ON ENTROPIC REDUCTION OF FLUCTUATIONS

THIERRY BODINEAU, GIAMBATTISTA GIACOMIN, AND YVAN VELENIK

Abstract. We point out that there is no general relation between ground state degeneracy and finite-temperature fluctuations for tilted interfaces.

1. Introduction

In a celebrated paper [D], Dobrushin proved that at low temperature, the horizontal interface of the three dimensional Ising model is rigid. The uniqueness of the ground state plays a predominant role in the proof because the rigidity comes from the fact that the fluctuations of the interface are not strong enough to destroy the ground state. This picture of a unique ground state slightly deformed is expected to fail when the temperature increases : entropy should take over, leading to a rough interface above the roughening transition. The analysis of ground states for tilted interfaces in 3D was a long standing open problem which has been solved recently by R. Kenyon for the interface orthogonal to the vector $(1,1,1)$ (see [Ke1] Theorem 15). In fact, the works of Kenyon concern the combinatorics of dimer models and the implications of his results are going far beyond the characterization of the 3D interfaces for Ising model at zero temperature (see [Ke2] and references therein). The ground states associated to the interface orthogonal to the vector $(1,1,1)$ are degenerate, and Kenyon proved in [Ke2] that a related model (domino tiling) converges in the thermodynamic limit to a Gaussian field (the same proof should also imply a similar statement for the ground states). Therefore, it is tempting to argue that an increase of the temperature, i.e. an addition of entropy, would lead to more fluctuations. One could believe that the fluctuations at low temperature are mainly driven by the degeneracy of the ground states and that there should exist some general monotonicity principle which would answer this question. For example, one could be tempted to use correlation inequalities, in the spirit of the proof of rigidity of the horizontal interface due to H. van Beijeren [vB].

Nevertheless such a picture does not seem to hold in full generality. Consider now the interface orthogonal to the vector $(1,1,0)$ obtained by imposing the boundary conditions

$$\overline{\sigma}_i = \begin{cases} 1 & \text{if } i\cdot(1,1,0) \geqslant 0 \text{ and } |i_3| \leqslant N, \\ -1 & \text{if } i\cdot(1,1,0) < 0 \text{ and } |i_3| \leqslant N, \\ 0 & \text{if } |i_3| > N, \end{cases}$$

outside the cube $\{i \in \mathbb{Z}^3 : |i_k| \leqslant N,\ k = 1,2,3\}$. By $\overline{\sigma}_i = 0$ we mean that the boundary condition at the site i is free. Notice that the interface is tighten only on two opposite edges of the cube, and free on the two other sides. Then at zero temperature this model reduces to a 2 dimensional model for which the fluctuations are known to be of order $\sqrt{N}$. However, the physical intuition would say that at finite temperature the system should

Date: September 8, 2000.

behave completely differently and fluctuate like $\sqrt{\ln(N)}$. This would say that the ground state fluctuations should play only a limited role in the fluctuations of the interface which, at least in this example, should be driven only by entropy. In fact the effect of the entropy seems to be even more drastic in dimension 4 where no fluctuations should pertain at positive temperature for the interface orthogonal to the vector $(1,1,0,0)$. Notice that in dimension 4 (for a different choice of boundary conditions), the rigidity of the tilted interface has been derived by Messager and Miracle-Sole [MM] by means of correlations inequalities.

There are many examples of systems with infinitely degenerated ground-states, only a finite number of which survive at positive temperature, see e.g. [BS], but the mechanism at play in the situation we consider in this note is very different.

In the low temperature regime and for small tilt, the interface of the 3D Ising model can be approximated, at least on a heuristic level, by a gas of lines which cannot intersect. For this model, the logarithmic fluctuations of the correlations were derived by Prähofer and Spohn in [PS]. This would indicate that at positive temperature, the transversal excitations reduce the amplitude of the fluctuations of the interface (see also [S1, S2] for related phenomena). In order to emphasize the lack of a general monotonicity property for the amplitude of the fluctuations, we are going to prove this entropic stabilization phenomenon for an effective interface model in dimension $d+1 \geqslant 3$ which shares features similar to the Ising model (including FKG inequalities ...).

For any finite domain $\Lambda \subset \mathbb{Z}^d$, we define the Hamiltonian

$$H_\Lambda^{\Phi_{\Lambda^c}}(\Phi_\Lambda) = \sum_{i \sim j} \nu(\varphi_i - \varphi_j), \tag{1}$$

where $\nu : \mathbb{R} \to \mathbb{R}^+$ is a convex function, $\Phi_\Lambda = \{\varphi_i\}_{i\in\Lambda}$. The heights Φ_{Λ^c} outside Λ are fixed. The Gibbs measure associated to this Hamiltonian will be denoted by

$$\mu_{\Lambda,\beta}^{\Phi_{\Lambda^c}}(d\Phi_\Lambda) = \frac{1}{Z_{\Lambda,\beta}^{\Phi_{\Lambda^c}}} \exp\big(-\beta H_\Lambda^{\Phi_{\Lambda^c}}(\Phi_\Lambda)\big)\, \Pi_{i\in\Lambda}\, d\varphi_i\,, \tag{2}$$

where the parameter β plays the role of the inverse of the temperature and $d\varphi$ denotes the Lebesgue measure. As explained before, we are mainly interested in the interface orthogonal to the vector $(1,1,0,\ldots,0)$. We denote by $\mu_{\beta,N}$ the Gibbs measure on the domain $\Lambda_N = \{-N+1,\ldots,N-1\}^d$ with tilted boundary conditions on the sides $|i_1| = N$

$$\forall (i_2,\ldots,i_d), \qquad \varphi_{N,i_2,\ldots,i_d} = N, \quad \text{and} \quad \varphi_{-N,i_2,\ldots,i_d} = -N.$$

and free (or periodic) boundary conditions on the other sides.

To give core to the previous heuristic, the most natural effective model should have been the SOS model ($\nu(x) = |x|$). However, only few results have been obtained about the fluctuations of this model, because the singularity of the interaction does not allow to use the techniques based on strict convexity of the potential. In the case of 0 boundary conditions, Bricmont, Fontaine, Lebowitz [BFL] analyzed the fluctuations by means of infrared bounds, but the latter estimates rely on a transfer matrix method which does not seem to be suitable for our choice of boundary conditions. Therefore, we will consider an alternative model which has the same features but with a quadratic potential at infinity.

Let ν be

$$\nu(x) = \begin{cases} |x|, & \text{if } x \in [-2,2], \\ x^2/2, & \text{if } x \in [-2,2]^c. \end{cases} \tag{3}$$

The ground state of this model is described by the limit of (2) as $\beta \to \infty$, i.e. by the uniform measure on all configurations of the field compatible with the boundary conditions, and satisfying

- $\varphi_i = \varphi_j$ when $i_1 = j_1$;
- $\varphi_i - 2 \leqslant \varphi_j \leqslant \varphi_i$, when $i_1 = j_1 + 1$.

Before stating the result we observe that, by the symmetries of the model $\mu_{\beta,N}(\varphi_0) = 0$ for any $\beta \in (0,\infty)$.

Theorem 1.1. *1. For $\beta = \infty$, in any dimension greater or equal to 2,*

$$\mu_{\infty,N}({\varphi_0}^2) = \frac{N}{2} + O(\sqrt{N}).$$

2. For any $0 < \beta < \infty$, there exists $C_1 > 0$ and $C_2 = C_2(d) < \infty$ such that

$$\mu_{\beta,N}({\varphi_0}^2) \leqslant C_1 \log N\,,$$

in dimension 2, and

$$\mu_{\beta,N}({\varphi_0}^2) \leqslant C_2\,,$$

in higher dimensions.

Acknowlegments: We are indebted to H. Spohn for sharing many insights on the problem of interface fluctuations. We also would like to thank R. Kenyon for his comments on the manuscript. GG has been partially supported by M.U.R.S.T. (Cofin99) and by GNAFA. YV is supported by a Swiss National Science Foundation grant #8220-056599. Finally, TB and YV acknowledge the kind hospitality of Technion, where part of this work was made.

2. Zero-temperature fluctuations

We prove here part 1 in Theorem 1.1. In this case the only coordinate along which a nontrivial behavior appears is the first one and we will employ the reduced description of the model $\{\varphi_i\}_{i=-N,\dots,N}$, interpreted as a collection of random variables (under the uniform measure introduced above).

Let $\{X_i\}_{i\in\mathbb{Z}}$ be an IID sequence of variables which are uniformly distributed on $[-1,+1]$ and for $n > -N$ set $S_n = \sum_{i=-N}^{n} X_i$ and $S_{-N} = 0$. Let us denote by f_I, I a finite subset of $\{-N+1, -N+2, \dots\}$, the density of $\{S_n\}_{n\in I}$. The random vector $\{S_n\}_{n=-N+1,\dots,N-1}$ admits a regular conditional density given S_N: it is

$$f(s_{-N+1},\dots,s_{N-1}|s_N) \equiv \mathbf{1}_{\{f_N(s_N)\neq 0\}} \frac{f_{-N+1,\dots,N}(s_{-N+1},\dots s_N)}{f_N(s_N)}. \tag{4}$$

We now observe that $f(s_{-N+1},\dots,s_{N-1}|0)$ is the density of the random vector $\{\varphi_i - i\}_{i=-N+1,\dots,N-1}$. By the Markov property and by the symmetry of the X_i's we obtain that the density of φ_0 is equal to $[f_0(\cdot)]^2/f_N(0)$. We now apply a Local Limit Theorem for densities ([P], page 214) to obtain that $\sqrt{N} f_N(0) = (1/2\sqrt{\pi}) + O(1/\sqrt{N})$ and that there exists c such that

$$\left| \sqrt{N} f_0(x\sqrt{N}) - g(x) \right| \leq \frac{c}{\sqrt{N}(1+x^2)}, \tag{5}$$

where g is the density of the standard Gaussian. This in particular implies that the variance of φ_0 is $N/2 + O(\sqrt{N})$.

3. Upper Bound at finite β

We adapt the decimation procedure introduced in [BLL] in the case of the interface orthogonal to the vector $(1, 1, 0, \dots, 0)$. The key step is contained in the following Theorem based on Brascamp-Lieb inequality [BL]

Theorem 3.1. [BLL] *Let $\nu : \mathbb{R} \to \mathbb{R}^+$ be a convex function such that there are positive constants A, B, C, D, E*

1. $0 < A \leqslant \nu''(x) \leqslant B < \infty, \qquad \forall |x| \geq E.$
2. $|\nu(x) - Cx^2| \leqslant D < \infty, \qquad \forall x.$

Then the function g defined by

$$g(y_1, \dots, y_k) = -\ln \left(\int_{\mathbb{R}} \exp \left(- \sum_{i=1}^{k} \nu(y_i - z) \right) dz \right) \tag{6}$$

is of the form

$$g(y_1, \dots, y_k) = a \sum_{i,j} (y_i - y_j)^2 + h(y_1, \dots, y_k), \tag{7}$$

where a is a positive constant and the function h is convex. Notice that the quadratic interaction involves every pair (i, j).

The potential ν in (3) has been designed to satisfy the previous assumptions. Following [BLL], we partition the domain Λ_N into 2 sets $\Lambda_N^{\text{even}}, \Lambda_N^{\text{odd}}$ containing the even and the odd sites. A site $i = (i_1, \dots, i_d)$ is said to be odd (resp. even) if $\sum_{k=1}^{d} i_k$ is odd (resp. even). We will also denote by $Y_{\Lambda_N^{\text{even}}} = \{y_i\}_{\Lambda_N^{\text{even}}}$ the heights Φ_{Λ_N} restricted to the even lattice.

A straight application of Theorem 3.1 implies that the measure restricted to even sites is a Gibbs measure with Hamiltonian

$$H_{\Lambda_N}^{\text{even}}(Y_{\Lambda_N^{\text{even}}}) = a \sum_{i \sim j} (y_i - y_j)^2 + F(Y_{\Lambda_N^{\text{even}}}),$$

where the constant a is positive and F is a convex function. The interactions concern only the nearest neighbors on the sublattice Λ_N^{even}, i.e. those which are $\star$-connected on $\mathbb{Z}^d$. One can choose F such that the boundary conditions remain unchanged: just include in F the interactions between points of the even lattice and the boundary. We now use the Brascamp–Lieb inequality to dominate the variance $\mu_{\beta,N}({\varphi_0}^2)$ with the variance at the origin of the Gaussian model on even sites with Hamiltonian $H_{\Lambda_N}^{\text{even}}(Y_{\Lambda_N^{\text{even}}}) = a \sum_{i \sim j} (y_i - y_j)^2$ and boundary conditions which are $y_{\pm N, i_2, \dots, i_d} = 0$ (when $(\pm N, i_2, \dots, i_d)$ is even) and free (or periodic) on the other sides. The variance of this Gaussian field has of course the stated behavior.

References

[vB] H. van Beijeren, *Interface sharpness in the Ising system*, Commun. Math. Phys. 40, 1–6 (1975).

[BL] H. Brascamp, E. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis 22 , no. 4, 366–389 (1976).

[BLL] H. Brascamp, E. Lieb, J. Lebowitz, *The statistical mechanics of anharmonic lattices*, Proceedings of the 40th Session of the International Statistical Institute (Warsaw, 1975), Vol. 1. Invited papers. Bull. Inst. Internat. Statist. 46 (1975), no. 1, 393–404 (1976).

[BFL] J. Bricmont, J.-R. Fontaine, J. Lebowitz, *Surface tension, percolation, and roughening*, J. Statist. Phys. 29, no. 2, 193–203 (1982).

[BS] J. Bricmont, J. Slawny, *Phase transitions in systems with a finite number of dominant ground states*, J. Statist. Phys. 54, no. 1, 89–161 (1989).

[D] R.L. Dobrushin, *Gibbs states describing coexistence of phases for a three-dimensional Ising model*, Th. Prob. Appl. 17, 582–600 (1972).

[Ke1] R. Kenyon, *Local statistics of lattice dimers*, Ann. Inst. H. Poincaré Probab. Statist, 33 , no. 5, 591–618 (1997).

[Ke2] R. Kenyon, *Dominos and the Gaussian free field*, preprint (2000).

[MM] A. Messager, S. Miracle-Sole, *Correlation functions and boundary conditions in the Ising ferromagnet*, J. Statist. Phys. 17, no. 4, 245–262 (1977).

[P] V.V. Petrov, *Sums of independent random variables*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82. Springer-Verlag (1975).

[PS] M. Prähofer, H. Spohn, *An exactly solved model of three-dimensional surface growth in the anisotropic KPZ regime*, J. Statist. Phys. 88, no. 5-6, 999–1012 (1997).

[S1] H. Spohn, *Dyson's model of interacting Brownian motions at arbitrary coupling strength*, Markov Process. Related Fields 4, no. 4, 649–661 (1998).

[S2] H. Spohn, *Bosonization, vicinal surfaces, and hydrodynamic fluctuation theory*, Phys. Rev. E, (3) 60, no. 6, part A, 6411–6420 (1999).

Université Paris 7, Départment de Mathématiques, Case 7012, 2 Place Jussieu, F-75251 Paris, France

E-mail address: `Thierry.Bodineau@gauss.math.jussieu.fr`

Dipartimento di Matematica, Università degli studi di Milano, (I) and Université Paris 7, Départment de Mathématiques, Case 7012, 2 Place Jussieu, F-75251 Paris, France

E-mail address: `giacomin@math.jussieu.fr`

Faculty of Industrial Engineering, Technion, Haifa 32000, Israel

E-mail address: `velenik@tx.technion.ac.il`