RENORMALIZATION GROUP ANALYSIS OF HAMILTONIAN FLOWS

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ABSTRACT. This paper is a summary of some recent work by the authors on the renormalization of Hamiltonian systems. Applications include the construction of invariant tori and related sequences of closed periodic orbits. We also discuss problems related to the breakup of invariant tori, and some numerical results.

The renormalization group transformation.

In this paper we describe some recent results [9–12] on the renormalization of Hamiltonian systems with two or more periodic degrees of freedom $q = (q_1, \ldots, q_d)$. One of the goals is to study invariant tori with certain arithmetically interesting rotation vectors, for the flow $(\dot{q}, \dot{p}) = (\nabla_2 H, -\nabla_1 H)$ associated with a Hamiltonian $H = H(q,p)$. Here, $\nabla_1 H$ and $\nabla_2 H$ denote the partial gradients of H with respect to the first (angle) variable q and the second (action) variable $p = (p_1, \ldots, p_d)$, respectively. Unless specified otherwise, a rotation vector $\omega \in \mathbb{R}^d$ is assumed to be nonzero, and parallel rotation vectors are identified. An invariant torus will be called an ω–torus, if the motion on it is conjugate to the linear flow $q_j(t) = q_j(0) + tc\omega_j \pmod{2\pi}$, for some nonzero constant c.

We are particularly interested in rotation vectors $\omega = (1, \omega_2, \ldots, \omega_d)$ whose components span an algebraic number field of degree d. They can also be characterized by a "self–similarity" property [11]: There exists an integer $d \times d$ matrix T, with determinant ± 1 and $d-1$ simple eigenvalues of modulus less than 1, for which ω is an eigenvector with a real eigenvalue $\vartheta_1 > 1$. The best known example is $\omega = (1, \vartheta_1)$ with ϑ_1 the golden mean, and $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. We note that the matrix T can also be used to construct sequences of rational approximants to ω, by defining $w_n = c_n T^n w$ for some nonzero $w \in \mathbb{Q}^d$, where c_n is a suitable normalization.

In order to study ω–tori and w_n–orbits, we would like to define a renormalization group (RG) transformation $\mathcal{R}$, acting on a space of Hamiltonians, such that

$$\mathcal{R}(H) = H \circ \mathcal{T}_1 \pmod{\mathcal{G}}\,, \tag{1}$$

with $\mathcal{T}_1$ defined by the equation $\mathcal{T}_\mu(q,p) = \bigl(Tq, \mu(T^*)^{-1}p\bigr)$, for $\mu = 1$. Here, T^* denotes the transpose of T, and $\mathcal{G}$ is some group (ideally) of transformations that act on Hamiltonians without changing the rotation vectors of their orbits.

1991 *Mathematics Subject Classification.* Primary 58F27; Secondary 58F22, 70K50..

HK was supported in part by the National Science Foundation under grants no DMS-9705095 and DMS-0088935.

PW was supported in part by the Swiss National Science Foundation.

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Approximate RG transformations of this type have been studied mainly in connection with the breakup of invariant tori, first in [1–2], and more recently in [5–10]. Notice that $\mathcal{T}_1$ is a canonical transformation. Since it is not homotopic to the identity, $\mathcal{R}$ can change rotation vectors: To a closed closed orbit for $\mathcal{R}(H)$ with rotation vector w corresponds a closed orbit for H with rotation vector Tw. Furthermore, if $\mathcal{R}(H)$ has an invariant ω–torus, then so does H. This is analogous to a property of the well known doubling transformation $R(f) = f \circ f \pmod{G}$ for maps: To a periodic orbit for $R(f)$ of length ℓ corresponds a periodic orbit for f of length 2ℓ. Furthermore, if $R(f)$ has has an invariant cantor set of type 2^∞, then so does f.

Natural candidates for transformations to be included in the "group" $\mathcal{G}$ are

- coordinate changes: $H \mapsto H \circ \mathcal{U}$, with $\mathcal{U}$ canonical, homotopic to the identity.
- scaling of time or energy: $H \mapsto \tau H - \epsilon$, with $\tau \neq 0$.
- scaling of the action variable: $H \mapsto \mu^{-1} H(\,.\,, \mu\,.\,)$, with $\mu \neq 0$.

This suggests the following form for $\mathcal{R}$.

$$\mathcal{R}(H) = \frac{\tau}{\mu} H \circ \mathcal{U}_H \circ \mathcal{T}_\mu - \epsilon\,, \tag{2}$$

where τ, μ and ϵ are allowed to depend on H. We choose $\tau = \tau_H$ in such a way that at $p = 0$, the torus average of $\omega' \cdot \nabla_2 \mathcal{R}(H)$ becomes 1, where ω' is the eigenvector of T^* for the eigenvalue ϑ_1 , with normalization $\omega \cdot \omega' = 1$. Unless specified otherwise, ϵ is taken to be zero, and μ is assumed to be a fixed positive constant less than $|\vartheta_1^{-1} \vartheta_d^2|$. Here, and in what follows, ϑ_j denotes the j-th largest eigenvalue (in modulus) of T.

The domain of $\mathcal{R}$ — near-resonant Hamiltonians.

The proper choice of $\mathcal{U}_H$ requires more work, except e.g. in a case of integrable Hamiltonians, which we shall describe first. Let $\mathcal{H}$ be some Banach space of functions $H(q,p)$ that are analytic in $|p| < \rho$ and do not depend on q. Consider an open set B in $\mathcal{H}$ where the function $H \mapsto \omega' \cdot \nabla_2 H(0)$ is bounded away from zero. Then it is easy to verify that $\mathcal{R}$ maps B into $\mathcal{H}$, and that the following statements (that will be generalized later) hold.

Proposition 1. *$\mathcal{R}$ is compact and analytic on B. The function $H^\circ(q,p) = \omega \cdot p$ is a hyperbolic fixed point for $\mathcal{R}$, and the spectrum of the derivative $D\mathcal{R}(H^\circ)$ lies inside the open unit disk, except for d eigenvalues $\delta_0 = \vartheta_1/\mu$ and $\delta_j = \vartheta_1/\vartheta_j$, $j = 2, \ldots d$, which are of modulus > 1. The local unstable manifold of $\mathcal{R}$ at H° consists of all functions $(q,p) \mapsto c + w \cdot p$ with $w \cdot \omega' = 1$. Every Hamiltonian H on the local stable manifold at H° has an analytic invariant ω–torus Γ_H which lies on the energy level $H^{-1}(0)$, and is "centered at $p = 0$", in the sense that the integral*

$$K(\gamma) = \frac{1}{2\pi} \oint_\gamma p \cdot dq \tag{3}$$

is zero for every closed curve γ on the torus Γ_H .

Our next goal is to extend $\mathcal{R}$ as a dynamical system to a larger space, which we take to be a Banach space $\mathcal{A}_\rho$ of analytic functions

$$H(q,p) = \sum_{(\nu,\alpha)\in I} H_{\nu,\alpha} e^{i\nu\cdot q} p^\alpha\,, \qquad \|H\|_\rho = \sum_{(\nu,\alpha)\in I} |H_{\nu,\alpha}| e^{\rho|\nu|} \rho^{|\alpha|} < \infty\,, \tag{4}$$

where $I = \mathbb{Z}^d \times \mathbb{N}^d$. Unless stated otherwise, the norm on a product space X^n is defined by $|x| = |x_1| + \ldots + |x_n|$. Notice that the common domain for functions in $\mathcal{A}_\rho$ is the set $\mathcal{D}_\rho$ characterized by $|\mathrm{Im}\ q| < \rho$ and $|p| < \rho$.

When trying to satisfy (1), the main problem is that for a general $H \in \mathcal{A}_\rho$, the function $H \circ \mathcal{T}_\mu$ is no longer analytic in $\mathcal{D}_\rho$. The situation is better for what we call "resonant" Hamiltonians. To be more precise, assume that σ and κ are given positive constants. Let

$$I^+ = \{(\nu, \alpha) \in I : \ |\omega \cdot \nu| \le \sigma|\nu| + \kappa|\alpha| \ \}, \qquad I^- = I \setminus I^+, \tag{5}$$

and define $\mathbb{I}^\pm H$ by restricting the sum in (4) to the corresponding index set $I^\pm$. We will call H resonant if it belongs to the range of $\mathbb{I}^+$, or equivalently, if $\mathbb{I}^- H = 0$. It is straightforward to verify the following.

Proposition 2. *If $\rho - \rho'$, σ and κ are positive and sufficiently small, then $H \mapsto H \circ \mathcal{T}_\mu$ defines a compact linear map from $\mathbb{I}^+\mathcal{A}_{\rho'}$ to $\mathcal{A}_\rho$.*

In fact, the "sufficiently small" can be replaced by explicit inequalities [11,12].

Motivated by this result, we try to define the canonical transformation $\mathcal{U}_H$ in equation (2) in such a way that for every H in some suitable subset of $\mathcal{A}_\rho$,

$$H \circ \mathcal{U}_H \in \mathcal{A}_{\rho'}, \qquad \mathbb{I}^-\bigl(H \circ \mathcal{U}_H\bigr) = 0\,. \tag{6}$$

This is compatible with choosing $\mathcal{U}_H = \mathrm{I}$ whenever $H(q,p)$ depends on p only, since by definition, such integrable Hamiltonians are resonant. For non-resonant Hamiltonians, the equation for $\mathcal{U}_H$ is rather nontrivial. This is quite different from the doubling transformation R mentioned earlier: In the standard case, it suffices to re-normalize $f \circ f$ with a linear change of coordinates Λ_f. And this change of coordinates happens to be a contraction, so that R becomes automatically analyticity improving. In particular, Λ_f is determined by a finite number of (scalar) normalization conditions, while the condition (6) for $\mathcal{U}_H$ is infinite dimensional.

Denote by $\mathcal{A}'_\rho$ the space of all functions in $\mathcal{A}_\rho$ whose first partial derivatives belong to $\mathcal{A}_\rho$. Define $\widehat{F}\phi = \nabla_1 F \cdot \nabla_2 \phi - \nabla_2 F \cdot \nabla_1 \phi$.

Theorem 3. *Let $0 < \rho' < \rho$. Then the following holds for some $c > 0$. Let $F \in \mathcal{A}_\rho$. Assume that the operator $\mathbb{I}^-\widehat{F} : \mathbb{I}^-\mathcal{A}'_r \to \mathbb{I}^-\mathcal{A}_r$ is invertible and satisfies*

$$\bigl\|\mathbb{I}^-\widehat{F}\phi\bigr\|_r \ge c\,\bigl\|\mathbb{I}^-F\bigr\|_\rho^{1/2}\|F\|_\rho^{1/2}\|\nabla\phi\|_r\,, \qquad \forall \phi \in \mathbb{I}^-\mathcal{A}'_r\,, \tag{7}$$

for every r between ρ' and $\rho'/9 + 8\rho/9$. Then there exists an open set $B \ni F$ in $\mathcal{A}_\rho$, and for each $H \in B$ a canonical transformation $\mathcal{U}_H : \mathcal{D}_{\rho'} \to \mathcal{D}_\rho$ satisfying (6), such that $H \mapsto H \circ \mathcal{U}_H$ is analytic as a map from B to $\mathcal{A}_{\rho'}$.

A version of this theorem, which allows for more general operators $\mathbb{I}^-$, was proved in [12]. Notice that the inequality (7) holds if $\|\mathbb{I}^-F\|_\rho$ is sufficiently small, i.e., if F is near-resonant.

Near-integrable Hamiltonians.

From the definition (5) of $I^{\pm}$ it follows immediately that $F = H^{\circ}$ satisfies the hypothesis of Theorem 3. Thus, by Proposition 2, if $\sigma, \tau > 0$ are sufficiently small, $\mathcal{R}$ is well defined near H°. In addition, we have the following result [11].

Theorem 4. *Proposition 1 generalizes to $\mathcal{R} : B \to \mathcal{A}_\rho$, with B some open neighborhood of H° in $\mathcal{A}_\rho$.*

We should add that the transformations $\mathcal{U}_H$ described in Theorem 3 are (by construction) composed of canonical transformations $(q,p) \mapsto (q+Q, p+P)$ for which the one-form $P\cdot dq + p\cdot dQ$ is exact. This excludes e.g. translations $J_\beta(q,p) = (q, p+\beta)$, which is the reason why $D\mathcal{R}(H^{\circ})$ has d unstable directions.

If H lies on the local stable manifold $\mathcal{W}_s$ of $\mathcal{R}$ at H°, then the torus Γ_H is obtained from the canonical transformations

$$\mathcal{V}_n(H) = V_0(H) \circ V_1(H) \circ \ldots \circ V_{n-1}(H)\,, \qquad V_k(H) = \mathcal{T}_\mu^k \circ \mathcal{U}_{\mathcal{R}^k(H)} \circ \mathcal{T}_\mu^{-k}\,,$$

by restricting $\mathcal{V}_n(H)$ to $p = 0$ and taking the limit $n \to \infty$. If H does not lie on $\mathcal{W}_s$, then we need to assume that H is non-degenerate, i.e., that the family $\beta \mapsto H_\beta = H \circ J_\beta$ intersects $\mathcal{W}_s$ transversally. Consider e.g. a fixed Hamiltonian $F(q,p) = f(p)$ close to H°, with $\nabla f(0) = \omega$, so that F lies on $\mathcal{W}_s$. Assume that the Hessian of f at zero is of maximal rank. Then F is non-degenerate, and the same is true for every Hamiltonian H sufficiently close to F. Thus, H has an invariant torus $\Gamma' = J_{\beta'} \circ \Gamma_{H_{\beta'}}$, where β' is the value of the parameter β for which H_β belongs to $\mathcal{W}_s$.

The renormalization group formalism can also be used to construct sequences of periodic orbits accumulating at Γ'. To simplify the task, we restrict the analysis to a subspace that is invariant under $\mathcal{R}$: the space $\mathcal{B}_\rho$ of all Hamiltonians in $\mathcal{A}_\rho$ that are even functions of q. The starting point is the following result [12].

Proposition 5. *Given $w \in \mathbb{R}\mathbb{Z}^d$ sufficiently close to ω, there exists an analytic manifold $\Sigma(w)$ of codimension d, intersecting the local unstable manifold $\mathcal{W}_u$ transversally, such that every Hamiltonian $H \in \Sigma(w)$ has a periodic orbit γ with rotation vector w, which lies on the energy level $H^{-1}(0)$ and is centered at $p = 0$.*

To be more precise: For a closed orbit γ to be "centered at $p = 0$", we require not only that the integral $K(\gamma)$, defined in (3), be zero. Another $d-1$ conditions are needed in order to distinguish γ from its translates $J_\beta \circ \gamma$. But since a curve (orbit of a near-integrable Hamiltonian) does not have enough useful invariants, we have made an ad hoc choice in [12], which we shall not specify here.

Consider now the manifolds $\Sigma_n(w) = \mathcal{R}^{-n}(\Sigma(w))$, and a non-degenerate Hamiltonian H close to H°. Then the λ–Lemma implies (roughly) that for large n, the family $\beta \mapsto H_\beta$ intersects $\Sigma_n(w)$ at a parameter value β_n, with $|\beta_n - \beta'|$ decreasing like $|\delta_2|^{-n}$. By a fundamental property of $\mathcal{R}$ mentioned earlier, H_{β_n} has a closed orbit γ_n' with rotation vector parallel to $T^n w$. Translating this orbit by β_n yields an orbit γ_n for H. By keeping track of the position of γ_n', and of the translation parameter β_n, we obtain the following theorem [12].

Theorem 6. *Let $r > \rho$ and $w \in \mathbb{RZ}^d$ be given, $w \neq 0$. If F is a non-degenerate q–independent Hamiltonian sufficiently close to H° in $\mathcal{B}_r$, then there exists an open neighborhood B of F in $\mathcal{B}_r$, and an integer $N > 0$, such that for all $H \in B$, and for all $n \geq N$, the Hamiltonian H has a closed orbit γ_n on $H^{-1}(0)$ with frequency vector $w_n = (\vartheta_1^{-1}T)^n w$. The sequence of these orbits satisfies*

$$-\frac{1}{n}\ln\bigl|\gamma_n(0) - \Gamma'(0)\bigr| = \ln|\delta_2| + \mathcal{O}\bigl(\tfrac{1}{n}\bigr)\,. \tag{8}$$

Concerning nontrivial fixed points.

One of the future goals is to find non-integrable fixed points for a transformation like $\mathcal{R}$. Numerical results [1–10] suggest that such fixed points exists for $d = 2$, and maybe even $d > 2$ in some cases, and that they are related to the breakup of invariant tori. We describe here some results and insights obtained in [9,10].

Assume that T has a real eigenvalue ϑ_m of modulus < 1, and let Ω be an eigenvector for this eigenvalue, normalized such that one of its components is 1. By construction [11,12] of the transformations $\mathcal{U}_H$, if a Hamiltonian H is of the form

$$H(q,p) = \omega\cdot p + \sum_{(\nu,k)\in I} H_{\nu,k}\cos(\nu\cdot q)(\Omega\cdot p)^k\,, \tag{9}$$

then so is $\mathcal{R}(H)$, if defined, and $\tau_H = \vartheta_1$ independently of H. Here, $I = \mathbb{Z}^d \times \mathbb{N}$. Thus, our fixed point problem can be simplified by restricting $\mathcal{R}$ to a space of such Hamiltonians. Let us now choose $\epsilon = \epsilon_H$ in (2) in such a way that $H'' = \mathcal{R}(H)$ has no constant term. Furthermore, in order to allow for a nontrivial scaling of the actions, we no longer fix μ, but determine $\mu = \mu_H$ by a normalization $H''_{0,2} = 1/2$. Then $\mathcal{R}$ has $H^\circ(q,p) = \omega\cdot p + \frac{1}{2}(\Omega\cdot p)^2$ as its trivial fixed point, with $\mu_{H^\circ} = \vartheta_m^2/\vartheta_1$, and $D\mathcal{R}(H^\circ)$ has a single expanding direction, given by the function $(q,p)\mapsto\Omega\cdot p$.

For a study of Hamiltonians far from H°, but near a given function G, it is convenient to consider a modified RG transformation of the form

$$\mathcal{N}(H) = H'\circ\mathcal{U}_{H'}\,,\quad H' = \frac{\tau}{\mu}H\circ\mathcal{T}_\mu\circ U - \epsilon\,, \tag{10}$$

where U is a fixed canonical transformation, homotopic to the identity, and chosen in such a way that $\bar{\mathbb{I}}G' \approx 0$, assuming that such a U can be found. This opens the possibility of applying Theorem 3. Notice that $H \mapsto \mathcal{U}_H$ is now the last step in the RG transformation, and not the first, as in (2). As a result, $\mathcal{N}(H)$ is always resonant. Thus, the analysis of $\mathcal{N}$ can be restricted to a subspace of resonant Hamiltonians.

Notice that for the flow of a Hamiltonian of type (9), the angles q change non-trivially only in the direction Ω. The same is true for the canonical transformations $\mathcal{U}_{H'}$ appearing in (10). Thus, given a Hamiltonian H of the form (9), consider

$$(\mathcal{L}H)(s,z) = H(s\Omega,p) - \omega\cdot p\,,\qquad \Omega\cdot p = z\,. \tag{11}$$

This defines a function $\mathcal{L}H$ which is quasiperiodic in the first variable, with frequencies in the number field $\mathbb{Q}[\vartheta_m]$. By using the abovementioned property of $\mathcal{U}_{H'}$, and assuming that U has been chosen of the same form, one finds that formally, $h' = \mathcal{L}(H')$ and $h'' = \mathcal{L}(\mathcal{N}(H))$ are determined by $h = \mathcal{L}(H)$ alone. In other words, $\mathcal{N}$ can be reduced to a RG transformation $h \mapsto h''$ for functions of two variables. To be more specific, define a function u on $\mathbb{Q}[\vartheta_m]$ by setting $u(\Omega \cdot \nu) = \omega \cdot \nu$, for all integer vectors $\nu \in \mathbb{Z}^d$. Then the reduced RG transformation can be written as

$$h''(s,z) = h'\Bigl(s + \Bigl[\frac{\partial_2}{u(\partial_1)}\psi\Bigr]\bigl(s, Z(s,z)\bigr), Z\Bigr) - \psi\bigl(s, Z(s,z)\bigr)\,, \tag{12}$$

with the (quasiperiodic) functions Z and $\psi = \mathbb{I}^{-}\psi$ determined by the equation

$$Z(s,z) + \Bigl[\frac{\partial_1}{u(\partial_1)}\psi\Bigr]\bigl(s, Z(s,z)\bigr) = z\,, \qquad \mathbb{I}^{-}h'' = 0\,. \tag{13}$$

The function ψ is related to the generating function of $\mathcal{U}_{H'}$. Equation (13) can be solved e.g. if $F = H'$ satisfies the hypothesis of Theorem 3. In fact, an analogous theorem [10] holds in this situation, for spaces of less regular functions, where

$$h(s,z) = \sum_{(\lambda,k)\in I} h_{\lambda,k}\cos(\lambda s) z^k\,, \quad \|h\| = \sum_{(\lambda,k)\in I} |h_{\lambda,k}|\cosh(\varrho\lambda)\rho^k < \infty\,, \tag{14}$$

with $I = \mathbb{Q}[\vartheta_m] \times \mathbb{N}$. We note that in the case $d = 2$, the resulting function h'' nevertheless defines an analytic Hamiltonian H''. This is due to the fact that $\mathbb{I}^{-}h'' = 0$, which for $d = 2$ implies e.g. that for fixed k, there are at most $\mathcal{O}(n^2)$ nonzero coefficients $h''_{\lambda,k}$ with $|\lambda| < n$. The same no longer holds for $d > 2$. Thus, in cases where $d > 2$, the RG transformation $h \mapsto h''$ could have an analytic fixed point corresponding to a non-differentiable fixed point for $\mathcal{N}$.

Some numerical results.

The RG transformations described above have been implemented numerically in [9,10], in the case $d = 2$, with ϑ_1 the golden mean. All functions involved are approximated by finite Fourier–Taylor series such as (9). In [9], the cutoff is of the form $|\nu| + k \le N$, and the largest "degree" considered is $N = 18$. (The cutoffs used in [10] are more complicated, and the degrees are larger by a factor two or more.) In order to find a good approximation for the expected non-trivial fixed point H^*, we started with some initial guess, obtained by a standard bisection procedure, and iterated (the numerical implementation of) a contraction that has the same fixed point as $\mathcal{R}$. The resulting Hamiltonian H satisfies the fixed point equation up to an error $\sup_{\nu,k} |(\mathcal{R}_{\rm num}(H) - H)_{\nu,k}| \exp|2\nu| < 2 \times 10^{-15}$.

Numerically, $D\mathcal{R}(H^*)$ has two expanding directions, as expected. One of them is associated with translations in the action variable, and thus irrelevant. The second eigenvalue $\delta > 1$ describes one of the universality phenomena associated with the breakup of ω–tori: the geometric accumulation of parameter values for which the corresponding w_n–orbit has some given residue. The value of δ found in [9], and more recent values [10] for $\mu^* = \mu_{H^*}$ and λ_2 (explained below) are

$$\mu^* = 0.2304601966125\ldots\,, \ \lambda_2 = -0.760795669179\ldots\,, \ \delta = 1.6279502\ldots\,.$$

These numbers are consistent with those found in [13] for area preserving maps.

Another object of interest is the scaling transformation $\mathcal{S} = \mathcal{U}_{H^*} \circ \mathcal{T}_{\mu^*}$. It maps a closed orbit for H^* to another, with the rotation vector w changed to $\vartheta_1^{-1} T w$. For parity reasons, $\mathcal{S}$ leaves the section $q = 0$ invariant. Numerically, $\mathcal{S}$ is a contraction on this section, with a fixed point $(0, p_\infty)$ close to the origin. This implies e.g. that if H^* has a closed w–orbit passing through $(0, p_0)$, then it has a sequence of closed orbits with rotation numbers $w_n = (\vartheta_1^{-1} T)^n w$. And these orbits pass through points $(0, p_n)$ that approach $(0, p_\infty)$ at a geometric rate, as $n \to \infty$, given by an eigenvalue (see below) of the matrix $D\mathcal{S}(0, p_\infty)$. For parity reasons,

$$D\mathcal{S}(0, p_\infty) = \begin{bmatrix} S & 0 \\ 0 & \mu^* (S^*)^{-1} \end{bmatrix}, \tag{15}$$

where S is some real 2×2 matrix. From the fixed point equation for H^*, one finds that S has a trivial eigenvalue $\lambda_1 = \vartheta_1$, with eigenvector $v_1 = \nabla_2 H^*(0, p_\infty)$. A second eigenvalue λ_2 of S was determined numerically, as indicated earlier. The remaining eigenvalues of $D\mathcal{S}(0, p_\infty)$ are clearly $\lambda_3 = \mu^*/\lambda_2$ and $\lambda_4 = \mu^*/\vartheta_1$. Thus, given that λ_2 is of modulus < 1, the abovementioned accumulation of orbits should be governed by λ_3. This agrees very well with numerical observations.

Consistent with the above, we also find numerically an invariant torus for H^* with rotation vector ω, passing through the scaling fixed point $(0, p_\infty)$. The fact that the unstable eigenvector v_1 of $D\mathcal{S}(0, p_\infty)$ is tangent to the flow at $(0, p_\infty)$ suggests that this torus is an attractor for $\mathcal{S}$. This in turn indicates that the invariant torus for H^* is "critical", with a degree of differentiability of at most $\ln(-\lambda_2)/\ln(1/\vartheta_1) \approx 0.72$.

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