GLOBAL H FOURIER INTEGRAL OPERATORS WITH COMPLEX-VALUED PHASE FUNCTIONS

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ABSTRACT. We consider globally defined h Fourier integral operators (h F.I.O.) with complex-valued phase functions. Symbolic calculus of h F.I.O. is considered and, using a new complex Gauss transform, composition of h pseudodifferential operators (h P.D.O.) and h F.I.O. is considered. For a self-adjoint h P.D.O. $A(h)$ and h P.D.O. $P(h)$ and $Q(h)$ with compactly supported symbols, we apply the results to approximate the kernel of the operator

$$U_{P,Q}(t;h) := P(h)e^{-ih^{-1}tA(h)}Q(h)^*,\ t \in \mathbb{R},\ h > 0,$$

by a single, globally defined h oscillatory integral.

1. INTRODUCTION AND MAIN RESULTS

1.1 Introduction.

Let $A(h), P(h)$ and $Q(h)$ denote admissible h pseudodifferential operators, or h P.D.O. for short, as defined in [HR]. (The notions of h P.D.O., h F.I.O., etc. will be made precise in the next subsection.) Suppose that $A(h)$ is (essentially) self-adjoint when considered as an operator on $L^2(\mathbb{R}^n)$, and let $P(h)$ and $Q(h)$ have compactly supported symbols. Here, for example, we can think of $A(h)$ as being a Schrödinger operator $A(h) = -h^2\Delta + V(x)$ with suitable real-valued, smooth potential. $P(h)$ and $Q(h)$ can be thought of as compactly supported functions of the operator $A(h)$, ie. $P(h) = Q(h) = f(A(h))$ with $f \in C_c^\infty(\mathbb{R})$.

By adapting methods developed in [SV] to the semi-classical situation, we obtain an approximation to the operator

$$U_{P,Q}(t;h) := P(h)e^{-ih^{-1}tA(h)}Q(h)^*,\ t \in \mathbb{R},\ h > 0, \tag{1}$$

in terms of an h Fourier integral operator, or h F.I.O. for short, with complex-valued phase function. The advantage of allowing the h F.I.O. to have a complex-valued phase is that we can approximate the kernel of $U_{P,Q}(t;h)$ in terms of an h oscillatory integral with one, globally defined phase function. This, amognst other things, serves to greatly simplify the analysis when two-term semi-classical spectral asymptotics of the counting function or spectral function is considered, see [B2] for example.

A disadvantage of allowing the h F.I.O. to have a complex-valued phase is that we are forced to consider symbol terms of the operator $A(h)$ evaluated at complex

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values of ξ. Of course this poses no problem when the symbol terms of $A(h)$ are analytic in ξ—this being the case when, for example, $A(h)$ is a Schrödinger operator.

To overcome this problem in the general case, firstly we find it convenient to generalise the notion of Gauss transforms—used for example in [H] to study composition of P.D.O.—to the case when the quadratic form is complex-valued (ie. when the quadratic form comes from a not necessarily self-adjoint matrix). We then use these *generalised* complex Gauss transforms to prove a composition formula for h P.D.O. with symbols from the general Hörmander class $S(m)$ and h F.I.O. with complex-valued phase functions.

The results here are similar to those proved recently in [LS]. However, there are two main differences between the approach taken here and that used in [LS]. Firstly, we find it convenient to define the notion of h F.I.O. in a more analagous way to the definition of admissible h P.D.O. given in [HR]. Secondly, the approach here, using complex Gauss transforms to prove a general composition result, allows us to consider general h P.D.O. $A(h)$ with symbol terms lying in the Hörmander class $S(m)$.

1.2 h P.D.O.

We briefly recall the notion of an admissible h P.D.O. as defined in [HR]. For $x, \xi \in \mathbb{R}^n$ we denote by $\langle x, \xi \rangle$ and $\langle x \rangle$ the positive quantities $\langle x, \xi \rangle := (1+|x|^2+|\xi|^2)^{\frac{1}{2}}$ and $\langle x \rangle := \langle x, 0 \rangle$. According to [H], in the case when $g = g_0$ is the standard Euclidean metric on $\mathbb{R}^{2n}$, a positive, continuous function m defined on $\mathbb{R}^{2n}$ is said to be σ temperate if there exist positive constants $C, N > 0$ such that,

$$m(x, \xi) \leq Cm(y, \eta)\langle x - y, \xi - \eta \rangle^N$$

for all $(x, \xi), (y, \eta) \in \mathbb{R}^{2n}$. For a σ temperate function m, the Hörmander space of symbols $S(m)$, see [H], is then defined to be the set of all functions $a \in C^\infty(\mathbb{R}^{2n})$ such that, for all mutli-indices $\alpha, \beta \in \mathbb{Z}^n_+$, there exists a constant $C_{\alpha,\beta} > 0$ such that,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} m(x, \xi), \quad \text{for all } (x, \xi) \in \mathbb{R}^{2n}.$$

For a C^∞ function a and $h > 0$, we denote by $a^w(x, hD_x)$ the Weyl quantisation of the function a. That is, for a Schwartz function $u \in \mathcal{S}(\mathbb{R}^n)$, (ie. a function u which, along with all its derivatives, is rapidly decreasing at infinity) we denote by $a^w(x, hD_x)$ the operator which acts on u:

$$(2) \qquad (a^w(x, hD_x)u)(x) := (2\pi h)^{-n} \int a\left(\frac{x+y}{2}, \xi\right) e^{ih^{-1}\xi\cdot(x-y)} u(y)dyd\xi.$$

We note that by a standard Fourier transform argument, for all $h > 0$, $a^w(x, hD_x)$ is an operator which maps the Schwartz space of functions $\mathcal{S}(\mathbb{R}^n)$ into itself.

From [HR] we have the following definition.

Definition 1. An operator valued map $A : (0, h_0] \mapsto \mathcal{L}(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n))$, some $h_0 > 0$, is said to be an admissible h P.D.O. if there exists a sequence of functions $\{a_j\}_{j\geq 0}$, each $a_j \in S(m)$, and an integer $N_0 > 0$ such that, for all $N \geq N_0$,

$$\left\| A(h) - \sum_{0\leq j\leq N} h^j a_j^w(x, hD_x) \right\| = O(h^{N+1}), \ h \to 0,$$

where $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ denotes the space of linear operators acting from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, and $\|\cdot\|$ denotes L^2 operator norm.

In Definition 1, the formal series $a(h) := \sum_{j\geq 0} h^j a_j$ is said to be the Weyl symbol of $A(h)$ and we shall refer to the functions a_j as the Weyl symbol terms of $A(h)$. In particular, a_0 will be referred to as the principal symbol and $a_{\text{sub}} := a_1$ as the sub-principal symbol. Also, we shall say that an h P.D.O. has compactly supported symbol if all its symbol terms lie in the space $C_c^\infty(\mathbb{R}^{2n})$ of smooth compactly supported functions.

In [HR], sufficient conditions are provided under which $A(h)$ is essentially self-adjoint when considered as an operator on $L^2(\mathbb{R}^n)$. For completeness we include the (now standard) conditions here. Denoting:

(H_1) there exists an $h_0 > 0$ such that $a(h)$ is real-valued for all $h \in (0, h_0]$ and $(x,\xi) \in \mathbb{R}^{2n}$,

(H_2) there exists a constant $\gamma_0 \in \mathbb{R}$ such that $a_0(x,\xi) \geq \gamma_0$ for all $(x,\xi) \in \mathbb{R}^{2n}$,

(H_3) for some fixed $\gamma_1 > \gamma_0$, $a_0(x,\xi) - \gamma_1$ is a σ temperate function,

(H_4) with the same fixed γ_1, $a_j \in S(a_0 - \gamma_1)$ for all $j \geq 1$,

we have the following theorem.

Theorem 2. *(see* [HR]*) Suppose that an admissible h P.D.O. $A(h)$ satisfies hypotheses (H_1) to (H_4). Then there exists an $h_1 \in (0, h_0]$ such that $A(h)$ is essentially self-adjoint for all $h \in (0, h_1]$ when considered as an operator on $L^2(\mathbb{R}^n)$.*

1.3 h F.I.O.

Next we define the class of h F.I.O. we shall work with. We consider a particular class of h F.I.O. with complex-valued phase functions. Let $a \in C^\infty(\mathbb{R}^{2n})$ denote an arbitrary smooth function and denote by $\exp(tH_a)$ the Hamiltonian flow generated by a. For the trajectory starting at the point $(y,\eta) \in \mathbb{R}^{2n}$ we denote:

$$(x^*, \xi^*) = (x^*(t,y,\eta), \xi^*(t,y,\eta)) := \exp(tH_a)(y,\eta). \tag{3}$$

A function $\phi \in C^\infty(\mathbb{R}^{3n+1})$ is said to be a phase function (associated with a) if there exists a real, symmetric, positive definite matrix $\Theta > 0$ such that

$$\phi(t,x,y,\eta) = \int_0^t \xi^*(s)\cdot\dot{x}^*(s)ds - ta(x^*,\xi^*) + \xi^*\cdot(x-x^*) + \frac{i}{2}\Theta(x-x^*)\cdot(x-x^*), \tag{4}$$

where Θ is independent of t, x, y and η. The (convex) set of all such functions will be denoted $\Psi(a)$.

Remark. The phase functions used here have general positive quadratic forms for their imaginary part. We carry these general phase functions (general in the sense that Θ need not only be equal to the identity matrix) through our analysis as in applications it is often necessary to vary Θ. The technique of varying the quadratic form was used in [B2] for example where, to overcome difficulties experienced with the stationary phase method in the case of operators with non-homogeneous symbols, it was necessary to take Θ large.

A function $u \in C^\infty(\mathbb{R}^{3n+1})$ is said to be a symbol if u has compact support in the y and η variables and the function $x \mapsto u(t,x,y,\eta)$ and all its derivatives have polynomially bounded growth, uniformly, for all $y, \eta \in \mathbb{R}^n$ and all t in any bounded interval.

Definition 3. An operator valued map $U(t;h) : \mathbb{R}\times(0,h_0] \to \mathcal{L}(L^2(\mathbb{R}^n))$ is said to be an h F.I.O. if for all $T>0$, there exist a phase function ϕ, a sequence of symbols $\{u_j\}_{j\geq 0}$ and an integer $N_0 \geq 0$ such that, denoting by $\mathcal{V}_N(t;h)$ the operator with integral kernel

$$\nu_N(t,x,y;h) = (2\pi h)^{-n} \int \sum_{0\leq j\leq N} h^j u_j(t,x,y,\eta) e^{ih^{-1}\phi(t,x,y,\eta)} d\eta,$$

we have $\max_{t\in[-T,T]} \|U(t;h) - \mathcal{V}_N(t;h)\| = O(h^{N+1})$, $h\to 0$ for all $N \geq N_0$.

In Definition 3, analagously to Definition 1, the function ϕ is said to be a phase function of $U(t;h)$ and the (formal) series $u \sim \sum_{j\geq 0} h^j u_j$ is said to be a symbol of $U(t;h)$.

1.4 Main result.

Defining, for any vector-valued function $f(\eta)$, the matrix of derivatives f_η to be the matrix with j,k th entry $(f_\eta)_{jk} := \frac{\partial f_k}{\partial \eta_j}$, we set

$$Z(t,y,\eta) := \xi^*_\eta - i x^*_\eta \Theta. \tag{5}$$

According to [LS], section 4 we have the following lemma.

Lemma 4. *([LS], section 4) For all $\Theta > 0$ and all t, y and η, the matrix Z has non-zero determinant.*

The matrix Z will play a central rôle in our theory and Lemma 4 is of crucial importance. Amongst other things, Lemma 4 allows us to define the square root of the determinant of Z. From here on, as $Z(0) = I$, we interpret $(\det Z)^{\frac{1}{2}}$ as the branch of the square root which is continuous for all t and equal to 1 when $t=0$.

Remark. The matrix Z may also be used to define a Maslov index. For a point $(y,\eta)\in\mathbb{R}^{2n}$ such that at some positive time $T>0$ we have $x^*_\eta(T) = 0$, we define a Maslov index of the trajectory $\{\exp(tH_a)(y,\eta) | 0 \leq t \leq T\}$ as follows. Since $x^*_\eta(T) = 0$, as t evolves from 0 to T, $(\det Z)^2$ makes an integer number of revolutions about the origin in the complex plane. This number (here we use the standard convention that one complete anti-clockwise revolution counts $+1$ and one complete clockwise revolution counts -1) counted modulo 4 defines a useful Maslov index (see [B2] for a demonstration of where this Maslov index enters semi-classical spectral asymptotic formulae).

We note that due to it being integer-valued and the fact that $\Psi(a)$ is a convex set, this Maslov index is independent of the choice of phase function in $\Psi(a)$. Also we refer the reader to [SV] Appendix D6 for proof, in the classical case (i.e. when $a(x,\xi)$ is homogeneous in the ξ variables) of the fact that this Maslov index coincides with other Maslov indices defined in various, more traditional ways.

To state our main result we first need to define an operator acting on Schwartz functions. This operator will be similar to the generalised complex Gauss transforms we shall define below but, in order to state the main theorem, we need only consider this operator acting on Schwartz functions. We therefore define the operator:

$$e^{-\frac{ih}{2}(D_x\cdot D_\xi + i\Theta D_\xi \cdot D_\xi)},$$

acting on Schwartz functions, via the Fourier transform. That is for $u \in \mathcal{S}(\mathbb{R}^{2n})$,

(6)
$$\begin{aligned} e^{-\frac{ih}{2}(D_x\cdot D_\xi + i\Theta D_\xi \cdot D_\xi)}u(x,\xi) \\ := (2\pi)^{-2n}\int e^{-\frac{ih}{2}(s\cdot\zeta + i\Theta\zeta\cdot\zeta)}e^{i(s\cdot(x-y)+\zeta\cdot(\xi-\eta))}u(y,\eta)dyd\eta ds d\zeta. \end{aligned}$$

With this notation we are ready to state the main theorem of this paper.

Theorem 5. *Let $A(h)$ denote a self-adjoint admissible h P.D.O. (see Definition 1 and Theorem 2) and let $P(h)$ and $Q(h)$ denote arbitrary h P.D.O. with compactly supported symbols. Then the operator-valued map $U_{P,Q}(t;h)$ defined in (1) is an h F.I.O. Furthermore, for any phase $\phi \in \Psi(a_0)$, $U_{P,Q}(t;h)$ has (x independent) symbol $\sum_{j\geq 0} h^j u_j$ with,*

$$u_j(t,y,\eta) = (\det Z)^{\frac{1}{2}} e^{-i\int_0^t a_{\mathrm{sub}}(x^*,\xi^*)ds}\nu_j(t,y,\eta) \quad \text{for all } j \geq 0. \tag{7}$$

In particular,

$$\begin{aligned} \nu_0(t,y,\eta) &= p_0(x^*,\xi^*)\overline{q_0}(y,\eta) \quad \text{for all } t, \\ \text{and} \quad \sum_{j\geq 0} h^j \nu_j(0,y,\eta) &= e^{-\frac{ih}{2}(D_x\cdot D_\xi + i\Theta D_\xi\cdot D_\xi)}(p\sharp\overline{q})\Big|_{(x,\xi)=(y,\eta)}, \end{aligned} \tag{8}$$

where $(p\sharp\overline{q})(x,\xi;h)$ denotes the (full) Weyl symbol of the composition $P(h)Q(h)^*$.

Remark. In the theorem we note that the $\{u_j\}_{j\geq 0}$ and $\{\nu_j\}_{j\geq 1}$ may depend on the choice of $\phi \in \Psi(a_0)$. This dependence is evident for u_0 if we recall that the matrix Z itself depends on Θ, but is omitted from notations. On the other hand, in (8) we note that ν_0 is independent of the choice of ϕ.

In fact we obtain rather better information about the operator $U_{P,Q}$. Denoting the kernel of $U_{P,Q}$ by $u_{P,Q}(t,x,y;h)$ and denoting the kernel of the h F.I.O. with phase $\phi \in \Psi(a_0)$ and symbol $\sum_{0\leq j\leq N} h^j u_j$, (the u_j being those in Theorem 5) by $u_N(t,x,y;h)$ we have:

Corollary 6. *For all $T \geq 0$ and integer $N \geq 0$,*

$$\sup_{x,y\in\mathbb{R}^n}\max_{t\in[-T,T]}|u_{P,Q}(t,x,y;h) - u_N(t,x,y;h)| = O(h^{N+1}), \; h \to 0.$$

2. Proof of Theorem 5

2.1 Symbolic calculus for h F.I.O.

We start the proof of Theorem 5 with a brief discussion of symbolic calculus for h F.I.O. Plainly, as defined in Definition 3, a particular h F.I.O. may have more than one phase or symbol. It turns out that, due to the fact that:

$$(x - x^*)e^{ih^{-1}\phi} = -ihZ^{-1}\nabla_\eta e^{ih^{-1}\phi}, \tag{9}$$

all h F.I.O. have at least one x independent symbol. This fact is central to our global construction—allowing us always to 'get rid' of any x dependence in symbols—and is established essentially by expanding the symbol in a Taylor polynomial in $(x-x^*)$, and using (9) and integration by parts to reduce any x dependence in the symbol to lower order (in h) terms.

In view of Lemma 4, we define the differential operators

$$D_1 = \sum_{j,k} \left(\left(\frac{\partial}{\partial \eta_k} (Z^{-1})_{jk} \right) \frac{\partial}{\partial x_j} + (Z^{-1})_{jk} \frac{\partial^2}{\partial x_j \partial \eta_k} + (Z^{-1} x^*_\eta)_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right),$$

$$D_2 = - \sum_{j,k} (Z^{-1} x^*_\eta)_{jk} \frac{\partial^2}{\partial x_j \partial x_k}.$$

Then we have the following theorem.

Theorem 7. *Let $U(t;h)$ denote an h F.I.O. with phase ϕ and symbol $\sum_{j\geq 0} h^j u_j$. Then with the same phase ϕ, $U(t;h)$ has an x independent symbol $\sum_{j\geq 0} h^j \tilde{u}_j$. The $\tilde{u}_j$ are given by*

$$\tilde{u}_j(t,y,\eta) = \sum_{\substack{0\leq l\leq j \\ (j-l)\leq k\leq 2(j-l)}} D_{k,j-l} u_l|_{x=x^*}$$

where the $D_{k,j}$ are differential operators such that, in particular,

$$\begin{aligned} D_{0,0} &= \text{the identity operator} \\ D_{k,k} &= \frac{(i)^k}{k!} D_1^k \qquad \text{for all } k\geq 1 \\ D_{k,\frac{k}{2}} &= \frac{(i)^{\frac{k}{2}} 1.3\ldots(k-1)}{k!} D_2^{\frac{k}{2}} \qquad \text{for all even } k\geq 2. \end{aligned} \tag{10}$$

If $\nu(x,\eta)\in C^\infty$ is an arbitrary function with a zero of order m in x at $x=x^*$ (i.e. $\partial_x^\alpha|_{x=x^*}\nu = 0$ for all $|\alpha|\leq m$) then

$$\sum_{j\leq k\leq 2j} D_{k,j}\nu|_{x=x^*} = 0 \tag{11}$$

for all $0\leq j\leq \left[\frac{m}{2}\right]$ where $[\cdot]$ denotes integer part.

Proof. Let $k\geq 2$ denote an integer. Using (9) and integration by parts in η, the h F.I.O. with phase ϕ and symbol

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} (\partial_x^\alpha u_j)(t, \mu_1 x^* + \mu_2 x, y, \eta)(x-x^*)^\alpha$$

is equal to the h F.I.O. with the same phase and symbol

(12)

$$\frac{ih}{k}\sum_{|\alpha|=k-1}\frac{1}{\alpha!}(\partial_x^\alpha D_1 u_j)(t,\mu_1 x^*+\mu_2 x,y,\eta)(x-x^*)^\alpha$$
$$+\frac{ih}{k}\sum_{|\alpha|=k-2}\frac{1}{\alpha!}(\partial_x^\alpha D_2 u_j)(t,\mu_1 x^*+\mu_2 x,y,\eta)(x-x^*)^\alpha$$
$$+\frac{ih}{k}(\mu_1-1)\sum_{\substack{|\alpha|=k-1,\\ l,m=1}}^{n}\frac{1}{\alpha!}\left(Z^{-1}x^*_\eta\right)_{ml}\left(\partial_x^{\alpha+e_m+e_l}u_j\right)(t,\mu_1 x^*+\mu_2 x,y,\eta)(x-x^*)^\alpha$$

where $\mu_1,\mu_2\in\mathbb{R}$ denote arbitrary parameters. In the case when $k=1$ we obtain (12) without the second sum.

Choose any integer $N\geq 0$ and let $j\leq 2N$. Then expanding u_j in a Taylor polynomial of order $2(N-j)$ we have

(13)

$$u_j=\sum_{0\leq|\alpha|\leq 2(N-j)}\frac{1}{\alpha!}\,\partial_x^\alpha u_j|_{x=x^*}(x-x^*)^\alpha+(2(N-j)+1)\times$$
$$\times\sum_{|\alpha|=2(N-j)+1}\frac{1}{\alpha!}\left(\int_0^1(1-s)^{2(N-j)}\,\partial_x^\alpha u_j|_{x=sx+(1-s)x^*}\,ds\right)(x-x^*)^\alpha.$$

Using (12) (with $\mu_1=(1-s)$ and $\mu_2=s$) inductively to remove the $(x-x^*)^\alpha$ factor, and denoting the h F.I.O. with phase ϕ and symbol given by the second sum in the right-hand side of (13) by $R_j(t;h)$, we have, for all $T>0$,

$$\max_{t\in[-T,T]}\|R_j(t;h)\|=O(h^{N-j+1}),\ \ h\to 0. \tag{14}$$

(Here, to establish the operator norm estimate in (14) we have used a classical lemma of Schur, see [H], Lemma 18.1.12 for example, and the following facts: i) u_j has compact support in y and η; ii) as u_j and all its derivatives have polynomially bounded growth in x, $(\partial_x^\alpha u_j)e^{ih^{-1}\phi}$ is Schwartz in x for all α and $h>0$.)

Again using (12) inductively (this time with $\mu_1=1$ and $\mu_2=0$) to remove the $(x-x^*)^\alpha$ factor, the h F.I.O. with phase ϕ and symbol given by the first sum in the right-hand side of (13) is equal to the h F.I.O. with the same phase and symbol $\sum_{l=0}^{2(N-j)}h^l\nu_l^{(j)}$. The $\nu_l^{(j)}$ denote x independent h F.I.O. symbols and, for $0\leq l\leq N$,

$$\nu_l^{(j)}(t,y,\eta)=\sum_{l\leq k\leq 2l}D_{k,l}u_j|_{x=x^*},$$

the operators $D_{k,l}$ being compositions of D_1 and D_2 and having the properties in (10). The proof of the first statement in the theorem is completed by summing the terms $\nu_l^{(j)}$ for $0\leq l\leq N$.

The property in (11) follows by noting that, in the above proof, if we take $u_0=\nu$ with $\partial_x^\alpha|_{x=x^*}\nu=0$ for all $|\alpha|\leq m$ and $u_j=0$ for all $j\geq 1$ then we must have $2N\geq(m+1)$ to obtain any contribution to the Taylor polynomials. Thus the $\tilde{u}_j=0$ for all $0\leq j\leq\left[\frac{m}{2}\right]$ which implies (11). $\square$

2.2 Complex Gauss transforms.

Next we consider *generalised* complex Gauss transforms similar to those studied in [H], section 18.4. Let C denote the matrix

$$C = \begin{pmatrix} 0 & C_1 \\ C_1 & iC_2 \end{pmatrix} \tag{15}$$

where C_1, C_2 denote $n \times n$, real, symmetric, positive definite matrices. Let $B(x,\xi)$ denote an n vector-valued function which is such that the matrix of derivatives B_x is independent of x and $B(x_0,\xi_0) = 0$ at a point $(x_0,\xi_0) \in \mathbb{R}^{2n}$.

Writing $A(s,\zeta) = \frac{1}{2}C(s,\zeta)\cdot(s,\zeta)$ and denoting $D_x = -i\partial_x$ (the same for D_ξ), the operator $e^{i(hA(D_x,D_\xi)-iB(x,\xi)\cdot D_\xi)}$ may be defined, for all $h > 0$, on Schwartz functions via the Fourier transform. Explicitly,

$$e^{i(hA(D_x,D_\xi)-iB(x,\xi)\cdot D_\xi)}a(x,\xi) := (2\pi)^{-2n}\int e^{i(hA(s,\zeta)-iB(x,\xi)\cdot\zeta)}\hat{a}(s,\zeta)dsd\zeta, \tag{16}$$

for $a \in \mathcal{S}(\mathbb{R}^{2n})$, the conditions on C ensuring that $e^{i(hA(s,\zeta)-iB(x,\xi)\cdot\zeta)}\hat{a}(s,\zeta)$ is Schwartz in s,ζ, for all $h > 0$ and all x and ξ. (Here $\hat{a}$ denotes the Fourier transform of a, defined $\hat{a}(s,\zeta) := \int a(x,\xi)e^{-i(x\cdot s+\xi\cdot\zeta)}dxd\xi$.)

We now extend the definition of the operator in (16) to the case when the function $a \in S(m)$ belongs to the general Hörmander class of symbols $S(m)$.

We define the constant $C_n = (2\pi)^{-n}(\det(-iC)^{-1})^{\frac{1}{2}}$, where the square root is understood in the sense of [H], section 3.4 for example, and we define the function $\psi(x,\xi;s,\zeta) = \frac{i}{2}Ds\cdot s - C_1^{-1}(\zeta - iB(x,\xi))\cdot s$ where $D > 0$ is the positive definite matrix $D = C_1^{-1}C_2C_1^{-1}$. Then we define

$$e^{i(hA(D_x,D_\xi)-iB(x,\xi)\cdot D_\xi)}a(x,\xi) := h^{-n}C_n\int e^{ih^{-1}\psi(x,\xi;s,\zeta)}a(x-s,\xi-\zeta)dsd\zeta \tag{17}$$

for all functions a such that the convolution in (17) is finite. We note that when a is Schwartz the Fourier transform implies that (16) and (17) are equal.

Theorem 8. *Suppose that A and B are as defined above. Then for all multi-indices $\alpha \geq 0$, the map*

$$\left(\partial_x^\alpha e^{i(hA(D_x,D_\xi)-iB(x,\xi)\cdot D_\xi)}\ \cdot\ \right)\Big|_{(x,\xi)=(x_0,\xi_0)} : S(m) \longrightarrow \mathbb{C} \tag{18}$$

is a continuous linear functional. Furthermore, for all $\alpha \geq 0$,

$$\left(\partial_x^\alpha e^{i(hA(D_x,D_\xi)-iB(x,\xi)\cdot D_\xi)}u\right)\Big|_{(x,\xi)=(x_0,\xi_0)} = O_\alpha(1),\ \ h \to 0,$$

uniformly for all u lying in any bounded subset of $S(m)$.

Proof. Let $a \in S(m)$, denote by g_α the functional in (18) applied to a and denote $\psi_0 = \psi(x_0,\xi_0;s,\zeta)$. From (17), modulo constants, g_α is equal to the sum for multi-indices $\beta \leq \alpha$ of terms

$$h^{-|\beta|-n}\int(-B_xC_1^{-1}s)^\beta e^{ih^{-1}\psi_0}(\partial_x^{\alpha-\beta}a)(x_0-s,\xi_0-\zeta)dsd\zeta$$

each of which, using intgeration by parts, is equal to

$$h^{-n}\int e^{ih^{-1}\psi_0}(\partial_x^{\alpha-\beta}(-iB_x\partial_\xi)^\beta a)(x_0-s,\xi_0-\zeta)dsd\zeta. \tag{19}$$

To handle the ζ integration in (19) we employ a method similar to that appearing in the proof of Proposition 7.6 in [DS]. First we split the ζ integration in (19) by introducing the functions $\chi(h^{-\frac{1}{2}}\zeta)$ and $1-\chi(h^{-\frac{1}{2}}\zeta)$ where $\chi\in C_c^\infty(\mathbb{R}^n)$ is equal to 1 near 0.

As χ has compact support, the first integral is easy to estimate. Defining $\tilde{\psi}=-C_1^{-1}\zeta\cdot s$ we use the identity $h\frac{\tilde{\psi}_s}{|\tilde{\psi}_s|^2}\cdot D_se^{ih^{-1}\tilde{\psi}}=e^{ih^{-1}\tilde{\psi}}$ and integration by parts to estimate the second integral. Then changing integration variables $\zeta\mapsto h^{\frac{1}{2}}\tilde{\zeta}$ and $s\mapsto h^{\frac{1}{2}}\tilde{s}$ we obtain that the second integral is, in absolute value, less than or equal to a constant times the integral $d\tilde{s}d\tilde{\zeta}$ of:

$$\left|\left(D_{\tilde{s}}\cdot\frac{\tilde{\psi}_s}{|\tilde{\psi}_s|^2}\right)^N\left\{e^{-\frac{1}{2}D\tilde{s}\cdot\tilde{s}}(\partial_x^{\alpha-\beta}(B_x\partial_\xi)^\beta a)(x_0-h^{\frac{1}{2}}\tilde{s},\xi_0-h^{\frac{1}{2}}\tilde{\zeta})\right\}(1-\chi(\tilde{\zeta}))\right|. \tag{20}$$

Using the fact that $a\in S(m)$ there exist constants $C_{\alpha,\beta},N_0>0$ such that the object in (20) is less than or equal to

$$C_{\alpha,\beta}\langle\tilde{s}\rangle^N\langle\tilde{\zeta}\rangle^{-N}\langle h^{\frac{1}{2}}\tilde{s},h^{\frac{1}{2}}\tilde{\zeta}\rangle^{N_0}e^{-\frac{1}{2}D\tilde{s}\cdot\tilde{s}}m(x_0,\xi_0). \tag{21}$$

Choosing N large enough ensures integrability of the object in (21) with respect to $\tilde{\zeta}$. Finally, as the object in (21) is Schwartz in $\tilde{s}$, it is always integrable with respect to $\tilde{s}$, and this completes the proof. □

2.3 Composition of h P.D.O. and h F.I.O.

Theorems 7 and 8 yield the following corollary.

Corollary 9. *Let $A(h)$ denote an arbitrary, admissible h P.D.O. (see Definition 1) with symbol $a(x,\xi;h)$ and let $U(t;h)$ denote an h F.I.O. with phase $\phi\in\Psi(b)$ (b any C^∞ function) and finite, x independent symbol $u=\sum_{j\le N}h^ju_j$. Then the compostion $A(h)U(t;h)$ is an h F.I.O with the same phase ϕ and symbol given by the formal expression*

$$e^{i(\frac{h}{2}(D_x+i\Theta D_\xi)\cdot D_\xi+i\Theta(x-x^*)\cdot D_\xi)}a(x,\xi;h)\Big|_{\xi=\xi^*}u(t,y,\eta;h) \tag{22}$$

where (22) can be made sense of by applying Theorems 7 and 8. Here, (x^*,ξ^*) is the trajectory in the flow generated by b, ie. $(x^*,\xi^*)=\exp(tH_b)(y,\eta)$.

Proof. Denoting by $\{a_j\}$ the terms in the symbol of $A(h)$ we consider the composition $a_j^w(x,hD_x)U_k(t;h)$ where U_k is the h F.I.O. with phase ϕ and symbol u_k.

Applying Theorem 8 with $C_1 = \frac{1}{2}I, C_2 = \Theta$ and $B = -\Theta(x - x^*)$, for all integer $l \geq 0$,

$$(23) \qquad \sum_{|\alpha|=l} \frac{1}{\alpha!} \left(\partial_x^\alpha e^{i(\frac{h}{2}(D_x + i\Theta D_\xi)\cdot D_\xi + i\Theta(x-x^*)\cdot D_\xi)} a_j \right)\Big|_{(x,\xi)=(x^*,\xi^*)} (x - x^*)^\alpha u_k$$

is an h F.I.O. symbol uniformly with respect to $h \in (0, h_0]$.

In view of Theorem 7 and (11), the h F.I.O. with phase ϕ and symbol in (23) is equal to the h F.I.O. with the same phase ϕ and symbol $\sum_{m \geq [\frac{l+1}{2}]} h^m \nu_m$ where $\{\nu_m\}$ denotes a sequence of x independent symbols. Applying Taylor's Theorem we have thus established that the operator with integral kernel given by the formal expression (understood in the sense demonstrated here in the proof):

$$(24) \qquad (2\pi h)^{-n} \int e^{i(\frac{h}{2}(D_x + i\Theta D_\xi)\cdot D_\xi + i\Theta(x-x^*)\cdot D_\xi)} a_j \Big|_{\xi=\xi^*} u_k e^{ih^{-1}\phi} d\eta$$

is an h F.I.O.

In the case when $a_j \in \mathcal{S}(\mathbb{R}^{2n})$, according to Fourier's inversion formula, the compostition $a_j^w U_k$ has kernel equal to (24). Hence in the general case we have established that $a_j^w U_k$ is an h F.I.O. and, noting that the finiteness of the symbol u ensures $\mathcal{L}(L^2(\mathbb{R}^n))$ boundedness of any remainder operators, we conclude that $A(h)U(t;h)$ is an h F.I.O. $\square$

Remark 10. In view of the proof of Corollary 9, as the h F.I.O. with phase ϕ and symbol in (23) is equal to the h F.I.O. with the same phase and symbol $\sum_{m \geq [\frac{l+1}{2}]} h^m \nu_m$, to calculate the first N terms in the symbol of $A(h)U(t;h)$ it suffices to consider the first $2N$ terms in the Taylor polynomial (in $(x - x^*)$) of (22) and then use Theorem 7.

2.4 Proof of Theorem 5.

Before commencing the proof of Theorem 5, we require the following additional lemma concerning the initial condition of an h F.I.O. An application of Fourier's inversion formula in (6) gives the following:

Lemma 11. *Let $U(t;h)$ be an h F.I.O. with x independent symbol $u(t, y, \eta; h)$ and let $P(h)$ be an h P.D.O. with compactly supported symbol $p(x, \xi; h)$. If*

$$u(0, y, \eta; h) = e^{-\frac{ih}{2}(D_x + i\Theta D_\xi)\cdot D_\xi} p \Big|_{(x,\xi)=(y,\eta)},$$

then $U(0;h) = P(h)$.

We are now prepared for the proof of Theorem 5.

Proof (of Theorem 5). Denote by U_N an h F.I.O. with phase $\phi \in \Psi(a_0)$ and symbol $\sum_{j \leq N} h^j u_j$, the u_j satisfying (7) and (8). Assume for the moment that $P(h) = I$, the identity operator. Observing that $\dot{\phi}$ is polynomial in $(x - x^*)$, differentiating under the integral sign and applying Corollary 9, $(-ih\frac{\partial}{\partial t} + A(h))U_N$ is an h F.I.O. with phase ϕ and symbol $\sum_{j \geq 0} h^j \tilde{u}_j$.

Now $\dot{\phi} = -a_0(x^*,\xi^*) + (\dot{\xi}^* - i\Theta\dot{x}^*)\cdot(x-x^*)$ and hence, according to Corollary 9 and Remark 10, $(-ih\frac{\partial}{\partial t} + A(h))U_N$ has symbol $\sum_{k\le N} h^{k+1}q_k$ where the q_k denote the terms:

(25)
$$\begin{aligned} q_k = -i\dot{u}_k + a_{\mathrm{sub}}(x^*,\xi^*)u_k + \frac{i}{2}\,(2D_1 + D_2)\left((e^{-\Theta(x-x^*)\cdot D_\xi}a_0)(x,\xi^*)u_k\right)\Big|_{x=x^*} \\ + i\,D_1\left((\dot{\xi}^* - i\Theta\dot{x}^*)\cdot(x-x^*)u_k\right)\Big|_{x=x^*} + \frac{1}{2}\left((-\Theta D_\xi + iD_x)\cdot D_\xi a_0\right)(x^*,\xi^*)u_k \\ + O_k(h), \end{aligned}$$

with each $O_k(h)$ depending only on u_k and being independent of all other terms in the symbol of U_N.

By straightforward calculations,

(26)
$$D_1\left((e^{-\Theta(x-x^*)\cdot D_\xi}a_0)(x,\xi^*)u_k\right)\Big|_{x=x^*} = -\,D_1\left((\dot{\xi}^* - i\Theta\dot{x}^*)\cdot(x-x^*)u_k\right)\Big|_{x=x^*},$$

and,

$$\begin{aligned} D_2\left((e^{-\Theta(x-x^*)\cdot D_\xi}a_0)(x,\xi^*)u_k\right)\Big|_{x=x^*} + \left((i\Theta D_\xi + D_x)\cdot D_\xi a_0\right)(x^*,\xi^*)u_k \\ = \mathrm{Tr}\,(Z^{-1}\dot{Z})u_k. \end{aligned} \tag{27}$$

Thus, from (25), (26) and (27), $(-ih\frac{\partial}{\partial t} + A(h))U_N$ has symbol equal to the sum for $k\le N$ of terms:

$$-ih^{k+1}\dot{u}_k + h^{k+1}a_{\mathrm{sub}}(x^*,\xi^*)u_k + \frac{ih^{k+1}}{2}\mathrm{Tr}\,(Z^{-1}\dot{Z})u_k + O_k(h^{k+2}). \tag{28}$$

Hence, $\tilde{u}_0 = 0$, $\tilde{u}_1 = -i\dot{u}_0 + \left(a_{\mathrm{sub}}(x^*,\xi^*) + \frac{i}{2}\mathrm{Tr}\,(Z^{-1}\dot{Z})\right)u_0$ and, as the $O_k(h^{k+2})$ depend only on u_k, there exist differential operators $\{L_{j,k}\}$ such that

$$\tilde{u}_j = -i\dot{u}_{j-1} + \left(a_{\mathrm{sub}}(x^*,\xi^*) + \frac{i}{2}\mathrm{Tr}\,(Z^{-1}\dot{Z})\right)u_{j-1} + \sum_{k\le(j-2)} L_{j,k}u_k$$

for all $2\le j\le (N+1)$. Differentiating and using Liouville's formula to re-write $\frac{\frac{d}{dt}\det Z}{\det Z} = \mathrm{Tr}\,(Z^{-1}\dot{Z})$ we find $\tilde{u}_1 = 0$ and

$$\tilde{u}_j = (\det Z)^{\frac{1}{2}} e^{-i\int_0^t a_{\mathrm{sub}}(x^*,\xi^*)ds}\dot{\nu}_{j-1} + \sum_{k\le(j-2)} L_{j,k}u_k$$

for all $2\le j\le (N+1)$.

The ν_j satisfy (8). However, we may choose the ν_j in such a way that $\tilde{u}_j = 0$ for all $2\le j\le (N+1)$. We have thus established that

$$\max_{t\in[-T,T]}\left\|\left(-ih\frac{\partial}{\partial t} + A(h)\right)(U_N - U_{I,Q})\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(h^{N+2}),\ h\to 0,$$

and from (8) and Lemma 11,

$$\|U_N(0;h) - Q(h)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(h^{N+1}),\ h \to 0.$$

Setting $R(t;h) := U_N(t;h) - U_{I,Q}(t;h)$, an application of the Duhamel formula:

$$R(t;h) = \int_0^t e^{ih^{-1}(s-t)A(h)} \left(\left(\frac{\partial}{\partial s} + ih^{-1}A(h) \right) R(s;h) \right) ds + e^{-ih^{-1}tA(h)} R(0;h),$$

then establishes that $e^{-ih^{-1}tA(h)}Q(h)^*$ is an h F.I.O. which, for any $\phi \in \Psi(a_0)$, has symbol satisfying (7) and (8).

The proof of Theorem 5 is completed by observing that, from Theorem 7, $e^{-ih^{-1}tA(h)}Q(h)^*$ has an x independent symbol, and then applying Corollary 9 and Lemma 11 to deduce that $U_{P,Q}$ is an h F.I.O. with symbol having the properties in (7) and (8). □

All that remains is to check Corollary 6. Towards that end, for $x \in \mathbb{R}^n$, $s > n$ we define the function

$$\rho_x^{(s)}(y;h) = (2\pi h)^{-n} \int e^{ih^{-1}\xi\cdot(y-x)} \langle \xi \rangle^{-s} d\xi.$$

Now $\rho_x^{(s)} \in L^2(\mathbb{R}^n)$ and $\|\rho_x^{(s)}\| = O(1), h \to 0$, the $O(1)$ being independent of x.

We denote by $L_{(s)}(h)$ the h P.D.O. with symbol $l_{(s)}(x,\xi) = \langle \xi \rangle^s$ and denoting by $U_N(t;h)$ the h F.I.O. with phase $\phi \in \Psi(a_0)$ and symbol $\sum_{j \le N} h^j u_j$, (the u_j being those in Theorem 5)

(29)

$$\begin{aligned} |u_{P,Q}(t,x,y;h) - u_N(t,x,y;h)| &= \left| \left(L_{(s)}(h)^*(U_P - U_N) L_{(s)}(h) \rho_y^{(s)}, \rho_x^{(s)} \right) \right| \\ &\le \|L_{(s)}(h)^*(U_P - U_N) L_{(s)}(h)\| \, \|\rho_y^{(s)}\| \, \|\rho_x^{(s)}\|. \end{aligned}$$

The right hand side of (29) is independent of x and y and, applying Theorem 5 and Corollary 9, is equal to $O(h^{N+1})$ as $h \to 0$ uniformly for $t \in [-T,T]$, all $T > 0$. Corollary 6 then follows.

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