{\frac{r-2}{2}}$. From the tree substeps, we deduce directly the result.

Proof of Theorem 5.
The proof is deduced by simple integration, from the theorem 4 proof's.

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